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DUALITY BETWEEN $D(X)$ AND $D(\hat{X})$ WITH ITS APPLICATION TO PICARD SHEAVES

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Introduction

As is well known, for a real vector space V, the Fourier transformation

$$\hat{f}(\alpha) = \int_V f(v)e^{2\pi i<v,\alpha>}dv \qquad \alpha \in V^\vee$$

gives an isometry between $L^2(V)$ and $L^2(V^\vee)$, where $V^\vee$ is the dual vector space of V and $\langle\ ,\ \rangle$: $V\times V^\vee \to \boldsymbol{R}$ is the canonical pairing.

In this article, we shall show that an analogy holds for abelian varieties and sheaves of modules on them: Let X be an abelian variety, $\hat{X}$ its dual abelian variety and $\mathscr{P}$ the normalized Poincaré bundle on $X\times\hat{X}$. Define the functor $\hat{\mathscr{S}}$ of $\mathcal{O}_X$-modules M into the category of $\mathcal{O}_{\hat{X}}$-modules by

$$\hat{\mathscr{S}}(M) = \pi_{\hat{X},*}(\mathscr{P}\otimes\pi_X^*M)\ .$$

Then the derived functor $\boldsymbol{R}\hat{\mathscr{S}}$ of $\hat{\mathscr{S}}$ gives an equivalence of categories between two derived categories $\boldsymbol{D}(X)$ and $\boldsymbol{D}(\hat{X})$ (Theorem 2.2).

In § 3, we shall investigate the relations between our functor $\boldsymbol{R}\hat{\mathscr{S}}$ and other functors, translation, tensoring of line bundles, direct (inverse) image by an isogeny, etc. The result (3.14) that if X is principally polarized then $\boldsymbol{D}(X)$ has a natural action of SL (2, Z) seems to be significant.

In §§ 4 and 5, we shall apply the duality between $\boldsymbol{D}(X)$ and $\boldsymbol{D}(\hat{X})$ to the study of Picard sheaves. We shall compute the cohomology of Picard sheaves (Proposition 4.4), determine the moduli of deformations of them (Theorem 4.8) and give a characterization of them in the case of dim $X = 2$ (Theorem 5.4). Other applications of the duality will be treated elsewhere.

After the original paper was written, the author learned by a letter from G. Kempf that Proposition 3.11 and some results in § 4 had also been proved independently by him.

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NOTATIONS. We denote by k a fixed algebraically closed field and mean by a *scheme* a scheme of finite type over k. For the product variety $X\times Y\times Z$, π_X (or p_1) and $\pi_{X,Y}$ (or p_{12}) are the projections of $X\times Y\times Z$ to X and $X\times Y$, respectively. For a coherent sheaf F on a variety X, $r(F)$ denotes the rank of F at the generic point of X. $F^\vee$ denotes $\mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

§1. Preliminary

Let X and Y be schemes and F an $\mathcal{O}_{X\times Y}$-module. We define the functor $\mathcal{S}_{X\to Y,F}$ from the category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules into $\mathrm{Mod}(Y)$ by

$$\mathcal{S}_{X\to Y,F}(?) = \pi_{Y,*}(F\otimes \pi_X^*?)\,, \tag{1.1}$$

where ? is an $\mathcal{O}_X$-module or an $\mathcal{O}_X$-homomorphism.

EXAMPLE 1.2. Let Γ_f be the graph of a morphism $f\colon X\to Y$ and F the structure sheaf $\mathcal{O}_{\Gamma_f}$ of Γ_f.
Then $\mathcal{S}_{X\to Y,F} = f_*$ and $\mathcal{S}_{Y\to X,F} = f^*$.

We denote by $\boldsymbol{D}(X)$ the derived category of $\mathrm{Mod}(X)$ and by $\boldsymbol{D}_{qc}(X)$ (resp. $\boldsymbol{D}_c(X)$) the full subcategory of $\boldsymbol{D}(X)$ consisting of the complexes whose i-th cohomologies are quasi-coherent (resp. coherent) for all i. $\boldsymbol{D}^-(X)$ (resp. $\boldsymbol{D}^b(X)$) is the full subcategory of $\boldsymbol{D}(X)$ consisting of the complexes bounded above (resp. bounded on both sides) and $\boldsymbol{D}^-_{qc}(X) = \boldsymbol{D}^-(X)\cap \boldsymbol{D}_{qc}(X)$, $\boldsymbol{D}^b_c(X) = \boldsymbol{D}^b(X)\cap \boldsymbol{D}_c(X)$, etc.

For an object F of $\boldsymbol{D}^-(X\times Y)$, we define the functor $\boldsymbol{R}\mathcal{S}_{X\to Y,F}$ from $\boldsymbol{D}^-(X)$ into $\boldsymbol{D}^-(Y)$ by

$$\boldsymbol{R}\mathcal{S}_{X\to Y,F}(?) = \boldsymbol{R}\pi_{Y,*}(F\overset{L}{\underset{=}{\otimes}} \pi_X^*?)\,. \tag{1.4}$$

If F is an $\mathcal{O}_X$-flat module, then $\boldsymbol{R}\mathcal{S}_{X\to Y,F}$ is the derived functor of $\mathcal{S}_{X\to Y,F}$. To consider the derived functors has the following advantage:

PROPOSITION 1.3. *Let Z be a scheme and G an object of $\boldsymbol{D}^-(X\times Y)$. Then there is a natural isomorphism of functors*:

$$\boldsymbol{R}\mathcal{S}_{Y\to Z,G}\circ \boldsymbol{R}\mathcal{S}_{X\to Y,F} \cong \boldsymbol{R}\mathcal{S}_{X\to Z,H}\,,$$

where $H = \boldsymbol{R}\pi_{X,Z,*}(\pi_{X,Y}^*F\overset{L}{\underset{=}{\otimes}} \pi_{Y,Z}^*G)$.

Proof. We use (1) the commutativity of $\boldsymbol{R}$ and the composition of functors, (2) the projection formula and (3) the base change theorem. (See

[2] Proposition 5.1, 5.3, 5.6, 5.12)
Let ? be an object or morphism in $D^-(X)$.

$$\begin{aligned}
&\boldsymbol{R}\mathscr{S}_{Y\to Z,G}(\boldsymbol{R}\mathscr{S}_{X\to Y,F}(?)) \\
&\quad\cong \boldsymbol{R}\pi_{Z,*}(G \overset{L}{\underset{=}{\otimes}} \pi_Y^*(\boldsymbol{R}\pi_{Y,*}(F \overset{L}{\underset{=}{\otimes}} \pi_X^*?))) \\
&\quad\cong \boldsymbol{R}\pi_{Z,*}(G \overset{L}{\underset{=}{\otimes}} \boldsymbol{R}\pi_{Y,Z,*}(\pi_{X,Y}^*(F \overset{L}{\underset{=}{\otimes}} \pi_X^*?))) && (3) \\
&\quad\cong \boldsymbol{R}\pi_{Z,*}\boldsymbol{R}\pi_{Y,Z,*}(\pi_{Y,Z}^*G \overset{L}{\underset{=}{\otimes}} \pi_{X,Y}^*F \overset{L}{\underset{=}{\otimes}} \pi_X^*?) && (2) \\
&\quad\cong \boldsymbol{R}\pi_{Z,*}\boldsymbol{R}\pi_{X,Z,*}(\pi_{Y,Z}^*G \overset{L}{\underset{=}{\otimes}} \pi_{X,Y}^*F \overset{L}{\underset{=}{\otimes}} \pi_{X,Z}^*\pi_X^*?) && (1) \\
&\quad\cong \boldsymbol{R}\pi_{Z,*}(H \overset{L}{\underset{=}{\otimes}} \pi_X^*?) = \boldsymbol{R}\mathscr{S}_{X\to Z,H}(?) && (2)
\end{aligned}$$

q.e.d.

PROPOSITION 1.4. (1) *If F has finite Tor-dimension as a complex of $\mathcal{O}_X$-modules, then we can extend the domain of definition of $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ to*

$$\boldsymbol{R}\mathscr{S}_{X\to Y,F}\colon \boldsymbol{D}(X) \longrightarrow \boldsymbol{D}(Y)$$

and $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}^b(X)$ into $\boldsymbol{D}^b(Y)$.

(2) *If F belongs to $\boldsymbol{D}^-_{qc}(X\times Y)$, then $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}^-_{qc}(X)$ into $\boldsymbol{D}^-_{qc}(Y)$.*

(3) *If X is proper and $F\in \boldsymbol{D}^-_c(X\times Y)$, then $\boldsymbol{R}\mathscr{S}_{X\to Y,F}$ maps $\boldsymbol{D}^-_c(X)$ into $\boldsymbol{D}^-_c(Y)$.*

Proof. For (1), see [2] Proposition 4.2 and Corollary 4.3. (2) and (3) follow from [EGA] III 1.4.10 and 3.2.1, respectively. q.e.d.

§ 2. Fourier functor

Let X be an abelian variety of dimension g (the business is similar for a complex torus) and $\hat{X}$ its dual abelian variety. Let $\mathscr{P}$ be the normalized Poincaré bundle on $X\times\hat{X}$. Here "normalized" means that both $\mathscr{P}|_{X\times\hat{0}}$ and $\mathscr{P}|_{0\times\hat{X}}$ are trivial. For $\hat{x}\in\hat{X}$ (resp. $x\in X$), $P_{\hat{x}}$ (resp. P_x) denotes $\mathscr{P}|_{X\times\hat{x}}$ (resp. $\mathscr{P}|_{x\times\hat{X}}$). We put $\mathscr{S}=\mathscr{S}_{\hat{X}\to X,\mathscr{P}}$ and $\hat{\mathscr{S}}=\mathscr{S}_{X\to\hat{X},\mathscr{P}}$. Since $\hat{X}$ is complete and $\mathscr{P}$ is $\mathcal{O}_{\hat{X}}$-flat, we have by Proposition 1.4,

PROPOSITION 2.1. *The derived functor $\boldsymbol{R}\mathscr{S}\colon \boldsymbol{D}(\hat{X})\to\boldsymbol{D}(X)$ of $\mathscr{S}$ can be defined. It maps $\boldsymbol{D}^b(\hat{X})$, $\boldsymbol{D}^-_{qc}(\hat{X})$ and $\boldsymbol{D}^-_c(\hat{X})$ into $\boldsymbol{D}^b(X)$, $\boldsymbol{D}^-_{qc}(X)$ and $\boldsymbol{D}^-_c(X)$ respectively.*

The following theorem is fundamental:

THEOREM 2.2. *There are isomorphisms of functors*:

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_X)^*[-g]$$

and

$$R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{X}})^*[-g] ,$$

where $[-g]$ *denotes "shift the complex* g *places to the right". In other words,* $R\mathcal{S}$ *gives an equivalence of categories between* $D(\hat{X})$ *and* $D(X)$, *and its quasi-inverse is* $(-1_{\hat{X}})^* \circ R\hat{\mathcal{S}}[g]$.

