DUALITY FOR CROSSED PRODUCTS AND THE STRUCTURE OF VON NEUMANN ALGEBRAS OF TYPE III

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1. Introduction

Undoubtedly, the principal problem in many field of mathematics is to understand and describe precisely the structure of the objects in question in terms of simpler (or more tactable) objects. After the fundamental classification of factors into those of type I, type II and type III by F. J. Murray and J. von Neumann, [25], the structure theory of von Neumann algebras has remained untractable in general form. It seems that the complete solution to this question is still out of sight. In the previous papers [44, 45], however, the author obtained a structure theorem for certain von Neumann algebras of type III in terms of a von Neumann algebra of type II_1 and an endomorphism of this algebra. Also,

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A. Connes further classified the factors of type III into those of type III_λ, $0 \leqslant \lambda \leqslant 1$, based on his new algebraic invariant $S(\mathcal{M})$, and obtained a structure theorem for factors of type III_λ, $0 \leqslant \lambda < 1$, in terms of a von Neumann algebra of type II_∞ and its automorphism, [10]. These two structure theorems are closely related, and encourage us to obtain a general structure theorem for a von Neumann algebra of type III in terms of a von Neumann algebra of type II_∞ and its automorphism group. The present paper is devoted to this task. The structure theorems in the papers mentioned above were obtained by spectral analysis of the modular automorphism group associated with a *carefully chosen* state or weight. In contrast, a *general* structure theorem for a von Neumann algebra $\mathcal{M}$ of type III in terms of a von Neumann algebra of type II_∞ and a one parameter automorphism group will be obtained by constructing the crossed product, say $\mathcal{N}$, of a von Neumann algebra $\mathcal{M}$ of type III by the modular automorphism group of $\mathcal{M}$ associated with an *arbitrary* faithful semifinite normal weight, and then the crossed product of $\mathcal{N}$ by another one parameter automorphism group, see § 8.

Although the crossed product of operator algebras had been treated by F. J. Murray and J. von Neumann in their fundamental work as the so-called group measure space construction of a factor, it was M. Nakamura who proposed the investigation of crossed products of operator algebras, especially factors of type II_1, as a possible analogy of crossed products of simple algebras, with the aim of describing or constructing more factors of type II_1. In 1955, T. Turumaru gave a framework for crossed products of C^*-algebras, which was published in 1958 [51]. Soon after the work of Turumaru, Nakamura and Takeda began a serious study of crossed products of factors of type II_1, [27, 28, 29], and N. Suzuki worked also on this subject at the same time [38]. They considered, however, only *discrete* crossed products of factors of type II_1. Continuous crossed products of C^*-algebras were first proposed by mathematical physicists, S. Doplicher, D. Kastler and D. Robinson under the terminology "*covariance algebra*" in order to describe symmetries and the time evolution in a physical system, [12]. *Continuous* crossed products of *von Neumann algebras* have been, however, left untouched.

We shall give, in § 3, the definition as well as the construction of the crossed product of a von Neumann algebra by a general locally compact automorphism group, which is somewhat different from that of a C_*-algebra—it has no universal property as in the case of C_*-algebras. We then restrict ourselves to the case of *abelian* automorphism groups throughout most of this paper.

In § 4, we prove our main duality theorem for crossed products, which says that given a von Neumann algebra $\mathcal{M}$ equipped with a continuous action α of a locally compact abelian group G, the crossed product of $\mathcal{M}$ by G with respect to α, denoted by $\mathcal{R}(\mathcal{M}; \alpha)$, admits a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ so that the second crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ is isomorphic to the tensor product $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ of $\mathcal{M}$ and the factor $\mathcal{L}(L^2(G))$ of type I. This result, together with Connes' result concerning the unitary co-cycle Radon–Nikodym Theorem, [10; Théorème 1.21] will enable us to describe the structure of a von Neumann algebra of type III in § 8.

Sections 5 and 6 are devoted to analysis of weights on crossed products. The results obtained there will be used to show in § 8 that the crossed product of a von Neumann algebra of type III by the modular automorphism group is semifinite (actually of type II_∞).

We shall show in § 7 a Galois type correspondence between some intermediate von Neumann subalgebras of the crossed product and closed subgroups of the dual group $\hat{G}$.

Sections 8 and 9 are devoted to the study of von Neumann algebras of type III as mentioned above. In § 9, we examine the algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ introduced recently by Connes for a factor $\mathcal{M}$ of type III, in terms of the structure theorem in § 8.

In § 10, we shall discuss induced covariant systems for general locally compact automorphism groups, and prove that the crossed product of the induced covariant system is essentially isomorphic to the crossed product of the original smaller covariant system. We shall then apply the result to the structure of a von Neumann algebra of type III of a certin class in order to describe the algebra in question as the *discrete* crossed product of a von Neumann algebra of type II_∞ by an automorphism; hence by the additive group $\mathbf{Z}$ of integers. This description corresponds to the structure theorems obtained previously by A. Connes [10] and the author [45].

Section 11 is devoted to discussing the example of hyperfinite factors of type III_1. As an application of our theory, we prove that the fundamental group of a hyperfinite factor $\mathcal{F}$ of type II_1 in the sense of Murray–von Neumann [26] is represented by a continuous one parameter automorphism group of a hyperfinite factor of type II_∞ which is the tensor product of $\mathcal{F}$ and a factor of type I_∞.

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2. Prelimaries

For a locally compact space G and a topological vector space T, we denote by $\mathcal{K}(T;G)$ the vector space of all continuous T-valued functions on G with compact support. When the complex number field $\mathbf{C}$ is taken as T, then we write $\mathcal{K}(G)$ for $\mathcal{K}(\mathbf{C};G)$. When a von Neumann algebra $\mathcal{M}$ is taken as T, then we consider the σ-strong* topology in $\mathcal{M}$ for the space $\mathcal{K}(\mathcal{M}; G)$. If a positive Radon measure dg in G and a Hilbert space $\mathfrak{H}$ are given, then we consider the inner product in $\mathcal{K}(\mathfrak{H}; G)$ defined by

$$(\xi|\eta)=\int_G (\xi(g)|\eta(g))\,dg, \quad \xi,\eta\in\mathcal{K}(\mathfrak{H}; G), \tag{2.1}$$

which makes $\mathcal{K}(\mathfrak{H}; G)$ a pre Hilbert space. The completion of $\mathcal{K}(\mathfrak{H}; G)$ with respect to this inner product is denoted by $L^2(\mathfrak{H}; G, dg)$, or by $L^2(\mathfrak{H}; G)$ when the measure dg is fixed by the circumstances. Of course, the imbedding of $\mathcal{K}(\mathfrak{H}; G)$ into $L^2(\mathfrak{H}; G; dg)$ is not injective unless the support of dg is the whole space G. However in most cases, we consider only such measures. Each element ξ in $L^2(\mathfrak{H}; G, dg)$ is realized by an $\mathfrak{H}$-valued function $\xi(\cdot)$ with the properties:

(i) $g\in G\mapsto(\xi(g)|\eta_0)$ is dg-measurable for each $\eta_0\in\mathfrak{H}$;

(ii) for each compact subset K of G, there exists a separable subspace $\mathfrak{H}_1$ of $\mathfrak{H}$ such that $\xi(g)\in\mathfrak{H}_1$ for dg-almost every $g\in K$;

(iii) $$\int_G\|\xi(g)\|^2dg<+\infty.$$

The last integral is equal to $\|\xi\|^2$. Each essentially bounded dg-measurable function f on G acts on $L^2(\mathfrak{H}; G, dg)$ as a multiplication operator, i.e.,

$$(f\xi)(g)=f(g)\xi(g), \quad \xi\in L^2(\mathfrak{H}; G). \tag{2.2}$$

Such an operator f is called a *diagonal* operator on $L^2(\mathfrak{H}; G)$. The set $\mathcal{A}$ of all diagonal operators is an abelian von Neumann algebra isomorphic to $L^\infty(G, dg)$. If x is an operator in the commutant $\mathcal{A}'$ of $\mathcal{A}$, then there exists an $\mathcal{L}(\mathfrak{H})$-valued function $x(\cdot)$ on G with the following properties:

(iv) for each pair ξ,η in $\mathfrak{H}$, the function: $g\in G\mapsto(x(g)\xi|\eta)$ is dg-measurable;

(v) for each fixed $\xi\in\mathfrak{H}$ and compact subset $K\subset G$, there exists a separable subspace $\mathfrak{H}_1$ of $\mathfrak{H}$ such that $x(g)\xi$ falls in $\mathfrak{H}_1$ for dg-almost every $g\in K$;

(vi) $$\operatorname*{ess\,sup}_{g\in G}\|x(g)\|=\|x\|<+\infty.$$

Conversely, each $\mathcal{L}(\mathfrak{H})$-valued function $x(\cdot)$ satisfying the above properties gives rise to an operator x in $\mathcal{A}'$ by

$$(x\xi)(g)=x(g)\xi(g), \quad \xi\in L^2(\mathfrak{H}; G). \tag{2.3}$$

The operators of $\mathcal{A}'$ are called *decomposable*. For details, we refer to papers of Vesterstrøm & Wills [52] and Maréchal [22].

Given a faithful semifinite normal weight φ on a von Neumann algebra $\mathcal{M}_1$, we get a one parameter automorphism group $\{\sigma_t^\varphi\}$ of $\mathcal{M}$, which is uniquely determined by φ subject to the so-called Kubo–Martin–Schwinger condition: for each pair x, y in the definition hereditary subalgebra $\mathfrak{m}$ of φ there exists a bounded continuous function F on the strip, $0 \leqslant \operatorname{Im} z \leqslant 1$, which is holomorphic in the interior such that

$$F(t) = \varphi(\sigma_t^\varphi(x)\, y); \quad F(t+i) = \varphi(y\sigma_t^\varphi(x)).$$

The group $\{\sigma_t^\varphi\}$ is called the modular automorphism group of $\mathcal{M}$ associated with φ. The fixed point subalgebra of $\mathcal{M}$ under $\{\sigma_t^\varphi\}$ is called the *centralizer* of φ, and sometimes denoted by $\mathcal{M}_\varphi$. For the details of the theory of weights and modular automorphism groups, we refer to the articles, [9], [10], [33], [42], [43] and [50].

3. Construction of crossed products

Let $\mathcal{M}$ be a von Neumann algebra. We denote by Aut $(\mathcal{M})$ the group of all automorphisms (*-preserving) of $\mathcal{M}$ and by Int $(\mathcal{M})$ the group of all inner automorphisms of $\mathcal{M}$. Clearly, Int $(\mathcal{M})$ is a normal subgroup of Aut $(\mathcal{M})$. We denote by ι the identity in Aut $(\mathcal{M})$.

Definition 3.1. Given a topological group G, a *continuous action* of G on $\mathcal{M}$ means a homomorphism α: $G \ni g \mapsto \alpha_g \in \operatorname{Aut}(\mathcal{M})$ such that for each fixed $x \in \mathcal{M}$, the map: $g \in G \mapsto \alpha_g(x) \in \mathcal{M}$ is σ-strongly* continuous. The pair $\{\mathcal{M}, \alpha\}$ is sometimes called a *covariant system* on G. In particular a continuous action of the additive group $\mathbf{R}$ of real numbers is called a (continuous) *one parameter automorphism group* of $\mathcal{M}$.

The following easy proposition says that the continuity assumption imposed in the above definition is weakest as long as we intend to study the relation between unitary representations and actions of a topological group.

PROPOSITION 3.2. *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra acting on a Hilbert space. Every continuous unitary representation $\{U, \mathfrak{H}\}$ of a topological group G such that $U(g)\mathcal{M}U(g)^* = \mathcal{M}$, $g \in G$, gives rise to a continuous action α of G on $\mathcal{M}$ by*

$$\alpha_g(x) = U(g)xU(g)^*, \quad g \in G,\ x \in \mathcal{M}.$$

Suppose that $\{\mathcal{M}, \mathfrak{H})$ is a von Neumann algebra on a Hilbert space, equipped with a continuous action α of a locally compact group G. We denote a left invariant Haar measure of G by dg. On the Hilbert space $L^2(\mathfrak{H}; G)$, we define representations π_α of $\mathcal{M}$ and λ of G as follows

$$\left.\begin{aligned}(\pi_\alpha(x)\xi)(h) &= \alpha_h^{-1}(x)\xi(h), \quad h\in G;\\ (\lambda(g)\xi)(h) &= \xi(g^{-1}h), \quad g\in G, \quad \xi\in L^2(\mathfrak{H}; G).\end{aligned}\right\} \tag{3.1}$$

It is easily seen that π_α is a normal faithful representation and

$$\lambda(g)\pi_\alpha(x)\lambda(g)^* = \pi_\alpha\circ\alpha_g(x), \quad x\in\mathcal{M}, \, g\in G. \tag{3.2}$$

In general, if a pair $\{\pi_\alpha, \lambda\}$ of representations π of $\mathcal{M}$ and λ of G satisfies (3.2), then it is called a *covariant* representation of the covariant system $\{\mathcal{M}, \alpha\}$.

Definition 3.3. The von Neumann algebra on $L^2(\mathfrak{H}; G)$ generated by $\pi_\alpha(\mathcal{M})$ and $\lambda(G)$ is called the *crossed product* of $\mathcal{M}$ by G with respect to the action α, or simply the *crossed product of $\mathcal{M}$ by the action α* of G, and denoted by $\mathcal{R}(\mathcal{M}; \alpha)$.

Apparently, the crossed product $\mathcal{R}(\mathcal{M}; \alpha)$ depends also on the underlying Hilbert space $\mathfrak{H}$. However, the next proposition assures that the algebraic structure of $\mathcal{R}(\mathcal{M}; \alpha)$ is independent of the Hilbert space $\mathfrak{H}$.

PROPOSITION 3.4. *Let $\{\mathcal{M}, \mathfrak{H})$ and $\{\mathcal{N}, \mathfrak{K}\}$ be two von Neumann algebras equipped with continuous actions α and β of a locally compact group G respectively. If there exists an isomorphism $\varkappa$ of $\mathcal{M}$ onto $\mathcal{N}$ such that*

$$\varkappa\circ\alpha_g = \beta_g\circ\varkappa, \quad g\in G, \tag{3.3}$$

then there exists an isomorphism $\tilde{\varkappa}$ of $\mathcal{R}(\mathcal{M}; \alpha)$ onto $\mathcal{R}(\mathcal{N}; \beta)$ such that

$$\left.\begin{aligned}\pi_\beta\circ\varkappa(x) &= \tilde{\varkappa}\circ\pi_\alpha(x), \, x\in\mathcal{M};\\ \tilde{\varkappa}(\lambda_m(g)) &= \lambda_n(g), \, g\in G,\end{aligned}\right\} \tag{3.4}$$

where π_α, π_β, λ_m and λ_n mean the representations of $\mathcal{M}$, $\mathcal{N}$, G and G respectively which are used to construct $\mathcal{R}(\mathcal{M}; \alpha)$ and $\mathcal{R}(\mathcal{N}; \beta)$.

Proof. By the theorem of Dixmier [11, Théorèm 1.4.3], there exists a Hilbert space $\mathfrak{M}$ and a unitary U of $\mathfrak{H}\otimes\mathfrak{M}$ onto $\mathfrak{K}\otimes\mathfrak{M}$ such that $U(x\otimes 1)U^* = \varkappa(x)\otimes 1$, $x\in\mathcal{M}$. Since the assertion is trivially true for a spatial isomorphism $\varkappa$, we may assume that $\varkappa$ is an amplification: $x\in\mathcal{M}\mapsto x\otimes 1$. Namely, we assume that the Hilbert space $\mathfrak{K}$ and $\varkappa$ are of the forms:

$$\mathfrak{K} = \mathfrak{H}\otimes\mathfrak{M}, \; \varkappa(x) = x\otimes 1, \; x\in\mathcal{M},$$

for some Hilbert space $\mathfrak{M}$. It is then clear that

$$L^2(\mathfrak{K}; G) = L^2(\mathfrak{H}; G)\otimes\mathfrak{M};$$

so we define an amplification $\tilde{\varkappa}$ of $\mathcal{R}(\mathcal{M}; \alpha)$ by

$$\tilde{\varkappa}(x) = x \otimes 1, \quad x \in \mathcal{R}(\mathcal{M}; \alpha).$$

It is then straightforward to see that $\tilde{\varkappa}$ is the required isomorphism. Q.E.D.

PROPOSITON 3.5. *Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra equipped with two continuous actions α and β of a locally compact group G. If there exists a σ-strongly continuous function u: $g \in G \mapsto u_g \in \mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M})$ denotes the group of all unitaries in $\mathcal{M}$, such that*

$$u_{gh} = u_g \alpha_g(u_h), \quad g, h \in G; \tag{3.5}$$

$$\beta_g(x) = u_g \alpha_g(x) u_g^*, \quad x \in \mathcal{M}, \tag{3.6}$$

then there exists an isomorphism $\varkappa$ of $\mathcal{R}(\mathcal{M}; \alpha)$ onto $\mathcal{R}(\mathcal{M}; \beta)$ such that

$$\varkappa \circ \pi_\alpha(x) = \pi_\beta(x), \quad x \in \mathcal{M},$$

where π_α and π_β are the representations of $\mathcal{M}$ on $L^2(\mathfrak{H}; G)$ given by (3.1) *based on α and β respectively.*

Proof. First of all, we remark that $\mathcal{R}(\mathcal{M}; \alpha)$ and $\mathcal{R}(\mathcal{M}; \beta)$ both act on the same Hilbert space $L^2(\mathfrak{H}; G)$, and the representation λ of G does not depend on α or β. Define a unitary operator U on $L^2(\mathfrak{H}; G)$ by

$$(U\xi)(g) = u_{g^{-1}}\xi(g), \quad g \in G, \quad \xi \in L^2(\mathfrak{H}; G). \tag{3.7}$$

We have then for each $x \in \mathcal{M}$

$$\begin{aligned}(U\pi_\alpha(x) U^* \xi)(g) &= u_{g^{-1}}(\pi_\alpha(x) U^* \xi)(g) = u_{g^{-1}}\alpha_g^{-1}(x) (U^*\xi)(g) \\ &= u_{g^{-1}}\alpha_g^{-1}(x) u_{g^{-1}}^* \xi(g) = \beta_g^{-1}(x) \xi(g) = (\pi_\beta(x) \xi)(g),\end{aligned}$$

and

$$\begin{aligned}(U\lambda(h)U^*\xi)(g) &= u_{g^{-1}}(\lambda(h) U^* \xi)(g) \\ &= u_{g^{-1}}(U^*\xi)(h^{-1}g) = u_{g^{-1}}u_{g^{-1}h}^* \xi(h^{-1}g) = u_{g^{-1}}(u_{g^{-1}}\alpha_g^{-1}(u_h))^* \xi(h^{-1}g) \\ &= u_{g^{-1}}\alpha_g^{-1}(u_h^*) u_{g^{-1}}^* \xi(h^{-1}g) = \beta_g^{-1}(u_h^*) \xi(h^{-1}g) = (\pi_\beta(u_h^*) \lambda(h) \xi)(g).\end{aligned}$$

Therefore, we get

$$\begin{cases} U\pi_\alpha(x) U^* = \pi_\beta(x), & x \in \mathcal{M}; \\ U\lambda(g) U^* = \pi_\beta(u_g^*) \lambda(g), & g \in G. \end{cases}$$

Hence it follows that

$$U\mathcal{R}(\mathcal{M}; \alpha) U^* \subset \mathfrak{R}(\mathcal{M}; \beta).$$

Using the facts that

$$u_{gh}^* = u_g^* \beta_g(u_h^*), \quad g, h \in G$$

and

$$u_g^* \beta_g(x) u_g = \alpha_g(x), \quad x \in \mathcal{M},$$

we can show $U^*\mathcal{R}(\mathcal{M};\beta)U\subset\mathcal{R}(\mathcal{M};\alpha)$. Therefore, putting $\varkappa(x)=UxU^*$, $x\in\mathcal{R}(\mathcal{M};\alpha)$, we obtain the required isomorphism $\varkappa$. Q.E.D.

Combining Propositions 3.4 and 3.5, we obtain the following result:

COROLLARY 3.6. *Let $\{\mathcal{M},\alpha\}$ and $\{\mathcal{N},\beta\}$ be two covariant systems on the same locally compact group G. If there exist an isomorphism $\varkappa$ of $\mathcal{M}$ onto $\mathcal{N}$ and a strongly continuous function u: $g\in G\mapsto u_g\in\mathcal{U}(\mathcal{M})$ satisfying* (3.5) *such that*

$$\varkappa^{-1}\circ\beta_g\circ\varkappa(x)=u_g\alpha_g(x)u_g^*,\quad x\in\mathcal{M},\ g\in G,$$

then there exist an isomorphism $\tilde{\varkappa}$ of $\mathcal{R}(\mathcal{M};\alpha)$ onto $\mathcal{R}(\mathcal{N};\beta)$ such that $\tilde{\varkappa}\circ\pi_\alpha(x)=\pi_\beta\circ\varkappa(x)$, $x\in\mathcal{M}$.

Given two continuous actions α and β of a locally compact group G on $\mathcal{M}$, let $\{u_g^{\beta,\alpha}: g\in G\}$ be a σ-strongly continuous $\mathcal{U}(\mathcal{M})$-valued function on G satisfying (3.5) and (3.6). If γ is another continuous action of G on $\mathcal{M}$ such that there exists $\{u_g^{\gamma,\beta}: g\in G\}$ satisfying (3.5) and (3.6) for β and γ, then the function $u^{\gamma,\alpha}$: $g\in G\mapsto u_g^{\gamma,\alpha}=u_g^{\gamma,\beta}u_g^{\beta,\alpha}\in\mathcal{U}(\mathcal{M})$ satisfies (3.5) and (3.6) with respect to α and γ. Furthermore, the function $u^{\alpha,\beta}$: $g\in G\mapsto u_g^{\alpha,\beta}=(u_g^{\beta,\alpha})^*$ satisfies (3.5) and (3.6) for β and α. Therefore, if we write $\alpha\sim\beta$ when the assumption of Proposition 3.5 is satisfied, then the relation "$\alpha\sim\beta$" is an equivalence relation among the actions of G on $\mathcal{M}$. We say that α and β are *equivalent* if $\alpha\sim\beta$, and α and β are *weakly equivalent* if there exists an automorphism $\varkappa$ of $\mathcal{M}$ such that $\varkappa\circ\alpha\circ\varkappa^{-1}\sim\beta$. More generally, two covariant systems $\{\mathcal{M},\alpha\}$ and $\{\mathcal{N},\beta\}$ on a locally compact group G are said to be *weakly equivalent* if the assumption of Corollary 3.6 is satisfied.

4. Duality

In this section, we shall show a duality for crossed products of von Neumann algebras by locally compact *abelian* groups. We consider throughout most of this section locally compact *abelian* groups only and denote by *addition* the group operation. Given a locally compact abelian group G, we denote by $\hat{G}$ the dual group. We fix Haar measures dg in G and dp in $\hat{G}$ so that the Plancherel formula holds.

Let $\{\mathcal{M},\mathfrak{H}\}$ be a von Neumann algebra equipped with a continuous action α of G. Consider the crossed product $\mathcal{R}(\mathcal{M};\alpha)$ of $\mathcal{M}$ by α on $L^2(\mathfrak{H};G)$. We then define a unitary representation μ of $\hat{G}$ on $L^2(\mathfrak{H};G)$ by

$$\mu(p)\xi(g)=\overline{\langle g,p\rangle}\xi(g),\quad \xi\in L^2(\mathcal{R};G),\ g\in G,\ p\in\hat{G},\tag{4.1}$$

where $\langle g,p\rangle$ denotes the value of $p\in\hat{G}$ at $g\in G$. Clearly we have

$$\mu(p)\pi_\alpha(x)\mu(-p)=\pi_\alpha(x),\quad x\in\mathcal{M},\ p\in\hat{G};\tag{4.2}$$

$$\mu(p)\lambda(g)\mu(-p) = \overline{\langle g, p\rangle}\lambda(g), \quad g \in G, \tag{4.3}$$

so that

$$\mu(p)\mathcal{R}(\mathcal{M}; \alpha)\mu(-p) = \mathcal{R}(\mathcal{M}; \alpha), \quad p \in \hat{G}. \tag{4.4}$$

Hence we can define a continuous action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{R}(\mathcal{M}; \alpha)$ by

$$\hat{\alpha}_p(x) = \mu(p)x\mu(-p), \quad x \in \mathcal{R}(\mathcal{M}; \alpha),\ p \in \hat{G}. \tag{4.5}$$

Definition 4.1. We call $\hat{\alpha}$ the *dual* action of $\hat{G}$ on $\mathcal{R}(\mathcal{M}; \alpha)$, or more specifically we say that the action $\hat{\alpha}$ of $\hat{G}$ ob $\mathcal{R}(\mathcal{M}; \alpha)$ is *dual* to the action α of G on $\mathcal{M}$.

PROPOSITION 4.2. *If $\{\mathcal{M}, \alpha\}$ and $\{\mathcal{N}, \beta\}$ are weakly equivalent covariant systems on a locally compact abelian group G, then the isomorphism $\tilde{\varkappa}$ in Corollary 3.6 intertwines the dual actions, $\hat{\alpha}$ on $\mathcal{R}(\mathcal{M}; \alpha)$ and $\hat{\beta}$ on $\mathcal{R}(\mathcal{N}; \beta)$, of $\hat{G}$ in the sense that*

$$\tilde{\varkappa} \circ \hat{\alpha}_p(x) = \hat{\beta}_p \circ \tilde{\varkappa}(x), \quad x \in \mathcal{R}(\mathcal{M}; \alpha). \tag{4.6}$$

Proof. As in the previous section, we can easily reduce the problem to the case where $\mathcal{M}$ and $\mathcal{N}$ are the same von Neumann algebra, denoted again by $\mathcal{M}$, and $\varkappa$ is the identity automorphism. Let U be the unitary operator on $L^2(\mathfrak{H}; G)$ defined by (3.7). Put $\tilde{\varkappa}(x) = UxU^*$, $x \in \mathcal{R}(\mathcal{M}; \alpha)$. We have then for each $x \in \mathcal{M}$, $g \in G$ and $p \in \hat{G}$,

$$\begin{aligned}\tilde{\varkappa} \circ \hat{\alpha}_p \circ \pi_\alpha(x) &= \tilde{\varkappa} \circ \pi_\alpha(x) = \pi_\beta(x) = \hat{\beta}_p \circ \tilde{\varkappa} \circ \pi_\alpha(x);\\ \tilde{\varkappa} \circ \hat{\alpha}_p \circ \lambda(g) &= \overline{\langle g, p\rangle}\, \tilde{\varkappa} \circ \lambda(g) = \overline{\langle g, p\rangle}\, \pi_\beta(u_g^*)\, \lambda(g)\\ &= \hat{\beta}_p(\pi_\beta(u_g^*)\, \lambda(g)) = \hat{\beta}_p \circ \tilde{\varkappa} \circ \lambda(g).\end{aligned}$$

Hence we have $\tilde{\varkappa} \circ \hat{\alpha}_p = \hat{\beta}_p \circ \tilde{\varkappa}$, $p \in \hat{G}$. Q.E.D.

