DUALITY AND MULTIPLIERS FOR MIXED NORM SPACES

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Introduction. If $p>0$, $q>0$, $\alpha>-1$, a function f, holomorphic in the unit disk U, is said to belong to the space $A^{p,q,\alpha}$ if

$$\|f\|_{p,q,\alpha}^p=\int_0^1 (1-r)^\alpha M_q(r,f)^p\,dr<\infty, \quad \text{where}$$

$$M_q(r,f)^q=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q\,d\theta,$$

with the usual understanding if p or $q=\infty$.

In the first section we deal with the dual space of $A^{p,q,\alpha}$. We list some known results and calculate the dual in two cases that, up to now, had been left unsettled.

In the second section we use the duality results of the first section along with a method of J. Anderson and A. Shields, [3], to calculate the coefficient multipliers of some of the spaces $A^{p,q,\alpha}$. It seems that the multiplier theory for $A^{p,q,\alpha}$ is similar to that for H^q. We shall see that if $2\leqslant q\leqslant\infty$ then the multipliers for $A^{p,q,\alpha}$ are the same as for $A^{p,2,\alpha}$. We can find the multipliers for $A^{p,1,\alpha}$ but not for $A^{p,q,\alpha}$, $1<q<2$.

In [2], J. Anderson has calculated the multipliers for Lipschitz spaces Λ_α^p, $2\leqslant p\leqslant\infty$, and asks about similar results for $1\leqslant p<2$. We are able to calculate multipliers for Λ_α^1.

In the third section we introduce some special mixed norm spaces $D^{p,q}$. We show that when $q=2$ these are exactly the spaces D^p introduced by F. Holland and B. Twomey [11]. They showed, using the Hardy–Stein identity, that for $p\leqslant 2$, $H^p\subset D^p$, and for $p\geqslant 2$, $D^p\subset H^p$. Using only a classical result of Hardy and Littlewood we generalize this to show that $H^p\subset D^{p,q}$ for $p<q$ and $D^{p,q}\subset H^p$ for $q<p$. We also give some results on fractional integrals and derivatives for functions in the spaces $D^{p,q}$, as well as a result on multiplication by bounded functions.

1. Duality. If X and Y are spaces of functions holomorphic in U, the statement "$X^*=Y$" means that for every continuous linear form, φ, on X there is a unique $g(z)=\sum_{k=0}^\infty g_k z^k\in Y$ such that if $f(z)=\sum_{k=0}^\infty f_k z^k\in X$ then

$$\varphi(f)=\lim_{r\to 1}\sum_{k=0}^\infty f_k\bar g_k r^k,$$

and conversely if f,g are given as above then $\lim_{r\to 1}\sum_{k=0}^\infty f_k\bar g_k r^k$ exists and defines a bounded linear form on X.

If $g(z)=\sum_{k=0}^\infty g_k z^k$ is holomorphic in U, and α is any real number then we define $(D^\alpha g)(z)=\sum_{k=0}^\infty (k+1)^\alpha g_k z^k$.

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We note that for $\alpha > 0$, $0 \leqslant \rho < 1$, we have

$$(1.1) \qquad \sum_{k=0}^{\infty} f_k \bar{g}_k \rho^k = \frac{2^\alpha}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{r}\right)^{\alpha-1} \left(\int_0^{2\pi} f(re^{i\theta}) \overline{(D^\alpha g)(r\rho e^{i\theta})} \frac{d\theta}{2\pi}\right) r\, dr.$$

Here we are following Flett's treatment of fractional integrals and derivatives, [7].

We now state some duality results.

1. If $1 < p < \infty$, $1 < q < \infty$, then $(A^{p,q,\alpha})^* = \{g : D^{\alpha+1} g \in A^{p',q',\alpha}\}$ where

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1.$$

This can be found in M. Taibleson [17]. See also J. Shapiro [16].

2. If $p = 1$, $1 \leqslant q \leqslant \infty$, then $(A^{1,q,\alpha})^* = \{g : M_{q'}(r, D^{\alpha+2} g) = O((1-r)^{-1})\}$. For $q = 1$, this is the result of P. Duren, B. Romberg and A. Shields [6]. For $q > 1$ it was proved by T. Flett [7].

3. If $0 < p < 1$, $1 \leqslant q$, then $(A^{p,q,\alpha})^* = (A^{1,q,(\alpha+1)/p-1})^*$. This was proved by J. Shapiro [16], when $q = 1$, and by Flett [7] in the general case.