Proof. It suffices to show that $(R\mathcal{S}|_{D^-(\hat{X})}) \circ (R\hat{\mathcal{S}}|_{D^-(X)}) = (-1_X)^*[-g]$. By (1.3) the left side is isomorphic to $R\mathcal{S}_{X\to X,H}$ with $H = Rp_{12,*}(p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P})$, where p_{ij} are projections of $X\times X\times \hat{X}$. Since $p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P} \cong (m\times 1)^*\mathcal{P}$ (which is easily verified by the seesaw principle), $H \cong Rp_{12,*}(m\times 1)^*\mathcal{P} \cong m^*Rp_{1,*}\mathcal{P}$. As was shown in the course of the proof of the theorem in [6] § 13, $R^ip_{1,*}\mathcal{P} = 0$ for every $i \neq g$ and $R^gp_{1,*}\mathcal{P} \cong k(0)$, i.e., $Rp_{1,*}\mathcal{P} \cong k(0)[-g]$. Hence H is isomorphic to $\mathcal{O}_E[-g]$, where E is the graph of -1_X: $X \to X$. Therefore $R\mathcal{S}_{X\to X,H} \cong (-1_X)^*[-g]$ (see Example 1.2). q.e.d.

In order to apply the theorem, we need

DEFINITION 2.3. We say that W.I.T. (weak index theorem) holds for a coherent sheaf F on X if $R^i\hat{\mathcal{S}}(F) = 0$ for all but one i. This i is denoted by $i(F)$ and called the index of F. We denote the coherent sheaf $R^{i(F)}\hat{\mathcal{S}}(F)$ on $\hat{X}$ by $\hat{F}$ and call it the Fourier transform of F.

We say that I.T. (index theorem) holds for F if $H^i(X, F\otimes P) = 0$ for all $P \in \mathrm{Pic}^\circ X$ and all but one i.

Since $(\mathcal{P} \otimes \pi_{\hat{X}}^*F)|_{X\times\hat{x}} \cong P_{\hat{x}} \otimes F$, we see by virtue of the base change theorem, that I.T. implies W.I.T. and $\hat{F}$ is locally free if I.T. holds for F. We always identify $\mathcal{O}_X$-module F with the complex consisting of F in degree 0, and 0 elsewhere. Hence if W.I.T. holds for F, then $R\mathcal{S}(F)$ is isomorphic to $\hat{F}[-i(F)]$. Hence we have

COROLLARY 2.4. *If W.I.T. holds for* F, *then so does for* $\hat{F}$ *and* $i(\hat{F}) = g - i(F)$. *Moreover* $\hat{\hat{F}}$ *is isomorphic to* $(-1_X)^*F$.

COROLLARY 2.5. *Assume that W.I.T. holds for* F *and* G. *Then* $\mathrm{Ext}^i_{\mathcal{O}_X}(F, G) \cong \mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{G})$ *for every integer* i, *where* $\mu = i(F) - i(G)$. *Especially, we have an isomorphism* $\mathrm{Ext}^i_{\mathcal{O}_X}(F, F) \simeq \mathrm{Ext}^i_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{F})$ *for every* i.

Proof.
$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{O}_X}(F, G) &\cong \operatorname{Hom}_{D(X)}(F, G[i]) \\ &\cong \operatorname{Hom}_{D(\hat{X})}(\boldsymbol{R}\mathscr{S}(F), \boldsymbol{R}\mathscr{S}(G)[i]) \\ &\cong \operatorname{Hom}_{D(\hat{X})}(\hat{F}[-i(F)], \hat{G}[i - i(G)]) \\ &\cong \operatorname{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{G})\end{aligned}$$
q.e.d.

EXAMPLE 2.6. Let $k(\hat{x})$ be the one dimensional sky-scraper sheaf supported by $\hat{x} \in \hat{X}$. Since $H^i(X, k(\hat{x}) \otimes P) = 0$ for every $i > 0$ and $P \in \operatorname{Pic}^\circ \hat{X}$, I.T. holds for $k(\hat{x})$, $i(k(\hat{x})) = 0$ and $\widehat{k(\hat{x})} \simeq P_{\hat{x}}$. Hence by Corollary 2.4, W.I.T. holds for $P_{\hat{x}}$, $i(P_{\hat{x}}) = g$ and $\widehat{P_{\hat{x}}} \simeq k(-x)$. Note that I.T. does not hold for $P_{\hat{x}}$.

Combining the above with Corollary 2.5, we have

PROPOSITION 2.7. *Assume that W.I.T. holds for a coherent sheaf F on X. Then we have*

$$H^i(X, F \otimes P_{\hat{x}}) \cong \operatorname{Ext}^{g-i(F)+i}_{\mathcal{O}_{\hat{X}}}(k(\hat{x}), \hat{F})$$

and

$$\operatorname{Ext}^i_{\mathcal{O}_X}(k(x), F) \cong H^{i-i(F)}(\hat{X}, \hat{F} \otimes P_{-x}) .$$

Proof. By Corollary 2.4, it suffices to show the first isomorphism. Since $P_{\hat{x}}$ is locally free, $H^i(X, F \otimes P_{\hat{x}})$ is isomorphic to $\operatorname{Ext}^i_{\mathcal{O}_X}(P_{-\hat{x}}, F)$. Hence by Corollary 2.5, it is isomorphic to $\operatorname{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{P}_{-\hat{x}}, \hat{F}) \cong \operatorname{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(k(\hat{x}), \hat{F})$, where $\mu = i(P_{-\hat{x}}) - i(F) = g - i(F)$. q.e.d.

COROLLARY 2.8. *The Euler-Poincaré characteristic of F is equal to $(-1)^{i(F)}r(F)$.*

Proof.
$$\begin{aligned}\chi(X, F) &= \sum_i (-1)^i h^i(X, F) \\ &= \sum_i (-1)^i \dim \operatorname{Ext}^{i+g-i(F)}_{\mathcal{O}_X}(k(\hat{x}), \hat{F}) \\ &= (-1)^{i(F)} r(\hat{F}) .\end{aligned}$$
q.e.d.

EXAMPLE 2.9 ([4] § 4). *A* vector bundle U on X is said to be unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong \mathcal{O}_X$ for $i = 1, 2, \cdots, n$. Since the functor $R^i\mathscr{S}$ is semi-exact for all i, W.I.T. holds for U, $i(U) = g$ and the coherent sheaf $\hat{U}$ is supported by $\hat{0} \in \hat{X}$. Hence $R^g\mathscr{S}$ gives an equivalence of the categories

(Unipotent vector bundles on X) and (Coherent sheaves on $\hat{X}$ supported by $\hat{0}$) = (Artinian B-modules), where B is the local ring $\mathcal{O}_{\hat{X},\hat{0}}$ of $\hat{X}$ at $\hat{0}$. Moreover we have by Proposition 2.7.

$$H^i(X, U) \cong \mathrm{Ext}^i_B(k(\hat{0}), \hat{U}) .$$

§ 3. Relations between $R\mathcal{S}$ and other functors

The properties of the Poincaré bundle $\mathcal{P}$ give relations between $\mathcal{S}$ and other functors. From this we obtain by the universal property of $R\mathcal{S}$, relations between $R\mathcal{S}$ and other functors. For example, from the isomorphism $T^*_{(0,\hat{x})}\mathcal{P} \cong \mathcal{P} \otimes \pi^*_X P_{\hat{x}}$, we obtain the isomorphism of functors $\mathcal{S} \circ T^*_{\hat{x}} \cong (\otimes P_{-\hat{x}}) \circ \mathcal{S}$ because $\mathcal{S}(T^*_{\hat{x}}?) = \pi_{X,*}(\mathcal{P} \otimes T^*_{(0,\hat{x})}\pi^*_{\hat{X}}?) \cong \pi_{X,*}T^*_{(0,\hat{x})}\ (T^*_{(0,-\hat{x})}\mathcal{P} \otimes \pi^*_{\hat{X}}?) \cong \pi_{X,*}(\mathcal{P} \otimes \pi^*_X P_{-\hat{x}} \otimes \pi^*_{\hat{X}}?) \cong \mathcal{S}(?) \otimes P_{-\hat{x}}$.
Hence we have

(3.1) (Exchange of translations and $\otimes\,\mathrm{Pic}^\circ$)

$$R\mathcal{S} \circ T^*_{\hat{x}} \cong (\otimes P_{-\hat{x}}) \circ R\mathcal{S}$$
$$R\mathcal{S} \circ (\otimes P_x) \cong T^*_x \circ R\mathcal{S} .$$

EXAMPLE 3.2. W.I.T. holds for every homogeneous vector bundle H on X. The index $i(H)$ is equal to g and $\hat{H}$ is a coherent sheaf supported by a finite set of points. Hence $R^g\hat{\mathcal{S}}$ gives an equivalence of categories between $\boldsymbol{H}_X$ = (Homogeneous vector bundles on X) and $\boldsymbol{C}^f_{\hat{X}}$ = (Coherent sheaves on $\hat{X}$ supported by a finite set of points).

Proof. If a coherent sheaf M on $\hat{X}$ is supported by a finite set of points, then $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^\circ \hat{X}$ and hence $\mathcal{S}(M)$ is a homogeneous vector bundle by (3.1). Therefore it suffices to show the first statement. Put $M_i = R^i\hat{\mathcal{S}}(H)$. Since $T^*_x H \cong H$ for all $x \in X$, $M_i \otimes P \cong M_i$ for all $P \in \mathrm{Pic}^\circ \hat{X}$ by (3.1). Hence by the lemma (3.3) below M_i is supported by a finite set of points. By Theorem 2.2, there is a spectral sequence whose E_2 term is $R\mathcal{S}^j(M_i)$ and which converges to zero when $i + j \neq g$. Since $R\mathcal{S}^j(M_i) = 0$ if $j \neq 0$, the spectral sequence degenerates and M_i is zero for every $i \neq g$. q.e.d.