Suppose now $\{\mathcal{M}, \mathfrak{H}\}$ is a von Neumann algebra equipped with a continuous action α of a locally compact abelian group G. We shall show that the second crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha})$ is isomorphic to the tensor product $\mathcal{M} \otimes \mathcal{L}(L^2(G))$.

Put $\mathcal{M}_0 = \mathcal{R}(\mathcal{M}; \alpha)$ and $\mathcal{N} = \mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha})$. By construction, $\mathcal{N}$ acts on the Hilbert space $L^2(\mathfrak{H}; G \times \hat{G})$, and is generated by the operators of the following three types:

$$\left.\begin{aligned}(\pi_{\hat{\alpha}} \circ \pi_\alpha(x)\, \xi)\,(g, p) &= \alpha_g^{-1}(x)\, \xi(g, p), \quad x \in \mathcal{M};\\ \pi_{\hat{\alpha}}(\lambda(h))\, \xi(g, p) &= \langle h, p\rangle\, \xi(g - h, p), \quad h \in G;\\ \lambda(q)\, \xi(g, p) &= \xi(g, p - q), \quad q \in \hat{G};\end{aligned}\right\} \tag{4.7}$$

We consider the operator F on $\mathcal{K}(\mathfrak{H}; G \times \hat{G})$ defined by

$$F\xi(g, h) = \int_{\hat{G}} \overline{\langle h, p\rangle}\, \xi(g, p)\, dp, \quad \xi \in \mathcal{K}(\mathfrak{H}; G \times \hat{G}). \tag{4.8}$$

It is well-known that F may be extended to a unitary of $L^2(\mathfrak{H}; G\times\hat{G})$ onto $L^2(\mathfrak{H}; G\times G)$, which is also denoted by F, and that the inverse F^* of F is given on $\mathcal{K}(\mathfrak{H}; G\times G)$ by

$$(F^*\xi)(g,p)=\int_G \langle h,p\rangle\, \xi(g,h)\,dh, \quad \xi\in\mathcal{K}(\mathfrak{H}; G\times G). \tag{4.9}$$

We consider the von Neumann algebra $F\mathcal{N}F^*$, say $\mathcal{P}$, instead of $\mathcal{N}$ itself. We put

$$\left.\begin{aligned} \tilde{x}&=F\pi_{\hat{\alpha}}\circ\pi_\alpha(x)F^*, && x\in\mathcal{M};\\ v(g)&=F\pi_{\hat{\alpha}}\circ\lambda(g)F^*, && g\in G;\\ u(p)&=F\lambda(p)F^*, && p\in G.\end{aligned}\right\}$$

We have then

$$\left.\begin{aligned} (\tilde{x}\xi)(g,h)&=\alpha_g^{-1}(x)\,\xi(g,h), && x\in\mathcal{M};\\ (v(k)\,\xi)(g,h)&=\xi(g-k,h-k), && k\in G;\\ (u(p)\,\xi)(g,h)&=\overline{\langle h,p\rangle}\,\xi(g,h), && p\in\hat{G}\end{aligned}\right\} \tag{4.10}$$

for every $\xi\in L^2(\mathfrak{H}; G\times G)$. The von Neumann algebra $\mathcal{P}$, which is isomorphic to $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\alpha)$, is generated by $\tilde{x}$, $x\in\mathcal{M}$, $v(g)$, $g\in G$, and $u(p)$, $p\in\hat{G}$. These operators satisfy the following equation:

$$(v(g)\tilde{x}v(-g)=(\alpha_g(x))^{\sim}, \quad x\in\mathcal{M},\ g\in G; \tag{4.11}$$

$$u(p)\tilde{x}u(-p)=\tilde{x}, \quad x\in\mathcal{M},\ p\in\hat{G}; \tag{4.12}$$

$$v(g)u(p)v(-g)u(-p)=\langle g,p\rangle 1, \quad g\in G,\ p\in\hat{G}. \tag{4.13}$$

The last equation (4.13) is known as the (generalized) *Heisenberg commutation relation*. It is then known, see [18, 19, 30], that the von Neumann algebra $\mathcal{B}$ generated by $\{v(g), u(p): g\in G, p\in\hat{G}\}$ is isomorphic to the algebra $\mathcal{L}(L^2(G))$ of all bounded operators on $L^2(G)$, which is a factor of type I. Therefore, we have $\mathcal{P}\cong(\mathcal{P}\cap\mathcal{B}')\otimes\mathcal{B}$. For each $x\in\mathcal{M}$, we define

$$\pi(x)\,\xi(g,h)=\alpha_{g-h}^{-1}(x)\,\xi(g,h), \quad \xi\in L^2(\mathfrak{H}; G\times G). \tag{4.14}$$

LEMMA 4.3. *The map π: $x\in\mathcal{M}\to\pi(x)\in\mathcal{L}(L^2(\mathfrak{H}; G\times G))$ is a normal isomorphism of $\mathcal{M}$ into $\mathcal{P}\cap\mathcal{B}'$.*

Proof. It is easy to see that π is a normal isomorphism of $\mathcal{M}$ into $\mathcal{B}'$, i.e., $\pi(x)$, $x\in\mathcal{M}$, commutes with $v(g)$ and $u(p)$, $g\in G$ and $p\in\hat{G}$. Hence we have only to show the inclusion $\pi(\mathcal{M})\subset\mathcal{P}$. Let x be a fixed arbitrary element of $\mathcal{M}$. Put $y=\pi(x)$. For each $\varphi\in\mathcal{K}(G)$ and $\psi\in\mathcal{K}(\hat{G})$, we define

$$y_{\varphi,\psi}=\int_{G\times\hat{G}}\overline{\langle g,p\rangle}\,\varphi(g)\,\psi(p)\,\alpha_g^{-1}(x)^{\sim}\,u(p)\,dg\,dp.$$

It is clear that $y_{\varphi,\psi}$ belongs to $\mathcal{D}$. For each $\xi, \eta \in L^2(\mathfrak{H}; G \times G)$, we have

$$
\begin{aligned}
(y_{\varphi,\psi}\xi|\eta) &= \iint_{G\times\hat{G}} \overline{\langle g,p\rangle}\,\varphi(g)\,\psi(p)\,(\alpha_g^{-1}(x)^{\sim}\,u(p)\,\xi|\eta)\,dg\,dp \\
&= \iiiint_{\hat{G}\times G\times G\times G} \overline{\langle g,p\rangle}\,\varphi(g)\,\psi(p)\,\overline{\langle k,p\rangle}\,(\alpha_{g+h}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk\,dp \\
&= \iiiint_{\hat{G}\times G\times G\times G} \overline{\langle g+k,p\rangle}\,\varphi(g)\,\psi(p)\,(\alpha_{g+h}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk\,dp \\
&= \iiint_{G\times G\times G} \hat{\psi}(g+k)\,\varphi(g)\,(\alpha_{g+h}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk \\
&\qquad\qquad \text{(by Fubini's Theorem)} \\
&= \iiint_{G\times G\times G} \hat{\psi}(g)\,\varphi(g-k)\,(\alpha_{g+h-k}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk,
\end{aligned}
$$

where $\hat{\psi}$ means, of course, the Fourier transform of ψ on $\hat{G}$, i.e.,

$$\hat{\psi}(g) = \int_{\hat{G}} \overline{\langle g,p\rangle}\,\psi(p)\,dp, \quad g \in G.$$

Put
$$F(g,h,k) = \varphi(g)\,(\alpha_{g+h}^{-1}(x)\,\xi(h,k)|\eta(h,k)), \quad g,h,k \in G.$$

The map: $g \in G \mapsto F(g,\cdot,\cdot) \in L^1(G\times G)$ is then continuous and has compact support. Therefore, when the measure $\hat{\psi}(g)dg$ converges to the Dirac measure δ_0 at the origin $0 \in G$, the above integral converges to

$$\iint_{G\times G} \varphi(-k)\,(\alpha_{h-k}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dh\,dk,$$

and this converges to $(y\xi|\eta)$ as φ converges to the identity constant function uniformly on each compact subset of G. Therefore, y is well approximated weakly by $y_{\varphi,\psi}$, so that y belongs to $\mathcal{D}$. Q.E.D.

LEMMA 4.4. *The von Neumann algebra $\mathcal{D}$ is generated by $\pi(\mathcal{M})$ and $\mathcal{B}$; hence*

$$\mathcal{D} \cong \mathcal{M} \otimes \mathcal{B}.$$

Proof. Let x be a fixed arbitrary element of $\mathcal{M}$. For each $\varphi \in \mathcal{K}(G)$ and $\psi \in \mathcal{K}(\hat{G})$, we put

$$x_{\varphi,\psi} = \iint_{\hat{G}\times G} \overline{\langle g,p\rangle}\,\varphi(g)\,\psi(p)\,\pi(\alpha_g(x))\,u(p)^*\,dg\,dp.$$

It follows then that $x_{\varphi,\psi}$ belongs to $(\pi(\mathcal{M}) \cup \mathcal{B})''$. For each $\xi, \eta \in L^2(\mathfrak{H}; G \times G)$, we have

$$\begin{aligned}(x_{\varphi,\psi}\xi|\eta) &= \iint_{\hat{G}\times G} \overline{\langle g,p\rangle}\,\varphi(g)\,\psi(p)\,(\pi(\alpha_g(x))\,u(p)^*\xi|\eta)\,dg\,dp \\ &= \iiiint_{\hat{G}\times G\times G\times G} \overline{\langle g+k,p\rangle}\,\varphi(g)\,\psi(p)\,(\alpha_{h-k-g}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk\,dp \\ &= \iiint_{G\times G\times G} \hat{\psi}(g+k)\,\varphi(g)\,(\alpha_{h-k-g}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk \\ &= \iiint_{G\times G\times G} \hat{\psi}(g)\,\varphi(g-k)\,(\alpha_{h-g}^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dg\,dh\,dk.\end{aligned}$$

For the same reason as before, when $\hat{\psi}(g)\,dg$ converges vaguely to the Dirac measure δ_0, the last integral converges to

$$\iint_{G\times G} \varphi(-k)\,(\alpha_h^{-1}(x)\,\xi(h,k)|\eta(h,k))\,dh\,dk,$$

which converges to $(\tilde{x}\xi|\eta)$ as φ tends to 1 in an appropriate sense. Thus every $\tilde{x}$, $x\in\mathcal{M}$, belongs to $(\pi(\mathcal{M})\cup\mathcal{B})''$, so that

$$\mathcal{D}\subset(\pi(\mathcal{M})\cup\mathcal{B})''. \qquad \text{Q.E.D.}$$

Combining Lemmas 4.3 and 4.5, we obtain the following duality theorem.

THEOREM 4.5 (Duality). *Let $\mathcal{M}$ be a von Neumann algebra equipped with a continuous action α of a locally compact abelian group G. Then the crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ of $\mathcal{R}(\mathcal{M};\alpha)$ by the dual action $\hat{\alpha}$ of the dual group $\hat{G}$ of G is isomorphic to the tensor product $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ of $\mathcal{M}$ itself and the factor $\mathcal{L}(L^2(G))$ of type I of all bounded operators on $L^2(G)$. Therefore, if $\mathcal{M}$ is properly infinite and if G is separable, then $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ is isomorphic to the original algebra $\mathcal{M}$ itself.*

We now consider the action $\tilde{\alpha}$ of G on the second crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M};\ \alpha);\hat{\alpha})$, say $\mathcal{N}$, which is dual to the action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{R}(\mathcal{M};\alpha)$. The relevant unitary representation $\tilde{\mu}$ of G on $L^2(\mathfrak{H};G\times\hat{G})$, which gives rise to the action $\tilde{\alpha}$ of G on $\mathcal{N}$, is defined by

$$\tilde{\mu}(g)\xi(h,p)=\langle g,p\rangle\xi(h,p),\quad g\in G,\ \xi\in L^2(\mathfrak{H};G\times\hat{G}). \tag{4.15}$$

The action $\tilde{\alpha}$ of G on $\mathcal{N}$ is now given by

$$\tilde{\alpha}_g(x)=\tilde{\mu}(g)x\tilde{\mu}(-g),\quad g\in G,\ x\in\mathcal{N} \tag{4.16}$$

Let $w(g)=F\tilde{\mu}(g)F^*$, $g\in G$. We have

$$w(g)\xi(h,k)=\xi(h,k+g),\quad \xi\in L^2(\mathfrak{H};G\times G). \tag{4.17}$$

If we identify $\mathcal{N}$ and $\mathcal{D}=F\mathcal{N}F^*$, then the corresponding action $\tilde{\alpha}$ of G on $\mathcal{D}$ should be given by

$$\tilde{\alpha}_g(x)=w(g)xw(-g), \quad x\in\mathcal{D},\ g\in G. \tag{4.18}$$

We have then, for each $g, h\in G$ and $p\in\hat{G}$,

$$\left.\begin{aligned}\tilde{\alpha}_g(v(h))&=v(h)\\ \tilde{\alpha}_g(u(p))&=\overline{\langle g,p\rangle}\,u(p).\end{aligned}\right\} \tag{4.19}$$

Hence the action $\tilde{\alpha}$ of G leaves $\mathcal{B}$ invariant and its restriction to $\mathcal{B}$ is induced by the unitary representation v of G, that is

$$\tilde{\alpha}_g(x)=v(g)^*xv(g), \quad x\in\mathcal{B},\ g\in G. \tag{4.20}$$

We have next, for each $x\in\mathcal{M}$,

$$\begin{aligned}(\tilde{\alpha}_g\circ\pi(x)\,\xi)\,(h,k)&=(w(g)\,\pi(x)\,w(g)^*\,\xi)\,(h,k)\\ &=(\pi(x)\,w(g)^*\,\xi)\,(h,k+g)=\alpha^{-1}_{h-k-g}(x)\,(w(g)^*\,\xi)\,(h,k+g)\\ &=\alpha^{-1}_{h-k}\circ\alpha_g(x)\,\xi(h,k)=\pi\circ\alpha_g(x)\,\xi(h,k),\end{aligned}$$

so that

$$\tilde{\alpha}_g\circ\pi(x)=\pi\circ\alpha_g(x), \quad x\in\mathcal{M}, \quad g\in G. \tag{4.21}$$

Therefore, we have

$$\tilde{\alpha}_g=\alpha_g\otimes\mathrm{Ad}\,(v(g)^*), \quad g\in G, \tag{4.22}$$

under the identification of $\mathcal{N}$ and $\mathcal{M}\otimes\mathcal{B}$, where, for any unitary u, Ad (u) means the inner automorphism defined by

$$\mathrm{Ad}\,(u)x=uxu^*.$$

Thus we obtain the following result:

THEOREM 4.6. *Under the same assumptions as in Theorem 4.5, the isomorphism of* $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ *onto* $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ *in Theorem 4.5 transforms the action* $\tilde{\alpha}$ *of* G *on* $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ *into the action of* G *on* $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ *given by* $\alpha_g\otimes\mathrm{Ad}\,(v(g)^*)$, $g\in G$, *where* $v(\cdot)$ *is the regular representation of* G *on* $L^2(G)$ *defined by*

$$v(g)\xi(h)=\xi(h-g), \quad g,h\in G,\ \xi\in L^2(G). \tag{4.23}$$

Suppose now that $\mathcal{M}$ is properly infinite and G is separable. By Theorem 4.5, we may identify $\mathcal{M}$ and $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$. We consider here the problem of how the original action α of G on $\mathcal{M}$ and the bidual action $\tilde{\alpha}$ of G on $\mathcal{M}$ are related under the above identification.

Let $\mathcal{M}$ be a *properly infinite* von Neumann algebra equipped with a continuous action α of a locally compact group G. Let $\mathcal{B}$ be a factor of type I with *separable* predual. Let

$\{e_n: n=1, 2, \ldots\}$ be an orthogonal family of projections in $\mathcal{M}$ with $e_n \sim 1$ and $\sum_{n=1}^{\infty} e_n = 1$. Let $\{v_n: n=1, 2, \ldots\}$ be a sequence of partial isometries in $\mathcal{M}$ with $v_n^* v_n = 1$ and $v_n v_n^* = e_n$. Let $\{u_{i,j}: i, j=1, 2, \ldots\}$ be matrix units in $\mathcal{B}$. For each $x \in \mathcal{M}$, put

$$\left.\begin{aligned} x_{i,j} &= v_i^* x v_j, \quad i, j = 1, 2, \ldots; \\ \sigma(x) &= \sum_{i,j=1}^{\infty} x_{i,j} \otimes u_{i,j}. \end{aligned}\right\} \tag{4.24}$$

It is then well-known that $\sigma: x\mathcal{M} \mapsto \sigma(x) \in \mathcal{M} \otimes \mathcal{B}$ is an isomorphism of $\mathcal{M}$ onto $\mathcal{M} \otimes \mathcal{B}$. Define an action β of G on $\mathcal{M} \otimes \mathcal{B}$ by

$$\beta_g(x) = \sigma \circ \alpha_g \circ \sigma^{-1}(x), \quad x \in \mathcal{M} \otimes \mathcal{B}. \tag{4.25}$$

We shall show that the actions β and $\{\alpha_g \otimes \iota: g \in G\}$ of G on $\mathcal{M} \otimes \mathcal{B}$ are equivalent.

Put $w_n = \sigma(v_n)$, $n=1, 2, \ldots$, and

$$u_g = \sum_{n=1}^{\infty} w_n \beta_g(w_n^*), \quad g \in G. \tag{4.26}$$

We have then

$$\begin{aligned} u_g^* u_g &= \left(\sum_{n=1}^{\infty} \beta_g(w_n) w_n^*\right)\left(\sum_{n=1}^{\infty} w_m \beta_g(w_n^*)\right) \\ &= \sum_{n,m=1}^{\infty} \beta_g(w_n) w_n^* w_m \beta_g(w_m^*) = \sum_{n=1}^{\infty} \beta_g(w_n) w_n^* w_n \beta_g(w_n^*) = \beta_g\left(\sum_{n=1}^{\infty} w_n w_n^*\right) = \beta_g(1) = 1; \end{aligned}$$

$$u_g u_g^* = \left(\sum_{n=1}^{\infty} w_n \beta_g(w_n^*)\right)\left(\sum_{m=1}^{\infty} \beta_g(w_m) w_m^*\right) = \sum_{n=1}^{\infty} w_n \beta_g(w_n^* w_n) w_n^* = \sum_{n=1}^{\infty} w_n w_n^* = 1;$$

$$\begin{aligned} u_{gh} &= \sum_{n=1}^{\infty} w_n \beta_{gh}(w_n^*) = \sum_{n=1}^{\infty} w_n \beta_g \circ \beta_h(w_n^*) = \sum_{n=1}^{\infty} w_n \beta_g(w_n^*) \beta_g(w_n) \beta_g \circ \beta_h(w_n^*) \\ &= \left[\sum_{n=1}^{\infty} w_n \beta_g(w_n^*)\right] \beta_g\left[\sum_{m=1}^{\infty} w_m \beta_h(w_m^*)\right] = u_g \beta_g(u_h) \end{aligned}$$

Noticing that $1 \otimes u_{i,j} = w_i w_j^*$, $i, j = 1, 2, \ldots$, we have

$$\begin{aligned} u_g \beta_g(1 \otimes u_{i,j}) u_g^* &= \left(\sum_{n=1}^{\infty} w_n \beta_g(w_n^*)\right) \beta_g(w_i w_j^*) \left(\sum_{m=1}^{\infty} \beta_g(w_m) w_m^*\right) \\ &= w_i \beta_g(w_i^* w_i w_j^* w_j) w_j^* = w_i w_j^* = 1 \otimes u_{i,j}. \end{aligned}$$

Let x be an arbitrary element of $\mathcal{M}$, and put $y = x \otimes 1$. We have then

$$\sigma^{-1}(y) = \sum_{n=1}^{\infty} v_n x v_n^*;$$

$$\begin{aligned} u_g\beta_g(y)\,u_g^* &= u_g\sigma\left(\sum_{n=1}^{\infty}\alpha_g(v_n x v_n^*)\right)u_g^* \\ &= \left[\sum_{i=1}^{\infty} w_i\beta_g(w_i^*)\right]\sigma\left[\sum_{n=1}^{\infty}\alpha_g(v_n x v_n^*)\right]\left[\sum_{j=1}^{\infty}\beta_g(w_j)\,w_j^*\right] \\ &= \sigma\left(\left[\sum_{i=1}^{\infty} v_i\alpha_g(v_i^*)\right]\left[\sum_{n=1}^{\infty}\alpha_g(v_n x v_n^*)\right]\left[\sum_{j=1}^{\infty}\alpha_g(v_j)\,v_j^*\right]\right) \\ &= \sigma\left(\sum_{n=1}^{\infty} v_n\alpha_g(x)\,v_n^*\right) = \alpha_g(x)\otimes 1. \end{aligned}$$

Hence we get $u_g\beta_g(x\otimes 1)u_g^* = \alpha_g(x)\otimes 1$. Thus we have proved the following lemma.

LEMMA 4.7. *In the above situation the actions β and $\{\alpha_g\otimes\iota : g\in G\}$ of G on $\mathcal{M}\otimes\mathcal{B}$ are equivalent.*

THEOREM 4.8. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra equipped with a continuous action α of a separable locally compact abelian group G. Let σ be the isomorphism of $\mathcal{M}$ onto $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ given by* (4.24). *Identifying $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ and $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ by Theorem 4.5, the action β of G given by* (4.25) *on $\mathcal{M}\otimes\mathcal{L}(L^2(G))$, and the second dual action $\tilde{\alpha}$ of G on $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ are equivalent.*

Proof. By Theorem 4.6, the action $\tilde{\alpha}$ of G is given by $\{\alpha_g\otimes\mathrm{Ad}\,(v(g)^*) : g\in G\}$; so it is equivalent to $\{\alpha_g\otimes\iota : g\in G\}$. Thus Lemma 4.7 implies the equivalence of $\tilde{\alpha}$ and β. Q.E.D.

Thus, the actions α of G on $\mathcal{M}$ and $\tilde{\alpha}$ of G on $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$ are weakly equivalent.

5. Dual weight

In this section, given a von Neumann algebra $\mathcal{M}$ equipped with a continuous action α of a locally compact abelian group G, we shall establish a canonical way of constructing a faithful semifinite normal weight on $\mathcal{R}(\mathcal{M};\alpha)$ from a faithful semifinite normal weight on $\mathcal{M}$ which is relatively invariant under the action α of G.

Since we consider only *faithful semifinite normal* weights throughout this paper, we shall omit the adjectives "faithful semifinite normal". Namely, a weight means always in this paper a faithful semifinite normal one.

Suppose φ is a weight on a von Neumann algebra $\mathcal{M}$. Let $\mathfrak{n} = \{x\in\mathcal{M} : \varphi(x^*x) < +\infty\}$ and $\mathfrak{m} = \mathfrak{n}^*\mathfrak{n}$, the space spanned linearly by the elements x^*y, $x, y\in\mathfrak{n}$. The following facts are then known:

(i) $\mathfrak{n}$ is a left ideal of $\mathcal{M}$;

(ii) $\mathfrak{m}$ is linearly spanned by its positive part $\mathfrak{m}_+$, and $\mathfrak{m}_+ = \{x \in \mathcal{M}_+ : \varphi(x) < +\infty\}$, and φ may be extended to a linear functional $\dot{\varphi}$ on $\mathfrak{m}$;

(iii) $\mathfrak{n}$ is a pre-Hilbert space with inner products: $(x, y) \in \mathfrak{n} \times \mathfrak{n} \mapsto \dot{\varphi}(y^*x)$;

(iv) Denoting the completion of $\mathfrak{n}$ by $\mathfrak{H}$ and the imbedding of $\mathfrak{n}$ into $\mathfrak{H}$ by η, $\mathcal{M}$ is faithfully represented as a von Neumann algebra on $\mathfrak{H}$ in such a way that

$$a\eta(x) = \eta(ax), \quad a \in \mathcal{M}, \ x \in \mathfrak{n};$$

(v) The image $\mathfrak{A} = \eta(\mathfrak{n} \cap \mathfrak{n}^*)$ of $\mathfrak{n} \cap \mathfrak{n}^*$ in $\mathfrak{H}$ turns out naturally to be a full left Hilbert algebra such that $\mathcal{M}$ is the left von Neumann algebra $\mathcal{L}(\mathfrak{A})$ of $\mathfrak{A}$;

(vi) The modular operator Δ of $\mathfrak{A}$ gives rise to the modular automorphism group $\{\sigma_t^\varphi\}$ of $\mathcal{M}$ associated with φ in such a way that

$$\left.\begin{aligned} \Delta^{it}\eta(x) &= \eta(\sigma_t^\varphi(x)), \quad x \in \mathfrak{n}, \quad t \in \mathbf{R}; \\ \sigma_t^\varphi(x) &= \Delta^{it} x \Delta^{-it}, \quad x \in \mathcal{M}; \end{aligned}\right\} \tag{5.1}$$

(vii) The set of all analytic elements in $\mathfrak{A}$ with respect to $\{\Delta^{it} : t \in \mathbf{R}\}$, or more precisely the image of all analytic elements in $\mathfrak{n} \cap \mathfrak{n}^*$ with respect to $\{\sigma_t^\varphi\}$, form the maximal Tomita algebra $\mathfrak{A}_0$ contained in $\mathfrak{A}$ and $\mathcal{L}(\mathfrak{A}_0) = \mathcal{M}$. For each $\xi \in \mathfrak{A}_0$ (sometimes $\xi \in \mathfrak{A}$ or $\xi \in \mathfrak{A}'$), $\pi_l(\xi)$ and $\pi_r(\xi)$ mean respectively the left and right multiplication operators by ξ.

Let α be a continuous action of a locally compact abelian group G on $\mathcal{M}$.

Definition 5.1. A weight φ, (faithful semifinite normal), on $\mathcal{M}$ is said to be *relatively invariant* under the action of α of G if there exist a continuous positive character χ of G such that

$$\varphi \circ \alpha_g = \chi(g)\varphi, \quad g \in G. \tag{5.2}$$

In order to construct canonically a weight $\tilde{\varphi}$ on the crossed product $\mathcal{R}(\mathcal{M}; \alpha)$ of $\mathcal{M}$ by α, neither the relative invariance of φ nor the commutativity of G is essential. But the presentation of the theory of crossed products in full generality is not our purpose in this paper, while it should certainly be done. We shall treat it on another occasion. We assume instead the relative invariance of φ as well as the commutativity, in order to reach quickly a structure theorem of von Neumann algebras of type III.