4. The dual of $A^{p,q,\alpha}$ when $0 < p \leqslant 1$, $0 < q \leqslant 1$ has been calculated by J. Shapiro [16].

The cases $q = 1$, $1 < p < \infty$, and $q = \infty$, $1 < p < \infty$, will be treated below. The proof of the next lemma follows closely to one of the proofs of Hilbert's inequality, [9].

LEMMA 1. *If* $1 < p < \infty$ *there is a constant* $c_p > 0$ *such that if* $f \geqslant 0$ *and* $\int_0^1 f(r)^p\, dr < \infty$ *we have*

$$\int_0^1 \left(\int_0^1 f(\rho)(1 - r\rho)^{-1}\, d\rho\right)^p dr \leqslant c_p \int_0^1 f(r)^p\, dr.$$

Proof. It is sufficient to show that if $g \geqslant 0$, $\int_0^1 g(r)^{p'}\, dr < \infty$, then

$$\int_0^1 \int_0^1 g(r) f(\rho)(1 - r\rho)^{-1}\, d\rho\, dr < \infty.$$

$$\int_0^1 \int_0^1 g(r) f(\rho)(1 - r\rho)^{-1}\, d\rho\, dr$$

$$= \int_0^1 \int_0^1 g(r) \left(\frac{1-r}{1-\rho}\right)^{1/pp'} f(\rho) \left(\frac{1-\rho}{1-r}\right)^{1/pp'} (1 - r\rho)^{-1}\, d\rho\, dr.$$

By Holder's inequality this last integral is at most

$$\left[\int_0^1 \int_0^1 g(r)^{p'} \left(\frac{1-r}{1-\rho}\right)^{1/p} (1 - r\rho)^{-1}\, d\rho\, dr\right]^{1/p'}$$

$$\times \left[\int_0^1 \int_0^1 f(\rho)^p \left(\frac{1-\rho}{1-r}\right)^{1/p'} (1 - r\rho)^{-1}\, d\rho\, dr\right]^{1/p}.$$

The first factor is

$$\int_0^1 g(r)^{p'}(1-r)^{1/p}\left(\int_0^1 (1-\rho)^{-1/p}(1-r\rho)^{-1}\,d\rho\right)dr$$

and the second factor is

$$\int_0^1 f(\rho)^p(1-\rho)^{1/p'}\left(\int_0^1 (1-r)^{-1/p'}(1-r\rho)^{-1}\,dr\right)d\rho.$$

The result now follows from the inequality

$$\int_0^1 (1-x)^{-\alpha}(1-tx)^{-1}\,dx \leqslant c_\alpha(1-t)^{-\alpha},$$

valid for $0<\alpha<1$. □

THEOREM 1. *If* $1<p<\infty$, $(A^{p,1,\alpha})^*=\{g: D^{\alpha+1}g\in A^{p',\infty,\alpha}\}$.

Proof. First we assume $\alpha=0$. Take g such that $D^1g\in A^{p',\infty,0}$ and let f be a polynomial, then by (1.1) we have

$$\sum f_k\bar{g}_k = 2\int_0^1\int_0^{2\pi} f(re^{i\theta})\overline{(D^1g)}(re^{i\theta})\,\frac{r}{2\pi}\,dr\,d\theta.$$

So

$$\begin{aligned}\left|\sum f_k\bar{g}_k\right| &\leqslant 2\int_0^1 M_1(r,f)M_\infty(r,D^1g)\,dr\\ &\leqslant 2\left[\int_0^1 M_1(r,f)^p\,dr\right]^{1/p}\left[\int_0^1 M_\infty(r,D^1g)^{p'}\,dr\right]^{1/p'}\\ &= 2\|f\|_{p,1,0}\|D^1g\|_{p',\infty,0}.\end{aligned}$$

Since the polynomials are dense in $A^{p,1,0}$, the mapping $\varphi(f)=\sum f_k\bar{g}_k$ extends to be a bounded linear form on $A^{p,1,0}$. If f is holomorphic in a neighborhood of the closed unit disc the partial sums of the Taylor's series for f converge to f in $A^{p,1,0}$ and hence $\varphi(f)=\sum f_k\bar{g}_k$ in this case. In general, if $f\in A^{p,1,0}$ we know that $f_r\to f$ in $A^{p,1,0}$ where $f_r(z)=f(rz)$, it follows that

$$\varphi(f)=\lim_{r\to 1}\sum_{k=0}^{\infty} f_k\bar{g}_k r^k.$$

Now, take $\varphi\in(A^{p,1,0})^*$. By the Hahn–Banach theorem φ extends to be a bounded linear form Φ on the space $L^{p,1}$ of all measurable complex valued functions F defined on U such that $\int_0^1 M_1(r,F)^p\,dr<\infty$. By a theorem of A. Benedek and R. Panzone [4] there is a complex valued function G defined on U such that $\int_0^1 M_\infty(r,G)^{p'}\,dr<\infty$ and

$$\Phi(F)=\int_0^1\int_0^{2\pi} F(re^{i\theta})\overline{G(re^{i\theta})}\,\frac{r}{2\pi}\,d\theta\,dr \quad\text{for all}\quad F\in L^{p,1}.$$