LEMMA 3.3. *Let M be a coherent sheaf on an abelian variety $\hat{X}$. If $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^\circ \hat{X}$, then* Supp M *is finite.*

Proof. Suppose that $\dim \mathrm{Supp}\, M \geq 1$. Take a curve C contained in Supp M and let $\tilde{C}$ be its normalization. Put $N = M \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{O}_{\tilde{C}}$ and $L = N/$ "the torsion part of N". Then N is a vector bundle on $\tilde{C}$ and $N \otimes f^*P$

$\cong N$ for all $P \in \operatorname{Pic}^\circ \hat{X}$, where f is the natural morphism $\tilde{C} \to C \subsetneq X$. Therefore, taking the determinant of both sides, we see that $(f^*P)^{\otimes r(N)}$ is trivial for all $P \in \operatorname{Pic}^\circ \hat{X}$. This is a contradiction because the morphism $f^*\colon \operatorname{Pic}^\circ \hat{X} \to \operatorname{Pic}^\circ \tilde{C}$ is not zero. q.e.d.

Combining Example 2.9 and 3.2, we have

THEOREM (Matsushima, Morimoto, Miyanishi, Mukai). *A vector bundle F on X is homogeneous if and only if F is isomorphic to $\bigoplus_{i=1}^n P_i \otimes U_i$ for some $P_1, \cdots, P_n \in \operatorname{Pic}^\circ X$ and unipotent vector bundles $U_1, \cdots, U_n$.*

Let Y be an abelian variety, $\varphi\colon Y \to X$ an isogeny and $\hat{\varphi}\colon \hat{X} \to \hat{Y}$ the dual isogeny of φ.

(3.4) (Exchange of the direct image and the inverse image)

$$\varphi^* \circ \boldsymbol{R}\mathscr{S}_X \cong \boldsymbol{R}\mathscr{S}_Y \circ \hat{\varphi}_*$$
$$\varphi_* \circ \boldsymbol{R}\mathscr{S}_Y \cong \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \, .$$

Proof. The second isomorphism is obtained from the first in the following manner. Replacing φ by $\hat{\varphi}$ in the first isomorphism, we have $\hat{\varphi}^* \circ \boldsymbol{R}\hat{\mathscr{S}}_Y \cong \boldsymbol{R}\hat{\mathscr{S}}_X \circ \varphi_*$. By Theorem 2.2,

$$\begin{aligned} \varphi_* \circ \boldsymbol{R}\mathscr{S}_Y &\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \boldsymbol{R}\hat{\mathscr{S}}_X \circ \varphi_* \circ \boldsymbol{R}\mathscr{S}_Y[g] \\ &\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \circ \boldsymbol{R}\hat{\mathscr{S}}_Y \circ \boldsymbol{R}\mathscr{S}_Y[g] \\ &\cong (-1_X)^* \circ \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \circ (-1_{\hat{Y}})^* \\ &\cong \boldsymbol{R}\mathscr{S}_X \circ \hat{\varphi}^* \, . \end{aligned}$$

Hence it suffices to show $\varphi^* \circ \mathscr{S}_X \cong \mathscr{S}_Y \circ \hat{\varphi}_*$. By the definition of $\hat{\varphi}$, $(\varphi \times 1)^*\mathscr{P}_X \cong (1 \times \hat{\varphi})^*\mathscr{P}_Y$. Hence we have

$$\begin{aligned} \varphi^*\mathscr{S}_X(?) &= \varphi^*\pi_{X,*}(\mathscr{P}_X \otimes \pi_{\hat{X}}{}^*?) \\ &\cong \pi_{Y,*}((\varphi \times 1)^*\mathscr{P}_X \otimes \pi_{\hat{X}}{}^*?) \\ &\cong \pi_{Y,*}(1 \times \hat{\varphi})_*((1 \times \hat{\varphi})^*\mathscr{P}_Y \otimes \pi_{\hat{X}}^*?) \\ &\cong \pi_{Y,*}(\mathscr{P}_Y \otimes (1 \times \hat{\varphi})_*\pi_{\hat{X}}^*?) \\ &\cong \mathscr{S}_Y(\hat{\varphi}_*?) \, . \end{aligned}$$

$$\begin{array}{ccccc} & & Y & \xrightarrow{\varphi} & X \\ & & \uparrow{\scriptstyle \pi_X} & & \uparrow{\scriptstyle \pi_X} \\ \hat{X} & \xleftarrow{\pi_{\hat{X}}} & Y \times \hat{X} & \xrightarrow{\varphi \times 1} & X \times \hat{X} \\ {\scriptstyle \hat{\varphi}}\downarrow & & {\scriptstyle 1 \times \hat{\varphi}}\downarrow & & \downarrow \\ \hat{Y} & \xleftarrow[\pi_{\hat{Y}}]{} & Y \times \hat{Y} & \longrightarrow & X \times \hat{Y} \end{array}$$

q.e.d.

Remark 3.5. The second isomorphism can be also proved in the same way as the first by the isomorphism $(1\times\hat{\varphi})_*\mathscr{P}_X \cong (\varphi\times 1)_*\mathscr{P}_Y$ which was proved in [7].

EXAMPLE 3.6. If H is a homogeneous vector bundle on X (resp. Y), so is φ^*H (resp. φ_*H). Moreover the following diagram is (quasi-)commutative.

$$\begin{array}{ccc} C^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}_X} & H_X \\ {\scriptstyle\hat{\varphi}_*}\downarrow & & \downarrow{\scriptstyle\varphi^*} \\ C^f_{\hat{Y}} & \xrightarrow[\mathscr{S}_Y]{\sim} & H_Y \end{array} \qquad\qquad \begin{array}{ccc} C^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}_X} & H_X \\ {\scriptstyle\hat{\varphi}^*}\uparrow & & \uparrow{\scriptstyle\varphi_*} \\ C^f_{\hat{Y}} & \xrightarrow[\mathscr{S}_Y]{\sim} & H_Y \end{array}$$

Now we investigate other properties of the Fourier functor $\boldsymbol{R}\mathscr{S}$. Let $m\colon X\times X\to X$ be the group law of X. For $\mathcal{O}_X$-modules M and N, we define the Pontrjagin product $M*N$ of M and N by $M*N = m_*(p_1^*M\otimes p_2^*N)$. $*$ is a bifunctor from $\mathrm{Mod}(X)\times\mathrm{Mod}(X)$ into $\mathrm{Mod}(X)$. We denote its derived functor by $\overset{R}{\underset{=}{*}}$.

(3.7) (Exchange of the Pontrjagin product and the tensor product)

$$\boldsymbol{R}\mathscr{S}\left(F\overset{R}{\underset{=}{*}}\,?\right) \cong \boldsymbol{R}\mathscr{S}(F)\overset{L}{\underset{=}{\otimes}}\boldsymbol{R}\mathscr{S}(?)$$

$$\boldsymbol{R}\mathscr{S}\left(F\overset{L}{\underset{=}{\otimes}}\,?\right) \cong \boldsymbol{R}\mathscr{S}(F)\overset{R}{\underset{=}{*}}\boldsymbol{R}\mathscr{S}(?)\ [g]$$

where $F\in \boldsymbol{D}(\hat{X})$ and ? is an object or a morphism in $\boldsymbol{D}(\hat{X})$.

Proof. It suffices to show the first isomorphism. We use the isomorphism $(1\times m)^*\mathscr{P}\cong p_{12}^*\mathscr{P}\otimes p_{13}^*\mathscr{P}$, where p_{ij}'s are projections of $X\times\hat{X}\times\hat{X}$.

$$\begin{aligned} \boldsymbol{R}\mathscr{S}(F\overset{R}{\underset{=}{*}}?) &\cong \boldsymbol{R}\pi_{X,*}(\mathscr{P}\otimes\pi_{\hat{X}}^*(\boldsymbol{R}m_*(p_1^*F\otimes p_2^*?))) \\ &\cong \boldsymbol{R}\pi_{X,*}(\mathscr{P}\otimes\boldsymbol{R}(1\times m)_*p_{23}^*(p_1^*F\otimes p_2^*?)) \\ &\cong \boldsymbol{R}\pi_{X,*}\boldsymbol{R}(1\times m)_*((1\times m)^*\mathscr{P}\otimes p_2^*F\otimes p_3^*?) \\ &\cong \boldsymbol{R}p_{1,*}(p_{12}^*\mathscr{P}\otimes p_{13}^*\mathscr{P}\otimes p_2^*F\otimes p_3^*?) \\ &\cong \boldsymbol{R}p_{1,*}(p_{12}^*(\mathscr{P}\otimes\pi_{\hat{X}}^*F)\otimes p_{13}^*(\mathscr{P}\otimes\pi_{\hat{X}}^*?)) \\ &\cong \boldsymbol{R}\mathscr{S}(F)\overset{L}{\underset{=}{\otimes}}\boldsymbol{R}\mathscr{S}(?) \end{aligned}$$

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X\times\hat{X} & \xleftarrow{1\times m} & X\times\hat{X}\times\hat{X} \\ & & \downarrow{\scriptstyle\pi_{\hat{X}}} & & \downarrow{\scriptstyle p_{23}} \\ & & \hat{X} & \xleftarrow{m} & \hat{X}\times\hat{X} \end{array}$$

q.e.d.

Let Δ_X be the dualizing functor. Since the canonical module of X is trivial, $\Delta_X(?) = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(?, \mathcal{O}_X)\,[g]$.

(3.8) (Skew commutativity of $\boldsymbol{R}\mathcal{S}$ and Δ)

$$\Delta_X \circ \boldsymbol{R}\mathcal{S} \cong ((-1_X)^* \circ \boldsymbol{R}\mathcal{S} \circ \Delta_{\hat{X}})\,[g]\,.$$

Proof. We use the isomorphism $\mathcal{P}^{-1} \cong ((-1_X) \times 1_{\hat{X}})^*\mathcal{P}$ and the Grothendieck duality.

$$\begin{aligned}\Delta_X(\boldsymbol{R}\mathcal{S}(?)) &= \Delta_X \boldsymbol{R}\pi_{X,*}(\mathcal{P} \otimes \pi_{\hat{X}}^* ?)\\ &\cong \boldsymbol{R}\pi_{X,*}\Delta_{X\times\hat{X}}(\mathcal{P} \otimes \pi_{\hat{X}}{}^* ?)\\ &\cong \boldsymbol{R}\pi_{X,*}(\mathcal{P}^{-1} \otimes \pi_{\hat{X}}^*\Delta_{\hat{X}} ?)\,[g]\\ &\cong \boldsymbol{R}\pi_{X,*}(((-1_X) \times 1_{\hat{X}})^*\mathcal{P} \otimes \pi_{\hat{X}}^*\Delta_{\hat{X}} ?)\,[g]\\ &\cong (-1_X)^*\boldsymbol{R}\mathcal{S}(\Delta_{\hat{X}} ?)\,[g]\end{aligned}$$

q.e.d.