Suppose that φ is a relatively invariant weight on $\mathcal{M}$. Since the modular automorphism groups associated with φ and $\chi(g)\varphi$ are the same, the modular automorphism group $\{\sigma_t^\varphi\}$ associated with φ and the action α of G commute. In fact, we have

$$\sigma_t^\varphi = \sigma_t^{\chi(g)\varphi} = \sigma_t^{\varphi \circ \alpha_g} = \alpha_g^{-1} \circ \sigma_t^\varphi \circ \alpha_g, \quad g \in G.$$

For each $g \in G$ we define operators $T(g)$ and $U(g)$ by

$$\left.\begin{aligned} T(g)\,\eta(x) &= \eta\circ\alpha_g(x);\\ U(g)\,\eta(x) &= \chi(g)^{-\frac{1}{2}}\eta\circ\alpha_g(x), \quad x\in\mathfrak{n}.\end{aligned}\right\} \tag{5.3}$$

The operators $T(g)$ and $U(g)$ are then extended to bounded operators on $\mathfrak{H}$ which are denoted by the same symbols.

LEMMA 5.2. *The map $U\colon g\in G\mapsto U(g)$ is a continuous unitary representation of G on $\mathfrak{H}$ and the map $T\colon g\in G\mapsto T(g)$ is a strongly continuous representation of G by bounded invertible operators.*

Proof. By construction, it is sufficient to prove the first assertion. The assertions for T follow automatically from the first. Since the group property is obvious, we have only to prove the strong continuity of U. By the normality of φ, there exists an increasing net $\{\omega_i\}_{i\in I}$ of normal positive linear functionals on $\mathcal{M}$ such that

$$\varphi(x) = \lim_i w_i(x), \quad x\in\mathcal{M}_+.$$

There exists then an increasing net $\{h_i\}_{i\in I}$ in $(\mathcal{M}')^+$ converging strongly to the identity 1 such that

$$\omega_i(y^*x) = (\eta(x)\,|\,h_i\eta(y)), \quad x, y\in\mathfrak{n}, i\in I.$$

We have then

$$(U(g)\eta(x)\,|\,h_i\eta(y)) = \chi(g)^{-\frac{1}{2}}(\eta\circ\alpha_g(x)\,|\,h_i\eta(y)) = \chi(g)^{-\frac{1}{2}}\,\omega_i(y^*\alpha_g(x)), \quad x, y\in\mathfrak{n},\ i\in I.$$

Hence the function: $g\in G\mapsto(U(g)\eta(x)\,|\,h_i\eta(y))$ is continuous. Since $\eta(\mathfrak{n})$ and $\bigcup_{i\in I}h_i\eta(\mathfrak{n})$ are both dense $\mathfrak{H}$ and $U(g)$, $g\in G$, are unitaries, U is strongly continuous. Q.E.D.

The commutativity of $\{\sigma_t^\varphi\}$ and the action α of G entails the following:

$$\left.\begin{aligned} T(g)\,\Delta^{it} &= \Delta^{it}\,T(g), \quad g\in G, \quad t\in\mathbf{R};\\ U(g)\,\Delta^{it} &= \Delta^{it}\,U(g).\end{aligned}\right\} \tag{5.4}$$

Furthermore, being an algebraic *-automorphism of $\mathfrak{A}$, $T(g)$ commutes with the involution S of $\mathfrak{A}$, the closure of the $\sharp$-operation in $\mathfrak{A}$; hence $T(g)$ and the unitary involution J in $\mathfrak{H}$ associated with $\mathfrak{A}$ comute; that is,

$$\left.\begin{aligned} T(g) &= JT(g), \quad g\in G;\\ U(g)\,J &= JU(g).\end{aligned}\right\} \tag{5.5}$$

Thus, $T(g)$, and hence $U(g)=\chi(g)^{-\frac{1}{2}}T(g)$, leave the Tomita algebra $\mathfrak{A}_0$ invariant.

To consider $\mathcal{K}(\mathfrak{A}_0; G)$, we equip the Tomita algebra $\mathfrak{A}_0$ with the locally convex topology induced by the family $\{p_K: K$ runs over all compact subsets of $\mathbf{C}\}$ of seminorms on $\mathfrak{A}_0$ defined as follows:

$$p_K(\xi) = \sup_{\omega \in K} \{\|\Delta^\omega \xi\| + \|\pi_l(\Delta^\omega \xi)\| + \|\pi_r(\Delta^\omega \xi)\|\}, \quad \xi \in \mathfrak{A}_0. \tag{5.6}$$

In $\mathcal{K}(\mathfrak{A}_0; G)$, we define an inner product by

$$(\xi|\eta) = \int_G (\xi(g)|\eta(g))\, dg, \quad \xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G). \tag{5.7}$$

The completion $\mathfrak{K}$ of $\mathcal{K}(\mathfrak{A}_0; G)$ is nothing but $L^2(\mathfrak{H}; G)$. We consider the algebraic structure in $\mathcal{K}(\mathfrak{A}_0; G)$ defined by the following:

$$\left.\begin{aligned} (\xi\eta)(g) &= \int_G [T(-h)\, \xi(g-h)]\, \eta(h)\, dh; \\ \xi^\#(g) &= T(-g)\, \xi(-g)^\#; \\ [\tilde{\Delta}(\omega)\, \xi](g) &= \chi(g)^\omega \Delta^\omega \xi(g), \quad \omega \in \mathbf{C}. \end{aligned}\right\} \tag{5.8}$$

It is clear that

$$\tilde{\Delta}(\omega)\, \mathcal{K}(\mathfrak{A}_0; G) = \mathcal{K}(\mathfrak{A}_0; G), \quad \omega \in \mathbf{C};$$

$$\mathcal{K}(\mathfrak{A}_0; G)^\# = \mathcal{K}(\mathfrak{A}_0; G).$$

LEMMA 5.3. *If ξ and η are elements of $\mathcal{K}(\mathfrak{A}_0; G)$, then the product $\xi\eta$ falls in $\mathcal{K}(\mathfrak{A}_0; G)$.*

Proof. Putting $\xi_\omega(g) = \Delta^\omega \xi(g)$ and $\eta_\omega(g) = \Delta^\omega \eta(g)$ for each $\omega \in \mathbf{C}$ and $g \in G$, we have

$$\begin{aligned} (\xi_\omega \eta_\omega)(g) &= \int_G [T(-h)\, \Delta^\omega \xi(g-h)]\, [\Delta^\omega \xi(h)]\, dh \\ &= \int_G [\Delta^\omega T(-h)\, \xi(g-h)]\, [\Delta^\omega \eta(h)]\, dh = \int_G \Delta^\omega \{[T(-h)\, \xi(g-h)]\, \eta(h)\}\, dh \\ &= \Delta^\omega \int_G [T(-h)\, \xi(g-h)]\, \eta(h)\, dh, \end{aligned}$$

where the last step is justified by the closedness of the operator Δ^ω. Hence $\xi\eta(g)$ belongs to the domain $\mathcal{D}(\Delta^\omega)$ of Δ^ω for each $g \in G$.

For each compact subset K of $\mathbf{C}$, we put

$$\gamma_K(g) = \sup_{\omega \in K} \int_G \|\pi_l(\xi_\omega(g-h))\|\, \|\pi_l(\eta_\omega(h))\|\, dh.$$

We have then for any $\zeta \in \mathfrak{A}_0$ and $\omega \in K$,

$$\pi_r(\zeta)\{(\xi_\omega\eta_\omega)(g)\} = \pi_r(\zeta)\int_G [T(-h)\,\xi_\omega(g-h)]\,\eta_\omega(h)\,dh$$

$$= \int_G \pi_r(\zeta)\,[T(-h)\,\xi_\omega(g-h)]\,\eta_\omega(h)\,dh = \int_G \pi_l(T(-h)\,\xi_\omega(g-h)\,\pi_l(\eta_\omega(h))\,\zeta\,dh$$

$$= \left\{\int_G \alpha_h^{-1}\circ\pi_l(\xi_\omega(g-h))\,\pi_l(\eta_\omega(h))\,dh\right\}\zeta,$$

so that $$\|\pi_r(\zeta)\,[\Delta^\omega\xi\eta(g)]\| \leqslant \gamma_K(g)\,\|\zeta\|, \quad \omega\in K,\ \zeta\in\mathfrak{A}_0.$$

Hence $\Delta^\omega\xi\eta(g)$ is left bounded for every $\omega\in\mathbf{C}$. Hence $\xi\eta(g)$ belongs to $\mathfrak{A}_0$ for every $g\in G$. We have next for each $g, g_0\in G$,

$$\sup_{\omega\in K}\|\Delta_\omega\{\xi\eta(g)-\xi\eta(g_0)\}\| = \sup_{\omega\in K}\|\xi_\omega\eta_\omega(g)-\xi_\omega\eta_\omega(g_0)\|$$

$$= \sup_{\omega\in K}\left\|\int_G \{T(-h)\,\Delta^\omega[\xi(g-h)-\xi(g_0-h)]\}\,[\Delta^\omega\eta(h)]\,dh\right\|$$

$$\leqslant \sup_{\omega\in K}\int_G \|\pi_l[\Delta^\omega(\xi(g-h)-\xi(g_0-h))]\|\,\|\Delta^\omega\eta(h)\|\,dh$$

$$\leqslant \int_G p_K(\xi(g-h)-\xi(g_0-h))\,p_K(\eta(h))\,dh \to 0$$

as g tends to g_0. Similar arguments show that

$$\lim_{g\to g_0}\sup_{\omega\in K}\|\pi_l(\Delta^\omega\xi\eta(g))-\pi_l(\Delta^\omega\xi\eta(g_0))\| = 0;$$

$$\lim_{g\to g_0}\sup_{\omega\in K}\|\pi_r(\Delta^\omega\xi\eta(g))-\pi_r(\Delta^\omega\xi\eta(g_0))\| = 0.$$

Hence the function: $g\in G\mapsto\xi\eta(g)\in\mathfrak{A}_0$ is continuous. The fact that $\xi\eta$ has compact support follows from the usual arguments for convolution. Q.E.D.

LEMMA 5.4. *For each ξ, η, $\zeta\in\mathcal{K}(\mathfrak{A}_0; G)$, we have $(\xi\eta)\zeta=\xi(\eta\zeta)$ and $(\xi\eta\,|\,\zeta)=(\eta\,|\,\xi^\#\zeta)$.*

Proof. The usual arguments of changing the order in integration based on Fubini's theorem verify the equalities, so we leave it to the reader. Q.E.D.

Thus $\mathcal{K}(\mathfrak{A}_0; G)$ is an involutive algebra over $\mathbf{C}$.

LEMMA 5.5. *For each fixed $\xi\in\mathcal{K}(\mathfrak{A}_0; G)$, the left multiplication operator $\pi_l(\xi)$: $\eta\in\mathcal{K}(\mathfrak{A}_0; G)\mapsto\xi\eta\in\mathcal{K}(\mathfrak{A}_0; G)$ is bounded.*

Proof. Taking an arbitrary $\eta\in\mathcal{K}(\mathfrak{A}_0; G)$, we compute as follows:

$$\begin{aligned}
|(\xi\eta|\zeta)| &= \left|\int_G\left(\int_G [T(-h)\,\xi(g-h)]\,\eta(h)\,dh\,|\,\zeta(g)\right)dg\right| \\
&\leqslant \iint_{G\times G} |([T(-h)\,\xi(g-h)]\,\eta(h)\,|\,\zeta(g))|\,dg\,dh \\
&\leqslant \iint_{G\times G} \|[T(-h)\,\xi(g-h)]\,\eta(h)\|\,\|\zeta(g)\|\,dg\,dh \\
&\leqslant \iint_{G\times G} \|\pi_l(\xi(g-h))\|\,\|\eta(h)\|\,\|\zeta(g)\|\,dg\,dh \\
&= \iint_{G\times G} \|\pi_l(\xi(g))\|\,\|\eta(h)\|\,\|\zeta(g-h)\|\,dg\,dh \\
&\leqslant \int_G \|\pi_l(\xi(g))\|\left[\left(\int_G \|\eta(h)\|^2 dh\right)^{\frac12}\left(\int_G \|\zeta(g-k)\|^2\,dk\right)^{\frac12}\right]dg \\
&= \|\eta\|\,\|\zeta\|\int_G \|\pi_l(\xi(g))\|\,dg.
\end{aligned}$$

Hence we have
$$\|\xi\eta\| = \sup_{\|\zeta\|\leqslant 1}|(\xi\eta|\zeta)| \leqslant \|\eta\|\int_G \|\pi_l(\xi(g))\|\,dg.$$

Thus $\pi_l(\xi)$ is bounded Q.E.D.

LEMMA 5.6. *Given a function $f\in\mathcal{K}(G)$, we put, for each $\xi\in L^2(\mathfrak{H};G)$,*

$$(f*\xi)(g) = \int_G f(g-h)\,\xi(h)\,dh. \tag{5.9}$$

We conclude then that

(i) *$f*\xi$ is an $\mathfrak{H}$-valued square integrable continuous function on G;*

(ii) *for each $\xi\in\mathcal{K}(\mathfrak{A}_0;G)$ $f*\xi$ belongs to $\mathcal{K}(\mathfrak{A}_0;G)$;*

(iii) *for each $\xi,\eta\in\mathcal{K}(\mathfrak{A}_0;G)$,*

$$(f*\xi)\eta = f*(\xi\eta) \text{ and } (f*\xi|\eta) = (\xi|f^**\eta),$$

where $f^*(g)=\overline{f(-g)}$.

Proof. Let $\{\xi_n\}$ be a sequence in $\mathcal{K}(\mathfrak{A}_0;G)$ with $\lim_{n\to\infty}\|\xi-\xi_n\|=0$. We have

$$\begin{aligned}
\|f*\xi(g) - f*\xi_n(g)\| &\leqslant \int_G |f(g-h)|\,\|\xi(h)-\xi_n(h)\|\,dh \\
&\leqslant \left\{\int_G |f(g-h)|^2 dh\right\}^{\frac12}\left\{\int_G \|\xi(h)-\xi_n(h)\|^2\,dh\right\}^{\frac12} \\
&= \left[\int |f(h)|^2\,dh\right]^{\frac12}\|\xi-\xi_n\| \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence $f*\xi$ is the uniform limit of $\{f*\xi_n\}$, so that it is continuous since $f*\xi_n$ is continuous. The square integrability of $f*\xi$ is seen by an argument similar to that in Lemma 5.5. It is also obvious that $f*\xi$ belongs to $\mathcal{K}(\mathfrak{A}_0; G)$ if ξ does. The last equalities are verified by the usual arguments of changing order of integration based on Fubini's Theorem Q.E.D.

LEMMA 5.7. *The set of all products $\xi\eta$, $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G)$, is total in $L^2(\mathfrak{H}; G)$.*

Proof. Let ζ be an element of $L^2(\mathfrak{H}; G)$ orthogonal to every $\xi\eta$, $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G)$ which means that

$$\int_G (\xi\eta(g)\,|\,\zeta(g))\,dg = 0, \quad \xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G).$$

By Lemma 5.6, we have, for each $f \in \mathcal{K}(G)$.

$$(\xi\eta\,|\,f*\zeta) = ((f^* * \xi)\eta\,|\,\zeta) = 0.$$

Let f_1 and f_2 be elements of $\mathcal{K}(G)$. Since the functions: $g \in G \mapsto f_1(g)\xi(g) = (f_1\xi)(g)$ and $g \in G \mapsto f_2(g)\eta(g) = (f_2\eta)(g)$, $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G)$, belong to $\mathcal{K}(\mathfrak{A}_0; G)$, we have

$$\begin{aligned} 0 = ((f_1\xi)(f_2\eta)\,|\,f*\zeta) &= \int_G (\{(f_1\xi)(f_2\eta)\}(g)\,|\,f*\zeta(g))\,dg \\ &= \iint_{G\times G} f_1(g-h) f_2(h)([T(-h)\,\xi(g-h)]\,\eta(h)\,|\,f*\zeta(g))\,dg\,dh \\ &= \iint_{G\times G} f_1(g) f_2(h)([T(-h)\,\xi(g)]\,\eta(h)\,|\,f*\zeta(g+h))\,dg\,dh. \end{aligned}$$

Since f_1 and f_2 are arbitrary, and the function:

$$(g, h) \in G \times G \mapsto ([T(-h)\xi(g)]\eta(h)\,|\,(f*\zeta)(g+h)) \in \mathbf{C}$$

is continuous, we have

$$(T(-h)\xi(g)\eta(h)\,|\,(f*\zeta)(g+h)) = 0, \quad g, h \in G.$$

Putting $h = 0$, we have $(\xi(g)\eta(0)\,|\,f*\zeta(g)) = 0$ for every $g \in G$. Since the values of elements of $\mathcal{D}(\mathfrak{A}_0; G)$ at a fixed point in G exhaust all of $\mathfrak{A}_0$, we have $(\xi\eta\,|\,f*\zeta(g)) = 0$ for every $\xi, \eta \in \mathfrak{A}_0$, and $g \in G$, so that $f*\zeta(g) = 0$ for every $g \in G$. For each compact neighborhood K of 0 in G, we choose a positive $f_K \in \mathcal{K}(G)$ with $\int_G f_K(g)\,dg = 1$. The net $\{f_K * \zeta\}$ converges in norm to ζ. Hence we have

$$\zeta = \lim f_K * \zeta = 0. \qquad \text{Q.E.D.}$$

LEMMA 5.8. *For each* $\omega \in \mathbf{C}$ *and* $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G)$, *we have*

$$\left.\begin{aligned} &\tilde{\Delta}(\omega)(\xi\eta) = (\tilde{\Delta}(\omega)\,\xi)\,(\tilde{\Delta}(\omega)\,\eta);\\ &(\tilde{\Delta}(\omega)\,\xi)^{\#} = \tilde{\Delta}(-\bar{\omega})\,\xi^{\#}. \end{aligned}\right\} \tag{5.10}$$

Proof. The first equality is seen by the following:

$$\begin{aligned}
[\tilde{\Delta}(\omega)\,\xi]\,[\tilde{\Delta}(\omega)\,\eta]\,(g) &= \int_G [T(-h)\,(\tilde{\Delta}(\omega)\,\xi)\,(g-h)]\,[\tilde{\Delta}(\omega)\,\eta]\,(h)\,dh \\
&= \int_G \chi(g-h)^{\omega}\,[T(-h)\,\Delta^{\omega}\xi(g-h)]\,[\chi(h)^{\omega}\Delta^{\omega}\eta(h)]\,dh \\
&= \chi(g)^{\omega}\int_G \Delta^{\omega}\{[T(-h)\,\xi(g-h)]\,\eta(h)\}\,dh \\
&= \chi(g)^{\omega}\Delta^{\omega}\int_G [T(-h)\,\xi(g-h)]\,\eta(h)\,dh = \tilde{\Delta}(\omega)\,(\xi\eta)\,(g),
\end{aligned}$$

where the last step follows from the closedness of Δ^{ω}.

The second equality follows from the calculation:

$$\begin{aligned}
(\tilde{\Delta}(\omega)\,\xi)^{\#}\,(g) &= T(-g)\,(\tilde{\Delta}(\omega)\,\xi)\,(-g)^{\#} = T(-g)\,[\chi(g)^{-\omega}\Delta^{\omega}\xi(-g)]^{\#} \\
&= T(-g)\,\chi(g)^{-\bar{\omega}}\Delta^{-\bar{\omega}}\xi(-g)^{\#} = \chi(g)^{-\bar{\omega}}\Delta^{-\bar{\omega}}T(-g)\,\xi(-g)^{\#} = [\tilde{\Delta}(-\bar{\omega})\,\xi^{\#}]\,(g).
\end{aligned}$$

Q.E.D.

LEMMA 5.9. *For every pair* ξ, η *in* $\mathcal{K}(\mathfrak{A}_0; G)$, *we have*

$$(\tilde{\Delta}(1)\,\xi\,|\,\eta) = (\eta^{\#}\,|\,\xi^{\#}). \tag{5.11}$$

Proof. We compute as follows:

$$\begin{aligned}
(\eta^{\#}\,|\,\xi^{\#}) &= \int_G (T(-g)\,\eta(-g)^{\#}\,|\,T(-g)\,\xi(-g)^{\#})\,dg \\
&= \int_G (\chi(g)^{-\frac{1}{2}}U(-g)\,\eta(-g)^{\#}\,|\,\chi(g)^{-\frac{1}{2}}U(-g)\,\xi(-g)^{\#})\,dg \\
&= \int_G \chi(g)^{-1}(U(-g)\,\eta(-g)^{\#}\,|\,U(-g)\,\xi(-g)^{\#})\,dg \\
&= \int_G \chi(g)^{-1}(\eta(-g)^{\#}\,|\,\xi(-g)^{\#})\,dg = \int_G \chi(g)^{-1}(\Delta\xi(-g)\,|\,\eta(-g))\,dg \\
&= \int_G \chi(g)\,(\Delta\xi(g)\,|\,\eta(g))\,dg = (\tilde{\Delta}(1)\,\xi\,|\,\eta).
\end{aligned}$$

Q.E.D.

LEMMA 5.10. *For every $t \in \mathbf{R}$, $\tilde{\Delta}(t)$ is essentially self-adjoint on $\mathcal{K}(\mathfrak{A}_0; G)$.*

Proof. The algebraic tensor product $\mathfrak{A}_0 \otimes \mathcal{K}(G)$ is canonically imbedded in $\mathcal{K}(\mathfrak{H}; G)$. Let χ be the (not bounded unless $\chi = 1$) self-adjoint operator on $L^2(G)$ defined by

$$(\chi f)(g) = \chi(g) f(g), \quad f \in L^2(G).$$

We have then $(1 + \chi^t)\mathcal{K}(G) = \mathcal{K}(G)$. Identifying $L^2(\mathfrak{H}; G)$ and the Hilbert space tensor product $\mathfrak{H} \otimes L^2(G)$, we define

$$H(t) = (1 - \Delta^t)^{-1} \otimes (1 + \chi^t)^{-1} + \Delta^t (1 + \Delta^t)^{-1} \otimes \chi^t (1 + \chi^t)^{-1}.$$

on $L^2(\mathfrak{H}; G)$. It is then clear that $H(t)$ is a bounded positive operator on $L^2(\mathfrak{H}; G)$. Since $(1 + \Delta^t)^{-1} \otimes (1 + \chi^t)^{-1}$ is nonsingular and $H(t) \geqslant (1 + \Delta^t)^{-1} \otimes (1 + \chi^t)^{-1}$, $H(t)$ is nonsingular too, i.e., the range of $H(t)$ is dense in $L^2(\mathfrak{H}; G)$. We have

$$\begin{aligned}(1 + \tilde{\Delta}(t))\mathcal{K}(\mathfrak{A}_0; G) &\supset (1 + \Delta^t \otimes \chi^t)(\mathfrak{A}_0 \otimes \mathcal{K}(G)) \\ &= H(t)[1 + \Delta^t)\mathfrak{A}_0 \otimes (1 + \chi^t)\mathcal{K}(G)) = H(t)[(1 + \Delta^t)\mathfrak{A}_0 \otimes \mathcal{K}(G)],\end{aligned}$$

where the last two tensor products mean the algebraic ones. Since $(1 + \Delta^t)\mathfrak{A}_0$ is dense in $\mathfrak{H}$, the last expression in the above equality is dense in $L^2(\mathfrak{H}; G)$; hence so is $(1 + \tilde{\Delta}(t))\mathcal{K}(\mathfrak{A}_0; G)$ in $L^2(\mathfrak{H}; G)$. This means that $\tilde{\Delta}(t)$ is essentially self-adjoint. Q.E.D.

It is now clear that the function:

$$w \in \mathbf{C} \to (\tilde{\Delta}(\omega)\,\xi \,|\, \eta) = \int_G (\chi(g)^{\omega} \Delta^{\omega} \xi(g) \,|\, \eta(g))\, dg$$

is an entire function for every pair ξ, η in $\mathcal{K}(\mathfrak{A}_0; G)$. Thus combining this with Lemmas 5.3 through 5.10, we have obtained the following result.

THEOREM 5.11. *The involutive algebra $\mathcal{K}(\mathfrak{A}_0; G)$ is a Tomita algebra.*

The associated unitary involution $\tilde{J}$ in $L^2(\mathfrak{H}; G)$ is given by

$$(\tilde{J}\xi)(g) = U(-g)J\xi(-g) = JU(-g)\xi(-g), \quad \xi \in L^2(\mathfrak{H}; G). \tag{5.12}$$

This is seen by the following, with $\xi \in \mathcal{K}(\mathfrak{A}_0; G)$:

$$\begin{aligned}(\tilde{J}\xi)(g) &= (\tilde{\Delta}(\tfrac{1}{2})\,\xi^{\#})(g) = \chi(g)^{\frac{1}{2}}\Delta^{\frac{1}{2}}\xi^{\#}(g) = \chi(g)^{\frac{1}{2}}\Delta^{\frac{1}{2}}T(-g)\,\xi(-g)^{\#} \\ &= U(-g)\,\Delta^{\frac{1}{2}}J\Delta^{\frac{1}{2}}\xi(-g) = U(-g)\,J\xi(-g) = JU(-g)\,\xi(-g).\end{aligned}$$

THEOREM 5.12. *The left von Neumann algebra $\mathfrak{L}(\mathcal{K}(\mathfrak{A}_0; G))$ of the Tomita algebra $(\mathcal{K}\mathfrak{A}_0; G)$ coincides with the crossed product $\mathcal{R}(\mathcal{M}; \alpha)$ of $\mathcal{M}$ by the action α of G.*

Proof. Let ξ be an element of $\mathcal{K}(\mathfrak{A}_0; G)$. For each $g \in G$, put

$$x(g) = \alpha_g \pi_l(\xi(g)) \in \mathcal{M};$$

$$x = \int_G \pi_\alpha(x(g))\, \lambda(g)\, dg \in \mathcal{R}(\mathcal{M}; \alpha).$$

We have then for each $\eta, \zeta \in \mathcal{K}(\mathfrak{A}_0; G)$,

$$\begin{aligned}(x\eta\,|\,\zeta) &= \int_G (\pi_\alpha(x(g))\,\lambda(g)\,\eta\,|\,\zeta)\, dg = \int_{G\times G} (\alpha_h^{-1}(x(g))\,\eta(h-g)\,|\,\zeta(h))\, dh\, dg \\ &= \iint_{G\times G} (\alpha_{h-g}^{-1}\circ\pi_l(\xi(g))\,\eta(h-g)\,|\,\zeta(h))\, dg\, dh = \iint_G ([T(g-h)\,\xi(g)]\,\eta(h-g)\,|\,\zeta(h))\, dg\, dh \\ &= \iint_{G\times G} ([T(-g)\,\xi(h-g)]\,\eta(g)\,|\,\zeta(h))\, dg\, dh - (\xi\eta\,|\,\zeta) = (\pi_l(\xi)\,\eta\,|\,\zeta).\end{aligned}$$

Hence $\pi_l(\xi) = x$ belongs to $\mathcal{R}(\mathcal{M}; \alpha)$, so that $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G)) \subset \mathcal{R}(\mathcal{M}; \alpha)$.