In particular

$$\varphi(f)=\int_0^1\int_0^{2\pi} f(re^{i\theta})\overline{G(re^{i\theta})}\,\frac{r}{2\pi}\,d\theta\,dr \quad\text{for}\quad f\in A^{p,1,0}.$$

Let

$$H(z)=\frac{1}{\pi}\int_0^1\int_0^{2\pi}G(re^{i\theta})(1-zre^{-i\theta})^{-2}r\,dr\,d\theta$$

be the Bergman projection of G. If f is a holomorphic polynomial, then

$$\begin{aligned}\varphi(f)&=\int_0^1\int_0^{2\pi}f(re^{i\theta})\overline{G(re^{i\theta})}\,\frac{r}{2\pi}\,dr\,d\theta\\&=\int_0^1\int_0^{2\pi}f(re^{i\theta})\overline{H(re^{i\theta})}\,\frac{r}{2\pi}\,dr\,d\theta\\&=\int_0^1\int_0^{2\pi}f(re^{i\theta})\overline{D^1g(re^{i\theta})}\,\frac{r}{2\pi}\,dr\,d\theta,\end{aligned}$$

where g is defined to be $D^{-1}H$. The proof will be complete if we can show that $\int_0^1 M_\infty(r,H)^{p'}\,dr<\infty$. Now

$$H(re^{i\theta})=\frac{1}{\pi}\int_0^1\int_0^{2\pi}G(\rho e^{i\varphi})(1-r\rho e^{i(\theta-\varphi)})^{-2}\rho\,d\rho\,d\varphi$$

and hence

$$\begin{aligned}M_\infty(r,H)&\leqslant\frac{1}{\pi}\int_0^1 M_\infty(\rho,G)\left(\int_0^{2\pi}\frac{d\theta}{|1-r\rho e^{i\theta}|^2}\right)d\rho\\&\leqslant C\int_0^1 M_\infty(\rho,G)(1-r\rho)^{-1}\,d\rho,\end{aligned}$$

where we have used the well-known inequality

$$\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^\alpha}\leqslant C_\alpha(1-r)^{1-\alpha},\quad \alpha>1.$$

Now apply Lemma 1 to finish the proof of the case $\alpha=0$.

By the theorem of Hardy and Littlewood on fractional integrals [10] we know that $\int_0^1(1-r)^\alpha M_1(r,f)^p\,dr<\infty$ if and only if $\int_0^1 M_1(r,D^{-\alpha/p}f)^p<\infty$. This fact combined with the case $\alpha=0$ gives the general case of Theorem 1, if we use the observation that

$$\sum_{k=0}^\infty f_k\bar{g}_k r^k=\sum_{k=0}^\infty (D^{-\alpha}f)_k\overline{(D^\alpha g)_k}r^k.\qquad\square$$

THEOREM 2. *If* $1<p<\infty$, $(A^{p,\infty,\alpha})^*=\{g: D^{\alpha+1}g\in A^{p',1,\alpha}\}$.

Proof. The proof that $\{g: D^{\alpha+1}g\in A^{p',1,\alpha}\}\subseteq(A^{p,\infty,\alpha})^*$ goes just like the first part of the proof of Theorem 1. In the other direction take $\varphi\in(A^{p,\infty,\alpha})^*$. If we define $\bar{g}_k=\varphi(z^k)$ it is easy to see that $g\in H(U)$ and

$$\varphi(f)=\sum_{k=0}^\infty f_k\bar{g}_k=\lim_{r\to1}\sum_{k=0}^\infty f_k\bar{g}_k r^k$$

for any function f, holomorphic in a neighborhood of the closed disc $\bar{U}$. Fix r, $0<r<1$, then

$$\begin{aligned}\|D^{\alpha+1}g_r\|_{A^{p',1,\alpha}} &= \sup\{|\psi(D^{\alpha+1}g_r)|: \psi\in (A^{p',1,\alpha})^*,\ \|\psi\|\leqslant 1\}\\ &\leqslant C\sup\left\{\left|\lim_{\rho\to 1}\sum_{k=0}^{\infty}(D^{\alpha+1}g_r)_k\bar{f}_k\rho^k\right| : \|D^{\alpha+1}f\|_{A^{p,\infty,\alpha}}\leqslant 1\right\}\\ &= C\sup\left\{\left|\lim_{\rho\to 1}\sum_{k=0}^{\infty}g_k\overline{(D^{\alpha+1}f_r)_k}\rho^k\right| : \|D^{\alpha+1}f\|_{A^{p,\infty,\alpha}}\leqslant 1\right\}\\ &= C\sup\{|\varphi(D^{\alpha+1}f_r)|: \|D^{\alpha+1}f\|_{A^{p,\infty,\alpha}}\leqslant 1\}\\ &\leqslant C\|\varphi\|.\end{aligned}$$

