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DUALITY AND VON NEUMANN ALGEBRAS[1]

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The aim of this paper is to announce the results of the author's lecture given in Tulane University for the Fall of 1970 under the same title. Since the Pontryagin duality theorem was shown, a series of duality theorems for nonabelian groups has been discovered, Tannaka duality theorem [**14**], Stinespring duality theorem [**10**], Eymard-Saito duality theorem [**5**], [**8**] and Tatsuuma duality theorem [**15**]. Motivated by the Stinespring duality theorem, Kac [**7**] introduced the notion of "ring-groups" in order to clarify the duality principle for *unimodular* locally compact groups. Sharpening and generalizing Kac's postulate for the "ring-group," the author [**11**] gave a characterization of the group algebra of a *general* locally compact group as an involutive abelian Hopf-von Neumann algebra with left invariant measure.

Let G be a locally compact group with left Haar measure ds. Let $\mathfrak{H}$ denote the Hilbert space $L^2(G, ds)$. Define a unitary operator W on $\mathfrak{H} \otimes \mathfrak{H}$ by $(Wf)(s, t) = f(s, st)$, $f \in \mathfrak{H} \otimes \mathfrak{H}$, $s, t \in G$. Let $\mathcal{A}(G)$ be the von Neumann algebra on $\mathfrak{H}$ consisting of all multiplication operators $\rho(f)$ by $f \in L^\infty(G)$. The algebras $\mathcal{A}(G)$ and $L^\infty(G)$ will be identified. Let $\mathfrak{M}(G)$ denote the von Neumann algebra on $\mathfrak{H}$ generated by left regular representation λ of G. The fundamental facts of all duality arguments for groups are the following: the map $\delta_G: x \mapsto W(x \otimes 1)W^*$ is an isomorphism of $\mathcal{A}(G)$ into $\mathcal{A}(G) \mathbin{\bar{\otimes}} \mathcal{A}(G)$ such that $(\delta_G \otimes i) \circ \delta_G = (i \otimes \delta_G) \circ \delta_G$; the map $\gamma_G: x \mapsto W^*(x \otimes 1)W$ is an isomorphism of $\mathfrak{M}(G)$ into $\mathfrak{M}(G) \mathbin{\bar{\otimes}} \mathfrak{M}(G)$ such that $(\gamma_G \otimes i) \circ \gamma_G = (i \otimes \gamma_G) \circ \gamma_G$ and $\sigma \circ \gamma_G = \gamma_G$ where σ denotes the automorphism of $\mathcal{B}(\mathfrak{H}) \mathbin{\bar{\otimes}} \mathcal{B}(\mathfrak{H})$ defined by $\sigma(x \otimes y) = y \otimes x$. According to these facts, the preduals $\mathcal{A}_*(G)$ and $\mathfrak{M}_*(G)$ of $\mathcal{A}(G)$ and $\mathfrak{M}(G)$ turn out to be Banach algebras by the following multiplications: $\langle x, \psi * \varphi \rangle = \langle \delta(x), \psi \otimes \varphi \rangle$, $x \in \mathcal{A}(G)$, $\psi, \varphi \in \mathcal{A}_*(G)$ and $\langle x, \varphi * \psi \rangle = \langle \gamma(x), \varphi \otimes \psi \rangle$, $x \in \mathfrak{M}(G)$, $\varphi, \psi \in \mathfrak{M}_*(G)$. The Banach algebra $\mathcal{A}_*(G)$ is nothing but the usual group algebra $L^1(G)$ and the duality theorems mention that the Banach algebra $\mathfrak{M}_*(G)$ is semisimple and the spectrum space

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of $\mathfrak{M}_*(G)$ is the given group G itself by the map: $\varphi \in \mathfrak{M}_*(G) \mapsto \langle \lambda(s)^*, \varphi \rangle$, $s \in G$.