EXAMPLE 3.9. Let U and V be unipotent vector bundles on X. As we saw in Example 2.9, $\hat{U}$ and $\hat{V}$ are artinian B-modules. $U \otimes V$ and $U^\vee$ are also unipotent vector bundles. $\widehat{U \otimes V}$ is isomorphic to $\hat{U} * \hat{V}$ and $\widehat{U^\vee}$ is isomorphic to $(-1_B)^*\Delta(\hat{U})$. $\hat{U} * \hat{V}$ is $\hat{U} \otimes_k \hat{V}$ regarded as a $\hat{B}$-modules via the co-multiplication μ: $\hat{B} \to \hat{B} \hat{\otimes} \hat{B}$ of the formal group $\hat{B}$. -1_B is an automorphism of B induced by $-1_{\hat{X}}$: $\hat{X} \to \hat{X}$ and Δ is the dualizing functor of Mod (B).

Next we investigate the relation between $\boldsymbol{R}\hat{\mathcal{S}}$ and $\otimes N$ for a line bundle N on X. In the rest of this section we always assume that N is nondegenerate, i.e., $\chi(N) \neq 0$. Hence ϕ_N ([6] p. 59, p. 131) is an isogeny.

(3.10) $\quad ? \overset{R}{\underset{=}{*}} N \cong (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^*)\,(?)$

where ? is an object or a morphism in $\boldsymbol{D}(X)$.

Proof. Consider the isomorphism ψ: $X \times X \to X \times X$ such that $\psi(x, y) = (x, x + y)$. The morphisms p_1, p_2 and m is sent by ψ to p_1, μ and p_2, respectively, where μ: $X \times X \to X$, $\mu(x, y) = y - x$. Hence $?*N = m_*(p_1^* ? \otimes p_2^* N)$ is isomorphic to $p_{2,*}(p_1^* ? \otimes \mu^* N)$. By the definition of the morphism ϕ_N: $X \to \hat{X}$, we have $m^*N \cong p_1^*N \otimes p_2^*N \otimes (1 \times \phi_N)^*\mathcal{P}$ and hence $\mu^*N \cong p_1^*(-1_X)^*N \otimes p_2^*N \otimes (-1_X \times \phi_N)^*\mathcal{P}$. Therefore the functor $?*N$ is isomorphic to $(\otimes N) \circ \mathcal{S}_{X \to X,\, (-1_X \times \phi_N)^*\mathcal{P}} \circ (\otimes\, (-1_X)^*N)$. By our assumption on N, ϕ_N is an isogeny, hence a flat morphism. Hence $\mathcal{S}_{X \to X,\, (-1_X \times \phi_N)^*\mathcal{P}} = \phi_N^* \circ \hat{\mathcal{S}} \circ (-1_X)^*$.

q.e.d.

Since I.T. holds for N ([6] § 16), $\hat{N}$ is a vector bundle on $\hat{X}$. $\hat{N}$ is simple, i.e., $\mathrm{End}_{\mathcal{O}_{\hat{X}}}(\hat{N}) \cong k$ by Corollary 2.5.

PROPOSITION 3.11. (1) $\phi_N^* \hat{N} \cong (N^{-1})^{\oplus|\chi(N)|}$

(2) $\hat{N}^{\oplus|\chi(N)|} \cong \phi_{N,*} N^{-1}$

(3) *If* $|\chi(N)| = 1$, *e.g., N is a principal polarization of X, then* $\hat{N} \cong (\phi_N^{-1})^* N^{-1}$.

(4) *There is an isogeny* $\pi: X \to Y$ *of degree* $|\chi(N)|$ *and a line bundle L on Y such that* $N \cong \pi^* L$. *Since* $\mathrm{Ker}(\pi) \subset K(N)$, *there is an isogeny* $\tau: Y \to \hat{X}$ *such that* $\tau \circ \pi = \phi_N$. *Then* $\hat{N}$ *is isomorphic to* $\tau_* L^{-1}$.

Proof. (1) is obtained from (3.10) by putting $? = \mathcal{O}_X$, because then the left side is $\mathcal{O}_X \overset{R}{\underset{=}{*}} N \cong \boldsymbol{R}p_{2,*}(p_1^* N) \cong \mathcal{O}_X \otimes_k H^i(X, N)\,[-i]$ and the right side is $N \otimes \phi_N^* \hat{N}\,[-i]$, where $i = i(F)$. Replacing N by N^{-1} in (1), we have $N^{\oplus|\chi(N)|} \cong (-\phi_N)^* \widehat{N^{-1}}$. Operating $\wedge$ on both sides, we have (2) because $\hat{N}^{\oplus|\chi(N)|} \cong (-\phi_N)_*(-1_X)^* N^{-1} \cong \phi_{N,*} N^{-1}$ by (3.4). Since $\deg \phi_N = |\chi(N)|^2$, ϕ_N is an isomorphism if $|\chi(N)| = 1$. Hence (3) is a special case of (1) or (2). For the first half of (4), see [6] § 23. It suffices to show the last statements. Since $|\chi(L)| = 1$, we have by (3), $\hat{N} \cong \widehat{\pi^* L} \cong \hat{\pi}_* \hat{L} \cong \hat{\pi}_* \phi_{L,*} L^{-1}$. On the other hand, since $N \cong \pi^* L$, we have $\phi_N = \hat{\pi} \circ \phi_L \circ \pi$. Since $\phi_N = \tau \circ \pi$ and π is an isogeny, we have $\tau = \hat{\pi} \circ \phi_L$. Hence $\hat{N} \cong \tau_* L^{-1}$. q.e.d.

(3.10) gives us an interesting relation between two functors $\boldsymbol{R}\mathcal{S}$ and $\otimes N$.

(3.12) $(\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\,[g + i(N)] \cong (\otimes \mathcal{O}_X^{\oplus|\chi(N)|}) \circ \phi_N^* \circ \phi_{N,*}$.

Especially, when the group scheme $K(N)$ is discrete, e.g., when $\chi(N)$ is prime to the characteristic exponent p of the ground field, then we have

(3.12′) $(\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\,[g + i(N)] \cong (\bigoplus_{x \in K(N)} T_x^*)^{\oplus|\chi(N)|}$.

Proof. First operate $\boldsymbol{R}\mathcal{S}$ on both sides of (3.10). By (3.7), we have

$$\boldsymbol{R}\mathcal{S}(?) \overset{L}{\underset{=}{\otimes}} \boldsymbol{R}\mathcal{S}(N) \cong (\boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^*)\,(?)$$

i.e.,

$$\otimes \hat{N} \circ \boldsymbol{R}\hat{\mathcal{S}}[-i(N)] \cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ (-1_X)^* .$$

Operating $(-1_X)^* \circ \boldsymbol{R}\mathcal{S} \circ \phi_{N,*}$ from the right, we have

$$\begin{aligned} \otimes \hat{N} \circ \varphi_{N,*}[-g - i(N)] &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \boldsymbol{R}\mathcal{S} \circ \phi_{N,*} \\ &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \circ \otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}} \\ &\cong \boldsymbol{R}\hat{\mathcal{S}} \circ (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^2 . \end{aligned}$$

Hence $\otimes N \circ \phi_N^* \circ \otimes \hat{N} \circ \phi_{N,*} \cong (\otimes N \circ \phi_N^* \circ \boldsymbol{R}\hat{\mathcal{S}})^3\,[g + i(N)]$. By (1) of Proposition 3.11, we have $\phi_N^*\hat{N} \cong (N^{-1})^{\oplus|\chi(N)|}$ and hence $\phi_N^* \circ \otimes \hat{N} \cong (\otimes\, \phi_N^*\hat{N}) \circ \phi_N^* \cong (\otimes\,(N^{-1})^{\oplus|\chi(N)|}) \circ \phi_{N,*}$, which proves our assertion. q.e.d.

In the case (X, L) is a principally polarized abelian variety, $\hat{X}$ is identified with X by the isomorphism $\phi_L\colon X \to \hat{X}$. Hence $\boldsymbol{R}\mathscr{S}$ is considered to be an automorphism of $\boldsymbol{D}(X)$. We summarize the results derived in this section for this case.

THEOREM 3.13. *Let (X, L) be a principally polarized abelian variety of dimension g. Then we have*

(1) $(\boldsymbol{R}\mathscr{S})^2 \cong (-1_X)^*[-g]$,

(2) $\boldsymbol{R}\mathscr{S} \circ \otimes P_x \cong T_x^* \circ \boldsymbol{R}\mathscr{S}$ *for* $x \in X$,

(3) $\boldsymbol{R}\mathscr{S} \circ \varphi \cong \hat{\varphi} \circ \boldsymbol{R}\mathscr{S}$ *for an isogeny* $\varphi\colon X \to X$.

(4) $\boldsymbol{R}\mathscr{S} \circ \Delta \cong ((-1_X)^* \circ \Delta \circ \boldsymbol{R}\mathscr{S})[g]$, *where Δ is the dualizing functor of* $\boldsymbol{D}(X)$,

(5) $\hat{L} \cong L^{-1}$ *and* $\widehat{L^{-1}} \cong (-1_X)^*L$,

(6) $(\otimes L \circ \boldsymbol{R}\mathscr{S})^3 \cong [-g]$.

(1) and (6) implies that the relation modulo the shift [ ] between two automorphisms $\boldsymbol{R}\mathscr{S}$ and $\otimes L$ is same as the relation between the generators $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of $\mathrm{SL}(2, \boldsymbol{Z})$. In other words,

(3.14) if X is principally polarized, then $\mathrm{SL}(2, \boldsymbol{Z})$ acts on $\boldsymbol{D}(X)$ modulo the shift.