It follows that $D^{\alpha+1}g\in A^{p',1,\alpha}$. So, we have shown that there exists g, $D^{\alpha+1}g\in A^{p',1,\alpha}$, such that $\varphi(f)=\lim_{r\to 1}\sum_{k=0}^{\infty}f_k\bar{g}_kr^k$, for all f, holomorphic in a neighborhood of the closed unit disc. The theorem will follow if we show such functions are dense in $A^{p,\infty,\alpha}$. We show that if $f\in A^{p,\infty,\alpha}$ then $f_r\to f$ in $A^{p,\infty,\alpha}$.

$$\begin{aligned}\|f-f_r\|_{p,\infty,\alpha}^p &= \int_0^1(1-\rho)^\alpha M_\infty(\rho,f-f_r)^p\,d\rho\\ &= \int_0^\lambda(1-\rho)^\alpha M_\infty(\rho,f-f_r)^p\,d\rho+\int_\lambda^1(1-\rho)^\alpha M_\infty(\rho,f-f_r)^p\,dr\\ &\leqslant \int_0^\lambda(1-\rho)^\alpha M_\infty(\rho,f-f_r)^p\,d\rho+2^p\int_\lambda^1(1-\rho)^\alpha M_\infty(\rho,f)^p\,d\rho.\end{aligned}$$

First we choose λ so that the second term is as small as we please, independent of r, then the first term goes to zero as $r\to 1$. □

2. Multipliers. We use an approach for finding coefficient multipliers due to J. Anderson and A. Shields [3]. If A and B are sequence spaces (A,B) denotes the multipliers from A to B. If A is a sequence space A^a is defined to be the set of sequences $\{\lambda_n\}$ such that $\lim_{r\to 1}\sum_{n=0}^{\infty}\lambda_na_nr^n$ exists for all $\{a_n\}\in A$, and $s(A)$ is defined to be (l^∞,A). It is shown that $s(A)\subseteq A$ and if $\{a_n\}\in s(A)$ and $|b_n|\leqslant|a_n|$ then $\{b_n\}\in s(A)$, and $s(A)$ is the largest subspace of A with those properties. A special case of one of their results is that $(A,l^s)=(s(A^a)^a,l^s)$, $1\leqslant s\leqslant\infty$.

We will need some other sequence spaces, namely

$$l(p,q)=\left\{a=\{a_n\}: \|a\|_{p,q}^q=\sum_{n=0}^{\infty}\left(\sum_{k\in I_n}|a_k|^p\right)^{q/p}<\infty\right\},$$

$$1\leqslant p<\infty,\qquad 1\leqslant q<\infty,$$

where $I_n=\{k: k \text{ is an integer, } 2^{n-1}\leqslant k<2^n\}$, $n=1,2,\dots,I_0=\{0\}$. In the case where p or q is infinite, replace the corresponding sum by a supremum.

We will also use the following result of M. Mateljević and M. Pavlović [15].

THEOREM A. *If* $p, \alpha > 0$ *there are positive constants* $A_{p,\alpha}$ *and* $B_{p,\alpha}$ *such that if* $a_k \geqslant 0,\ k = 0, 1, 2, \ldots,$

$$A_{p,\alpha} \sum_{n=0}^{\infty} 2^{-n\alpha}\Big(\sum_{k\in I_n} a_k\Big)^p \leqslant \int_0^1 (1-r)^{\alpha-1}\Big(\sum_{k=0}^{\infty} a_k r^k\Big)^p dr$$
$$\leqslant B_{p,\alpha} \sum_{n=0}^{\infty} 2^{-n\alpha}\Big(\sum_{k\in I_n} a_k\Big)^p.$$

THEOREM 3. *If* $p > 1$, $(A^{p,1,\alpha}, l^s) = \{\{\lambda_k\} : \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in l(s,t)\}$ *where* $t = \infty$ *if* $s \geqslant p$ *and* $1/t = 1/s - 1/p$ *if* $s < p$.