Noticing this scheme of duality principle for groups, we develop the duality theory for more general subjects, Hopf-von Neumann algebras, based on the theory of von Neumann algebras. A *Hopf-von Neumann algebra* is a pair $\{\mathfrak{A}, \delta\}$ of a von Neumann algebra $\mathfrak{A}$ and a normal isomorphism δ of $\mathfrak{A}$ into $\mathfrak{A} \overline{\otimes} \mathfrak{A}$ such that $(\delta \otimes i) \circ \delta = (i \otimes \delta) \circ \delta$, where i denotes the identity automorphism of $\mathfrak{A}$. The isomorphism δ is called a *comultiplication* of $\mathfrak{A}$. If $\mathfrak{A}$ is abelian, then $\{\mathfrak{A}, \delta\}$ is said to be *abelian*. If $\{\mathfrak{A}, \delta\}$ is a Hopf-von Neumann algebra, then the predual $\mathfrak{A}_*$ of $\mathfrak{A}$ turns out to be a Banach algebra with the product, called the convolution, given by $\langle x, \varphi * \psi \rangle = \langle \delta(x), \varphi \otimes \psi \rangle$, $x \in \mathfrak{A}$, $\varphi, \psi \in \mathfrak{A}_*$. In the tensor product $\mathfrak{A} \overline{\otimes} \mathfrak{A}$, let σ denote the automorphism defined by $\sigma(x \otimes y) = y \otimes x$, $x, y \in \mathfrak{A}$. If $\sigma \circ \delta = \delta$, then δ is said to be *symmetric*. The resulted Banach algebra $\mathfrak{A}_*$ is abelian if and only if δ is symmetric. The *involution* of $\{\mathfrak{A}, \delta\}$ is an anti-automorphism ν of $\mathfrak{A}$, such that $\nu^2 = i$ and $(\nu \otimes \nu) \circ \delta = \sigma \circ \delta \circ \nu$. The triplet $\{\mathfrak{A}, \delta, \nu\}$ is called an involutive Hopf-von Neumann algebra. In this case, the Banach algebra $\mathfrak{A}_*$ admits the involution: $\varphi \in \mathfrak{A}_* \mapsto \varphi^{\#} \in \mathfrak{A}_*$ defined by $\langle x, \varphi^{\#} \rangle = \langle \nu(x^*), \varphi \rangle^-$, $x \in \mathfrak{A}$, $\varphi \in \mathfrak{A}_*$. In the following, we write $\nu(a) = a^{\vee}$, $\nu(a^*) = a^{\flat}$ for $a \in \mathfrak{A}$ and ${}^t\nu(\varphi) = \varphi^{\wedge}$ for $\varphi \in \mathfrak{A}_*$. If a faithful, semifinite, normal trace τ of $\mathfrak{A}$ satisfies the equation:

$$(\tau \otimes \tau)((a \otimes b)\delta(c)) = (\tau \otimes \tau)((a^{\wedge} \otimes c)\delta(b))$$

for every a, b and c in $L^1(\tau) \cap \mathfrak{A}$, then τ is called a *left invariant measure* on $\{\mathfrak{A}, \delta, \nu\}$. A right invariant measure is also defined in the similar way. If ν leaves τ invariant, then τ is said to be *unimodular*.

For a locally compact group G, $\mathfrak{A}(G)$ and $\mathfrak{M}(G)$ both admit the comultiplications δ_G and γ_G respectively. The involutions ν_G of $\{\mathfrak{A}(G), \delta_G\}$ and κ_G of $\{\mathfrak{M}(G), \gamma_G\}$ are defined respectively by $\nu_G(f)(s) = f(s^{-1})$, $f \in \mathfrak{A}(G)$, $s \in G$, and $\kappa_G(x) = Cx^*C$, $x \in \mathfrak{M}(G)$, where C denotes the conjugation $C: \xi \in \mathfrak{H} \mapsto \bar{\xi} \in \mathfrak{H}$. The trace μ_G on $\mathfrak{A}(G)$ defined by $\mu(f) = \int_G f(s)\, ds$ is indeed a left invariant measure of $\{\mathfrak{A}(G), \delta_G, \nu_G\}$. If G is unimodular, then the canonical measure φ_G defined on $\mathfrak{M}(G)$ is a unimodular measure of $\{\mathfrak{M}(G), \gamma_G, \kappa_G\}$.