Remark 3.15. The relation between automorphisms of $\boldsymbol{D}(X)$ and semi-homogeneous vector bundles on X will be discussed in [5]. Some applications of (3.14) to the vector bundles on an abelian surface will be treated in a forthcoming paper.

§4. Picard sheaves

In this section as an application of Fourier functor, we calculate the cohomology of Picard sheaves and determine the moduli of deformations of Picard sheaves.

Let C be a nonsingular complete curve of genus ≥ 2. We fix a point c of C and put $\xi_n = \mathcal{O}_C(n(c))$. We identify C with the subvariety $\{(x)-(c) \mid x \in C\}$ of the Jacobian variety $X = J(C)$ and also identify a sheaf on C with a sheaf on X supported by C. The subvariety $W_i = \overbrace{C + \cdots + C}^{i}$ of X is said to be the distinguished subvariety of dimension i, for $0 \leq i \leq g-1$.

W_{g-1} is a divisor of X and (X, L) is a principally polarized abelian variety of dimension g, where $L = \mathcal{O}_X(W_{g-1})$. We denote the canonical point of (X, L) by κ, that is, $\kappa - W_{g-1} = W_{g-1}$.

DEFINITION 4.1. The sheaf $F_n = R^1\mathcal{S}(\xi_n)$ is called a Picard sheaf of rank $g - n - 1$.

Our definition of F_n is same as that in [8], because a normalized Poincaré bundle $\mathcal{L}$ on $C \times X$ is isomorphic to $\mathcal{P}|_{C\times X}$. Replacing c by another point $c' \in C$, we get another Picard sheaf F'_n.

PROPOSITION 4.2. $F'_n \cong T^*_{n(c'-c)}F_n \otimes P_{c-c'}$

Proof.
$$\begin{aligned} F'_n &= R^1\mathcal{S}(T^*_{c'-c}\xi'_n) \\ &\cong R^1\mathcal{S}(\xi'_n) \otimes P_{c-c'} \\ &\cong R^1\mathcal{S}(\xi_n \otimes P_{nc'-nc}) \otimes P_{c-c'} \\ &\cong T^*_{n(c'-c)}F_n \otimes P_{c-c'}\ . \end{aligned}$$
q.e.d.

We summarize some fundamental properties of F_n.

THEOREM 4.2 (See [8].)

(1) *F_n is zero for $n > 2g-2$.* Supp F_n *is* $\kappa - W_{2g-2-n}$ *for* $g - 1 \leqq n \leqq 2g-2$. Supp F_n *is X and the rank of F_n at the generic point of X is $g - n - 1$ for $n < g - 1$. F_n is locally free for $n < 0$.*

(2) *The i-th Chern class $c_i(F_n)$ is rationally equivalent to W_{g-i} for $i \leqq g - 1$. Especially,* $\det F_n \cong L$ *for $n \leq g - 1$.*

(3) *The projective fibre space $\boldsymbol{P}(\alpha^*F_n)$ associated with α^*F_n is isomorphic to the $(2g-2-n)$-th symmetric product* $\mathrm{Sym}^{2g-2-n}(C)$. *Where α is the automorphism of X for which $\alpha(x) = \kappa - x$.*

By the following proposition, we can apply the theory of Fourier functor to Picard sheaves.

PROPOSITION 4.3. (1) *For $n \leq g - 1$, F_n is $\hat{\xi}_n$, W.I.T. holds for F_n, $i(F_n) = g - 1$ and $\hat{F}_n \cong (-1_X)^*\xi_n$.*

(2) *For $n \geq g - 1$, F_n is isomorphic to $\alpha^* \mathcal{E}xt^1_{\mathcal{O}_X}(F_{2g-2-n}, \mathcal{O}_X)$ and $\mathcal{S}(\xi_n) \cong \alpha^* \mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n}, \mathcal{O}_X)$.*

(3) *$\mathcal{E}xt^i_{\mathcal{O}_X}(F_n, \mathcal{O}_X)$ is zero for $i \geq 2$, $n \geq g - 1$.*

Proof. Since $\dim \mathrm{Supp}\, \xi_n = 1$, $R^i\mathcal{S}(\xi_n)$ is zero for $i > 1$. On the other hand, $\mathcal{S}(\xi_n)$ is zero for $n < g$ ([8] § 3). Hence, when $n < g$, W.I.T. holds for

ξ_n and $i(\xi_n)=1$. Therefore (1) follows from Corollary 2.4. Since $\Delta(\mathcal{O}_C)$ is isomorphic to $K_C[1]\cong\xi_{2g-2}\otimes P_\kappa[1]$, ξ_n is isomorphic to $\Delta(\xi_{2g-2-n}\otimes P_\kappa)[-1]$. Hence, by (3.8), we have

$$\begin{aligned}\boldsymbol{R}\mathcal{S}(\xi_n) &\cong \boldsymbol{R}\mathcal{S}(\Delta(\xi_{2g-2-n}\otimes P_\kappa)[-1])\\ &\cong ((-1_X)^*\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}\otimes P_\kappa))[-g-1]\\ &\cong (-1_X)^*T_\kappa^*(\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}))[-g-1]\\ &\cong \alpha^*(\Delta\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n}))[-g-1]\ .\end{aligned}$$

When $n\geq g-1$, $\boldsymbol{R}\mathcal{S}(\xi_{2g-2-n})$ is isomorphic to $F_n[-1]$ by (1). Hence we have

$$\begin{aligned}\boldsymbol{R}\mathcal{S}(\xi_n) &\cong \alpha^*(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n}[-1],\mathcal{O}_X)[g])[-g-1]\\ &\cong \alpha^*\boldsymbol{R}\,\mathcal{H}om_{\mathcal{O}_X}(F_{2g-2-n},\mathcal{O}_X)\ .\end{aligned}$$

Therefore, $R^i\mathcal{S}(\xi_n)$ is isomorphic to $\alpha^*\,\mathcal{E}xt^i_{\mathcal{O}_X}(F_{2g-2-n},\mathcal{O}_X)$, which shows (2) and (3). q.e.d.

Applying the result in § 3 and § 4, we have the following three propositions.

PROPOSITION 4.4 (Cohomology of Picard sheaf). *Assume that* $n\leq g-1$.

(1) $h^g(X,F_n\otimes P_x)=0$ *for all* $x\in X$. *When* $0\leq i\leq g-1$, *we have*

$$h^i(X,F_n\otimes P_X)=\begin{cases}\binom{g-1}{i} & \text{if } -x\in C\\ 0 & \text{if } -x\notin C\end{cases}$$

(2) $h^i(X,F_n\otimes L^{-1}\otimes P_x)=h^{i-g+1}(C,\xi_{n+g}\otimes P_{\kappa+x})$ *for all* $x\in X$

(3) $h^i(X,F_n\otimes L\otimes P_x)=\begin{cases}2g-n-1 & \text{for } i=0\\ 0 & \text{for } i>0\end{cases}$.

Proof. By Proposition 2.7, $H^i(X,F_n\otimes P_x)$ is isomorphic to $\mathrm{Ext}^{i+1}_{\mathcal{O}_X}(k(x),(-1_X)^*\xi_n)$, which shows (1). By Corollary 2.5 and (5) of Theorem 3.13, $H^i(X,F_n\otimes L^{-1}\otimes P_x)\cong\mathrm{Ext}^i_{\mathcal{O}_X}(L\otimes P_{-x},F_n)$ is isomorphic to $\mathrm{Ext}^{i-g+1}_{\mathcal{O}_X}(\widehat{L\otimes P_{-x}},\hat{F}_n)\cong\mathrm{Ext}^{i-g+1}_{\mathcal{O}_X}(L^{-1}\otimes P_x,(-1_X)^*\xi_n)$. Since $L|_C\cong\xi_g$, we have $H^i(X,F_n\otimes L^{-1}\otimes P_x)\cong H^{i-g+1}(X,L\otimes P_{-x}\otimes(-1_X)^*\xi_n)\cong H^{i-g+1}(C,\xi_n\otimes(-1_X)^*L|_C\otimes P_x)\cong H^{i-g+1}(C,\xi_{n+g}\otimes P_{\kappa+x})$, which shows (2). In a similar manner, we have $H^i(X,F_n\otimes L\otimes P_x)\cong\mathrm{Ext}^{i+1}_{\mathcal{O}_X}((-1_x)^*(L\otimes P_x),(-1_X)^*\xi_n)\cong H^{i+1}(C,\xi_{n-g}\otimes P_{-x})$. Since $\deg\xi_{n-g}=n-g<0$, we have by Riemann-Roch theorem, $h^0(C,\xi_{n-g}\otimes P_{-x})=0$ and $h^1(C,\xi_{n-g}\otimes P_{-x})=2g-n-1$. Hence we have proved (3). q.e.d.

PROPOSITION 4.5 (Local property of Picard sheaf).

$$\mathrm{Tor}_i^{\mathcal{O}_X}(F_n, k(x)) \cong \begin{cases} H^1(C, \xi_n \otimes P_x) & i = 0 \\ H^0(C, \xi_n \otimes P_x) & i = 1 \\ \mathrm{Tor}_{i-2}^{\mathcal{O}_X}(\mathcal{S}(\xi_n), k(x)) & i \geq 2 \end{cases}$$

Proof. Assume that $n \leq g-1$. Then we have by Proposition 2.7, $\mathrm{Ext}^i_{\mathcal{O}_X}(k(x), F_n) \cong H^{i-g+1}(X, (-1_X)^*\xi_n \otimes P_{-x})$. Hence by the duality theorem, $\mathrm{Tor}^i_{\mathcal{O}_X}(F_n, k(x))$ is isomorphic to $\mathrm{Ext}^{g-i}_{\mathcal{O}_X}(k(x), F_n) \cong H^{1-i}(C, \xi_n \otimes P_x)$, which proves our assertion for $n \leq g-1$ because $\mathcal{S}(\xi_n)$ is zero for $n \leq g-1$. By what we have shown, the minimal resolution of $F_n \otimes \mathcal{O}_{X,x}$ is

$$0 \longleftarrow F_n \otimes \mathcal{O}_{X,x} \longleftarrow \mathcal{O}_{X,x} \underset{k}{\otimes} H^1(C, \xi_n \otimes P_x) \longleftarrow \mathcal{O}_{X,x} \otimes H^0(C, \xi_n \otimes P_x) \longleftarrow 0.$$

By (2) of Proposition 4.3, the sequence

(4.6) $0 \leftarrow F_{2g-2-n} \otimes \mathcal{O}_{X,\alpha(x)} \leftarrow \mathcal{O}_{X,\alpha(x)} \otimes H^0(C, \xi_n \otimes P_x)^\vee \leftarrow \mathcal{O}_{X,\alpha(x)} \otimes H^1(C, \xi_n \otimes P_x)^\vee \leftarrow \mathcal{S}(\xi_{2g-2-n}) \otimes \mathcal{O}_{X,\alpha(x)} \leftarrow 0$ is exact.