Proof. By Theorem 1, $(A^{p,1,\alpha})^a = \{g : D^{\alpha+1}g \in A^{p',\infty,\alpha}\}$. Now

$$\int_0^1 (1-r)^\alpha M_\infty(r, D^{\alpha+1}g)^{p'} dr \leqslant \int_0^1 (1-r)^\alpha \Big(\sum_{k=0}^{\infty} (k+1)^{\alpha+1}|g_k| r^k\Big)^{p'} dr$$
$$\leqslant C \sum_{n=0}^{\infty} 2^{-n(\alpha+1)}\Big(\sum_{k\in I_n} (k+1)^{\alpha+1}|g_k|\Big)^{p'}$$

by Theorem A. This last expression is bounded by constant times

$$\sum_{n=0}^{\infty}\Big(\sum_{k\in I_n} (k+1)^{(\alpha+1)/p}|g_k|\Big)^{p'}.$$

In other words $\{g : \{(k+1)^{(\alpha+1)/p}g_k\} \in l(1,p')\} \subseteq s((A^{p,1,\alpha})^a)$. On the other hand if $g \in s((A^{p,1,\alpha})^a)$ then so is $G(z) = \sum_{k=0}^{\infty} |g_k| z^k$. But $M_\infty(r, D^{\alpha+1}G) = \sum_{k=0}^{\infty} (k+1)^{\alpha+1}|g_k| r^k$, and it follows from Theorem A that $\{(k+1)^{(\alpha+1)/p}g_k\} \in l(1,p')$. In other words we have shown that

$$s((A^{p,1,\alpha})^a) = \{g : \{(k+1)^{(\alpha+1)/p}g_k\} \in l(1,p')\}.$$

Now it is easy to see that $l(1,p')^a = l(\infty,p)$ and from this it follows that

$$(s(A^{p,1,\alpha})^a)^a = \{\{\lambda_k\} : \{(k+1)^{-(\alpha+1)/p}\lambda_k\} \in l(\infty,p)\}.$$

We conclude that

$$\begin{aligned}(A^{p,1,\alpha}, l^s) &= ((s(A^{p,1,\alpha})^a)^a, l^s) \\ &= \{\{\lambda_k\} : \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in (l(\infty,p), l^s)\} \\ &= \{\{\lambda_k\} : \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in (l(\infty,p), l(s,s))\} \\ &= \{\{\lambda_k\} : \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in (l(s,t)\}\end{aligned}$$

where $t = \infty$ if $p \leqslant s$ and $1/t = 1/s - 1/p$, if $s < p$. Here we have used the result of C. Kellog, [13] to calculate $(l(\infty,p), l(s,s))$.

To calculate the multipliers of $A^{p,q,\alpha}$, $2 \leqslant q \leqslant \infty$, we need to find $s(A^{p,q,\alpha})$, $1 \leqslant q \leqslant 2$.

LEMMA 2. *If* $1 \leqslant p < \infty$ *and* $1 \leqslant q \leqslant 2$, *then* $s(A^{p,q,\alpha}) = A^{p,2,\alpha}$.

Proof. It is clear that $A^{p,2,\alpha} \subset s(A^{p,q,\alpha}) \subset s(A^{p,1,\alpha})$ for $1 \leqslant q \leqslant 2$. We use the Rademacher functions $\{r_k\}$ in a standard way, as in [3]. If $f \in s(A^{p,1,\alpha})$, then so is $f_t(z) = \sum_{k=0}^{\infty} r_k(t) f_k z^k$, moreover $\|f_t\|_{p,1,\alpha}^p \leqslant c_{p,\alpha}$, for every $t \in [0,1]$. By Khintchine's inequality we have

$$\left(\sum_{k=0}^{\infty} |f_k|^2 r^{2k}\right)^{1/2} \leqslant c\int_0^1 |f_t(re^{i\theta})|\,dt \quad \text{and hence} \quad M_2(r,f) \leqslant c\int_0^1 M_1(r,f_t)\,dt.$$

Since $p \geqslant 1$ we may use Jensen's inequality to conclude that

$$\begin{aligned}\infty > c_{p,\alpha} &\geqslant \int_0^1 \left(\int_0^1 (1-r)^\alpha \left(\frac{1}{2\pi}\int_0^{2\pi} |f_t(re^{i\theta})|\,d\theta\right)^p dr\right) dt\\ &= \int_0^1 (1-r)^\alpha \left(\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} |f_t(re^{i\theta})|\,d\theta\right)^p dt\right) dr\\ &\geqslant \int_0^1 (1-r)^\alpha \left(\int_0^1 M_1(r,f_t)\,dt\right)^p dr\\ &\geqslant c^{-p}\int_0^1 (1-r)^\alpha M_2^p(r,f)\,dr.\end{aligned}$$