Suppose now an involutive Hopf-von Neumann algebra $\{\mathfrak{A}, \delta, \nu, \tau\}$ with left invariant measure is given.

LEMMA 1. *There exists an ideal* $\mathfrak{a}$ *of* $\mathfrak{A}$ *contained in the definition ideal* $\mathfrak{m}_\tau$ *of the trace* τ *such that*

(i) $\mathfrak{a}$ *is invariant under both* ν *and the transpose* ${}^t\nu$ *of* ν;

(ii) $\mathfrak{a}$ *is dense in the Hilbert space* $L^2(\mathfrak{A}, \tau)$;

(iii) *the map*: $a \in \mathfrak{a} \mapsto {}^t\nu \circ \nu(a) \in \mathfrak{a}$ *is essentially selfadjoint in* $L^2(\mathfrak{A}, \tau)$ *and the closure* Δ *of this linear map is positive and nonsingular*;

(iv) $\mathfrak{a}$ *is closed under the convolution product.*

Regarding $\mathfrak{A}$ as a function system, we denote its elements by $f, g, \cdots$. A linear map: $\sum_{i=1}^n f_i \otimes g_i \in \mathfrak{a} \otimes \mathfrak{a} \mapsto \sum_{i=1}^n \delta(g_i)(f_i \otimes 1) \in \mathfrak{A} \otimes \mathfrak{A}$ is extended to a unitary operator W on $L^2(\mathfrak{A} \overline{\otimes} \mathfrak{A}, \tau \otimes \tau)$. Representing $\mathfrak{A}$ on $L^2(\tau)$, we have $\delta(a) = W(1 \otimes a)W^*$, $a \in \mathfrak{A}$. The convolution product and the involution: $f \mapsto f^\#$ makes $\mathfrak{a}$ into a left (generalized) Hilbert algebra in the sense of [**12**]. Let $\lambda(f)$, $f \in \mathfrak{a}$, denote the convolution operator: $g \mapsto f * g$. Then this map $\lambda: f \in \mathfrak{a} \mapsto \lambda(f)$ is extended uniquely to a *-representation of $\mathfrak{A}_* = L^1(\tau)$, which is also denoted by λ. Let $\mathfrak{M}(\lambda)$ denote the von Neumann algebra generated by $\lambda(f)$, $f \in L^1(\tau)$. Then $\mathfrak{M}(\lambda)$ is nothing but the left von Neumann algebra of the left Hilbert algebra. Using the selfadjoint unitary operator $V: f \mapsto \Delta^{-1/2} f^\sim$, we can define another representation $\lambda': f \in L^1(\tau) \mapsto V\lambda(f)V$, which is called the *right regular representation* of $L^1(\tau)$. The original one λ is called the *left* regular representation. Let $\mathfrak{M}(\lambda')$ denote the von Neumann algebra generated by $\lambda'(f)$, $f \in L^1(\tau)$. Then we have

THEOREM 1. $\mathfrak{M}(\lambda)' = \mathfrak{M}(\lambda')$.

THEOREM 2. $W \in \mathfrak{A} \overline{\otimes} \mathfrak{M}(\lambda)$.

THEOREM 3. *The map* $\gamma: x \mapsto \sigma(W^*(x \otimes 1)W)$ *is a comultiplication of* $\mathfrak{M}(\lambda)$.

Let C denote the conjugation: $f \in L^2(\tau) \mapsto f^* \in L^2(\tau)$. Then the map $\kappa: x \mapsto Cx^*C$ is an involution of the Hopf-von Neumann algebra $\{\mathfrak{M}(\lambda), \gamma\}$.