Let $x = \pi_l(\xi_0) \in \mathcal{M}$ for an arbitrary element $\xi_0 \in \mathfrak{A}_0$. For each $\eta \in \mathcal{K}(\mathfrak{A}_0; G)$, the function:

$$g \in G \mapsto (\pi_\alpha(x)\eta)(g) = \alpha_g^{-1}(x)\eta(g) = [T(-g)\xi_0]\eta(g) \in \mathfrak{A}_0$$

is continuous with respect to the locally convex topology in $\mathfrak{A}_0$ given by (5.6) and has compact support, so that $\pi_\alpha(x)\eta$ belongs to $\mathcal{K}(\mathfrak{A}_0; G)$. For each $\zeta \in \mathcal{K}(\mathfrak{A}_0; G)$, we have

$$\begin{aligned}[\pi_r(\zeta)\,\pi_\alpha(x)\,\eta](g) &= [(\pi_\alpha(x)\,\eta)\,\zeta](g) = \int_G [T(-h)\,(\pi_\alpha(x)\,\eta)\,(g-h)]\,\zeta(h)\, dh \\ &= \int_G (T(-h)\,\{[T(h-g)\,\xi_0]\,\eta(g-h)\}\,\zeta(h)\, dh \\ &= \int_G (T(-g)\,\xi_0)\,[T(-h)\,\eta(g-h)]\,\zeta(h)\, dh = \alpha_g^{-1}(x) \int_G [T(-h)\,\eta(g-h)]\,\zeta(h)\, dh \\ &= [\pi_\alpha(x)\,(\eta\zeta)](g) = [\pi_\alpha(x)\,\pi_r(\zeta)\,\eta](g).\end{aligned}$$

Hence $\pi_\alpha(x)$ commutes with $\pi_r(\zeta)$, $\zeta \in \mathcal{K}_0(\mathfrak{A}_0; G)$, so that it commutes with the right von Neumann algebra $\mathcal{R}(\mathcal{K}(\mathfrak{A}_0; G))$; Hence it belongs to $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G))$. Since $\pi_l(\mathfrak{A}_0)$ generates $\mathcal{M}$, $\pi_\alpha(\mathcal{M})$ is contained in $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G))$.

Let g be an arbitrary fixed element of G. For each $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G)$, we have

$$\begin{aligned}[\lambda(g)\,\pi_r(\eta)\,\xi](h) &= [\pi_r(\eta)\,\xi](h-g) = \int_G [T(-k)\,\xi(h-g-k)]\,\eta(k)\, dk \\ &= \int_G [T(-k)\,(\lambda(g)\,\xi)\,(h-k)]\,\eta(k)\, dk = [(\lambda(g)\,\xi)\,\eta](h) = [\pi_r(\eta)\,\lambda(g)\,\xi](h).\end{aligned}$$

Hence $\lambda(g)$ and $\pi_r(\eta)$ commute, so that $\lambda(g)$ falls in $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G))$. Therefore, $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G))$ contains the generators $\lambda(G)$ and $\pi_\alpha(\mathcal{M})$ of $\mathcal{R}(\mathcal{M}; \alpha)$; hence $\mathcal{R}(\mathcal{M}; \alpha) \subset \mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G))$. Thus we get

$$\mathcal{R}(\mathcal{M}; \alpha) = \mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G)). \qquad \text{Q.E.D.}$$

COROLLARY 5.13. *The commutant of $\mathcal{R}(\mathcal{M}; \alpha)$ is generated by the operators $\pi'(y)$, $y \in \mathcal{M}'$, and $\lambda'_\alpha(g)$, $g \in G$, which are defined as follows:*

$$\left.\begin{aligned} &(\pi'(y)\,\xi)\,(h) = y\xi(h)\,; \\ &(\lambda'_\alpha(g)\,\xi)\,(h) = U(g)\,\xi(h+g), \quad \xi \in L^2(\mathfrak{H}; G),\ h \in G. \end{aligned}\right\} \tag{5.13}$$

Remark. The action α of G on $\mathcal{M}$ is extended to an action of G on $\mathcal{L}(\mathfrak{H})$ induced by the unitary representation U of G, which in turn defines a continuous action of G on $\mathcal{M}'$, denoted also by α. We note here that the representation π' of $\mathcal{M}'$ does not depend on the action α, while the representation λ'_α of G depends on $U(g)$, hence on the action α of G, which is in contrast with the situation for the covariant representation $\{\pi_\alpha, \lambda\}$ of $\{\mathcal{M}, \alpha\}$.

Proof. By Theorem 5.12, we have

$$\tilde{J}\mathcal{R}(\mathcal{M}; \alpha)\tilde{J} = \mathcal{R}(\mathcal{M}; \alpha)'.$$

Hence $\mathcal{R}(\mathcal{M}; \alpha)'$ is generated by $\tilde{J}\pi_\alpha(\mathcal{M})\tilde{J}$ and $\tilde{J}\lambda(G)\tilde{J}$. For each $x \in \mathcal{M}$, we have by (5.12)

$$\begin{aligned}(\tilde{J}\pi_\alpha(x)\,\tilde{J}\xi)\,(g) &= JU(-g)\,(\pi_\alpha(x)\,\tilde{J}\xi)\,(-g) = JU(-g)\,\alpha_g^{-1}(x)\,(\tilde{J}\xi)\,(-g) \\ &= JU(-g)\,\alpha_g^{-1}(x)\,U(g)\,J\xi(g) = JxJ\xi(g) = \pi'(y)\,\xi(g),\end{aligned}$$

where $y = JxJ \in \mathcal{M}'$. We have next

$$\begin{aligned}(\tilde{J}\lambda(g)\tilde{J}\xi)\,(h) &= JU(-h)\,(\lambda(g)\,\tilde{J}\xi)\,(-h) = JU(-h)\,(\tilde{J}\xi)\,(-h-g) \\ &= U(g)\,\xi(h+g) = \lambda'_\alpha(g)\,\xi(h)\,;\end{aligned}$$

hence $\tilde{J}\lambda(g)\,\tilde{J} = \lambda'_\alpha(g), \quad g \in G.$ Q.E.D.

Definition 5.14. The canonical weight $\tilde{\varphi}$ on $\mathcal{R}(\mathcal{M}; \alpha)$ associated with the Tomita algebra $\mathcal{K}(\mathfrak{A}_0; G)$ is said to be *dual* to the original weight φ on $\mathcal{M}$.

The dual weight $\tilde{\varphi}$ is given by

$$\tilde{\varphi}(x) = \begin{cases} \|\xi\|^2 \ \text{ if } \ x = \pi_l(\xi)^*\pi_l(\xi), \quad \xi \in \tilde{\mathfrak{A}} \\ +\infty \ \text{otherwise} \end{cases} \tag{5.14}$$

where $\tilde{\mathfrak{A}}$ denotes the full left Hilbert algebra $\mathcal{K}(\mathfrak{A}_0; G)''$ obtained from the Tomita algebra $\mathcal{K}(\mathfrak{A}_0; G)$.

Let $\tilde{\Delta}$ denote the modular operator on $L^2(\mathfrak{H}; G)$ associated with $\mathcal{K}(\mathfrak{A}_0; G)$. We denote by $\{\sigma_t^{\tilde{\varphi}}\}$ the modular automorphism group of $\mathcal{R}(\mathcal{M}; \alpha)$ associated with the dual weight $\tilde{\varphi}$.

PROPOSITION 5.15. *The modular automorphism group $\{\sigma_t^{\tilde{\varphi}}\}$ acts on $\pi_\alpha(\mathcal{M})$ and $\lambda(G)$ in such a way that*

$$\left.\begin{aligned} &\sigma_t^{\tilde{\varphi}}\circ\pi_\alpha(x) = \pi_\alpha\circ\sigma_t^{\varphi}(x), \quad x\in\mathcal{M};\\ &\sigma_t^{\tilde{\varphi}}(\lambda(g)) = \chi(g)^{it}\lambda(g), \quad g\in G,\ t\in\mathbf{R}.\end{aligned}\right\} \tag{5.15}$$

Proof. Noticing that $(\tilde{\Delta}^{it}\xi)(g) = \chi(g)^{it}\Delta^{it}\xi(g)$ for $\xi\in L^2(\mathfrak{H}; G)$, we compute

$$\begin{aligned}\sigma_t^{\tilde{\varphi}}\circ\pi_\alpha(x)\,\xi(g) &= (\tilde{\Delta}^{it}\pi_\alpha(x)\,\tilde{\Delta}^{-it}\xi)\,(g)\\ &= \chi(g)^{it}\Delta^{it}(\pi_\alpha(x)\,\Delta^{-it}\xi)\,(g) = \chi(g)^{it}\Delta^{it}\alpha_g^{-1}(x)\,\chi(g)^{-it}\Delta^{-it}\xi(g)\\ &= \sigma_t^{\varphi}\circ\alpha_g^{-1}(x)\,\xi(g) = \alpha_g^{-1}\circ\sigma_t^{\varphi}(x)\,\xi(g) = (\pi_\alpha\circ\sigma_t^{\varphi}(x)\,\xi)\,(g);\end{aligned}$$

$$\begin{aligned}[\sigma_t^{\tilde{\varphi}}(\lambda(g))\,\xi]\,(h) &= (\tilde{\Delta}^{it}\lambda(g)\,\tilde{\Delta}^{-it}\xi)\,(h)\\ &= \chi(h)^{it}\Delta^{it}(\lambda(g)\,\tilde{\Delta}^{-it}\xi)\,(h) = \chi(h)^{it}\Delta^{it}(\tilde{\Delta}^{-it}\xi)\,(h-g)\\ &= \chi(h)^{it}\Delta^{it}\chi(h-g)^{-it}\Delta^{-it}\xi(h-g) = \chi(g)^{it}\xi(h-g) = \chi(g)^{it}(\lambda(g)\,\xi)\,(h).\end{aligned}$$ Q.E.D.

We now examine how the dual action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{R}(\mathcal{M}; \alpha)$ transforms the dual weight $\tilde{\varphi}$. To this end, we first observe that $\mu(p)$, $p\in\hat{G}$, is a *-automorphism of the Tomita algebra $\mathcal{K}(\mathfrak{A}_0; G)$. It is clear that $\mu(p)\mathcal{K}(\mathfrak{A}_0; G) = \mathcal{K}(\mathfrak{A}_0; G)$, $p\in\hat{G}$. For each $\xi, \eta\in\mathcal{K}(\mathfrak{A}_0; G)$, we get

$$\begin{aligned}[(\mu(p)\,\xi)\,(\mu(p)\,\eta)]\,(g) &= \int_G [T(-h)\,(\mu(p)\,\xi)\,(g-h)]\,(\mu(p)\,\eta)\,(h)\,dh\\ &= \int_G \overline{\langle g-h, p\rangle}\,[T(-h)\,\xi(g-h)]\,\overline{\langle h, p\rangle}\,\eta(h)\,dh\\ &= \int_G \overline{\langle g, p\rangle}\,[T(-h)\,\xi(g-h)]\,\eta(h)\,dh = \overline{\langle g, p\rangle}\,(\xi\eta)\,(g) = [\mu(p)\,\xi\eta]\,(g);\end{aligned}$$

$$\begin{aligned}[\mu(p)\,\xi]^{\#}(g) &= T(-g)\,(\mu(p)\,\xi)\,(-g)^{\#}\\ &= T(-g)\{\overline{\langle -g, p\rangle}\,\xi(-g)\}^{\#} = \overline{\langle g, p\rangle}\,T(-g)\,\xi(-g)^{\#} = [\mu(p)\,\xi^{\#}]\,(g).\end{aligned}$$

Therefore, $\mu(p)$ is a *-automorphism of $\mathcal{K}(\mathfrak{A}_0; G)$ which preserves the inner product as well, being unitary. Thus we have

$$\hat{\alpha}_p\circ\pi_l(\xi) = \pi_l(\mu(p)\,\xi), \quad p\in\hat{G},\ \xi\in\mathcal{K}(\mathfrak{A}_0; G). \tag{5.16}$$

Hence we can state the following result:

PROPOSITION 5.16. *The dual action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{R}(\mathcal{M};\alpha)$ leaves the dual weight $\tilde{\varphi}$ invariant.*

Proof. The assertion is seen immediately by the following:

$$\tilde{\varphi}\circ\hat{\alpha}_p(\pi_l(\xi)^*\pi_l(\xi)) = \tilde{\varphi}(\pi_l(\mu(p)\xi)^*\pi_l(\mu(p)\xi)) = \|\mu(p)\xi\|^2 = \|\xi\|^2 = \tilde{\varphi}(\pi_l(\xi)^*\pi_l(\xi)).$$

Q.E.D.

LEMMA 5.17. *For each element ξ of $L^2(\mathfrak{H};G)$ and $\omega\in\mathbf{C}$, the following two statements are equivalent:*

(i) *ξ belongs to the domain $\mathcal{D}(\tilde{\Delta}^\omega)$ of the closed operator $\tilde{\Delta}^\omega$;*

(ii) *$\xi(g)$ belongs to $\mathcal{D}(\Delta^\omega)$ for almost every $g\in G$ and*

$$\int_G |\chi(g)^\omega|^2 \|\Delta^\omega\xi(g)\|^2\,dg < +\infty.$$

If this is the case, then

$$\tilde{\Delta}^\omega\xi(g) = \chi(g)^\omega\Delta^\omega\xi(g) \tag{5.17}$$

for almost every $g\in G$.

Proof. (i) $\Rightarrow$ (ii): Suppose $\xi\in\mathcal{D}(\Delta^\omega)$. Let $\omega=s+it$, $s,t\in\mathbf{R}$. By Lemma 5.10, we can find a sequence $\{\xi_n\}$ in $\mathcal{K}(\mathfrak{A}_0;G)$ such that

$$\lim_{n\to\infty}\|\xi-\xi_n\|=0,$$

$$\lim_{n\to\infty}\|\tilde{\Delta}^\omega\xi-\tilde{\Delta}^\omega\xi_n\| = \lim_{n\to\infty}\|\tilde{\Delta}^s\xi-\tilde{\Delta}^s\xi_n\| = 0.$$

Choosing a subsequence, we may assume that

$$\sum_{n=0}^{\infty}\|\xi_{n+1}-\xi_n\|^2 < +\infty;$$

$$\sum_{n=0}^{\infty}\|\tilde{\Delta}^\omega(\xi_{n+1}-\xi_n)\|^2 < +\infty,$$

where $\xi_0=0$. Hence we have

$$\int_G\sum_{n=0}^{\infty}\|\xi_{n+1}(g)-\xi_n(g)\|^2\,dg = \sum_{n=0}^{\infty}\int_G\|\xi_{n+1}(g)-\xi_n(g)\|^2\,dg = \sum_{n=0}^{\infty}\|\xi_{n+1}-\xi_n\|^2 < +\infty;$$

$$\int_G\sum_{n=0}^{\infty}\|\chi(g)^\omega\Delta^\omega(\xi_{n+1}(g)-\xi_n(g))\|^2\,dg = \sum_{n=0}^{\infty}\|\tilde{\Delta}^\omega(\xi_{n+1}-\xi_n)\|^2 < +\infty.$$

Therefore, there exists a locally null subset N of G such that

$$\sum_{n=0}^{\infty} \|\xi_{n+1}(g) - \xi_n(g)\|^2 < +\infty$$

and
$$\sum_{n=0}^{\infty} \|\chi(g)^{\omega} \Delta^{\omega} (\xi_{n+1}(g) - \xi_n(g))\|^2 < +\infty$$

for every $g \notin N$. Hence $\{\xi_n(g)\}$ converges to $\xi'(g)$ and $\{\Delta^{\omega} \xi_n(g)\}$ converges to $\eta(g)$ in $\mathfrak{H}$ for every $g \notin N$, so that $\xi'(g) \in \mathcal{D}(\Delta^{\omega})$ and $\Delta^{\omega} \xi'(g) = \eta(g)$. For each $n < m$, we have

$$\|\xi_m(g) - \xi_n(g)\|^2 \leqslant \sum_{k \geqslant n} \|\xi_{k+1}(g) - \xi_k(g)\|^2;$$

hence
$$\|\xi'(g) - \xi_n(g)\|^2 \leqslant \sum_{k \geqslant n} \|\xi_{k+1}(g) - \xi_n(g)\|^2,$$

which implies that

$$\int_G \|\xi'(g) - \xi_n(g)\|^2 dg \leqslant \sum_{k \geqslant n} \int_G \|\xi_{k+1}(g) - \xi_k(g)\|^2 dg = \sum_{k \geqslant n} \|\xi_{k+1} - \xi_k\|^2.$$

Hence ξ' is an $\mathfrak{H}$-valued square integrable function and $\lim_{n \to \infty} \|\xi_n - \xi'\| = 0$. Therefore, we have $\xi = \xi'$ in $L^2(\mathfrak{H}; G)$. Similarly, the function: $g \in G \mapsto \chi(g)^{\omega} \Delta^{\omega} \xi'(g) = \chi(g)^{\omega} \eta(g)$ is square integrable and $(\tilde{\Delta}^{\omega} \xi)(g) = \chi(g)^{\omega} \Delta^{\omega} \xi(g)$ for almost every $g \in G$. Thus (ii) and (5.17) follow.

(ii) $\Rightarrow$ (i): Suppose condition (ii) is satisfied. For every $\eta \in \mathcal{K}(\mathfrak{A}_0; G)$, we have

$$\begin{aligned}
|(\xi | \tilde{\Delta}^{\bar{\omega}} \eta)| &= \left| \int_G (\xi(g) | \chi(g)^{\bar{\omega}} \Delta^{\bar{\omega}} \eta(g))\, dg \right| \\
&\leqslant \int_G |\chi(g)^{\omega}|\, |(\Delta^{\omega} \xi(g) | \eta(g))|\, dg \leqslant \int_G |\chi(g)^{\omega}|\, \|\Delta^{\omega} \xi(g)\|\, \|\eta(g)\|\, dg \\
&\leqslant \left\{ \int_G |\chi(g)^{\omega}|^2 \|\Delta^{\omega} \xi(g)\|^2 dg \right\}^{\frac{1}{2}} \left\{ \int_G \|\eta(g)\|^2 dg \right\}^{\frac{1}{2}} \\
&= \left\{ \int_G |\chi(g)^{\omega}|^2 \|\Delta^{\omega} \xi(g)\|^2 dg \right\}^{\frac{1}{2}} \|\eta\|,
\end{aligned}$$

thus ξ belongs to $\mathcal{D}(\Delta^{\omega})$ by Lemma 5.10. Q.E.D.

LEMMA 5.18. *Let η be an element of $\mathcal{K}(\mathfrak{A}_0; G)$. For every $\xi \in L^2(\mathfrak{H}; G)$, $\pi_r(\eta)\xi$ is an $\mathfrak{H}$-valued continuous function and given by the formula:*

$$\pi_r(\eta)\, \xi(g) = \int_G \pi_r(\eta(h))\, T(-h)\, \xi(g-h)\, dh. \tag{5.18}$$

Proof. Let $\{\xi_n\}$ be a sequence in $\mathcal{K}(\mathfrak{A}_0; G)$ such that $\lim_{n \to \infty} \|\xi - \xi_n\| = 0$. We have then

$$\left\| \int_G \pi_r(\eta(h))\, T(-h)\, \xi(g-h)\, dh - \int_G \pi_r(\eta(h))\, T(-h)\, \xi_n(g-h)\, dh \right\|$$
$$\leqslant \int_G \|\pi_r(\eta(h))\, T(-h)\, [\xi(g-h) - \xi_n(g-h)]\|\, dh$$
$$\leqslant \int_G \|\pi_r(\eta(h))\|\, \|T(-h)\|\, \|\xi(g-h) - \xi_n(g-h\|\, dh$$
$$\leqslant \left\{ \int_G \|\pi_r(\eta(h))\|^2 \chi(-h)\, dh \right\}^{\frac{1}{2}} \left\{ \int_G \|\xi(g-h) - \xi_n(g-h)\|^2\, dh \right\}^{\frac{1}{2}}$$
$$= \|\xi - \xi_n\| \left\{ \int_G \|\pi_r(\eta(h))\|^2 \chi(-h)\, dh \right\}^{\frac{1}{2}} \to 0$$

as $n \to \infty$.

Hence $\xi_n \eta(g)$ converges to the right hand side of (5.18) uniformly for $g \in G$. Hence equality (5.18) holds and the function: $g \in G \mapsto (\pi_r(\eta)\xi)(g)$ is continuous. Q.E.D.

Let ξ be an element of $\mathcal{K}(\mathfrak{A}_0; G)$ of the form $\xi = \eta\zeta$ with $\eta, \zeta \in \mathcal{K}(\mathfrak{A}_0; G)$. For each $g \in G$ put

$$x(g) = \alpha_g \circ \pi_l(\xi(g)), \quad y(g) = \alpha_g \circ \pi_l(\eta(g)), \quad z(g) = \alpha_g \circ \pi_l(\zeta(g)).$$

We have then

$$x(g) = \int_G y(h)\, \alpha_h(z(g-h))\, dh. \tag{5.19}$$

LEMMA 5.19. *In the above situation*

$$\dot{\tilde{\varphi}}(x) = \dot{\varphi}(x(0)). \tag{5.20}$$

Proof. The equality is seen as follows:

$$\dot{\tilde{\varphi}}(x) = (\zeta | \eta^\#) = \int_G (\zeta(g) | \eta^\#(g))\, dg = \int_G (\zeta(g) | T(-g)\, \eta(-g)^\#)\, dg$$
$$= \int_G \dot{\varphi}[\alpha_g^{-1} \circ \pi_l(\eta(-g))\, \pi_l(\zeta(g))]\, dg = \int_G \dot{\varphi}[y(-g)\, \alpha_g^{-1}(z(g))]\, dg$$
$$= \int_G \dot{\varphi}[y(g)\, \alpha_g(z(-g))]\, dg = \dot{\varphi}\left[\int_G y(g)\, \alpha_g(z(-g))\, dg \right] = \dot{\varphi}(x(0)),$$

where the last step is justified by arguments similar to the proof in [33; Lemma 3.1]. Q.E.D.

6. Bi-dual weight

We keep the basic assumptions and notations in the previous section. In this section, we examine the bi-dual weight $\tilde{\tilde{\varphi}}$ on $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha}))$.

By Lemma 5.17, the associated modular operator $\tilde{\tilde{\Delta}}$ and its complex power $\tilde{\tilde{\Delta}}^{\omega}$, $\omega \in \mathbf{C}$, are given as follows:

(i) The domain $\mathcal{D}(\tilde{\tilde{\Delta}}^{\omega})$ consists of all $\xi \in L^2(\mathfrak{H}; G\times\hat{G})$ such that $\xi(g, p) \in \mathcal{D}(\Delta^{\omega})$ for almost every $(g, p) \in G\times\hat{G}$ and

$$\int_G \|\chi(g)^{\omega}\Delta^{\omega}\xi(g, p)\|^2\, dg\, dp < +\infty;$$

(ii) Then $\Delta^{\omega}\xi$ is defined by

$$\tilde{\tilde{\Delta}}^{\omega}\xi(g, p) = \chi(g)^{\omega}\Delta^{\omega}\xi(g, p), \quad g \in G,\ p \in \hat{G}. \tag{6.1}$$

We first consider the Tomita algebra $\mathcal{K}(\mathfrak{A}_0; G\times\hat{G})$ which is defined by the same procedure as (5.7) and (5.8). Namely, we adapt the following structure in $\mathcal{K}(\mathfrak{A}_0; G\times\hat{G})$:

$$\left.\begin{aligned}
(\xi|\eta) &= \int_{G\times G}(\xi(g, p)|\eta(g, p))\, dg\, dp;\\
(\xi\eta)(g, p) &= \int_{G\times\hat{G}}\langle g-h, q\rangle\,[T(-h)\,\xi(g-h, p-q)]\,\eta(h, q)\, dh\, dq;\\
\xi^{\#}(g, p) &= \langle g, p\rangle\, T(-g)\,\xi(-g, -p)^{\#}.
\end{aligned}\right\} \tag{6.2}$$

However, as we have seen once in § 4, it is more convenient to express our Tomita algebra in terms of a function system over $G\times G$ instead of $G\times\hat{G}$.

LEMMA 6.1. *Let F denote the unitary operator of $L^2(\mathfrak{H}; G\times\hat{G})$ onto $L^2(\mathfrak{H}; G\times G)$ defined by* (4.8). *If ξ is an element of $\mathcal{K}(\mathfrak{A}_0; G\times\hat{G})$, then $F\xi(g, h)$ belongs to $\mathfrak{A}_0$ for every $g, h \in G$.*

Proof. Let η and ζ be elements in $\mathfrak{A}_0$. We have then

$$\begin{aligned}
(\pi_r(\eta)(F\xi)(g, h)|\zeta) = (F\xi(g, h)|\pi_r(\eta)^*\zeta) &= \int_{\hat{G}}\overline{\langle h, p\rangle}\,(\xi(g, p)|\pi_r(\eta)^*\zeta)\, dp\\
&= \int_{\hat{G}}\overline{\langle h, p\rangle}\,(\pi_r(\eta)\,\xi(g, p)|\zeta)\, dp = \int_{\hat{G}}\overline{\langle h, p\rangle}\,(\pi_l(\xi(g, p))\,\eta|\zeta)\, dp,
\end{aligned}$$

so that

$$\pi_r(\eta)(F\xi)(g, h) = \int_{\hat{G}}\overline{\langle h, p\rangle}\,\pi_l(\xi(g, p))\,\eta\, dp.$$

Hence $(F\xi)(g,h)$ is left bounded. For each $\omega \in \mathbf{C}$ we have

$$|(F\xi(g,h)|\Delta^{\bar{\omega}}\eta)| = \left|\int_{\hat{G}} \overline{\langle h,p\rangle}(\xi(g,p)|\bar{\Delta}^{\omega}\eta)\,dp\right| = \left|\int_{\hat{G}} \langle h,p\rangle(\Delta^{\omega}\xi(g,p)|\eta)\,dp\right|$$

$$\leqslant \int_{\hat{G}} |(\Delta^{\omega}\xi(g,p)|\eta)|\,dp \leqslant \|\eta\| \int_{\hat{G}} \|\Delta^{\omega}\xi(g,p)\|\,dp.$$

Since the function: $p \in G \to \|\Delta^{\omega}\xi(g,p)\|$ is integrable, $F\xi(g,h)$ belongs to $\mathcal{D}(\Delta^{\omega})$ for every $\omega \in \mathbf{C}$. Put $\xi_{\omega}(g,p) = \Delta^{\omega}\xi(g,p)$, $g \in G$, $p \in \hat{G}$. We have then

$$(F\xi_{\omega})(g,h) = \int_{\hat{G}} \overline{\langle h,p\rangle}\,\Delta^{\omega}\xi(g,p)\,dp = \Delta^{\omega}\int_{\hat{G}} \overline{\langle h,p\rangle}\,\xi(g,p)\,dp = \Delta^{\omega}(F\xi)(g,h),$$

where the second step is justified by the closedness of Δ^{ω}. Hence $\Delta^{\omega}(F\xi)(g,h)$ is left bounded for every $\omega \in \mathbf{C}$, so that $F\xi(g,h)$ falls in $\mathfrak{A}_0$. Q.E.D.