This shows that $s(A^{p,1,\alpha}) = A^{p,2,\alpha}$ and completes the proof. □

THEOREM 4. *If* $p > 1$, $2 \leqslant q \leqslant \infty$,

$$(A^{p,q,\alpha}, l^s) = \{\{\lambda_k\} : \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in l(u,v)\}$$

where

$$\frac{1}{u} = \frac{1}{s} - \frac{1}{2} \text{ if } s<2 \quad \text{and} \quad u=\infty \text{ if } 2 \leqslant s$$

and

$$\frac{1}{v} = \frac{1}{s} - \frac{1}{p} \text{ if } s<p \quad \text{and} \quad v=\infty \text{ if } p \leqslant s.$$

Proof. $(A^{p,q,\alpha})^a = \{g : D^{\alpha+1}g \in A^{p',q',\alpha}\}$ and hence

$$s((A^{p,q,\alpha})^a) = \{g : D^{\alpha+1}g \in A^{p',2,\alpha}\}.$$

If $D^{\alpha+1}g \in D^{p',2,\alpha}$ this just means

$$\int_0^1 (1-r)^\alpha \left(\sum_{k=0}^{\infty} (k+1)^{2(\alpha+1)} |g_k|^2 r^{2k}\right)^{p'/2} dr < \infty,$$

which is equivalent to

$$\sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{k\in I_n} (k+1)^{2(\alpha+1)} |g_k|^2\right)^{p'/2} < \infty.$$

This, in turn, is equivalent to

$$\sum_{n=0}^{\infty} \left(\sum_{k\in I_n} (k+1)^{2(\alpha+1)/p} |g_k|^2\right)^{p'/2} < \infty,$$

i.e. $\{(k+1)^{(\alpha+1)/p} g_k\} \in l(2,p')$. So we see that

$$(s(A^{p,q,\alpha})^a)^a = \{\{\lambda_k\}: \{(k+1)^{-(\alpha+1)/p}\lambda_k\} \in l(2,p)\}$$

and hence that

$$(A^{p,q,\alpha}, l^s) = \{\{\lambda_k\}: \{(k+1)^{(\alpha+1)/p}\lambda_k\} \in (l(2,p), l(s,s)) = l(u,v)\}$$

where $1/u = 1/s - 1/2$ if $s < 2$ and $u = \infty$ if $2 \leqslant s$ and $1/v = 1/s - 1/p$ if $s < p$ and $v = \infty$ if $p \leqslant s$. □

REMARK 1. Case $0 < p \leqslant 1$ in Theorem 3 is covered by known theorems on multipliers (B^q, l^s), where $B^q = A^{1,1,(1/q)-2}$, $0 < q < 1$. For example $(A^{1,1,\alpha}, l^s) = \{\{\lambda_n\}: \{(k+1)^{\alpha+1}\lambda_n\} \in l(s,\infty)\}$. If $0 < p < 1$, use Duality 4.

REMARK 2. Case $0 < p \leqslant 1$ in Theorem 4 can be treated in a similar way. Use Duality 3 instead of Theorem 1 and Theorem 2 and the fact that $s(\Lambda_\alpha^q) = \Lambda_\alpha^2$, if $1 \leqslant q \leqslant 2$. (Definition of Λ_α^q below.)

In [2], J. Anderson considers coefficient multipliers for the spaces

$$\Lambda_\alpha^p = \{g : g \in H(U),\ M_p(r, g') = O((1-r)^{\alpha-1})\},$$

$1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$. He is able to calculate (Λ_α^p, l^s) for $2 \leqslant p \leqslant \infty$, $0 < \alpha < 1$, $1 \leqslant s \leqslant \infty$, and asks about similar results for $1 \leqslant p < 2$.

Using the same methods as above we can calculate (Λ_α^1, l^s). Indeed we can do somewhat more. We start by noting that by a result of Hardy and Littlewood, if $0 < \alpha < 1$, the function g belongs to Λ_α^p if and only if $M_p(r, D^{\alpha+1}g) = O((1-r)^{-1})$.

With this in mind we may define Λ_α^p for all $\alpha > 0$ by this condition. As we noted above, T. Flett [7] has shown that, for $\alpha > 0$, $(A^{1,\infty,\alpha-1})^* = \Lambda_\alpha^1$ and, as he points out, it easily follows that $(\Lambda_\alpha^1)^a = A^{1,\infty,\alpha-1}$. As we showed in the proof of Theorem 3, $s(A^{1,\infty,\alpha-1}) = \{g : \{(k+1)^{-\alpha}g_k\} \in l(1,1)\}$. It follows from this that $(s(A^{1,\infty,\alpha-1}))^a = \{\{\lambda_k\}: \{(k+1)^{\alpha}\lambda_k\} \in l^\infty\}$. If we combine these observations with the fact that $(l^\infty, l^s) = l^s$ we have established the following.