The canonical weight φ of $\mathfrak{M}(\lambda)$ given by the left Hilbert algebra $\mathfrak{a}$, see [**1**], [**2**] and [**13**], behaves as if it were a unimodular measure on $\{\mathfrak{M}(\lambda), \gamma, \kappa\}$. Based on $\{\mathfrak{M}(\lambda), \gamma, \kappa, \varphi\}$, we can construct a Hilbert space $L^2(\varphi)$, the left regular representation ρ_0 of the involutive Banach algebra $\mathfrak{M}_*(\lambda)$ with convolution product, a comultiplication δ_0 of the von Neumann algebra $\mathfrak{M}(\rho_0)$ generated by $\rho_0(\varphi)$, $\varphi \in \mathfrak{M}_*(\lambda)$, an involution ν_0 of the Hopf-von Neumann algebra $\{\mathfrak{M}(\rho_0), \delta_0\}$ and the canonical trace τ_0 on $\mathfrak{M}(\rho_0)$. Then we get

THEOREM 4 (DUALITY). *There exists a unitary operator* Λ *of* $L^2(\tau)$ *onto* $L^2(\varphi)$ *which sets up an isomorphism of* $\{\mathfrak{A}, \delta, \nu, \tau\}$ *onto* $\{\mathfrak{M}(\rho_0), \delta_0, \nu_0, \tau_0\}$.

THEOREM 5. *The following statements* (Ci) *and* (Cii) (*resp.* (Di) *and* (Dii)) *are equivalent*:

(Ci) *The trace* τ *is finite*;
(Cii) *the convolution algebra* $\mathfrak{M}_*(\lambda)$ *admits an identity.*
(Di) *The canonical weight* φ *of* $\mathfrak{M}(\lambda)$ *is finite*;
(Dii) *the convolution algebra* $L^1(\tau) = \mathfrak{A}_*$ *admits an identity.*

Either (Ci) or (Cii) implies the unimodularity of τ.

If either (Ci) or (Cii) holds, then $\{\mathfrak{A}, \delta, \nu, \tau\}$ is called *compact.* On the contrary, if either (Di) or (Dii) holds, then $\{\mathfrak{A}, \delta, \nu, \tau\}$ is said to be *discrete.*

THEOREM 6 (PETER-WEYL). (i) *If* $\{\mathfrak{A}, \delta, \nu, \tau\}$ *is compact, then the left regular representation* λ *of* $L^1(\tau)$ *is decomposed into the direct sum of finite-dimensional irreducible representations with finite multiplicity*; *hence the von Neumann algebra* $\mathfrak{M}(\lambda)$ *is atomic and finite, so of type* I. *Furthermore, every irreducible representation of* $L^1(\tau)$ *is finite-dimensional and unitary equivalent to a component of* λ.

(ii) *If* $\{\mathfrak{A}, \delta, \nu, \tau\}$ *is discrete, then the dual statement of* (i) *for* $\mathfrak{M}_*(\lambda)$ *holds.*

THEOREM 7. (i) *If* $\{\mathfrak{A}, \delta, \nu, \tau\}$ *is abelian, then there exists uniquely a locally compact group G such that* $\{\mathfrak{A}(G), \delta_G, \nu_G, \tau_G\}$ *is isomorphic to* $\{\mathfrak{A}, \delta, \nu, \tau\}$.

(ii) *If* $\{\mathfrak{A}, \delta, \nu, \tau\}$ *is symmetric, then there exists uniquely a unimodular locally compact group G such that* $\{\mathfrak{M}(G), \gamma_G, \kappa_G, \varphi_G\}$ *is isomorphic to* $\{\mathfrak{A}, \delta, \nu, \tau\}$.

Applying our theory, we can get the Pontryagin, Tannaka, Stinespring, Eymard-Saito, Tatsuuma duality theorems for locally compact groups.

The whole theory announced above will be published as Tulane University Lecture Notes.

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