From the first part of the above proof, it follows that

$$\pi_l(F\xi(g,h)) = \int_{\hat{G}} \overline{\langle h,p\rangle}\,\pi_l(\xi(g,p))\,dp,\ \xi \in \mathcal{K}(\mathfrak{A}_0; G \times \hat{G}). \tag{6.3}$$

It is also seen similarly that

$$\pi_r(F\xi(g,h)) = \int_{\hat{G}} \overline{\langle h,p\rangle}\,\pi_r(\xi(g,p))\,dp.$$

Hence the function: $(g,h) \in G \times G \mapsto F\xi(g,h) \in \mathfrak{A}_0$ is continuous.

Let ξ and η be elements of $\mathcal{K}(\mathfrak{A}_0; G \times \hat{G})$. We compute $F(\xi\eta)$ as follows:

$$\begin{aligned}
F(\xi\eta)(g,h) &= \int_{\hat{G}} \overline{\langle h,p\rangle}(\xi\eta)(g,p)\,dp \\
&= \iiint_{\hat{G}\times\hat{G}\times G} \overline{\langle h,p\rangle}\langle g-k,q\rangle[T(-k)\xi(g-k,p-q)]\eta(k,q)\,dk\,dp\,dq \\
&= \iiint_{\hat{G}\times\hat{G}\times G} \overline{\langle h,p+q\rangle}\langle g-k,q\rangle[T(-k)\xi(g-k,p)]\eta(k,q)\,dk\,dp\,dq \\
&= \iint_{\hat{G}\times G} \langle g-h-k,q\rangle[T(-k)F\xi(g-k,h)]\eta(k,q)\,dk\,dq \\
&= \int_G \{T(-k)F\xi(g-k,h)\}\{F\eta(k,h+k-g)\}\,dk.
\end{aligned}$$

We have also

$$\begin{aligned}
F\xi^{\#}(g,h) &= \int_{\hat{G}} \overline{\langle h,p\rangle}\langle g,p\rangle T(-g)\xi(-g,-p)^{\#}\,dp \\
&= \int_{\hat{G}} \langle g-h,p\rangle T(-g)\xi(-g,-p)^{\#}\,dp = T(-g)(F\xi)(-g,h-g)^{\#}.
\end{aligned}$$

Based on the above observations, we define a Tomita algebra structure in the space $\mathcal{K}(\mathfrak{A}_0; G\times G)$ as follows:

$$\left.\begin{aligned}
(\xi\eta)(g,h) &= \int_G \{T(-k)\,\xi(g-k,h)\}\,\eta(k,h+k-g)\,dk;\\
\xi^{\#}(g,h) &= T(-g)\,\xi(-g,h-g)^{\#};\\
\tilde{\tilde{\Delta}}(\omega)\,\xi(g,h) &= \chi(g)^{\omega}\Delta^{\omega}\xi(g,h);\\
(\xi|\eta) &= \iint_{G\times G}(\xi(g,h)|\eta(g,h))\,dg\,dh.
\end{aligned}\right\} \tag{6.4}$$

The repetition of more or less the same arguments as in the previous section proves the following:

LEMMA 6.2. *With the above structure* (6.4), $\mathcal{K}(\mathfrak{A}_0; G\times G)$ *is a Tomita algebra.*

THEOREM 6.3. *The left von Neumann algebra* $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0; G\times G))$ *of the Tomita algebra* $\mathcal{K}(\mathfrak{A}_0; G\times G)$ *defined by* (6.4) *coincides with the von Neumann algebra* $\mathcal{P}=F\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})F^*$.

Proof. As seen in § 4, $\mathcal{P}$ is generated by the operators $\tilde{x}$, $x\in\mathcal{M}$, $v(g)$, $g\in G$, and $u(p)$, $p\in\hat{G}$, defined by (4.10). We denote by $\tilde{\mathcal{M}}$ the set of all $\tilde{x}$, $x\in\mathcal{M}$. The von Neumann algebra $\mathcal{Q}$ generated by $\tilde{\mathcal{M}}$ and $u(\hat{G})$, the image of $\hat{G}$ under u, is isomorphic to the tensor product $\mathcal{M}\otimes L^{\infty}(G)$. Hence if $x(\cdot)$ is a bounded strongly* continuous $\mathcal{M}$-valued function on G, then the operator x on $L^2(\mathfrak{H}; G\times G)$ defined by

$$(x\xi)(g,h)=\alpha_g^{-1}(x(h))\,\xi(g,h)$$

belongs to $\mathcal{Q}$. The set of such operators is a σ-weakly dense C^*-subalgebra of $\mathcal{Q}$.

Let ξ be an arbitrary element of $\mathcal{K}(\mathfrak{A}_0; G\times G)$. Put $x(g,h)=\alpha_g\circ\pi_l(\xi(g,h))$, $g,h\in G$. For each fixed $g\in G$, $x(g,\cdot)$ is an $\mathcal{M}$-valued strongly*, (even uniformly), continuous function on G with compact support, so that the operator $x(g)$ defined by

$$x(g)\,\eta(h,k)=\alpha_h^{-1}(x(g,k))\,\eta(h,k),\quad \eta\in L^2(\mathfrak{H}; G\times G),$$

belongs to $\mathcal{Q}$ and $x(\cdot)$ is a $\mathcal{Q}$-valued strongly* continuous function on G with compact support. Now, we compute, for each $\eta,\zeta\in\mathcal{K}(\mathfrak{H}; G\times G)$,

$$\begin{aligned}
(\pi_l(\xi)\eta|\zeta) &= \iiint_{G\times G\times G}(\alpha_k^{-1}\circ\pi_l(\xi(g-k,h))\,\eta(k,h+k-g)|\zeta(g,h))\,dk\,dh\,dg\\
&= \iiint_{G\times G\times G}(\alpha_k^{-1}\circ\alpha_{g-k}^{-1}(x(g-k,h))\,\eta(k,h+k-g)|\zeta(g,h))\,dg\,dh\,dk\\
&= \iiint_{G\times G\times G}(\alpha_g^{-1}(x(k,h))\,\eta(g-k,h-k)|\zeta(g,h))\,dg\,dh\,dk\\
&= \iiint_{G\times G\times G}(\alpha_g^{-1}(x(k,h))\,(v(k)\eta)(g,h)|\zeta(g,h))\,dg\,dh\,dk=\int_G(x(k)\,v(k)\,\eta|\zeta)\,dk.
\end{aligned}$$

Hence we get $$\pi_l(\xi)\,\eta = \int_G x(k)\,v(k)\,\eta\,dk, \quad \eta \in \mathcal{K}(\mathfrak{A}_0;\, G \times G).$$

so that $$\pi_l(\xi) = \int_G x(g)\,v(g)\,dg \in \mathcal{D}.$$

Therefore, we obtain $\mathcal{L}(\mathcal{K}(\mathfrak{A}_0;\, G \times G)) \subset \mathcal{D}$.

It is now straightforward to see that $\tilde{x}$, $x \in \mathcal{M}$, $v(g)$, $g \in G$, and $u(p)$, $p \in \hat{G}$, all commute with $\pi_r(\xi)$, $\xi \in \mathcal{K}(\mathfrak{A}_0;\, G \times G)$, which means that $\mathcal{D} \subset \mathcal{L}(\mathcal{K}(\mathfrak{A}_0;\, G \times G))$. Q.E.D.

LEMMA 6.4. (i) *The real power* $\tilde{\tilde{\Delta}}^t$, $t \in \mathbf{R}$, *of the modular operator on* $L^2(\mathfrak{H};\, G \times \hat{G})$ *associated with the bi-dual weight* $\tilde{\tilde{\varphi}}$ *is essentially self-adjoint on* $F^*\mathcal{K}(\mathfrak{A}_0;\, G \times G)$.

(ii) *For each* $t \in \mathbf{R}$, $\tilde{\tilde{\Delta}}(t)$ *is essentially self-adjoint on* $\mathcal{K}(\mathfrak{A}_0;\, G \times G)$, *and its closure coincides with* $F\tilde{\tilde{\Delta}}^t F^*$.

Proof. Let Φ be the Fourier transform on $L^2(\hat{G})$ defined by

$$(\Phi f)(g) = \int_{\hat{G}} \overline{\langle g, p\rangle}\, f(p)\,dp, \quad f \in \mathcal{K}(\hat{G}).$$

It's inverse Φ^* is given by

$$(\Phi^* f)(p) = \int_G \langle g, p\rangle\, f(g)\,dg, \quad f \in \mathcal{K}(G).$$

Then the algebraic tensor product $\mathcal{K}(\mathfrak{A}_0;\, G) \otimes \Phi^*\mathcal{K}(G)$ is contained in $F^*\mathcal{K}(\mathfrak{A}_0;\, G \times G)$, since $F = 1 \otimes \Phi$, and $\mathcal{K}(\mathfrak{A}_0;\, G) \otimes \mathcal{K}(G) \subset \mathcal{K}(\mathfrak{A}_0;\, G \times G)$. As seen in Lemma 5.10. $\tilde{\tilde{\Delta}}^t$ is essentially self-adjoint on $\mathcal{K}(\mathfrak{A}_0;\, G) \otimes \Phi^*\mathcal{K}(G)$ because $\Phi^*\mathcal{K}(G)$ is dense in $L^2(\hat{G})$. Hence the first assertion follows.

By construction, $F\tilde{\tilde{\Delta}}^t F^*$ and $\tilde{\tilde{\Delta}}(t)$ agree on $\mathcal{K}(\mathfrak{A}_0;\, G \times G)$. Since $\tilde{\tilde{\Delta}}(t) = \tilde{\Delta}(t) \otimes 1$ on the algebraic tensor product $\mathcal{K}(\mathfrak{A}_0;\, G) \otimes \mathcal{K}(G)$, and the latter are essentially self-adjoint on $\mathcal{K}(\mathfrak{A}_0;\, G) \otimes \mathcal{K}(G)$, so is $\tilde{\tilde{\Delta}}(t)$. Therefore, $F\tilde{\tilde{\Delta}}^t F^*$ is the closure of $\tilde{\tilde{\Delta}}(t)$. Q.E.D.

We denote the closure of $\tilde{\tilde{\Delta}}(1)$ by $\tilde{\tilde{\Delta}}$.

THEOREM 6.5. *The weight on* $\mathcal{D}$ *canonically associated with the Tomita algebra* $\mathcal{K}(\mathfrak{A}_0;\, G \times G)$ *defined by* (6.4) *is the image of the bi-dual weight* $\tilde{\tilde{\varphi}}$ *on* $\mathcal{R}(\mathcal{R}(\mathcal{M};\, \alpha);\, \hat{\alpha})$ *under the isomorphism:* $x \mapsto FxF^*$.

Proof. By Lemma 6.4, $F\mathcal{K}(\mathfrak{A}_0 G \times \hat{G})$ is dense in the domain $\mathcal{D}(\tilde{\tilde{\Delta}}^\omega)$ for each $\omega \in \mathbf{C}$ with respect to the norm in $\mathcal{D}(\tilde{\tilde{\Delta}}^\omega)$, (the graph norm). Furthermore, it is easily seen that each element of $F\mathcal{K}(\mathfrak{A}_0;\, G \times \hat{G})$ is left bounded with respect to the Tomita algebra $\mathcal{K}(\Delta_0;\, G \times G)$. Hence F is an isometric *-isomorphism of the Tomita algebra $\mathcal{K}(\mathfrak{A}_0;\, G \times \hat{G})$ into the full left Hilbert algebra associatec with $\mathcal{K}(\mathfrak{A}_0;\, G \times G)$. The image $F\mathcal{K}(\mathfrak{A}_0;\, G \times \hat{G})$ is equivalent to $\mathcal{K}(\mathfrak{A}_0;\, G \times G)$ in the sense of [42; Definition 5.1]. Hence the weights on $\mathcal{D}$

associated with $\mathcal{K}(\mathfrak{A}_0; G\times G)$ and $F\mathcal{K}(\mathfrak{A}_0; G\times \hat{G})$ are the same. But the weight associated with $F\mathcal{K}(\mathfrak{A}_0; G\times\hat{G})$ is nothing but the image of $\tilde{\tilde{\varphi}}$ under the isomorphism: $x\mapsto FxF^*$. Q.E.D.

In order to see the relation between the bi-dual weight $\tilde{\tilde{\varphi}}$ and the tensor product expression $\mathcal{M}\otimes\mathcal{L}(L^2(G))$ of the algebra $\mathcal{R}(\mathcal{R}(\mathcal{M};\alpha);\hat{\alpha})$, we shall further transform the Tomita algebra structure.

Define

$$(V\xi)(g,h)=T(h)\xi(h-g,-g),\quad \xi\in\mathcal{K}(\mathfrak{A}_0; G\times G). \tag{6.5}$$

Clearly V is a bijection of $\mathcal{K}(\mathfrak{A}_0; G\times G)$ to itself. The inverse V^{-1} is given by

$$(V^{-1}\xi)(g,h)=T(h-g)\xi(-h,g-h),\quad \xi\in\mathcal{K}(\mathfrak{A}_0; G\times G). \tag{6.5'}$$

For each pair $\xi,\eta\in\mathcal{K}(\mathfrak{A}_0; G\times G)$, and $\omega\in\mathbf{C}$, compute

$$\begin{aligned}
[V(\xi\eta)](g,h)&=T(h)\int_G[T(-k)\,\xi(h-g-k,\,-g)]\,\eta(k,k-h)\,dk\\
&=\int_G[T(h-k)\,\xi(h-k-g,\,-g)]\,[T(h)\,\eta(k,k-h)]\,dk\\
&=\int_G\{(V\xi)(g,h-k)\}\{(V\eta)(h-k,h)\}\,dk=\int_G[(V\xi)(g,k)][(V\eta)(k,h)]\,dk;
\end{aligned}$$

$$\begin{aligned}
V\xi^\#(g,h)&=T(h)\,\xi^\#(h-g,\,-g)=T(h)\,T(g-h)\,\xi(g-h,\,-h)^\#\\
&=T(g)\,\xi(g-h,\,-h)^\#=(V\xi)(h,g)^\#;
\end{aligned}$$

$$\begin{aligned}
(V\tilde{\tilde{\Delta}}^\omega\xi)(g,h)&=T(h)\,(\tilde{\tilde{\Delta}}^\omega\xi)(h-g,\,-g)\\
&=T(h)\,\chi(h-g)^\omega\Delta^\omega\xi(h-g,\,-g)=\chi(h-g)^\omega\Delta^\omega V\xi(g,h);
\end{aligned}$$

$$\begin{aligned}
(V^{-1}\xi|V^{-1}\eta)&=\iint_{G\times G}(T(h-g)\,\xi(-h,g-h)|T(h-g)\,\eta(-h,g-h))\,dg\,dh\\
&=\iint_{G\times G}\chi(h)^{-1}(\xi(g,h)|\eta(g,h))\,dg\,dh.
\end{aligned}$$

Therefore, we introduce the second Tomita algebra structure in $\mathcal{K}(\mathfrak{A}_0; G\times G)$ as follows;

$$\left.\begin{aligned}
(\xi\eta)(g,h)&=\int_G\xi(g,k)\,\eta(k,h)\,dk;\\
\xi^\#(g,h)&=\xi(h,g)^\#;\\
\Delta(\omega)\,\xi(g,h)&=\chi(g-h)^\omega\Delta^\omega\xi(g,h);\\
(\xi|\eta)&=\iint_{G\times G}\chi(h)^{-1}(\xi(g,h)|\eta(g,h))\,dg\,dh
\end{aligned}\right\} \tag{6.6}$$

for each $\xi, \eta \in \mathcal{K}(\mathfrak{A}_0; G \times G)$ and $\omega \in \mathbf{C}$. It is then obvious that the operator V is an isometric $*$-isomorphism of the first Tomita algebra $\mathcal{K}(\mathfrak{A}_0; G \times G)$ defined by (6.4) onto the second one $\mathcal{K}(\mathfrak{A}_0; G \times G)$ defined by (6.6). To distinguish them, we denote the first one by $\mathfrak{B}$ and the second one by $\mathfrak{C}$. We denote by $\mathfrak{K}$ the completion of $\mathfrak{C}$. It is clear by construction that the operator V is extended to a unitary of $L^2(\mathfrak{H}; G \times G)$ onto $\mathfrak{K}$, which we denote again by V. This unitary operator V gives rise to a spatial isomorphism of $\mathcal{P} = \mathcal{L}(\mathfrak{B})$ onto the left von Neumann algebra $\mathcal{L}(\mathfrak{C})$ of $\mathfrak{C}$ that transforms the canonical weight of $\mathcal{L}(\mathfrak{B})$ associated with $\mathfrak{B}$ into the canonical weight ψ on $\mathcal{L}(\mathfrak{C})$ associated with $\mathfrak{C}$. By virtue of Theorem 6.5, the spatial isomorphism of $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha})$ onto $\mathcal{L}(\mathfrak{C})$ induced by the unitary operator VF transforms the bi-dual weight $\tilde{\tilde{\varphi}}$ into ψ. Therefore, it suffices to study ψ on $\mathcal{L}(\mathfrak{C})$ instead of $\tilde{\tilde{\varphi}}$ on $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha)$: $\hat{\alpha})$.

We define a Tomita algebra structure in $\mathcal{K}(G \times G)$ by the following:

$$\left.\begin{aligned}
\xi\eta(g, h) &= \int_G \xi(g, k)\, \eta(k, h)\, dk\, \psi \\
\xi^\#(g, h) &= \overline{\xi(h, g)}; \\
\Delta(\omega)\, \xi(g, h) &= \chi(g-h)^\omega \xi(g, h); \\
(\xi \mid \eta) &= \int_{G \times G} \chi(h)^{-1} \xi(g, h)\, \overline{\eta(g, h)}\, dg\, dh
\end{aligned}\right\} \tag{6.7}$$

for each $\xi, \eta \in \mathcal{K}(G \times G)$ and $\omega \in \mathbf{C}$. A slight modification of the proof in Lemma 5.10 shows that the Tomita algebra $\mathfrak{C}$ is equivalent, in the sense of [42; Definition 5.1], to the algebraic tensor product $\mathfrak{A}_0 \otimes \mathcal{K}(G \times G)$. Hence the left von Neumann algebra $\mathcal{L}(\mathfrak{C})$ coincides with the tensor product $\mathcal{M} \otimes \mathcal{L}(\mathcal{K}(G \times G))$, and the canonical weight ψ on $\mathcal{L}(\mathfrak{C})$ agrees with the tensor product $\varphi \otimes \varrho$ of the original weight φ on $\mathcal{M}$ and the canonical weight ϱ of $\mathcal{L}(\mathcal{K}(G \times G))$ associated with $\mathcal{K}(G \times G)$.

LEMMA 6.6. *There exists an isomorphism of the left von Neumann algebra $\mathcal{L}(\mathcal{K}(G \times G))$ onto the algebra $\mathcal{L}(L^2(G))$ of all bounded operators on $L^2(G)$ which transforms the canonical weight ϱ on $\mathcal{L}(\mathcal{K}(G \times G))$ into the weight on $\mathcal{L}(L^2(G))$ defined by: $x \in \mathcal{L}(L^2(G)) \mapsto \mathrm{Tr}\,(H_\chi x)$, where H_χ is the nonsingular positive self-adjoint operator on $L^2(G)$ defined by*

$$(H_\chi \xi)(g) = \chi(g)^{-1} \xi(g), \quad \xi \in L^2(G).$$

Proof. It is clear that if $\chi(g) = 1$, $g \in G$, then $\mathcal{K}(G \times G)$ is a (unimodular) Hilbert algebra such that there exists an isomorphism γ of $\mathcal{L}(\mathcal{K}(G \times G))$ onto $\mathcal{L}(L^2(G))$ which maps ϱ into the usual trace Tr of $\mathcal{L}(L^2(G))$. The isomorphism γ is given by the following:

$$[\gamma \circ \pi_l(\xi)\, \zeta](g) = \int_G \xi(g, h)\, \zeta(h)\, dh, \quad \xi \in \mathcal{K}(G \times G), \zeta \in L^2(G).$$

Since $\xi \in \mathcal{K}(G \times G)$ is square integrable, $\gamma \circ \pi_l(\xi)$ is of Hilbert-Schmidt class and we have, for each $\xi, \eta \in \mathcal{K}(G \times G)$,

$$\mathrm{Tr}\,(\gamma \circ \pi_l(\eta)^* \gamma \circ \pi_l(\xi)) = \iint_{G \times G} \xi(g, h) \overline{\eta(g, h)}\, dg\, dh.$$

Now, we have, for each $\xi \in \mathcal{K}(G \times G)$ and $\zeta \in L^2(G)$,

$$[\gamma \circ \pi_l(\xi) H_\chi \zeta](g) = \int_G \xi(g, h) \chi(h)^{-1} \zeta(h)\, dh,$$

so that for each pair ξ, η in $\mathcal{K}(G \times G)$,

$$\mathrm{Tr}\,(\gamma(\pi_l(\eta)^* \pi_l(\xi)) H_\chi) = \iint_{G \times G} \chi(h)^{-1} \xi(g, h) \overline{\eta(g, h)}\, dg\, dh = \dot{\varrho}(\pi_l(\eta)^* \pi_l(\xi)).$$

Thus the canonical weight ϱ on $\mathcal{L}(\mathcal{K}(G \times G))$ is transformed by γ into the weight $\mathrm{Tr}\,(H_\chi \cdot)$ on $\mathcal{L}(L^2(G))$. Q.E.D.

After all these preparations we have proved the following result:

THEOREM 6.7. *The second dual weight* $\tilde{\tilde{\varphi}}$ *on* $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha})$ *is transformed into* $\varphi \otimes \mathrm{Tr}\,(H_\chi \cdot)$ *under the isomorphism of* $\mathcal{R}(\mathcal{R}(\mathcal{M}; \alpha); \hat{\alpha})$ *onto* $\mathcal{M} \otimes \mathcal{L}(L^2(G))$ *obtained in Theorem 4.5, where* H_χ *is defined by* (6.8). *In particular, if* φ *is invariant under the action* α *of* G, i.e., *if* $\chi = 1$, *then* $\tilde{\tilde{\varphi}}$ *is identified with* $\varphi \otimes \mathrm{Tr}$ *under the above isomorphism.*

7. Subgroups and subalgebras

Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a continuous action α of a locally compact abelian group G. In this section, we shall examine the fixed point subalgebra of $\mathcal{R}(\mathcal{M}; \alpha)$ under the restriction of the action of $\hat{\alpha}$ to a closed subgroup H of $\hat{G}$.

THEOREM 7.1. *Let* $\hat{H}$ *be a closed subgroup of* $\hat{G}$ *and* H *be the annihilator of* $\hat{H}$ *in* G, *that is,* $H = \{g \in G\colon \langle g, p \rangle = 1$ *for every* $p \in \hat{H}\}$. *If the action* α *of* G *on* $\mathcal{M}$ *admits a relatively invariant weight* φ *on* $\mathcal{M}$, *then the fixed point subalgebra* $\mathcal{N}$ *of* $\mathcal{R}(\mathcal{M}; \alpha)$ *under the action* $\{\hat{\alpha}_p\colon p \in \hat{H}\}$ *is generated by the canonical image* $\pi_\alpha(\mathcal{M})$ *of* $\mathcal{M}$ *and* $\{\lambda(g)\colon g \in H\}$; *hence it is isomorphic to* $\mathcal{R}(\mathcal{M}; \alpha|_H)$, *where* $\alpha|_H$ *means the restriction of* α *to* H.

Proof. Let $\mathfrak{H}$ be the Hilbert space constructed in § 5 based on the weight φ. We keep the notations established there.

We remark first that the covariant representations $\{\pi_\alpha, \lambda\}$ of $\{\mathcal{M}, G, \alpha\}$ is precisely the covariant representation induced, in the sense of [39], from the trivial covariant representation of $\{\mathcal{M}, \{0\}, \alpha\}$ on $\mathfrak{H}$. Therefore, the stage theorem of induction assures that the covariant representation $\{\pi_\alpha, \lambda\}$ of $\{\mathcal{M}, G, \alpha\}$ is identified with the covariant representation induced up to G from the covariant representation $\{\pi_\alpha^H, \lambda_H\}$ of $\{\mathcal{M}, H, \alpha|_H\}$ which is defined on $L^2(\mathfrak{H}; H)$ by the following:

$$\left.\begin{aligned} &\pi_\alpha^H(x)\,\xi(g) = \alpha_g^{-1}(x)\,\xi(g), \quad x \in \mathcal{M}, g \in H, \xi \in L^2(\mathfrak{H}; H); \\ &\lambda_H(h)\,\xi(g) = \xi(h^{-1}g), \quad g, h \in H. \end{aligned}\right\} \tag{7.1}$$

When necessary, we denote by $\{\pi_\alpha^G, \lambda_G\}$ the covariant representation of $\{\mathcal{M}, G, \alpha\}$ on $L^2(\mathfrak{H}; G)$.