THEOREM 5. *For* $\alpha > 0$, $1 \leqslant s \leqslant \infty$, $(\Lambda_\alpha^1, l^s) = \{\{\lambda_k\}: \{(k+1)^{-\alpha}\lambda_k\} \in l^s\}$.

3. $D^{p,q}$ spaces. In [11], F. Holland and J. Twomey introduced the spaces D^p as follows: a function f, holomorphic in the unit disc U is said to belong to D^p, $0 < p < \infty$, if

$$\|f\|_{D^p}^p = \int_0^1 \left(\int_0^r \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \frac{\rho}{\pi}\, d\rho\, d\theta \right)^{p/2} dr < \infty.$$

Of course

$$\|f\|_{D^p}^p = \int_0^1 \left(\sum_{k=1}^{\infty} k|f_k|^2 r^{2k} \right)^{p/2} dr.$$

Note that $\sum_{k=1}^\infty (k+1)|f_k|^2 r^{2k} = M_2^2(r, D^{1/2}f)$, if $f(0) = 0$, and hence that $f \in D^p$ if and only if $\int_0^1 M_2(r, D^{1/2}f)^p\, dr < \infty$. Now by the theorem of G. Hardy and J. Littlewood, [10], on fractional derivatives (or its extension due to Flett, [8], if $0 < p < 1$) we see that, for $0 < p \leqslant 2$, $f \in D^p$ if and only if

$$\int_0^1 (1-r)^{-p/2} M_2(r, f)^p\, dr < \infty,$$

and, in general, $f \in D^p$ if and only if $\int_0^1 (1-r)^{p/2} M_2(r,f')^p\,dr < \infty$, $0<p<\infty$.

Now, the inclusion $H^p \subseteq D^p$ for $0<p<2$ is just the classical inequality of Hardy and Littlewood (Theorem 5.11 in [5]). Indeed if $0<p<\infty$, $1 \leqslant q < \infty$, we are led to define $D^{p,q} = \{f\colon \int_0^1 (1-r)^{-(p/2)+p} M_q(r,f')^p\,dr < \infty\}$. By the theorem on the fractional derivative mentioned earlier, if $0<p<q$,

$$D^{p,q} = \left\{f\colon \int_0^1 (1-r)^{-p/q} M_q(r,f)^p\,dr < \infty\right\}.$$

THEOREM 6. (i) *If* $p<q$, $H^p \subseteq D^{p,q}$.

(ii) *If* $q<p$, $D^{p,q} \subset H^p$.

(iii) *If* $q<2$, $D^{q,q} \subset H^q$, *and* $H^q \subset D^{q,q}$ *if* $2 \leqslant q$.

Proof. (i) That $H^p \subset D^{p,q}$ for $p<q$ is just the result of Hardy and Littlewood (Theorem 5.11 in [5]) mentioned above.

(ii) If $q<p$ then the inclusion $D^{p,q} \subset H^p$ was proved by T. Flett (see [8]).

(iii) The last statement of the theorem is just a well-known result of Littlewood and Paley (see [10]). □

We point out that the duality and multiplier results of the first and second sections could be applied to the spaces $D^{p,q}$. We give some examples.

PROPOSITION 1.

(i) *If* $1<p,q<\infty$, *then* $(D^{p,q})^* = D^{p',q'}$.

(ii) *If* $0<p\leqslant 1$, *then* $(D^p)^* = \{g\colon M_2(r, D^{(1/p)+(1/2)}g) = O((1-r)^{-1})\}$.

PROPOSITION 2. *If* $0<p$, $s<\infty$, *then*

$$(D^p, l^s) = \{\{\lambda_k\}\colon \{(k+1)^{(1/p)-(1/2)}\lambda_k\} \in (l(2,p), l^s) = l(u,r)\}$$

where

$$\frac{1}{u} = \frac{1}{s} - \frac{1}{2} \quad \text{if } s<2, \quad \text{and} \quad u=\infty \quad \text{if } 2 \leqslant s,$$

and

$$\frac{1}{r} = \frac{1}{s} - \frac{1}{p} \quad \text{if } s<p, \quad \text{and} \quad r=\infty \quad \text{if } p \leqslant s.$$

COROLLARY 1 (Holland and Twomey, [11]). *If* $f \in D^p$, $0<p\leqslant 2$, *then*

$$\sum_{k=0}^{\infty} (k+1)^{p-2} |f_k|^p < \infty.$$

If $2 \leqslant p < \infty$, *and* $\sum (k+1)^{p-2}|f_k|^p < \infty$ *then* $f \in D^p$.