It is easy to see that the left three terms of (4.6) is the minimal resolution of $F_{2g-2-n} \otimes \mathcal{O}_{X,\alpha(x)}$. Hence $\mathrm{Tor}_i^{\mathcal{O}_X}(F_{2g-2-n}, k(\alpha(x)))$ is isomorphic to $H^i(C, \xi_n \otimes P_x)^\vee \cong H^{1-i}(C, K_C \otimes \xi_{-n} \otimes P_{-x}) \cong H^{1-i}(C, \xi_{2g-2-n} \otimes P_{\alpha(x)})$ for $i = 0, 1$ and isomorphic to $\mathrm{Tor}_{i-2}^{\mathcal{O}_X}(\mathcal{S}(\xi_{2g-2-n}), k(\alpha(x)))$. Hence our assertion has been proved for $n \geq g-1$, too. q.e.d.

PROPOSITION 4.7. *Assume that $n \leq g-1$. Then I.T. holds for $F_n \otimes L$, its index is zero and $\widehat{F_n \otimes L} \cong \alpha^* F_{n-g} \otimes L^{-1}$.*

Proof. The first half has been proved in (3) of Proposition 4.4. By (6) of Theorem 3.13, we have $(\otimes L \circ \boldsymbol{R}\mathcal{S} \circ \otimes L)(\widehat{F_n \otimes L}) = (\otimes L \circ \boldsymbol{R}\mathcal{S})^3(\xi_n)[1] \cong \xi_n[1-g]$. Hence $\widehat{F_n \otimes L}$ is isomorphic to $(\otimes L^{-1} \circ \boldsymbol{R}\mathcal{S}^{-1} \circ \otimes L^{-1})(\xi_n)[1-g] \cong ((-1_X)^* \boldsymbol{R}\mathcal{S}(\xi_n \otimes L^{-1}) \circ \otimes L^{-1})[1] \cong ((-1_X)^* \boldsymbol{R}\mathcal{S}(\xi_{n-g} \otimes P_\iota) \otimes L^{-1})[-1] \cong \alpha^* F_{n-g} \otimes L^{-1}$. q.e.d.

Next we consider the moduli of deformations of Picard sheaves. Define the functor $\mathcal{S}pl_X$ from the category of schemes (of finite type over k) into the category of sets by

$\mathcal{S}pl_X(T) = \{E \mid E$ is a T-flat coherent $\mathcal{O}_{X\times T}$-module and $E_t = E|_{X\times t}$ is simple for every $t \in T\}/_\sim$,

for every scheme T, where $E \sim E'$ if and only if $E \cong E' \otimes_{\mathcal{O}_T} L$ for some line bundle L on T, and $\mathcal{S}pl_X(f)$: $\mathcal{S}pl_X(T') \to \mathcal{S}pl_X(T)$ is the usual pull back for every morphism f: $T \to T'$. For every simple coherent sheaf F

on X, $\mathscr{S}pl_X^F$ denotes the connected component of $\mathscr{S}pl_X$ containing F. The following is the main theorem in this section.

THEOREM 4.8. *Assume that $n \leq g-1$ and $(*)$ $g(C)=2$ or C is not hyperelliptic. Then $\mathscr{S}pl_X^{F_n}$ is represented by $X \times X$ and the coherent sheaf $\tilde{F}_n = p_{12}^* m^* F_n \otimes p_{13}^* \mathscr{P}$ on $X \times (X \times X)$.*

Let $A_F\colon X \times \hat{X} \to \mathscr{S}pl_X^F$ be the morphism of functors such that $A_F(f, g) = T_f^* F_T \otimes P_g$ for every scheme T and T-valued point (f, g) of $X \times \hat{X}$, where we always identify a scheme S and the contravariant functor h_S on the category of schemes for which $h_S(T)$ is the set of T-valued points of S, i.e., morphisms from T to S. Theorem 4.8 says that A_F is an isomorphisms for $F = F_n$ $(n \leq g-1)$ under the assumption $(*)$. The following three lemmas are essential for the proof of the theorem.

LEMMA 4.9. *Picard sheaf F_n $(n \leq g-1)$ is simple and we have*

$$\begin{aligned} \dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(F_n, F_n) &= 3g-2 \qquad \textit{if } C \textit{ is hyperelliptic} \\ &= 2g \qquad \textit{otherwise}\,. \end{aligned}$$

Proof. By Corollary 2.5 and Proposition 4.3, it suffices to show the equality for $\dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n)$. Since there is a spectral sequence

$$H^i(X, \mathscr{E}xt^j_{\mathcal{O}_X}(\xi_n, \xi_n)) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{O}_X}(\xi_n, \xi_n)\,,$$

we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n)) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n) \longrightarrow H^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n)) \\ &\longrightarrow H^2(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n)) \longrightarrow 0\,. \end{aligned}$$

Since $\mathscr{E}nd_{\mathcal{O}_X}(\xi_n)$ is isomorphic to $\mathcal{O}_C$, $H^2(X, \mathscr{E}nd_{\mathcal{O}_X}(\xi_n))$ is zero and we have

$$\begin{aligned} \dim_k \operatorname{Ext}^1_{\mathcal{O}_X}(\xi_n, \xi_n) &= h^1(C, \mathcal{O}_C) + h^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n)) \\ &= g + h^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\xi_n, \xi_n))\,. \end{aligned}$$

SUBLEMMA. Let ξ be a line bundle on a subscheme C of X. Then there is a canonical isomorphism $\varphi\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \cong \mathscr{E}xt^i_{\mathcal{O}_X}(\xi, \xi)$ for every i.

Since $\mathscr{E}xt$ commutes with localizations, it suffices to give the canonical isomorphism in the case X is affine and $\xi \cong \mathcal{O}_C$. Let $f\colon \mathcal{O}_C \cong \xi$ be an isomorphism. Since $\mathscr{E}xt^i_{\mathcal{O}_X}(*, *)$ is a bifunctor, we have two isomorphisms

$$\begin{aligned} f_a &= \mathscr{E}xt^i_{\mathcal{O}_X}(\mathrm{id}, f)\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \cong \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \xi) \\ f_b &= \mathscr{E}xt^i_{\mathcal{O}_X}(f, \mathrm{id})\colon \mathscr{E}xt^i_{\mathcal{O}_X}(\xi, \xi) \cong \mathscr{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \xi)\,. \end{aligned}$$

Put $\varphi = f_b^{-1} \circ f_a$. If g: $\mathcal{O}_C \rightrightarrows \xi$ is another isomorphism, then there is a unit $\bar{u}$ of $\mathcal{O}_C$ such that $g = f \circ (\times \bar{u})$. There is an affine neighbourhood Y of C and a unit u of $\mathcal{O}_Y$ whose image by the natural homomorphism $\mathcal{O}_Y \to \mathcal{O}_C$ is $\bar{u}$. Since $(g_b^{-1} \circ g_a)|_Y = ((\times u) \circ f_b|_Y)^{-1} \circ (f_a|_Y \circ (\times u)) = \varphi|_Y$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \otimes \mathcal{O}_{X,x}$ is zero for every $x \notin Y$, φ does not depend on the choice of the isomorphism f. This proves the sublemma.

By this sublemma, we have only to compute the dimension of

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C)) \cong H^0(C, N_{C/X}) .$$

There is a natural exact sequence

$$0 \longrightarrow (N_{C/X})^\vee \longrightarrow \Omega_X \otimes \mathcal{O}_C \longrightarrow K_C \longrightarrow 0 .$$

Since Ω_X is trivial, tensoring K_C, we have the exact sequence

$$0 \longrightarrow (N_{C/X})^\vee \otimes K_C \longrightarrow K_C^{\oplus g} \longrightarrow K_C^{\otimes 2} \longrightarrow 0 .$$

In the long exact sequence

$$\begin{aligned} H^0(K_C)^{\oplus g} \xrightarrow{\alpha} H^0(K_C^{\otimes 2}) &\longrightarrow H^1((N_{C/X})^\vee \otimes K_C) \\ &\longrightarrow H^1(K_C)^{\oplus g} \longrightarrow H^1(K_C^{\otimes 2}) \longrightarrow 0 , \end{aligned}$$

the map α is just the natural map $H^0(K_C) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2})$. By Riemann-Roch theorem, we have $h^0(N_{C/X}) = h^1((N_{C/X})^\vee \otimes K_C) = \dim \operatorname{Coker} \alpha + gh^1(K_C) - h^1(K_C^{\otimes 2}) = \dim \operatorname{Coker} \alpha + g$. In the case C is hyperelliptic, $\dim \operatorname{Coker} \alpha$ is $g-2$ and otherwise α is surjective by a theorem due to Noether, [3] p. 502, which completes our proof. q.e.d.

LEMMA 4.10. *If* $n \leq g - 1$ *and* $T_x^* F_n \otimes P_y \cong T_{x'}^* F_n \otimes P_{y'}$ *for* x, x', y, $y' \in X$, then $x = x'$ and $y = y'$.

Proof. The assumption implies that $P_x \otimes T_{-y}^* \xi_n \cong P_{x'} \otimes T_{-y'}^* \xi_n$ by (3.1). Since $\operatorname{Supp} \xi_n = C$, y equals to y' and since $\operatorname{Pic}^\circ X \to \operatorname{Pic}^\circ C$ is injective, x is equal to x'. q.e.d.

We denote the tangential map of A_F at $(0, \hat{0})$ by α_F. Since the tangent spaces of X at 0, of $\hat{X}$ at $\hat{0}$ and of $\mathcal{S}pl_X$ at F are identified with $H^0(X, T_X)$, $H^1(X, \mathcal{O}_X)$ and $\operatorname{Ext}^1_{\mathcal{O}_X}(F, F)$, respectively, α_F is a k-linear map from $H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X)$ into $\operatorname{Ext}^1_{\mathcal{O}_X}(F, F)$.

LEMMA 4.11. α_{F_n} *is injective for the Picard sheaf* F_n $(n \leq g - 1)$.