It is clear that $\pi_\alpha^G(\mathcal{M})$ and $\lambda_G(H)$ are both contained in $\mathcal{N}$. We must show therefore that $\mathcal{N}$ is generated by $\pi_G(\mathcal{M})$ and $\lambda_G(H)$. Let $\mathcal{A}$ denote the von Neumann algebra on $L^2(\mathfrak{H}; G)$ generated by $\mu(\hat{G})$, where μ is the representation of $\hat{G}$ defined by (4.1). It is known that $\mathcal{A}$ consists of all multiplication operators $\mu(f)$, $f \in L^\infty(G)$, on $L^2(\mathfrak{H}; G)$ defined by

$$(\mu(f)\xi)(g) = f(g)\xi(g), \quad f \in L^\infty(G), \xi \in L^2(\mathfrak{H}; G). \tag{7.2}$$

Hence $\mathcal{A}$ is the canonical imprimitivity system associated with the induction of $\{\pi_\alpha^G, \lambda_G\}$ from $\{\pi^0, 1\}$, where π^0 means the identity representation of $\mathcal{M}$ on $\mathfrak{H}$ and 1 means the trivial representation of the trivial group $\{0\}$ on $\mathfrak{H}$. The canonical imprimitivity system $\mathcal{A}_H$ associated with the induction of $\{\pi_\alpha^G, \lambda_G\}$ from $\{\pi_\alpha^H, \lambda_H\}$ is the subalgebra of $\mathcal{A}$ consisting of all multiplication operators given by functions in $L^\infty(G)$, which are constant on every H-coset. Hence $\mathcal{A}_H$ is generated by $\mu(\hat{H})$. It is then clear that

$$\mathcal{N} = \mathcal{A}_H' \cap \mathcal{R}(\mathcal{M}; \alpha). \tag{7.3}$$

The Hilbert space $L^2(\mathfrak{H}; G)$ is regarded as the space of all measurable (in the sense of [7]) $\mathfrak{H}$-valued functions on $G \times H$ with the properties:

$$\xi(g-h, k) = \xi(g, k-h) \tag{7.4}$$

for every $h \in H$ and almost every $(g, k) \in G \times H$; and

$$\int_{G/H}\left(\int_H \|\xi(g, h)\|^2\, dh\right) d\dot{g} < +\infty, \tag{7.5}$$

where $d\dot{g}$ denotes the quotient Haar measure on G/H. Note that by virtue of (7.4), the first integration in (7.5) is constant on each H-coset so that it may be considered as a function on the quotient group G/H. The operators $\pi_\alpha^G(x)$, $x \in \mathcal{M}$, and $\lambda_G(g)$, $g \in G$, are defined by

$$\left.\begin{aligned}[\pi_\alpha^G(x)\xi](h,k)&=\alpha_h^{-1}(x)\xi(h,k), \quad h\in G, k\in H;\\ (\lambda_G(g)\xi)(h,k)&=\xi(h-g,k).\end{aligned}\right\} \tag{7.6}$$

By Corollary 5.13, the commutant $\mathcal{R}(\mathcal{M};\alpha)'$ is generated by the operators $\pi_G'(y)$, $y\in\mathcal{M}'$, and $\lambda_G'(g)$, $g\in G$, which are defined by

$$\left.\begin{aligned}\pi_G'(y)\xi(h,k)&=y\xi(h,k), \quad h\in G, k\in H;\\ \lambda_G'(g)\xi(h,k)&=U(g)\xi(h+g,k).\end{aligned}\right\} \tag{7.7}$$

Suppose x is an arbitrary element of $\mathcal{N}$. Put $y=\tilde{J}x\tilde{J}\in\mathcal{R}(\mathcal{M};\alpha)'$. We shall show that y belongs to the von Neumann algebra generated by $\pi_G'(\mathcal{M}')$ and $\lambda_G'(H)$. If this were done, then our assertion would follow automatically. Since $\tilde{J}\mu(p)\tilde{J}=\mu(-p)$, $p\in\hat{G}$, we have $\tilde{J}\mathcal{A}_H\tilde{J}=\mathcal{A}_H$, so that y commutes with $\mathcal{A}_H$. Namely, y belongs to $\mathcal{A}_H'\cap\mathcal{R}(\mathcal{M};\alpha)'$. By the modified Blattner–Mackey theorem for induced covariant representations, [39; Theorem 4.3] and [32], there exists a natural isomorphism γ of $\mathcal{R}(\mathcal{M};\alpha|_H)'$ onto $\mathcal{A}_H\cap\mathcal{R}(\mathcal{M};\alpha)'$. The construction of γ shows that $\gamma\circ\pi_H'(z)=\pi_G'(z)$, $z\in\mathcal{M}'$, and $\gamma\circ\lambda_H'(h)=\lambda_G'(h)$, $h\in H$, where π_H' and λ_H' should be naturally understood. Hence y belongs to $\gamma(\mathcal{R}(\mathcal{M};\alpha|_H)')$, which is generated by $\pi_G'(\mathcal{M}')$ and $\lambda_G'(H)$. Q.E.D.

Remark. The above arguments show that $\mathcal{R}(\mathcal{M};\alpha|_H)$ is canonically imbedded in $\mathcal{R}(\mathcal{M};\alpha)$ for any closed subgroup H of G.

THEOREM 7.2. *If H is a closed subgroup of G, then the subgroup $\hat{H}$ of $\hat{G}$ consisting of all elements $p\in\hat{G}$ which leave $\mathcal{R}(\mathcal{M};\alpha|_H)$ elementwise fixed is precisely the annihilator of H in $\hat{G}$.*

Proof. By definition, $\mathcal{R}(\mathcal{M};\alpha|_H)$ is generated by $\pi_\alpha^G(\mathcal{M})$ and $\lambda_G(H)$. Since $\hat{\alpha}_p(\pi_G(x))=\pi_G(x)$, $x\in\mathcal{M}$, and $\hat{\alpha}_p(\lambda_G(g))=\overline{\langle g,p\rangle}\lambda_G(g)$, an element $p\in\hat{G}$ belongs to $\hat{H}$ if and only if $\langle g,p\rangle=1$ for every $g\in H$. Q.E.D.

8. The structure of a von Neumann algebra of type III

We now come to the stage where we apply the theory established above to the structure theory of von Neumann algebras of type III. In this section, we shall see that every von Neumann algebra of type III is uniquely expressed as the crossed product of a von Neumann algebra of type II_∞ by a continuous one parameter automorphism group which leaves a trace relatively invariant (but not invariant).

Let $\mathcal{M}$ be a von Neumann algebra of type III. Let φ be an arbitrary (faithful semi-finite normal) weight on $\mathcal{M}$; the existence of such a weight is well-known; for example, the sum of a maximal family of positive linear functionals with mutually orthogonal supports. Let $\{\sigma_t^\varphi\}$ denote the modular automorphism group of $\mathcal{M}$ associated with φ. We consider the crossed product $\mathcal{R}(\mathcal{M}; \sigma^\varphi)$ of $\mathcal{M}$ by the action $\{\sigma_t^\varphi\}$ of $\mathbf{R}$. Trivially, the dual group of $\mathbf{R}$ is the additive group $\mathbf{R}$ itself. We denote by $\{\theta_t^\varphi\}$ the dual action of $\mathbf{R}$ on $\mathcal{R}(\mathcal{M}; \sigma^\varphi)$. An important feature of the covariant system is seen in the following:

THEOREM 8.1. *The covariant system $\{\mathcal{R}(\mathcal{M}; \sigma^\varphi), \theta^\varphi\}$ is independent of the choice of a weight φ on $\mathcal{M}$. In other words, $\{\mathcal{R}(\mathcal{M}; \sigma^\varphi), \theta^\varphi\}$ is determined uniquely, up to weak equivalence, by the algebraic type of $\mathcal{M}$.*

Proof. Let ψ be another weight on $\mathcal{M}$. By Connes' result, [10; Théorème 1.2.1], the actions σ^φ and σ^ψ of $\mathbf{R}$ on $\mathcal{M}$ are equivalent in the sense of § 3. Hence Proposition 4.2 immediately yields our assertion. Q.E.D.

By virtue of the above theorem, we denote the covariant system $\{\mathcal{R}(\mathcal{M}; \sigma^\varphi), \theta^\varphi\}$ simply by $\{\mathcal{M}_0, \theta\}$. By the duality theorem, Theorem 4.5, the second crossed product $\mathcal{R}(\mathcal{M}_0; \theta)$ of $\mathcal{M}_0$ by θ is isomorphic to $\mathcal{M} \otimes \mathcal{L}(L^2(\mathbf{R}))$, so that the original algebra $\mathcal{M}$, being purely infinite, is isomorphic to $\mathcal{R}(\mathcal{M}_0; \theta)$. Therefore, the algebraic structure of $\mathcal{M}$ is, in principle, completely determined by the covariant system $\{\mathcal{M}_0, \theta\}$. Hence the rest of the present section is devoted to studying $\{\mathcal{M}_0, \theta\}$.

We begin with the following lemma:

LEMMA 8.2. *The von Neumann algebra $\mathcal{M}_0$ is properly infinite but semifinite and admits a faithful semifinite normal trace τ such that*

$$\tau \circ \theta_t = e^{-t} \tau, \quad t \in \mathbf{R}. \tag{8.1}$$

Proof. We apply the results in § 5 to the crossed product $\mathcal{M}_0 = \mathbf{R}(\mathcal{M}; \sigma^\varphi)$. We use the notations established in § 5. The von Neumann algebra $\mathcal{M}$ acts on the Hilbert space $\mathfrak{H}$ which is obtained by the weight φ, and $\mathcal{M}_0$ acts on $L^2(\mathfrak{H}; \mathbf{R})$. We denote by π^φ the representation of $\mathcal{M}$ corresponding to π_α. The weight φ is invariant under σ^φ; so we do not have to bother with the character χ. The modular operator $\tilde{\Delta}$ associated with the dual weight $\tilde{\varphi}$ on $\mathcal{M}_0$ is given by

$$(\tilde{\Delta}\xi)(s) = \Delta\xi(s), \quad \xi \in L^2(\mathfrak{H}; \mathbf{R}).$$

The modular automorphism group $\{\sigma_t^{\tilde{\varphi}}\}$ of $\mathcal{M}_0$ associated with $\tilde{\varphi}$ is of the form:

$$\sigma_t^{\tilde{\varphi}}(x) = \tilde{\Delta}^{it} x \tilde{\Delta}^{-it}, \quad x \in \mathcal{M}_0, t \in \mathbf{G}.$$

Let x be an arbitrary element of $\mathcal{M}$. We have then

$$(\sigma_t^{\tilde{\varphi}}(\pi^\varphi(x))\xi)(s) = [\tilde{\Delta}^{it}\pi^\varphi(x)\tilde{\Delta}^{-it}\xi](s)$$
$$= \Delta^{it}(\pi^\varphi(x)\tilde{\Delta}^{-it}\xi)(s) = \Delta^{it}\sigma_{-s}^\varphi(x)(\tilde{\Delta}^{-it}\xi)(s)$$
$$= \Delta^{it}\sigma_{-s}^\varphi(x)\Delta^{-it}\xi(s) = \sigma_{t-s}^\varphi(x)\xi(s) = \pi^\varphi\circ\sigma_t^\varphi(x)\xi(s);$$

hence
$$\sigma_t^{\tilde{\varphi}}(\pi^\varphi(x)) = \pi^\varphi\circ\sigma_t^\varphi(x) = \lambda(t)\pi^\varphi(x)\lambda(t)^*, \quad x\in\mathcal{M}, t\in\mathbf{R},$$

where the last equality follows from (3.2). For each $r\in\mathbf{R}$, we have

$$(\sigma_t^{\tilde{\varphi}}(\lambda(r))\xi)(s) = [\tilde{\Delta}^{it}\lambda(r)\tilde{\Delta}^{-it}\xi](s)$$
$$= \Delta^{it}[\lambda(r)\tilde{\Delta}^{-it}\xi](s) = \Delta^{it}(\tilde{\Delta}^{-it}\xi)(s-r) = \Delta^{it}\Delta^{-it}\xi(s-r) = (\lambda(r)\xi)(s);$$

hence
$$\sigma_t^{\tilde{\varphi}}(\lambda(r)) = \lambda(r) = \lambda(t)\lambda(r)\lambda(t)^*, \quad r, t\in\mathbf{R}.$$

Therefore, the automorphisms $\{\sigma_t^{\tilde{\varphi}}\}$ and $\{\mathrm{Ad}(\lambda(t))\}$ agree on the generators $\pi^\varphi(\mathcal{M})$ and $\lambda(\mathbf{R})$ of $\mathcal{M}_0$, so that they agree on the whole algebra $\mathcal{M}_0$. Thus by [42; Theorem 14.1], (see also [33; Theorem 7.4]), $\mathcal{M}_0$ must be semifinite.

By Stone's Theorem, there exists a nonsingular positive selfadjoint operator h affiliated with $\mathcal{M}_0$ such that $\lambda(t) = h^{it}$, $t\in\mathbf{R}$. By the proof of [33; Theorem 7.4], the weight τ of $\mathcal{M}_0$ defined by $\tau(x) = \tilde{\varphi}(h^{-1}x)$ is a faithful semifinite normal trace. For each $s, t\in\mathbf{R}$, we have $\theta_s(\lambda(t)) = e^{-ist}\lambda(t)$ by (4.3) and (4.5), so that we have $\theta_s(h^{-1}) = e^s h^{-1}$, where $\theta_s(h^{-1}) = \mu(s)h^{-1}\mu(-s)$. Therefore, we have

$$\tau\circ\theta_s(x) = \tilde{\varphi}(h^{-1}\theta_s(x)) = \tilde{\varphi}\circ\theta_s(\theta_{-s}(h^{-1})x) = e^{-s}\tilde{\varphi}(h^{-1}x) = e^{-s}\tau(x), \quad x\in\mathcal{M}_0.$$

Thus, the trace τ satisfies equality (8.1).

The fact that $\mathcal{M}_0$ is properly infinite follows from the observation that $\mathcal{M}_0$ contains $\pi^\varphi(\mathcal{M})$, which is isomorphic to $\mathcal{M}$, and $\mathcal{M}$ contains an orthogonal infinite family $\{e_n\}$ of projections with $e_n\sim 1$. Q.E.D.

THEOREM 8.3. *Let $\mathcal{N}$ be a semifinite von Neumann algebra with a one parameter automorphism group $\{\theta_t\}$. If τ is a faithful semifinite normal trace of $\mathcal{N}$ with $\tau\circ\theta_t = e^{-t}\tau$, then the modular automorphism group $\{\sigma_t^{\tilde{\tau}}\}$ of the crossed product $\mathcal{R}(\mathcal{N};\theta)$ of $\mathcal{N}$ by the action θ associated with the dual weight $\tilde{\tau}$ is the action of $\mathbf{R}$ dual to the original one θ on $\mathcal{N}$. Hence, $\mathcal{N}$ is the centralizer $\mathcal{M}_{\tilde{\tau}}$ of the dual weight $\tilde{\tau}$.*

Proof. Let ψ denote the dual weight $\tilde{\tau}$ on $\mathcal{R}(\mathcal{N};\theta)$. Let $\mathfrak{H}$ denote the Hilbert space obtained from the trace τ on $\mathcal{N}$. The modular operator on $L^2(\mathfrak{H};\mathcal{R})$ associated with the weight ψ is then, by (5.17), of the form

$$(\tilde{\Delta}^{it}\xi)(s) = e^{-ist}\xi(s), \quad \xi \in L^2(\mathfrak{H}; \mathbf{R}).$$

Hence, in this context, the one parameter unitary group $\tilde{\Delta}^{it}$ coincides with the one parameter unitary group $\mu(t)$ defined by (4.1). Therefore, the modular automorphism group $\{\sigma_t^\psi\}$ of $\mathcal{R}(\mathcal{N}; \theta)$ associated with ψ coincides with the dual action $\hat{\theta}$ of $\mathbf{R}$. Q.E.D.

COROLLARY 8.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two properly infinite semifinite von Neumann algebras with one parameter automorphism groups $\{\theta_t^1\}$ and $\{\theta_t^2\}$ respectively. If $\mathcal{M}_2$ and $\mathcal{M}_2$ admit faithful semifinite normal traces τ_1 and τ_2 respectively, such that*

$$\tau_1 \circ \theta_t^1 = e^{-t}\tau_1$$

and

$$\tau_2 \circ \theta_t^2 = e^{-t}\tau_2,$$

then the following two statements are equivalent:

(i) *$\mathcal{R}(\mathcal{M}_1; \theta^1) \cong \mathcal{R}(\mathcal{M}_2; \theta^2)$;*

(ii) *The covariant systems $\{\mathcal{M}_1, \theta^1\}$ and $\{\mathcal{M}_2, \theta^2\}$ are weakly equivalent.*

Proof. The implication (ii) $\Rightarrow$ (i) is shown in Corollary 3.6.

Suppose $\mathcal{R}(\mathcal{M}_1; \theta^1) \cong \mathcal{R}(\mathcal{M}_2; \theta^2)$. The dual action $\hat{\theta}^i$ of $\mathbf{R}$ on $\mathcal{R}(\mathcal{M}_i; \theta^i)$, $i = 1, 2$, is the modular automorphism group $\{\sigma_t^i\}$ of $\mathbf{R}(\mathcal{M}_i; \theta^i)$ associated with the dual weight $\tilde{\tau}_i$, by Theorem 8.3. Hence we have, the Theorem 8.1,

$$\{\mathcal{R}(\mathcal{R}(\mathcal{M}_1; \theta^1); \sigma^1), \hat{\sigma}^1\} \cong \{\mathcal{R}(\mathcal{R}(\mathcal{M}_2; \theta^2); \sigma^2), \hat{\sigma}^2\}.$$

By Theorems 4.5 and 4.8, $\mathcal{M}_1$ and $\mathcal{M}_2$ are identified with $\mathcal{R}(\mathcal{R}(\mathcal{M}_1; \theta^1); \sigma^1)$ and $\mathcal{R}(\mathcal{R}(\mathcal{M}_2; \theta^2); \sigma^2)$ respectively, in such a way that the actions θ^1 and $\hat{\sigma}^1$ (resp. θ^2 and $\hat{\sigma}^2$) are equivalent. Hence $\{\mathcal{M}_1, \theta^1\}$ and $\{\mathcal{M}_2, \theta^2\}$ are weakly equivalent. Q.E.D.

Throughout the rest of this section, we consider a properly infinite semifinite von Neumann algebra $\mathcal{M}_0$ equipped with a one parameter automorphism group $\{\theta_t\}$ and a faithful semifinite normal trace τ satisfying (8.1). We denote by $\mathcal{Z}_0$ the center of $\mathcal{M}_0$ and by $\dot{\theta}_t$ the restriction of θ to $\mathcal{Z}_0$ for each $t \in \mathbf{R}$. Let $\mathcal{M} = \mathcal{R}(\mathcal{M}_0; \theta)$. We identify $\mathcal{M}_0$ and its canonical image $\pi_\theta(\mathcal{M}_0)$ in $\mathcal{M}$. We denote by $\{u(s)\}$ the one parameter unitary group $\{\lambda(t)\}$ defined in (3.1). We denote by φ the weight on $\mathcal{M}$ which is dual to τ, and by σ the modular automorphism group of $\mathcal{M}$ associated with φ. By Theorem 8.3, $\{\sigma_t\}$ is dual to $\{\theta_t\}$ on $\mathcal{M}_0$.

THEOREM 8.5. *In the above situation, the center $\mathcal{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of $\mathcal{Z}_0$ under $\{\dot{\theta}_t\}$. Hence $\mathcal{M}$ is a factor if and only if $\{\dot{\theta}_t\}$ is ergodic on $\mathcal{Z}_0$.*

Proof. Let $a \in \mathfrak{Z}_0$ be fixed by $\{\tilde{\theta}_t\}$. Then we have $u(t)au(t)^* = \tilde{\theta}_t(a) = a$, so that a commutes with $\mathcal{M}_0$ and $\{u(t)\}$; hence it commutes with every element in $\mathcal{M}$. Hence it is in the center $\mathfrak{Z}$.

Suppose a is an arbitrary element of $\mathfrak{Z}$. Then a is fixed by the modular automorphism group $\{\sigma_t\}$. By Theorem 7.1, a belongs to $\mathcal{M}_0$; hence $a \in \mathfrak{Z}_0$, being central in $\mathcal{M}$. Being central in $\mathcal{M}$, a commutes with $\{u(t)\}$, which means that a is left fixed under $\{\theta_t\}$. Q.E.D.

THEOREM 8.6. *In the same situation as before, the following two statements are equivalent:*

(i) *$\mathcal{M}$ is semifinite;*

(ii) *There exists a continuous one parameter unitary group $\{v(t)\}$ in $\mathfrak{Z}_0$ such that*

$$\tilde{\theta}_s(v(t)) = e^{ist}v(t), \quad s, t \in \mathbf{R}. \tag{8.2}$$

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathcal{M}$ is semifinite. There exists then a continuous one parameter unitary group $\{v(t)\}$ in $\mathcal{M}$ such that

$$\sigma_t(x) = v(t)xv(t)^*, \quad t \in \mathbf{R}.$$

Of course, we have

$$\sigma_t(v(t)) = v(s)v(t)v(s)^* = v(t), \quad s, t \in \mathbf{R}.$$

By Theorem 7.1, $v(t)$ is contained in $\mathcal{M}_0$. Since $\mathcal{M}_0$ is the fixed point subalgebra of $\mathcal{M}$ under $\{\sigma_t\}$ by Theorem 7.1 again, $v(t)$ and $\mathcal{M}_0$ commute, which means that $\{v(t)\}$ is contained in $\mathfrak{Z}_0$. For each, $s, t \in \mathbf{R}$, we have

$$e^{-ist}u(s) = \sigma_t(u(x)) = v(t)u(s)v(t)^*;$$

hence we have

$$\tilde{\theta}_s(v(t)) = u(s)v(t)u(s)^* = e^{ist}v(t).$$

(ii) $\Rightarrow$ (i): Suppose $\{v(t)\}$ is a continuous one parameter unitary group in $\mathfrak{Z}_0$ satisfying (8.2). We have then, for each $x \in \mathcal{M}_0$ and $s, t \in \mathbf{R}$,

$$\sigma_t(x) = x = v(t)xv(t)^*;$$

$$\sigma_t(u(s)) = e^{-ist}u(s) = v(t)u(s)v(t)^*;$$

hence σ_t and $\mathrm{Ad}(v(t))$ agree on the generators $\mathcal{M}_0$ and $\{u(s)\}$ of $\mathcal{M}$, so that $\sigma_t = \mathrm{Ad}(v(t))$. Thus $\mathcal{M}$ is semifinite by [42; Theorem 14.2]; see also [33; Theorem 7.4]. Q.E.D.

COROLLARY 8.7. *In the same situation as before, the following two statements are equivalent:*

(i) $\mathcal{M}$ *is of type* III;

(ii) *For any nonzero projection* $e \in \mathcal{Z}_0$ *which is fixed under* $\{\hat{\theta}_t\}$, *there exists no continuous one parameter unitary group* $\{v(t)\}$ *in* $\mathcal{Z}_0 e$ *satisfying* (8.2).

We are now going to show (in Theorem 8.11) that $\mathcal{M}_0$ must be of type II_∞ if $\mathcal{M}$ is of type III. To this end, we need a few lemmas. We owe the following lemma to H. Dye.

LEMMA 8.8. *Suppose* θ *is an automorphism of an abelian von Neumann algebra* $\mathcal{A}$. *If* $\mathcal{A}$ *admits a faithful semifinite normal trace* ψ *with* $\psi\circ\theta=\lambda\psi$ *for some* $0<\lambda<1$, *then there exists a projection* $e\in\mathcal{A}$ *such that* $\{\theta^n(e)\colon n\in\mathbf{Z}\}$ *are orthogonal and* $\sum_{n\in\mathbf{Z}}\theta^n(e)=1$.

Proof. Let $\mathcal{B}$ denote the fixed point subalgebra of $\mathcal{A}$ under θ. Suppose p is a nonzero projection in $\mathcal{A}$ with $\psi(p)<+\infty$. Let $q=\vee_{n\geq 0}\theta^n(p)$. We have then

$$\psi(q)\leq\sum_{n=0}^{\infty}\psi(\theta^n(p))=\sum_{n=0}^{\infty}\lambda^n\psi(p)=\frac{1}{1-\lambda}\psi(p)<+\infty.$$

Clearly, $\theta(q)\leq q$. Let $p_1=q-\theta(q)$. We have

$$\psi(p_1)=\psi(q)-\psi(\theta(q))=(1-\lambda)\psi(q)\neq 0$$

and $p_1\leq p$. Furthermore, $\{\theta(p_i)\colon n\in\mathbf{Z}\}$ are orthogonal and $\sum_{n\in\mathbf{Z}}\theta^n(p_i)$ belongs to $\mathcal{B}$. Let $\{p_i\colon i\in I\}$ be a maximal orthogonal family of projections in $\mathcal{A}$ such that $\{\theta^n(p_i)\colon n\in\mathbf{Z}, i\in I\}$ are orthogonal. By the maximality of $\{p_i\}$ and by the above arguments, we have $\sum_{i\in I}\sum_{n\in\mathbf{Z}}\theta^n(p_i)=1$. Putting $e=\sum_{i\in I}p_i$, we obtain the desired projection e. Q.E.D.

LEMMA 8.9. *Let* $\{\theta_t\}$ *be a continuous one parameter automorphism group of an abelian von Neumann algebra* $\mathcal{A}$. *If* $\mathcal{A}$ *admits a faithful semifinite normal trace* ψ *on* $\mathcal{A}$ *with* $\psi\circ\theta_t=e^{-t}\psi$, $t\in\mathbf{R}$, *then* $\mathcal{A}$ *has a faithful semifinite normal trace* ψ_0, *invariant under* $\{\theta_t\}$.

Proof. We apply the previous lemma to $\{\mathcal{A},\theta_1\}$. There exists then a projection e in $\mathcal{A}$ such that $\{\theta_n(e)\colon n\in\mathbf{Z}\}$ are orthogonal and $\sum_{n\in\mathbf{Z}}\theta_n(e)=1$. Let $\mathcal{B}$ be the fixed point subalgebra of $\mathcal{A}$ under $\{\theta_n\colon n\in\mathbf{Z}\}$. Let $e_n=\theta_n(e)$, $n\in\mathbf{Z}$. We have then

$$\mathcal{B}\cong\mathcal{B}e_n=\mathcal{A}e_n,\quad n\in\mathbf{Z};$$

$$\mathcal{A}=\sum_{n\in\mathbf{Z}}^{\oplus}\mathcal{B}e_n.$$

Let $\mathcal{A}_0$ be the algebraic direct sum $\sum'_{n\in\mathbf{Z}}\mathcal{B}e_n$. For each $x\in\mathcal{A}_0$, put

$$\varepsilon(x)=\sum_{n\in\mathbf{Z}}\theta_n(x).$$

Since $\{n\in\mathbf{Z}: \theta_n(x)e_m\neq 0\}$ is finite for each $m\in\mathbf{Z}$, ε is well-defined, and we have $\varepsilon(\mathcal{A}_0)\subset\mathcal{B}$. Let Γ be the spectrum of $\mathcal{B}$, which is a hyperstonean space. We identify $\mathcal{B}$ and $C(\Gamma)$, the algebra of all continuous functions on Γ. Let $\tilde{\mathcal{B}}_+$ denote the set of all $[0, \infty]$-valued lower semicontinuous functions on Γ. For each $n=1, 2, \ldots$, let $f_n=\sum_{|k|\leqslant n}e_k$. We define, for each $x\in\mathcal{A}_+$,

$$\varepsilon(x)=\sup_n \varepsilon(xf_n)\in\tilde{\mathcal{B}}_+.$$

The one parameter automorphism group $\{\theta_t\}$ induces naturally a one parameter automorphism group $\{\tilde{\theta}_t\}$ of the cone $\tilde{\mathcal{B}}_+$, which satisfies the equality:

$$\tilde{\theta}_t\circ\varepsilon(x)=\varepsilon\circ\theta_t(x),\quad x\in\mathcal{A}_+.$$

We first consider the case where $\mathcal{A}$ is σ-finite. Let ω be a faithful normal state on $\mathcal{A}$. Define

$$\omega_0(x)=\int_0^1 \omega\circ\psi_t(x)\,dt,\quad x\in\mathcal{A}.$$

We have then a faithful normal state ω_0 on $\mathcal{A}$, whose restriction to $\mathcal{B}$ is invariant under $\{\theta_t\}$ since $\theta_{n+t}(x)=\theta_t(x)$ for every $x\in\mathcal{B}$ and $n\in\mathbf{Z}$. The normal state ω_0 induces a normal measure on Γ; hence it is extended to a $[0, \infty]$-valued additive and positively homogeneous function on $\tilde{\mathcal{B}}_+$ which is also denoted by ω_0. Put

$$\psi_0(x)=\omega_0\circ\varepsilon(x),\quad x\in\mathcal{A}_+.$$

It is then clear that ψ_0 is a trace on $\mathcal{A}_+$. Since $\psi_0(x)<+\infty$ for each $x\in\mathcal{A}_0$, and $\mathcal{A}_0$ is σ-weakly dense in $\mathcal{A}$, ψ_0 is semifinite. If we define ψ_n on $\mathcal{A}_+$ by $\psi_n(x)=\omega_0(\varepsilon(xf_n))$, $n=1, 2, \ldots$, then $\{\psi_n\}$ are normal positive linear functionals on $\mathcal{A}$ and $\psi_0(x)=\sup\psi_n(x)$, $x\in\mathcal{A}_+$; hence ψ_0 is normal. Since ω_0 and ε are both faithful, so is ψ_0. Finally, we get

$$\psi_0\circ\theta_t(x)=\omega_0(\varepsilon\circ\theta_t(x))=\omega_0(\tilde{\theta}_t\circ\varepsilon(x))=\omega_0(\varepsilon(x))=\psi_0(x),\quad x\in\mathcal{A}_+,$$

so that ψ_0 is invariant.