PROPOSITION 3. *If* $0<p,q<\infty$, *then*

$$(D^p, D^q) = \{\{\lambda_k\}\colon \{(k+1)^{(1/p)-(1/q)}\lambda_k\} \in (l(2,p), l(2,q))\}.$$

Next we show that the spaces $D^{p,q}$ behave like H^p as far as fractional integrals are concerned and are much better for fractional derivatives.

THEOREM 7. *Suppose* $\alpha > 0$ *and* $D^\alpha f \in D^{p,q}$, *then* $f \in D^{p_1,q}$ *where* $\alpha = (1/p) - (1/p_1)$. *If* $f \in D^{p_1,q}$, *then* $D^\alpha f \in D^{s,q}$ *for all* $s < p$.

Proof. We are assuming that

$$\int_0^1 (1-r)^{(-p/q)+p} M_q(r, D^{\alpha+1}f)^p \, dr < \infty.$$

Hence

$$\int_0^1 (1-r)^{(-p/q)+p} M_q(r, D^{\alpha+1}f)^p \, dr \leqslant C.$$

Since $M_q(r, D^{\alpha+1}f)$ is increasing we find that
$M_q(\rho, D^{\alpha+1}f)^p(1-\rho)^{(-p/q)+p+1} \leqslant C$, i.e. $M_q(r, D^{\alpha+1}f) \leqslant C(1-r)^{(1/q)-(1/p)-1}$.
By the theorem on the fractional derivative

$$\begin{aligned}
&\int_0^1 (1-r)^{(-p_1/q)+p_1} M_q(r, D^1 f)^{p_1} \, dr \\
&\quad \leqslant C \int_0^1 (1-r)^{(-p_1/q)+p_1+\alpha p_1} M_q(r, D^{\alpha+1}f)^{p_1} \, dr \\
&\quad \leqslant C \int_0^1 (1-r)^{(-p_1/q)+p_1+\alpha p_1}(1-r)^{([1/q]-[1/p]-1)(p_1-p)} M_q(r, D^{\alpha+1}f)^p \, dr \\
&\quad = C \int_0^1 (1-r)^{(-p/q)+p} M_q(r, D^{\alpha+1}f)^p \, dr < \infty,
\end{aligned}$$

because of the equation relating α, p, p_1.

Now suppose that $f \in D^{p_1,q}$. This means that

$$\int_0^1 (1-r)^{(-p_1/q)+p_1} M_q(r, D^1 f)^{p_1} < \infty,$$

and hence that

$$\int_0^1 (1-r)^{(-p_1/q)+p_1+\alpha p_1} M_q(r, D^{\alpha+1}f)^{p_1} < \infty.$$

As in the first part of the proof, this implies that

$$M_q(r, D^{\alpha+1}f) \leqslant C(1-r)^{(1/q)-(1/p)-1},$$

and this estimate shows that

$$\int_0^1 (1-r)^{(-s/q)+s} M_q(r, D^{\alpha+1}f)^s \, dr < \infty, \quad \text{for all } s < p. \qquad \square$$

We point out that if

$$f(z) = (1-z)^{-1/p_1}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-1/p}$$

then using estimates found in [14], pages 93–95, it can be seen that $f \in D^{p_1,q}$ but $D^\alpha f \notin D^{p,q}$.

We finish with some remarks about pointwise multipliers for $D^{p,q}$. It is clear that if $f \in D^{p,q}$, $p<q$ and $g \in H^\infty$ that $fg \in D^{p,q}$. When $p>q$ the situation is much different. Note that if $f \in D^{p,q}$ then

$$M_1(r,f') \leqslant M_q(r,f') = O((1-r)^{(1/q)-(1/p)-1})$$

and hence

$$\int_0^1 M_1(r,f')\,dr < \infty.$$

However there are bounded functions, even functions, continuous in $\bar{U}$, for which

$$\int_0^1 M_1(r,f')\,dr = \infty,$$

also there are inner functions φ, for which

$$\int_0^1 M_1(r,\varphi')\,dr = \infty$$

and hence $\varphi \notin D^{p,q}$ for $p>q$. We point out that it follows from the results of [1], that if φ is inner and has a non-constant singular factor then $\varphi \notin D^{p,q}$ for $p \geqslant 2q$. On the other hand the results in [12] show that if φ is the atomic inner function then $\varphi \in D^{p,q}$ for all $p<2q$.

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