Assume that W.I.T. holds for F. By (3.1), we have $\widehat{T_x^* F \otimes P_y} \cong T_y^* \hat{F}$

$\otimes P_{-x}$. This is easily extended to scheme valued points and we have $T_f^* \widehat{F_S \otimes P_g} \cong T_g^* \hat{F}_S \otimes P_{-f}$ for every scheme S and S-valued point (f, g) of $X \times \hat{X}$. As a special case $S = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$, we have

PROPOSITION 4.12. *Assume that W.I.T. holds for a coherent sheaf F on X. Then the diagram*

$$\begin{array}{ccc} \alpha_F\colon H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) \\ \downarrow \wr\, j & & \downarrow \wr\, R\mathcal{S} \\ \alpha_{\hat{F}}\colon H^1(X, \mathcal{O}_X) \oplus H^0(X, T_X) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{F}) \\ \| \wr & & \\ H^0(\hat{X}, T_{\hat{X}}) \oplus H^1(\hat{X}, \mathcal{O}_{\hat{X}}) & & \end{array}$$

is commutative, where $j(a, b) = (b, -a)$.

By this proposition, the injectivity of α_{F_n} is equivalent to that of α_{ξ_n}. Let

$$0 \longrightarrow H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F)) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) \xrightarrow{\ \varepsilon\ } H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(F, F))$$

be the exact sequence obtained from the local-global spectral sequence with respect to Ext. The following proposition is easily verified.

PROPOSITION 4.13. (1) *$\alpha_F(H^1(X, \mathcal{O}_X))$ is contained in $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F))$.*

(2) *The diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & T_{X \times \hat{X}, (0, \hat{0})} & \longrightarrow & H^0(X, T_X) & \longrightarrow & 0 \\ & & \downarrow \beta_F & & \downarrow \alpha_F & & \downarrow \varepsilon \circ \gamma_F & & \\ 0 & \longrightarrow & H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(F)) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{O}_X}(F, F) & \xrightarrow{\ \varepsilon\ } & H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(F, F)) & & \end{array}$$

is commutative, where β_F and γ_F are the restrictions of α_F to $H^1(X, \mathcal{O}_X)$ and $H^0(X, T_X)$, respectively.

(3) *β_F is equal to $H^1(i)$, where i is the natural homomorphism from $\mathcal{O}_X$ into $\mathcal{E}nd_{\mathcal{O}_X}(F)$.*

(4) *$H^0(X, T_X)$ is the set of derivations of $\mathcal{O}_X$. For $D \in \operatorname{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$, $\gamma_F(D)$ is the extension class of*

$$0 \longrightarrow F \longrightarrow F_D \longrightarrow F \longrightarrow 0\,,$$

where F_D is $F \oplus F$ as a sheaf of abelian groups and regarded as an $\mathcal{O}_X$-module by $a(m, m') = (am + D(a)m', am')$ for every $a \in \mathcal{O}_X$ and $(m, m') \in F \oplus F$.

(5) *$\varepsilon \circ \gamma_F$ is equal to $H^0(\tilde{\gamma}_F)$, where $\tilde{\gamma}_F$ is an $\mathcal{O}_X$-homomorphism from T_X into $\mathcal{E}xt^1_{\mathcal{O}_X}(F, F)$ such that $\tilde{\gamma}_F|_U$ is equal to γ_{F_U}: $\mathrm{Der}_k(\mathcal{O}_U, \mathcal{O}_U) \to \mathrm{Ext}^1_{\mathcal{O}_U}(F_U, F_U)$ for every affine open subset U of X.*

(6) *If Y is a subscheme of X and F is a line bundle on Y, then $\tilde{\gamma}_F$ is the composition of the natural morphisms $T_X \to T_X \otimes \mathcal{O}_Y$ and $T_X \otimes \mathcal{O}_Y \to N_{Y/X} \cong \mathcal{E}xt^1_{\mathcal{O}_X}(F, F)$.*

In the case F is ξ_n, a line bundle on C, $\beta_F = H^1[\mathcal{O}_X \to \mathcal{O}_C]$ is an isomorphism, $H^0[T_X \to T_X \otimes \mathcal{O}_C]$ is also an isomorphism and by the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_X \otimes \mathcal{O}_C \longrightarrow N_{C/X} \longrightarrow 0\,,$$

$H^0[T_X \otimes \mathcal{O}_C \to N_{C/X}]$ is injective. Hence by (6) of Proposition 4.13, $\varepsilon \circ \gamma_F = H^0(\tilde{\gamma}_F)$ is an injection. Therefore by the diagram (2) of the proposition, α_F is an injection. This completes the proof of Lemma 4.11.

For the proof of Theorem 4.8, we need the following general facts about the flat deformation of a simple coherent sheaf.

(4.14) (Relative representability of $\mathcal{S}pl$) Let f: $V \to S$ be a proper integral morphism and F and G coherent $\mathcal{O}_V$-modules. Assume that F is S-flat and $F \otimes k(s)$ is simple for every $s \in S$. Then there exists a subscheme W of S such that for every morphism α: $T \to S$, F_T is isomorphic to $G_T \otimes_{\mathcal{O}_T} L$ with some line bundle L on T if and only if α factors through the inclusion $W \hookrightarrow S$. We call W the maximal subscheme over which F and G are isomorphic to each other.

(4.15) (Pro-representability of $\mathcal{S}pl$) Let F be a simple coherent $\mathcal{O}_X$-module. The functor $\mathcal{D}$ on artinian local rings A over k such that

$\mathcal{D}(A) = \{E | E$ is an A-flat coherent $\mathcal{O}_{X_A}$-module such that $E \otimes_A A/m$ is isomorphic to $F\}/\mathrm{isom}$.

is representable by a complete local ring R whose Zariski tangent space t_R is canonically isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F)$. We call R the local moduli of F.

(4.16) (Jumping never happens) Let E be an element of $\mathcal{S}pl_X(T)$. If $E|_{X \times t} \cong F$ for every closed point t of an open dence subset U of T, then $E|_{X \times t} \cong F$ for every $t \in T$.

The proofs are not so difficult and those of (4.14) and (4.15) are similar to the case of simple vector bundles. The stronger fact that the étale sheafification of $\mathscr{S}pl_{V/S}$ is representable by an algebraic space has been proved in [1]. Since the fact does not make our business so easy, we prove our theorem directly by (4.14), (4.15) and (4.16).

Step I. The functor A_{F_n} is injective.

Let f and g be two morphism from T to $X\times X$ such that $A_{F_n}\circ f = A_{F_n}\circ g$. Since $X\times X$ is a group scheme and A_{F_n} is an $X\times X$-morphism with respect to the natural action of $X\times X$ to $\mathscr{S}pl_X^{F_n}$, we may assume that g is the constant map to $(0,0)$. Let $\Phi(F_n)$ be the maximal subscheme of $X\times X$ over which $\tilde{F}_n$ and $p_1^*F_n$ on $X\times(X\times X)$ are isomorphic to each other. Since $A_{F_n}\circ f$ is the constant map to F_n by our assumption, f factors the inclusion $\Phi(F_n)\subsetneq X\times X$. By Lemma 4.10, $\Phi(F_n)$ is supported by the origin $(0,0)$ and by Lemma 4.11, the tangent space of $\Phi(F_n)$ is zero. Hence $\Phi(F_n)$ is $(0,0)$ and f is zero. (It is easily seen that $\Phi(F_n)$ is a group subscheme of $X\times X$. Hence Lemma 4.11 is not necessary for the proof of our assertion in the case $\operatorname{char} k = 0$.)

Step II. A_{F_n} is an open immersion.

A_{F_n} induces the homomorphism $f\colon R\to Q$ of complete local rings, where $(R,\mathfrak{m})$ is the local moduli of F_n and $(Q,\mathfrak{n})$ is the completion of $\mathcal{O}_{X\times X,(0,0)}$. Since A_{F_n} is injective, the fibre $Q/\mathfrak{m}Q$ of f is isomorphic to $Q/\mathfrak{n}$. Hence f is a surjection. By Lemma 4.9, we have

$$2g = \dim Q \leq \dim R \leq \dim t_R = 2g\ .$$

Hence $\dim R = 2g$, R is regular and f is a bijection. For every morphism $g\colon T\to\mathscr{S}pl_X^{F_n}$, by virtue of (4.14), $T\times_{\mathscr{S}pl_X^{F_n}}(X\times X)$ is representable by a scheme U. By what we have shown, $\hat{\mathcal{O}}_{T,h(u)}\to\hat{\mathcal{O}}_{U,u}$ is an isomorphism for every $u\in U$.

$$\begin{array}{ccc} X\times X & \xrightarrow{A_{F_n}} & \mathscr{S}pl_X^{F_n} \\ \uparrow{\scriptstyle \ell} & & {\scriptstyle g}\uparrow \\ U & \xrightarrow[h]{} & T \end{array} \qquad \text{cartesian}$$

Hence h is étale. By Step I, h is an open immersion.

Step III. A_{F_n} is a closed immersion.

In the above situation, we have to show that U is a union of connected components of T. Hence we may assume that T is irreducible and it suffices to prove that the set of k-rational points $U(k)$ of U is empty or equal to $T(k)$. Hence we may also assume that T is reduced. Assume that $U(k) \neq \phi$. Since $X \times X$ is an abelian variety, every rational map from T to $X \times X$ is a morphism. Hence there is a morphism $e = (e_1, e_2)$: $T \to X \times X$ whose restriction to U is equal to ℓ. Let μ: $X \times X \times \mathcal{S}pl_X^{F_n} \to \mathcal{S}pl_X^{F_n}$ be the natural action of $X \times X$ on $\mathcal{S}pl_X^{F_n}$. Put $c = [T \xrightarrow{(-e, g)} X \times X \times \mathcal{S}pl_X^{F_n} \xrightarrow{\mu} \mathcal{S}pl_X^{F_n}]$. Then $c(U(k)) = \{F_n\}$ and hence by virtue of (4.16), we have $c(T(k)) = \{F_n\}$, that is, $g(a) = T_{e_1(a)}^* F_n \otimes P_{e_2(a)}$ for every $a \in T(k)$. Hence $U(k)$ is equal to $T(k)$.

Step IV. A_{F_n} is an isomorphism.