We now drop the assumption that $\mathcal{A}$ is σ-finite. Let $\mathcal{C}$ denote the fixed point subalgebra of $\mathcal{A}$ under the whole group $\{\theta_t\}$. Let p be a nonzero σ-finite projection in $\mathcal{A}$. Put $q=\bigvee_{t\in\mathbf{Q}}\theta_t(p)$, where $\mathbf{Q}$ denotes the set of all rational numbers. The countability of $\mathbf{Q}$ implies that q is σ-finite. Since $\theta_t(q)=q$ for every $t\in\mathbf{Q}$, and $t\mapsto\theta_t(q)$ is strongly continuous, we have $\theta_t(q)=q$ for every $t\in\mathbf{R}$. Hence q falls in $\mathcal{C}$. Therefore, any σ-finite projection in $\mathcal{A}$ is majorized by a σ-finite projection in $\mathcal{C}$. Thus, we can find, by the usual exhaustion arguments, an orthogonal family $\{q_i: i\in I\}$ of σ-finite (in $\mathcal{A}$) projections in $\mathcal{C}$ with $\sum_{i\in I}q_i=1$. By the result for the σ-finite case, each $\mathcal{A}q_i$ admits a θ-invariant faithful semifinite normal trace ψ_i. Putting $\psi_0=\sum_{i\in I}\psi_i$, we obtain the desired trace ψ_0. Q.E.D.

LEMMA 8.10. *Under the same assumption as in Lemma 8.9, $\mathcal{A}$ contains a continuous one parameter unitary group $\{v(t)\}$ such that*

$$\theta_s(v(t)) = e^{ist}v(t), \quad s, t \in \mathbf{R}.$$

Proof. By the previous lemma, $\mathcal{A}$ has an invariant faithful semifinite normal trace ψ_0. By the Radon–Nikodym theorem for traces (in the abelian case), there exists a nonsingular positive self-adjoint operator h affiliated with $\mathcal{A}$ such that $\psi(x) = \psi_0(hx)$, $x \in \mathcal{A}$. We have then

$$\begin{aligned}\psi_0(e^{-t}hx) = e^{-t}\psi_0(hx) = e^{-t}\psi(x) = \psi(\theta_t(x)) = \psi_0(h\theta_t(x)) = \psi_0 \circ \theta_t(\theta_{-t}(h)x) \\ = \psi_0(\theta_{-t}(h)x), \quad x \in \mathcal{A},\ t \in \mathbf{R},\end{aligned}$$

so that we have $\theta_t(h) = e^t h$. Putting $v(t) = h^{it}$, $t \in \mathbf{R}$, we obtain the desired one parameter unitary group $\{v(t)\}$ in $\mathcal{A}$. Q.E.D.

Returning to the original situation, we have the following result.

THEOREM 8.11. *Under the same assumption as in Theorem 8.5, if $\mathcal{M}$ is of type* III, *then $\mathcal{M}_0$ must be of type* II_∞.

Proof. Since any automorphism of $\mathcal{M}_0$ leave invariant the greatest central type I projection of $\mathcal{M}_0$, we may, by virtue of Theorem 8.5, assume that $\mathcal{M}_0$ either of type I or type II. Suppose $\mathcal{M}_0$ is of type I. For the same reason as above, we may assume that $\mathcal{M}_0$ is homogeneous in the sense that $\mathcal{M}_0$ is isomorphic to the tensor product $\mathcal{Z}_0 \otimes \mathcal{B}$ of the center and a factor $\mathcal{B}$ of type I. So we identify $\mathcal{M}_0$ and $\mathcal{Z}_0 \otimes \mathcal{B}$. Let Tr denote the usual trace on $\mathcal{B}$. It is known that any faithful semifinite normal trace on $\mathcal{M}_0$ is of the form $\psi \otimes \mathrm{Tr}$ for some faithful semifinite normal trace ψ on $\mathcal{Z}_0$. Thus the trace τ is written as $\tau = \psi \otimes \mathrm{Tr}$. The one parameter automorphism group $\{\tilde{\theta}_t\}$ of $\mathcal{Z}_0$ is extended uniquely to a one parameter automorphism group of $\mathcal{M}_0$ leaving $\mathcal{B}$ elementwise fixed, which is obtained as $\tilde{\theta}_t \otimes \iota$ and denoted again by $\{\tilde{\theta}_t\}$. Since $\tilde{\theta}_{-t} \circ \theta_t$, $t \in \mathbf{R}$, leaves $\mathcal{Z}_0$ elementwise fixed and $\mathcal{M}_0$ is of type I, there exists a unitary w_t in $\mathcal{M}_0$ such that $\tilde{\theta}_{-t} \circ \theta_t(x) = w_t x w_t^*$, $x \in \mathcal{M}_0$. Hence we have

$$\theta_t(x) = \tilde{\theta}_t(w_t)\,\tilde{\theta}_t(x)\,\tilde{\theta}_t(w_t^*), \quad x \in \mathcal{M}_0,\ t \in \mathbf{R}.$$

Choose a positive nonzero $b \in \mathcal{B}$ with $\mathrm{Tr}\,(b) < +\infty$. We have for each $a \in \mathcal{Z}_0^+$, $t \in \mathbf{R}$,

$$\begin{aligned}e^{-t}\psi(a)\,\mathrm{Tr}\,(b) = e^{-t}\tau(ab) = \tau \circ \theta_t(ab) = \tau(\tilde{\theta}_t(w_t)\,\tilde{\theta}_t(ab)\,\tilde{\theta}_t w_t^*) = \tau(\tilde{\theta}_t(ab)) \\ = \tau(\tilde{\theta}_t(a)b) = [\psi \circ \tilde{\theta}_t(a)]\,\mathrm{Tr}\,(b),\end{aligned}$$

so that $\psi \circ \theta_t = e^{-t}\psi$. Hence Lemma 8.10 assures that there exists a continuous one parameter unitary group $\{v(t)\}$ in $\mathcal{Z}_0$ satisfying (8.2); hence $\mathcal{M}$ is semifinite by Theorem 8.6, a contradiction. Thus $\mathcal{M}_0$ has no direct summand of type I. Hence it must be of type II_∞. Q.E.D.

A more practical criterion for $\mathcal{M}$ to be of type III is given by the following result, although the proof will not be given until § 10.

THEOREM 8.12. *In the same situation as before, the following two statements are equivalent:*

(i) *$\mathcal{M}$ is a factor of type* III;

(ii) *The one parameter automorphism group $\{\hat{\theta}_t\}$ of $\mathcal{Z}_0$ is ergodic but not equivalent to the translation automorphism group on the abelian von Neumann algebra $L^\infty(\mathbf{R})$ of all essentially bounded measurable functions on* $\mathbf{R}$.

9. Algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ of A. Connes

We keep the notations, the terminologies and the basic assumptions of the previous section. In this sections, we shall examine the connection between the structure of a factor $\mathcal{M}$ of type III descrived in the previous section and the algebraic invariants $S(\mathcal{M})$ and $T(\mathcal{M})$ introduced recently by A. Connes, [10].

For each weight φ on a von Neumann algebra $\mathcal{M}$, let Δ_φ denote the associated modular operator. The following algebraic invariant was introduced by A. Connes, [10].

Definition 9.1. (A. Connes) The intersection $S(\mathcal{M})$ of the spectra of all possible Δ_φ is called the *modular spectrum* of $\mathcal{M}$. The intersection of the spectrum of Δ_φ when φ runs over all possible faithful positive normal linear functionals on $\mathcal{M}$ is denoted by $S_0(\mathcal{M})$ and called the *proper modular spectrum* of $\mathcal{M}$. Of course, $S_0(\mathcal{M})$ makes sense only for a σ-finite von Neumann algebra.

Both $S(\mathcal{M})$ and $S_0(\mathcal{M})$ are algebraic invariants of $\mathcal{M}$, and they are interesting when $\mathcal{M}$ is of type III. However, if $\mathcal{M}$ is σ-finite and of type III, then $S(\mathcal{M})=S_0(\mathcal{M})$. It is also easily seen that $S(\mathcal{M})=\{1\}$ if and only if $\mathcal{M}$ is semifinite; for a σ-finite von Neumann algebra $\mathcal{M}$, $S_0(\mathcal{M})=\{1\}$ if and only if $\mathcal{M}$ is finite.

A. Connes and van Daele have proved [54], that both $S(\mathcal{M})$ and $S_0(\mathcal{M})$, (with $\{0\}$ deleted), are closed subgroups of the multiplicative group $]0, \infty[$ of positive real numbers. Making use of $S(\mathcal{M})$, A. Connes further classified the factors of type III into those of type III_λ, $0 \leqslant \lambda \leqslant 1$.

Definition 9.2. (A. Connes) A factor $\mathcal{M}$ of type III is said to be of *type* III_λ, $0<\lambda<1$, if $S(\mathcal{M})=\{\lambda^n : n \in \mathbf{Z}\} \cup \{0\}$; *of type* III_0 if $S(\mathcal{M})=\{0, 1\}$; *of type* III_1 if $S(\mathcal{M})=\mathbf{R}_+$.

The following algebraic invariant is also due to A. Connes.

Definition 9.3. Given a von Neumann algebra $\mathcal{M}$, $T(\mathcal{M})$ is the set of all $t \in \mathbf{R}$ such that there exists a (faithful semifinite normal) weight φ on $\mathcal{M}$ with $\sigma_t^\varphi = \iota$. We call $T(\mathcal{M})$ *the modular period group* of $\mathcal{M}$.

He proved, [10; Théorème 1.3.2], that for any weight φ of $\mathcal{M}$,

$$T(\mathcal{M}) = \{t \in \mathbf{R} : \sigma_t^\varphi \in \operatorname{Int}(\mathcal{M})\} \tag{9.1}$$

where $\operatorname{Int}(\mathcal{M})$ means the group of all inner automorphisms of $\mathcal{M}$ as defined in § 3. Hence $T(\mathcal{M})$ is a subgroup of the additive group $\mathbf{R}$. However, by [45; Theorem 5.1], $T(\mathcal{M})$ need not be a closed subgroup of $\mathbf{R}$. However, it may be said that $S(\mathcal{M})$ and $T(\mathcal{M})$ are almost dual algebraic invariants; when the duality between $S(\mathcal{M})$ and $T(\mathcal{M})$ breaks down, they serve as complementary algebraic invariants.

THEOREM 9.4. *Under the same assumptions as in Theorem 8.5, a real number t falls in $T(\mathcal{M})$ if and only if there exists a unitary $v \in \mathcal{Z}_0$ with $\tilde{\theta}_s(v) = e^{ist} v$.*

The proof follows the same line as Theorem 8.6, so we leave it to the reader.

LEMMA 9.5. *Let $\mathcal{A}$ be an abelian von Neumann algebra equipped with a continuous one parameter automorphism group $\{\theta_t\}$. If $\theta_{t_0} \neq \iota$, $t_0 \in \mathbf{R}$, then there exist $\varepsilon > 0$ and a nonzero projection $e \in \mathcal{A}$ such that $\theta_t(e)e = 0$ for $|t - t_0| < \varepsilon$.*

Proof. Let $\mathcal{A}_0$ be the set of all $x \in \mathcal{A}$ such that $\lim_{t\to 0}\|\theta_t(x) - x\| = 0$. It is easily seen that $\mathcal{A}_0$ is a C^*-subalgebra of $\mathcal{A}$. For each $n > 0$ and $x \in \mathcal{A}$, put

$$x_n = n^{\frac{1}{2}}\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} \exp(-nt^2)\,\theta_t(x)\,dt.$$

Then x_n falls in $\mathcal{A}_0$ and $\{x_n\}$ converges σ-strongly* to x as $n \to \infty$. Hence $\mathcal{A}_0$ is σ-strongly* dense in $\mathcal{A}$. Let Ω be the spectrum of $\mathcal{A}_0$. Since $\mathcal{A}_0$ contains the identity 1, Ω is compact. We identify $\mathcal{A}_0$ with the algebra $C(\Omega)$ of all continuous functions on Ω. For each $t \in \mathbf{R}$, θ_t induces a homeomorphism θ_t^* of Ω such that

$$\theta_t^{-1}(a)(\omega) = a(\theta_t^*\omega), \quad a \in \mathcal{A}_0,\ \omega \in \Omega.$$

If $\{\omega_i\}$ is a net in Ω converging to ω and $\{s_n\}$ is a sequence in $\mathbf{R}$ converging to s, then we have, for every $a \in \mathcal{A}_0$,

$$
\begin{aligned}
|a(\theta^*_{s_n}\omega_i)-a(\theta^*_s\omega)| &= |\theta_{-s_n}(a)(\omega_i)-\theta_{-s}(a)(\omega)| \\
&\leqslant |\theta_{-s_n}(a)(\omega_i)-\theta_{-s}(a)(\omega_i)|+|\theta_{-s}(a)(\omega_i)-\theta_{-s}(a)(\omega)| \\
&\leqslant \|\theta_{-s_n}(a)-\theta_{-s}(a)\|+|\theta_{-s}(a)(\omega_i)-\theta_{-s}(a)(\omega)|\to 0;
\end{aligned}
$$

hence the map: $(t,\omega)\in\mathbf{R}\times\Omega\mapsto\theta^*_t\omega\in\Omega$ is continuous.

By the density of $\mathcal{A}_0$ in $\mathcal{A}$, we have $\theta_{t_0}|_{\mathcal{A}_0}\neq\iota$, so that $\theta^*_{-t_0}\neq\mathrm{id}.$ on Ω. Hence for some $\omega_0\in\Omega$, $\theta^*_{-t_0}\omega_0\neq\omega_0$; so there exist $\varepsilon>0$ and an open neighborhood U of ω_0 such that $U\cap\theta^*_{-t}(U)=\varnothing$ for $|t-t_0|<\varepsilon$. Let a be a positive element of $\mathcal{A}_0$ such that $a(\omega_0)=1$ and $a(U^c)=0$. We have then $\theta_t(a)a=0$ for $|t-t_0|<\varepsilon$. Let e be the spectral projection of a corresponding to the interval $[\frac{1}{2},1]$. We have then $a\geqslant e$, so that $\theta_t(e)e=0$ for $|t-t_0|<\varepsilon$.

Q.E.D.

THEOREM 9.6. *In the same situation as in Theorem 8.5, if $\mathcal{M}$ is a factor, or equivalently if $\{\tilde\theta_t\}$ is ergodic on $\mathcal{Z}_0$, then the following two statements for $t_0\in\mathbf{R}$ are equivalent;*

(i) $\tilde\theta_{t_0}=\iota$;

(ii) $e^{t_0}\in S(\mathcal{M})$.

Proof. Let $\mathfrak{H}_0$ be the Hilbert space constructed from the trace τ of $\mathcal{M}_0$ which is sometimes denoted by $L^2(\mathcal{M}_0;\tau)$. The Hilbert space $\mathfrak{H}$ on which $\mathcal{M}$ acts is $L^2(\mathfrak{H}_0;\mathbf{R})$. The modular operator $\tilde\Delta$ and the unitary involution $\tilde J$ associated with the dual weight $\tilde\tau$ on $\mathcal{M}$ are given, by (5.12) and (5.17), as follows:

$$
\left.\begin{aligned}
&\tilde\Delta\xi(s)=e^{-s}\xi(s),\ \xi\in L^2(\mathfrak{H}_0;\mathbf{R}),\ s\in\mathbf{R};\\
&\tilde J\xi(s)=\mu(-s)J\xi(-s),
\end{aligned}\right\}\tag{9.2}
$$

where $\{\mu(s)\}$ is the continuous one parameter unitary group on $\mathfrak{H}_0$ given by

$$\mu(s)\eta_\tau(x)=e^{\frac{1}{2}s}\eta_\tau(\theta_s(x)),\quad s\in\mathbf{R},\tag{9.3}$$

for every $x\in\mathcal{M}$ with $\tau(x^*x)<+\infty$, and J is the canonical unitary involution on $\mathfrak{H}_0$ associated with the trace τ, i.e., the "L^2-extension" of the involution: $x\in\mathcal{M}_0\mapsto x^*\in\mathcal{M}_0$.

By Theorem 8.3, $\mathcal{M}_0$ is the centralizer of the dual weight $\tilde\tau$ on $\mathcal{M}$. Hence by [10, Corollary 3.2.5(b)], we have

$$S(\mathcal{M})=\cap\{\mathrm{Sp}(\tilde\Delta e\tilde Je\tilde J): e \text{ runs over nonzero projections of } \mathcal{Z}_0\}.$$

For a nonzero projection $e\in\mathcal{Z}_0$, we have

$$(e\tilde Je\tilde J\xi)(s)=\theta_{-s}(e)(\tilde Je\tilde J\xi)(s)=\theta_{-s}(e)e\xi(s),\quad \xi\in L^2(\mathfrak{H}_0;\mathbf{R}),\ s\in\mathbf{R}.$$

Therefore, $\mathrm{Sp}(\tilde\Delta e\tilde Je\tilde J)$ consists of all e^t such that for any $\varepsilon>0$ $\theta_s(e)e\neq0$ for some $s\in[t-\varepsilon,t+\varepsilon]$. Thus, by virtue of Lemma 9.5, (i) and (ii) are equivalent. Q.E.D.

COROLLARY 9.7. *The following two statements are equivalent:*

(i) $\mathcal{M}_0$ *is a factor of type* II_∞;

(ii) $\mathcal{M}$ *is a factor of type* III_1, *i.e.,* $S(\mathcal{M}) = \mathbf{R}_+$.

Proof. If $\mathcal{M}_0$ is a factor, then $S(\mathcal{M}) = \mathrm{Sp}\,(\tilde{\Delta})$ because no nontrivial projection is in $\mathcal{Z}_0$; hence $S(\mathcal{M}) = \mathbf{R}_+$. If $S(\mathcal{M}) = \mathbf{R}_+$, then $\tilde{\theta}_t = \iota$ for every $t \in \mathbf{R}$ by Theorem 9.6, and $\{\tilde{\theta}_t\}$ is ergodic on $\mathcal{Z}_0$; this is impossible unless $\mathcal{Z}_0 = \mathbf{C}$. Q.E.D.

10. The crossed products by the induced actions

We have seen so far that every von Neumann algebra of type III is represented uniquely, up to weak equivalence, as the crossed product of a von Neumann algebra of type II_∞ by a one parameter automorphism group leaving a trace relatively invariant but not invariant. Since it is not such an easy task to analyze a continuous crossed product, (actually most of theories of crossed products of operator algebras, such as [13, 14, 27, 30, 38, 51, 53], have been restricted to the discrete case), it is, of course, desirable if we can further reduce the continuous crossed product to a discrete one based on the group $\mathbf{Z}$ of integers. Unfortunately, this is, however, not always the case. Nevertheless there are many cases where one can reconstruct a given von Neumann algebra of type III as the discrete crossed product of a von Neumann algebra of type II_∞ by a single automorphism, as seen in [2, 10, 45]. This section is devoted to the study of this problem. In the search for the solution we are eventually led to the comparison of the crossed product of a covariant system and that of a smaller covariant system. It turns out that this is closely related to Mackey's theory of induced representations of a locally compact group.

Suppose G is a locally compact *separable* group with a left Haar measure dg. The separability assumption here is *not* essential; one can get rid of this restriction at the cost of somewhat longer arguments, see for example [5, 6, 32]. For applications, we shall take the additive group $\mathbf{R}$ of real numbers anyway.

Let H be a closed subgroup of G. We denote by $d_G g$ and $d_H h$ left invariant Haar measures of G and H respectively, and by δ_G and δ_H the modular functions of G and H. It is known that (a) there exists a continuous function $\varrho(g) > 0$ on G such that

$$\varrho(gh) = \frac{\delta_H(h)}{\delta_G(h)}\varrho(g), \quad h \in H, g \in G; \tag{10.1}$$

(b) with such a function ϱ, there is associated a quasi-invariant measure $d\dot{g}$ on the left homogeneous space G/H such that

$$\int_G f(g)\,\varrho(g)\,dg = \int_{G/H}\left(\int_H f(gh)\,d_H h\right) d\dot{g}, \quad f \in \mathcal{K}(G); \tag{10.2}$$

(c) For each $g_1, g_2 \in G$, $\varrho(g_1 g_2)/\varrho(g_2)$ depends only on g_1 and $\dot{g}_2 = g_2 H$, so that we may define a continuous function χ on $G \times (G/H)$ by

$$\chi(g_1, \dot{g}_2) = \varrho(g_1 g_2)/\varrho(g_2). \tag{10.3}$$

We consider the Hilbert space $L^2(G, \nu)$ of square integrable functions on G with respect to the new measure $d\nu(g) = \varrho(g)dg$. Using the unitary operator U of $L^2(g, \nu)$ onto $L^2(G)$ defined by

$$(U\xi)(g) = \varrho(g)^{\frac{1}{2}}\xi(g), \quad \xi \in L^2(G, \nu), \tag{10.4}$$

we realize the left and right regular representations λ and λ' of G on $L^2(G, \nu)$ as follows:

$$\left.\begin{aligned} \lambda(g)\xi(h) &= \varrho(h)^{-\frac{1}{2}}\varrho(g^{-1}h)^{\frac{1}{2}}\xi(g^{-1}h), \quad \xi \in L^2(G, \nu); \\ \lambda'(g)\xi(h) &= \delta_G(g)^{\frac{1}{2}}\varrho(h)^{-\frac{1}{2}}\varrho(hg)^{\frac{1}{2}}\xi(hg), \quad g, h \in G. \end{aligned}\right\} \tag{10.5}$$

Since ν and the Haar measure are equivalent, we have $L^\infty(G, \nu) = L^\infty(G)$. In the von Neumann algebra $L^\infty(G)$ on $L^2(G, \nu)$, we consider the von Neumann subalgebra $L^\infty(G/H)$ of all functions in $L^\infty(G)$ which are constant on each left H-coset gH, $g \in G$. We denote it by $\mathcal{A}_H$.

By the separability assumption, G is identified, as a Borel space, with the cartesian product $(G/H) \times H$ and $L^2(G, \nu)$ is identified with the tensor product $L^2(G/H, dg) \otimes L^2(H)$ by equality (10.2).

LEMMA 10.1. *In the above situation, the von Neumann algebra $\mathcal{M}(G/H, \lambda(G))$ on $L^2(G, \nu)$ generated by $\mathcal{A}_H$ and $\lambda(G)$ coincides with the tensor product $\mathcal{L}(L^2(G/H, d\dot{g})) \otimes \mathcal{M}(H)$ of the algebra of all bounded operators on $L^2(G/H, d\dot{g})$ and the von Neumann algebra $\mathcal{M}(H)$ generated by the left regular representation $\lambda_H(H)$ of H on $L^2(H)$.*

Proof. By the separability of G, there exists a Borel subset E which meets with every left H-coset at one and only one point. Hence the homogeneous space G/H is identified with E and $G = EH$. By [41], the commutant $\mathcal{M}(G/H, \lambda(G))'$ of $\mathcal{M}(G/H, \lambda(G))$ is generated by $\lambda'(H)$. Hence $\mathcal{M}(G/H, \lambda(G))'$ coincides with $1 \otimes \mathcal{M}'(H)$, where $\mathcal{M}'(H)$ is the von Neumann algebra on $L^2(H)$ generated by the right regular representation of H. Thus we get

$$\mathcal{M}(H\diagdown G, \lambda(G)) = \mathcal{M}(H\diagdown G, \lambda(G))'' = \mathcal{L}(L^2(H\diagdown G), d\dot{g}) \otimes \mathcal{M}(H) \qquad \text{Q.E.D.}$$

Let $\{\mathcal{N}_0, \mathfrak{K}_0\}$ be a von Neumann algebra equipped with a continuous action β of a closed subgroup H of G. We consider the tensor product $L^\infty(G) \otimes \mathcal{N}_0$ of $\mathcal{N}_0$ and the abelian von Neumann algebra $L^\infty(G)$, whose elements are regarded as bounded $\mathcal{N}_0$-valued functions with the properties described in § 2. We define actions γ and α of H and G as follows

$$\left.\begin{aligned} \gamma_h(x)(g) &= \beta_h(x(gh)), \quad g\in G, h\in H; \\ \alpha_k(x)(g) &= x(k^{-1}g), \quad g, k\in G, x\in \mathcal{N}_0\otimes L^\infty(G). \end{aligned}\right\} \tag{10.6}$$

Let $\mathcal{M}_0$ denote the fixed point subalgebra of $L^\infty(G)\otimes\mathcal{N}_0$ under $\{\gamma_h: h\in H\}$. Since γ_h, $h\in H$, and α_g, $g\in G$, commute, $\mathcal{M}_0$ is invariant under α_g. The restriction of α_g to $\mathcal{M}_0$ is also denoted by α_g.

Definition 10.2. The action α of G on $\mathcal{M}_0$ is said to be *induced* up to G from the action β of H, and we write

$$\{\mathcal{M}_0, \alpha\} = \mathrm{Ind}_H^G\{\mathcal{N}_0, \beta\}.$$

THEOREM 10.3. *Given a von Neumann algebra $\mathcal{N}_0$ equipped with a continuous action β of a closed subgroup H of a locally compact separable group G, let*

$$\{\mathcal{M}_0, \alpha\} = \mathrm{Ind}_H^G\{\mathcal{N}_0, \beta\}.$$

Then the crossed product $\mathcal{R}(\mathcal{M}_0; \alpha)$ is isomorphic to $\mathcal{L}(L^2(G/H))\otimes\mathcal{R}(\mathcal{N}_0; \beta)$.