It suffices to show that $A_{F_n}(k)$: $(X \times X)(k) \to \mathcal{S}pl_X^{F_n}(k)$ is a surjection. By the definition, $\mathcal{S}pl_X^{F_n}$ is connected. Hence, for every $F \in \mathcal{S}pl_X^{F_n}(k)$, there exist a connected scheme T and a morphism g: $T \to \mathcal{S}pl_X^{F_n}$ such that $g(T(k))$ contains both F and F_n. By what we have shown in Step II and Step III, g factor through A_{F_n}. Hence F is contained in $\operatorname{Im} A_{F_n}(k)$.

We have completed the proof of Theorem 4.8.

Remark 4.17. Even if the condition (*) does not hold, $A_{F_n}(k)$ is bijective for $n \leq g-1$. But if C is hyperelliptic and $g(C) \geq 3$, then the dimension of the tangent space of $\mathcal{S}pl_X^{F_n}$ is greater than $2g$, hence $\mathcal{S}pl_X^{F_n}$ is not reduced.

§ 5. A characterization of Picard sheaf

In this section we give a characterization of the Picard sheaf in the case $g(C) = 2$.

Let ξ_n be the same as in the beginning of § 4. There is a natural exact sequence

$$0 \longrightarrow \xi_{n-1} \longrightarrow \xi_n \longrightarrow k(0) \longrightarrow 0 .$$

This gives the exact sequence

$$0 \longrightarrow \mathcal{S}(\xi_{n-1}) \longrightarrow \mathcal{S}(\xi_n) \longrightarrow \mathcal{O}_X \xrightarrow{f} F_{n-1} \longrightarrow F_n \longrightarrow 0 .$$

If $n \leq g-1$, then $\mathcal{S}(\xi_n)$ is zero ([8] § 3). Hence, for $n \leq g-1$, we have the exact sequence

$$(5.1) \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{f} F_{n-1} \longrightarrow F_n \longrightarrow 0 .$$

By (1) of Proposition 4.4, both $\dim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F_n) = h^0(F_n)$ and $\dim \mathrm{Ext}^1_{\mathcal{O}_X}(F_n, \mathcal{O}_X) = h^{g-1}(F_n)$ is equal to 1 for $n \leq g-1$. Hence we have

LEMMA 5.2. *Assume that $n \leq g-1$. Then f is the unique (up to constant multiplications) nonzero homomorphism from $\mathcal{O}_X$ into F_n and* (5.1) *is the unique nontrivial extension of F_n by $\mathcal{O}_X$.*

We denote the set $\{T_x^* F_n \otimes P_y | x, y \in X\}$ by Φ_n.
The above lemma is generalized for members of $\mathrm{Pic}^\circ X$ and Φ_n.

PROPOSITION 5.3. *Assume that $n \leq g-1$.*

(1) *Every nonzero homomorphism f from $P_x \in \mathrm{Pic}^\circ X$ to $F \in \Phi_{n-1}$ is injective and* Coker f *is isomorphic to a member of Φ_n.*

(2) *If $P_x \in \mathrm{Pic}^\circ X$, $F \in \Phi_n$ and the exact sequence*

$$0 \longrightarrow P_x \longrightarrow F' \longrightarrow F \longrightarrow 0$$

does not split, then F' is isomorphic to a member of Φ_{n-1}.

Proof. We prove only (2), because (1) can be proved in a quite similar manner. First we may assume that $F = F_n$. Since $\mathrm{Ext}^1_{\mathcal{O}_X}(F_n, P_x) \neq 0$, we have by (1) of Proposition 4.4, that x belongs to C and $\dim \mathrm{Ext}^1_{\mathcal{O}_X}(F_n, P_x)$ is equal to 1. Since $x \in C$, there is a surjection $\xi_n \to k(x)$ and we have the non-splitting exact sequence

$$0 \longrightarrow \xi_{n-1} \otimes P_x \longrightarrow \xi_n \longrightarrow k(x) \longrightarrow 0 .$$

Operating $\boldsymbol{R}\mathcal{S}$, we have the exact sequence

$$0 \longrightarrow P_x \longrightarrow T_x^* F_{n-1} \longrightarrow F_n \longrightarrow 0 .$$

Since this does not split, F' is isomorphic to $T_x^* F_{n-1}$. q.e.d.

For every nontorsion coherent sheaf F on X, let $\mu(F)$ denote the rational number $r(F)^{-1} \deg (\det F)|_C$. Umemura has showed that F_n is μ-stable for $n \leq g-1$ in the case $g(C) = 2$ ([9]). The following theorem says that the converse is also true.

THEOREM 5.4. *Assume that $g(C) = 2$ and F is a torsion free coherent sheaf with $r(F) = r \geq 1$, $\det F$ algebraically equivalent to $\mathcal{O}_X(C)$ and $\chi(F)$ zero. Then the following conditions are equivalent to one another:*

1) *F is μ-stable, i.e., $\mu(E) < \mu(F)$ for every $E \subsetneqq F$* with $r(E) < r$.

1′) *F is μ-semi-stable, i.e., $\mu(E) \le \mu(F)$ for every $E \subseteq F$.*

2) $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ *is zero for every $P \in \mathrm{Pic}^\circ X$. If H is a homogeneous vector bundle with $r(H) < r$ contained in F, then the quotient F/H is torsion free.*

3) *$F \cong T_x^* F_{1-r} \otimes P$ for some $x \in X$ and $P \in \mathrm{Pic}^\circ X$.*

Proof. Obviously 1) implies 1′). Assume that F is μ-semi-stable and H is a homogeneous vector bundle with $r(F) < r$ contained in F. Since $\mu(F) = 2/r$ is greater than $\mu(P)=0$, $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is zero for every $P \in \mathrm{Pic}^\circ X$. Let $f\colon F \to F/H$ be the projection and T the torsion part of F/H. Then $H' = f^{-1}(T)$ contains H and $r(H')$ is equal to $r(H)$. We have a nonzero homomorphism $\det H \to \det H'$. Hence $\det H' \cong \det H \otimes \mathcal{O}_X(D)$ for some divisor $D \ge 0$. Since $\det H \in \mathrm{Pic}^\circ X$ and F is μ-semi-stable, we have

$$\frac{(\mathcal{O}_X(D).\, \mathcal{O}_X(C))}{r(H)} = \frac{(\det H'.\, \mathcal{O}_X(C))}{r(H')} \le \mu(F) = \frac{2}{r}\,.$$

Since $D \ge 0$, $(\mathcal{O}_X(D).\, \mathcal{O}_X(C))$ is not less than zero and different from one ([9] Lemma 3.5). Hence by the inequality above $(\mathcal{O}_X(D).\, \mathcal{O}_X(C))$ is zero. Hence $D = 0$ and $\det H \to \det H'$ is an isomorphism. Since H is locally free, H' is isomorphic to H. Therefore T is zero. Hence 1′) implies 2). 3) implies 1), because if F is μ-stable, so is $T_x^* F \otimes P$ for every $x \in X$ and $P \in \mathrm{Pic}\, X$. Hence we have only to show that 2) implies 3). We prove it by induction on r.

Case $r = 1$. $\mathrm{Sym}^2 C \to X$ is the blowing up whose center is the canonical point κ. Hence, by (3) of Proposition 4.2, F_0 is isomorphic to $N \otimes \mathfrak{m}_{X,0}$ with some line bundle N, where $\mathfrak{m}_{X,0}$ is the maximal ideal of $\mathcal{O}_X$ at 0. By (2) of Proposition 4.2, N is isomorphic to $\mathcal{O}_X(C)$. Since $r(F) = 1$ and F is torsion free, F is contained in $\det F$. By the assumption, $\det F \cong \mathcal{O}_X(C) \otimes P$ for some $P \in \mathrm{Pic}^\circ X$. Since length $(\det F/F) = \chi(\det F) - \chi(F) - 1$, $\det F/F$ is isomorphic to the one dimensional sky-scraper sheaf $k(x)$ supported by a point $x \in X$. Hence F is isomorphic to $\det F \otimes \mathfrak{m}_{X,x} \cong T_x^* F_0 \otimes P \otimes P_{-x}$.

Case $r \ge 2$. We need the following easy but useful lemma.

LEMMA 5.5. *Let F be a nonzero coherent sheaf on an abelian surface. If $\chi(F)$ is zero, then* $\mathrm{Hom}_{\mathcal{O}_X}(P, F)$ *or* $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ *is not zero for some* $P \in \mathrm{Pic}^\circ X$.

Assume the contrary. Since $\dim \mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is equal to $h^2(F \otimes P^{-1})$

by virtue of the duality theorem and since $\chi(F\otimes P^{-1})$ is zero, $h^1(F\otimes P^{-1})$ is zero for all $P\in \mathrm{Pic}^\circ X$. Hence $R^i\mathscr{S}(F)$ is zero for every i. This means that $\boldsymbol{R}\mathscr{S}(F)$ is zero. Therefore by virtue of Theorem 2.2, F is zero. This shows Lemma 5.5.

By the assumption and the above lemma, $\mathrm{Hom}_{\mathcal{O}_X}(P, F)$ is not zero for some $P\in \mathrm{Pic}^\circ X$. Let $f\colon P\to F$ be a nonzero homomorphism. Since F is torsion free, f is injective. Since P is homogeneous, $F'=\mathrm{Coker}\, f$ is torsion free. We have the exact sequence

$$0\longrightarrow P\xrightarrow{f} F\xrightarrow{g} F'\longrightarrow 0\,.$$

Since $\mathrm{Hom}_{\mathcal{O}_X}(F, P)$ is zero, this exact sequence does not split. Hence by (2) of Proposition 5.3, it suffices to show $F'\cong T_x^*F_{2-r}\otimes Q$ for some $x\in X$ and $Q\in \mathrm{Pic}^\circ X$. F' is torsion free, $\det F'=\det F\otimes P^{-1}$ is algebraically equivalent to $\mathcal{O}_X(C)$ and $\chi(F')=\chi(F)-\chi(P)$ is equal to zero. By induction hypothesis, we have only to show that 2) holds for F'. Obviously $\mathrm{Hom}_{\mathcal{O}_X}(F', Q)$ is zero for every $Q\in \mathrm{Pic}^\circ X$. Let H' be a homogeneous vector bundle contained in F'. $H=g^{-1}(H')$ is an extension of H' by P. Hence by the theorem after Lemma 3.3, H is also homogeneous. By the assumption on F, $F'/H'\cong F/H$ is torsion free. Hence 2) holds for F', which completes the proof of Theorem 5.4. q.e.d.

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