Proof. We identify, as in the previous lemma, the homogeneous space G/H with the Borel cross section E. Suppose $\mathcal{N}_0$ acts on a Hilbert space $\mathfrak{K}_0$. Then the von Neumann algebra $\mathcal{M}_0$ is regarded as the algebra of bounded $\mathcal{N}_0$-valued functions on $G/H=E$ with the properties described in § 2, that is, $\mathcal{M}_0=L^\infty(G/H)\otimes\mathcal{N}_0$. The von Neumann algebra $L^\infty(G)\otimes\mathcal{N}_0$ acts on $L^2(\mathfrak{K}_0; G, \nu)$ in the canonical fashion. The crossed product $\mathcal{R}(L^\infty(G)\otimes\mathcal{N}_0; \alpha)$, hence $\mathcal{R}(\mathcal{M}_0; \alpha)$, is faithfully represented on $L^2(\mathfrak{K}_0; G\times G, \nu\otimes\nu)$. We shall represent $\mathcal{R}(L^\infty(G)\otimes\mathcal{N}_0; \alpha)$, hence $\mathcal{R}(\mathcal{M}_0; \alpha)$ too, on $L^2(\mathfrak{K}_0; G, \nu)$. The von Neumann algebra $\mathcal{R}(L^\infty(G)\otimes\mathcal{N}_0; \alpha)$ on $L^2(\mathfrak{K}_0; G\times G, \nu\otimes\nu)$ is generated by three kinds of operators:

$$\left.\begin{aligned} (\bar{x}\xi)(g, h) &= x\xi(g, h), \quad x\in\mathcal{N}_0, g, h\in G; \\ (a_f\xi)(g, h) &= f(gh)\xi(g, h), \quad f\in L^\infty(G); \\ (\lambda(k)\xi)(g, h) &= \varrho(g)^{-\frac{1}{2}}\varrho(k^{-1}g)^{\frac{1}{2}}\xi(k^{-1}g, h), \quad k\in G, \end{aligned}\right\} \tag{10.7}$$

for each $\xi\in L^2(\mathfrak{K}_0; G\times G, \nu\otimes\nu)$. We define a unitary operator W on $L^2(\mathfrak{K}_0; G\times G, \nu\otimes\nu)$ by

$$(W\xi)(g, h) = \delta_G(h)^{\frac{1}{2}}\varrho(g)^{-\frac{1}{2}}\varrho(gh)^{\frac{1}{2}}\xi(gh, h), \quad g, h\in G. \tag{10.8}$$

We have then

$$\left.\begin{aligned} W^*\bar{x}W &= \bar{x}, \quad x\in\mathcal{N}_0; \\ W^*a_fW &= \pi(f)\otimes 1, \quad f\in L^\infty(G); \\ W^*\lambda(k)W &= \lambda(k), \quad k\in G, \end{aligned}\right\} \tag{10.9}$$

where $\pi(f)$ denotes the multiplication operator on $L^2(G, \nu)$ defined by f. Hence $W^*\mathcal{R}(L^\infty(G)\otimes\mathcal{N}_0; \alpha)W$ is generated by the operators $\tilde{x}$, $x\in\mathcal{N}_0$, $\pi(f)\otimes 1$, $f\in L^\infty(G)$, and $\lambda(k)$, $k\in G$, which is obviously isomorphic to the von Neumann algebra on $L^2(\mathfrak{K}_0; G, \nu)$ generated by the following three kinds of operators:

$$\left.\begin{aligned} &\tilde{x}\xi(g) = x\xi(g), \quad x\in\mathcal{N}_0,\ \xi\in L^2(\mathfrak{K}_0; G, \nu);\\ &\pi(f)\xi(g) = f(g)\xi(g), \quad f\in L^\infty(G);\\ &\lambda(k)\xi(g) = \varrho(g)^{-\frac{1}{2}}\varrho(k^{-1}g)^{\frac{1}{2}}\xi(k^{-1}g), \quad g, k\in G.\end{aligned}\right\} \tag{10.10}$$

Therefore, the crossed product $\mathcal{R}(\mathcal{M}_0; \alpha)$ is isomorphic to the von Neumann algebra on $L^2(\mathfrak{K}_0; G, \nu)$ generated by the operators:

$$\left.\begin{aligned} &(x\xi)(g) = x(g)\xi(g), \quad x\in\mathcal{M}_0,\ \xi\in L^2(\mathfrak{K}_0; G, \nu);\\ &(\lambda(k)\xi)(g) = \varrho(g)^{-\frac{1}{2}}\varrho(k^{-1}g)^{\frac{1}{2}}\xi(k^{-1}g), \quad k, g\in G.\end{aligned}\right\} \tag{10.11}$$

According to the decomposition $G=EH$, we decompose $L^2(\mathfrak{K}_0; G)$ into the tensor product $L^2(G/H, d\dot{g})\otimes L^2(\mathfrak{K}_0; H)$. With respect to this decomposition, the algebra $\mathcal{M}_0$ on $L^2(\mathfrak{K}_0; G, \nu)$ is generated by two kinds of operators:

$$\left.\begin{aligned} &(\tilde{x}\xi)(\dot{g}, h) = \beta_h^{-1}(x)\,\xi(\dot{g}, h), \quad \dot{g}\in G/H=E,\ h\in H,\ x\in\mathcal{N}_0;\\ &\pi(f)\,\xi(\dot{g}, h) = f(\dot{g})\,\xi(\dot{g}, h), \quad f\in L^\infty(G/H).\end{aligned}\right\} \tag{10.12}$$

By the previous lemma, the von Neumann algebra on $L^2(\mathfrak{K}_0; G, \nu)$ generated by $\pi(f)$, $f\in L^\infty(G/H)$, and $\lambda(g)$, $g\in G$, coincides with $1\otimes\mathcal{L}(L^2(G/H, d\dot{g}))\otimes\mathcal{M}(H)$. Therefore, $\mathcal{R}(\mathcal{M}_0; \alpha)$ is isomorphic to the tensor product of $\mathcal{L}(L^2(G/H, d\dot{g}))$ and the von Neumann algebra on $L^2(\mathfrak{K}_0; H)$ generated by the operators:

$$\left.\begin{aligned} &(\tilde{x}\xi)(h) = \beta_h^{-1}(x)\xi(h); \quad x\in\mathcal{N}_0,\ h\in H,\ \xi\in L^2(\mathfrak{K}_0; H)\\ &(\lambda_H(k)\xi)(h) = \xi(k^{-1}h), \quad h, k\in H.\end{aligned}\right\}$$

But the latter is nothing but $\mathcal{R}(\mathcal{N}_0; \beta)$. Q.E.D.

We keep the notations in the previous theorem. Each elements x of $L^\infty(G)\otimes\mathcal{N}_0$ is regarded as a bounded $\mathcal{N}_0$-valued function on G with the properties described in § 2. An element x of $L^\infty(G)\otimes\mathcal{N}_0$ falls in $\mathcal{M}_0$ if and only if for every $h\in H$

$$x(gh) = \beta_h^{-1}(x(g)) \tag{10.13}$$

for almost every $g\in G$. The algebra $\mathcal{A}_H = L^\infty(G/H)$ is contained in the center of $\mathcal{M}_0$.

PROPOSITION 10.4. *In the above situation, if* $\mathcal{M}_1$ *is a von Neumann subalgebra of* $\mathcal{M}_0$ *such that* (i) $\mathcal{M}_1$ *is invariant under the action* α *of* G *and* (ii) $\mathcal{M}_1$ *contains* $\mathcal{A}_H$, *then* *there exists a unique von Neumann subalgebra $\mathcal{N}_1$ of $\mathcal{N}_0$ such that $\mathcal{N}_1$ is invariant under the action β of H and*

$$\{\mathcal{M}_1, \alpha\} = \mathrm{Ind}_H^G \{\mathcal{N}_1, \beta\}.$$

Proof. For each $f \in \mathcal{K}(G)$, we define

$$\alpha_f(x) = \int_G f(g)\,\alpha_g(x)\,dg, \quad x \in \mathcal{M}_0.$$

Let A be the C^*-subalgebra of all x such that $\lim_{g\to e}\|\alpha_g(x) - x\| = 0$. Clearly A contains $\alpha_f(x)$ for every $f \in \mathcal{K}(G)$ and $x \in \mathcal{M}_0$; hence $A \cap \mathcal{M}_1$ is σ-weakly dense in $\mathcal{M}_1$. Every element $x \in A$ is represented by a bounded norm continuous $\mathcal{N}_0$-valued function $x(\cdot)$ on G. Let $\mathcal{N}_1$ be the von Neumann subalgebra of $\mathcal{N}_0$ generated by $B = \{x(e)\colon x \in A \cap \mathcal{M}_1\}$. Let $y = x(e)$ with an $x \in A \cap \mathcal{M}_1$. We have then, for each $h \in H$, $\beta_h(y) = x(h^{-1}) = \alpha_h(x)(e)$. Since $\alpha_h(x) \in A \cap \mathcal{M}_1$, $\beta_h(y)$ belongs to $\mathcal{N}_1$. Hence $\mathcal{N}_1$ is invariant under the action β. If x is an element of $A \cap \mathcal{M}_1$, then $x(g) = \alpha_g^{-1}(x)(e)$ belongs to $\mathcal{N}_1$ since $\alpha_g(A \cap \mathcal{M}_1) = A \cap \mathcal{M}_1$. Hence each element x of $A \cap \mathcal{M}_1$ is represented by a bounded norm continuous $\mathcal{N}_1$-valued function $x(\cdot)$ with $x(gh) = \beta_h^{-1}(x(g))$, $g \in G$, $h \in H$. Let g_1 and g_2 be elements of G with $\dot{g}_1 \neq \dot{g}_2$, and y_1 and y_2 be two arbitrary elements in B. There exist two elements x_1 and x_2 in $A \cap \mathcal{M}_1$ such that $x_1(g_1) = y_1$ and $x_2(g_2) = y_2$. Let U_1 and U_2 be open neighborhoods of g_1 and g_2 respectively such that $U_1 \cap U_2 = \emptyset$, $U_1 H = U_1$ and $U_2 H = U_2$. Choose two functions f_1 and f_2 from $\mathcal{K}(G/H)$ such that $f_i(g_i) = 1$ and $f_i(\dot{U}_i^c) = 0$, $i = 1, 2$. Let $x = f_1 x_1 + f_2 x_2 \in A \cap \mathcal{M}_1$. Then we have $y_1 = x(g_1)$ and $y_2 = x(g_2)$. Hence if $x(\cdot)$ is a norm continuous B-valued function on G satisfying (10.13), and vanishes outside the inverse image of a compact set in G/H, then x belongs to $A \cap \mathcal{M}_1$. Hence we conclude that $\mathcal{M}_1$ is the subalgebra of $\mathcal{M}_0$ consisting all elements in $\mathcal{M}_0$ with values in $\mathcal{N}_1$. Thus we get $\{\mathcal{M}_1, \alpha\} = \mathrm{Ind}_H^G \{\mathcal{N}_1, \beta\}$.

Q.E.D.

This result indicates that the abelian von Neumann subalgebra $\mathcal{A}_H$ plays an important role in the analysis of the covariant system $\{\mathcal{M}_0, \alpha\}$. We call the action of G on $\mathcal{A}_H$ *canonical.*

THEOREM 10.5. *Let $\{\mathcal{M}, \alpha\}$ be a covariant system on a locally compact group G. Let H be a closed subgroup of G. If there exists an isomorphism of the abelian von Neumann algebra $L^\infty(G/H)$, say $\mathcal{A}_H$, onto an α-invariant von Neumann subalgebra $\mathcal{A}$ of the center $\mathcal{Z}$ of $\mathcal{M}$, which transforms the canonical action of G on $\mathcal{A}_H$ into α on $\mathcal{A}$, then there exists a covariant system $\{\mathcal{N}, \beta\}$ such that*

$$\{\mathcal{M}, \alpha\} \cong \mathrm{Ind}_H^G \{\mathcal{N}, \beta\}.$$

Proof. We may assume that $\mathcal{M}$ acts on a Hilbert space $\mathfrak{H}$ and α is implemented by a unitary representation U of G on $\mathfrak{H}$. We identify $\mathcal{A}$ and $\mathcal{A}_H$. Then $\mathcal{A}$ turns out to be a transitive imprimitivity system for the representation $\{U, \mathfrak{H}\}$ of G. By the Mackey-Blattner theorem for induced representations, there exists a unitary representation $\{L, \mathfrak{K}\}$ of H such that

$$\{U, \mathfrak{H}\} \cong \mathrm{Ind}_H^G \{L, \mathfrak{K}\}.$$

Therefore, the Hilbert space $\mathfrak{K}$ is identified with the Hilbert space of all $\mathfrak{H}$-valued measurable functions ξ such that

$$\left.\begin{aligned} &\xi(gh) = L(h)^{-1}\xi(g), \quad g \in G, h \in H; \\ &\int_{G/H} \|\xi(g)\|^2 d\dot{g} = \|\xi\|^2 < +\infty. \end{aligned}\right\} \tag{10.14}$$

The representation U is given by

$$(U(k)\xi)(g) = \varrho(g)^{-\frac{1}{2}}\varrho(k^{-1}g)^{\frac{1}{2}}\,\xi(k^{-1}g), \quad g, k \in G. \tag{10.15}$$

The arguments of O. Nielsen in [32] show that to each $x \in \mathcal{A}'$ there corresponds an essentially bounded $\mathcal{L}(\mathfrak{K})$-valued measurable, in the sense of § 2, function $x(\cdot)$ on G such that, for each $\xi \in \mathfrak{H}$ and $h \in H$,

$$\left.\begin{aligned} (x\xi)(g) &= x(g)\xi(g); \\ x(gh) &= L(h)^{-1}x(g)L(h) \end{aligned}\right\} \tag{10.16}$$

for almost every $g \in G$. It is then clear that for each $x \in \mathcal{M}$ and $g \in G$, we have

$$\alpha_g(x)(k) = x(g^{-1}k) \tag{10.17}$$

for almost every $k \in G$.

Let A be the set of all $x \in \mathcal{M}$ such that the function: $g \in G \to \alpha_g(x) \in \mathcal{M}$ is continuous in norm. As in the previous proposition, A is a σ-weakly dense C^*-subalgebra of $\mathcal{M}$. Each $x \in A$ is represented by a bounded norm continuous $\mathcal{L}(\mathfrak{K})$-valued function $x(\cdot)$ satisfying (10.16) and (10.17). Let $\mathcal{N}$ be the von Neumann algebra on $\mathfrak{K}$ generated by $B = \{x(e) : x \in A\}$. We have then for each $x \in A$

$$L(h)^{-1}x(e)L(h) = x(h) = \alpha_h^{-1}(x)(e), \quad h \in H.$$

Hence the unitary representation $\{L, \mathfrak{K}\}$ of H induces a continuous action β of H on $\mathcal{N}$. By the same reasoning as the last part of the previous proposition (or applying it to $\mathrm{Ind}_H^G(\mathcal{N}, \beta)$), we conclude that

$$\{\mathcal{M}, \alpha\} = \mathrm{Ind}_H^G(\mathcal{N}, \beta). \qquad \text{Q.E.D}$$

Now we apply Theorems 10.3 and 10.5 to the discrete crossed product description of a factor of type III of a certain class.

Let $\mathcal{M}_0$ be a von Neumann algebra of type II_∞ equipped with a continuous one parameter automorphism group $\{\theta_t\}$ such that $\tau\circ\theta_t=e^{-t}\tau$ for some faithful semifinite normal trace τ. Let $\mathcal{M}=\mathcal{R}(\mathcal{M}_0;\theta)$. We denote by $\mathcal{Z}_0$ the center of $\mathcal{M}_0$ and by $\{\tilde\theta_t\}$ the restriction of $\{\theta_t\}$ to $\mathcal{Z}_0$. Suppose there is a $T>0$ such that $\tilde\theta_T$ is not ergodic on $\mathcal{Z}_0$, that is, the fixed point subalgebra $\mathcal{A}$ of $\mathcal{Z}_0$ under $\tilde\theta_T$ is not reduced to the scalars. Of course, $\mathcal{A}$ is invariant under $\{\tilde\theta_t\}$, and the action $\{\tilde\theta_t\}$ on $\mathcal{A}$ is ergodic and periodic. Hence the action $\{\tilde\theta_t\}$ on $\mathcal{A}$ is transitive, in other words, there exists a $T_0>0$ such that the action $\{\tilde\theta_t\}$ on $\mathcal{A}$ is isomorphic to the canonical action of $\mathbf{R}$ on $L^\infty(\mathbf{R}/T_0\mathbf{Z})$. Hence there exists, by Theorem 10.5, a von Neumann algebra $\mathcal{N}_0$ equipped with an action ϱ of $T_0\mathbf{Z}$ such that

$$\{\mathcal{M}_0,\theta\}=\mathrm{Ind}^{\mathbf{R}}_{\mathbf{T}_0\mathbf{Z}}\{\mathcal{N}_0,\varrho\}.$$

Putting $\varrho=\varrho_{T_0}$, we have an automorphism of $\mathcal{N}_0$ with $\varrho_{nT_0}=\varrho^n$, $n\in\mathbf{Z}$. Since $\mathcal{M}_0\cong L^\infty(\mathbf{R}/T_0\mathbf{Z})\otimes\mathcal{N}_0$, $\mathcal{N}_0$ is also of type II_∞.

We assume, for the moment, that $\mathcal{M}_0$, is separable, that is, the predual of $\mathcal{M}_0$ is separable as a Banach space. In this case, there is no measure-theoretic difficulty in regarding $\mathcal{M}_0$ as the von Neumann algebra of all essentially bounded $\mathcal{N}_0$-valued σ-strongly* measurable functions on the half-open interval $[0, T_0)$. The action θ of $\mathbf{R}$ is given by

$$\theta_s(x)(t)=\begin{cases}\varrho^{n+1}(x(t-r+T_0)), & 0\leqslant t<r;\\ \varrho^n(x(t-r)), & r\leqslant t<T_0,\end{cases}\tag{10.18}$$

for $s=nT_0+r$, $0\leqslant r<T_0$. Let

$$\tau=\int^{\oplus}_{0\leqslant t<T_0}\tau_t\,dt\tag{10.19}$$

be the disintegration of the trace τ with respect to the diagonal algebra $\mathcal{A}=L^\infty(\mathbf{R}/T_0\mathbf{Z})=L^\infty(0,T_0)$. For each positive $x\in\mathcal{M}_0$, we have, for $s=nT_0+r, 0\leqslant r<T_0$,

$$\begin{aligned}e^{-s}\tau(x)=\tau\circ\theta_s(x)&=\int_0^r\tau_t\circ\varrho^{n+1}(x(t-r+T_0))\,dt+\int_r^{T_0}\tau_t\circ\varrho^n(x(t-r))\,dt\\&=\int_0^{T_0-r}\tau_{t+r}\circ\varrho^n(x(t))\,dt+\int_{T_0-r}^{T_0}\tau_{t+r-T_0}\circ\varrho^{n+1}(x(t))\,dt.\end{aligned}$$

Putting $s=nT_0$, we have

$$e^{-nT_0}\tau(x)=\int_0^{T_0}\tau_t\circ\varrho^n(x(t))\,dt,$$

so that we get
$$\tau_t \circ \varrho^n = e^{-nT_0}\tau_t, \quad n \in \mathbf{Z}, \tag{10.20}$$
for almost every $t \in [0, T_0)$. If $0 \leqslant s < T_0$, we get
$$\begin{aligned} e^{-s}\tau(x) &= \int_0^{T_0-s} \tau_{t+s}(x(t))\,dt + \int_{T_0-s}^{T_0} \tau_{t+s-T_0}(\varrho(x(t)))\,dt \\ &= \int_0^{T_0-s} \tau_{s+t}(x(t))\,dt + \int_{T_0-s}^{T_0} e^{-T_0}\tau_{t+s-T_0}(x(t))\,dt. \end{aligned}$$
Therefore, we get
$$\begin{cases} \tau_{s+t} = e^{-s}\tau_t, & 0 \leqslant t < T_0 - s \\ \tau_{s+t-T_0} = e^{T_0-s}\tau_t, & T_0 - s \leqslant t < T_0, \end{cases}$$
for each $s \in [0, T_0)$ and almost every $t \in [0, T_0)$. Hence we have
$$\tau_{s+t} = e^{-s}\tau_t, \quad -t \leqslant s < T_0 - t \tag{10.21}$$
for almost every $t \in [0, T_0)$. By Fubini's theorem for almost every $s \in [-t, T_0 - t)$, (10.21) holds. Hence we conclude that there is a unique faithful semifinite normal trace τ_0 on $\mathcal{N}_0$ such that
$$\tau(x) = \int_0^{T_0} e^{-t}\tau_0(x(t))\,dt, \quad x \in \mathcal{M}_0. \tag{10.22}$$
We have, by (10.20),
$$\tau_0 \circ \varrho = e^{-T_0}\tau_0. \tag{10.23}$$
Therefore, τ_0 is relatively invariant under ϱ. By Theorem 10.3, we have
$$\mathcal{M} = \mathcal{R}(\mathcal{M}_0; \theta) \cong \mathcal{R}(\mathcal{N}_0; \varrho).$$

Now, we drop the separability assumption for $\mathcal{M}_0$. Let $\{\mathcal{M}_i\}_{i\in I}$ be the family of all θ-invariant separable von Neumann subalgebras of $\mathcal{M}_0$ such that $\mathcal{M}_i$ contains $\mathcal{A}$ and the trace τ is semifinite on $\mathcal{M}_i$. Of course, $\{\mathcal{M}_i\}_{i\in I}$ is an increasing net with respect to the inclusion ordering, and $\mathcal{M}_0$ is generated by $\bigcup_{i\in I}\mathcal{M}_i$. By Proposition 10.4, to each $i \in I$, there corresponds uniquely a ϱ-invariant von Neumann subalgebra $\mathcal{N}_i$ of $\mathcal{N}_0$ such that
$$\{\mathcal{M}_i, \theta\} \cong \mathrm{Ind}_{T_0\mathbf{Z}}^{\mathbf{R}}\{\mathcal{N}_i, \varrho\}.$$
From the proof of Proposition 10.4, it follows that $\mathcal{N}_i \subset \mathcal{N}_j$ if and only if $\mathcal{M}_i \subset \mathcal{M}_j$. To each $i \in I$, there corresponds a unique faithful semifinite normal trace τ_i on $\mathcal{N}_i$ such that
$$\begin{cases} \tau(x) = \displaystyle\int_0^{T_0} e^{-t}\tau_i(x(t))\,dt, & x \in \mathcal{M}_i; \\ \tau_i \circ \varrho = e^{-T_0}\tau_i. \end{cases}$$

By the unicity of τ_i, τ_i is the restriction of τ_j if $\mathcal{N}_i \subset \mathcal{N}_j$. Hence there exists a unique faithful semifinite normal trace τ_0 on $\mathcal{N}_0$ such that $\tau_0|_{\mathcal{N}_i} = \tau_i$, $i \in I$. Hence we have

$$\begin{cases} \tau(x) = \int_0^{T_0} e^{-t}\tau_0(x(t))\,dt; \\ \tau_0 \circ \varrho = e^{-T_0}\tau_0. \end{cases}$$

Thus we have reached the following conclusion:

THEOREM 10.6. *Let $\mathcal{M}_0$ be a von Neumann algebra of type II_∞ equipped with a continuous one parameter automorphism group $\{\theta_t\}$ and a faithful semifinite normal trace τ such that $\tau \circ \theta_t = e^{-t}\tau$. Let $\mathcal{M} = \mathcal{R}(\mathcal{M}_0; \theta)$. If the restriction $\{\tilde{\theta}_t\}$ of $\{\theta_t\}$ to the center $\mathcal{Z}_0$ of $\mathcal{M}_0$ is ergodic, but $\tilde{\theta}_T$ is not ergodic for some $T > 0$, then there exists a von Neumann algebra $\mathcal{N}_0$ of type II_∞ equipped with an automorphism ϱ and a faithful semifinite normal trace τ_0 such that*

$$\begin{cases} \tau_0 \circ \varrho = e^{-T_0}\tau_0 \quad \textit{for some} \quad T_0 > 0; \\ \mathcal{M} \cong \mathcal{R}(\mathcal{N}_0; \varrho). \end{cases}$$

In particular, if $\mathcal{M}$ is a factor of type III_λ, $0 < \lambda < 1$, then there exists a factor $\mathcal{N}_0$ of type II_∞ equipped with an automorphism ϱ and a faithful semifinite normal trace τ_0 such that

$$\begin{cases} \mathcal{M} = \mathcal{R}(\mathcal{N}_0; \varrho); \\ \tau_0 \circ \varrho = e^{-T}\tau_0, \end{cases}$$

where $T = -\log\lambda$.

Proof. The first half of the theorem has been already proven. By Theorem 9.6, $\tilde{\theta}_T$ is the identity automorphism ι. Hence $\{\tilde{\theta}_t\}$ is periodic and ergodic on $\mathcal{Z}_0$ with period T. Hence $\{\tilde{\theta}_t\}$ must be transitive, that is, $\mathcal{Z}_0 \cong L^\infty(\mathbf{R}/T\mathbf{Z})$ and the action $\tilde{\theta}$ of $\mathbf{R}$ on $\mathcal{Z}_0$ is isomorphic to the canonical action on $L^\infty(\mathbf{R}/T\mathbf{Z})$. Therefore, there exists a von Neumann algebra $\mathcal{N}_0$ of type II_∞ equipped with an automorphism ϱ and a faithful semifinite normal trace τ_0 such that

$$\tau_0\varrho = e^{-T}\tau_0, \quad \{\mathcal{M}_0, \theta\} = \mathrm{Ind}_{T\mathbf{Z}}^{\mathbf{R}}\{\mathcal{N}_0, \varrho\}.$$

Hence we have

$$\mathcal{M} \cong \mathcal{R}(\mathcal{N}_0; \varrho).$$

Since $\mathcal{M}_0 \cong \mathcal{A} \otimes \mathcal{N}_0$, and $\mathcal{A} = \mathcal{Z}_0$ in this case, $\mathcal{N}_0$ must be a factor. Q.E.D.

It is now easy to prove Theorem 8.12. Suppose statement (ii) in Theorem 8.12 holds. By Theorem 8.5, $\mathcal{M}$ is a factor. Suppose that $\mathcal{M}$ is not of type III. Then $\mathcal{M}$ is semifinite, so it admits a faithful semifinite normal trace φ. The modular automorphism group $\{\sigma_t^\varphi\}$ of $\mathcal{M}$ associated with φ is the trivial group. Hence we have $\mathcal{R}(\mathcal{M}; \sigma^\varphi) \cong \mathcal{M} \otimes L^\infty(\mathbf{R})$. However, $\{\mathcal{M}_0, \theta\}$ is weakly equivalent to $\{\mathcal{M} \otimes L^\infty(\mathbf{R}), \iota \otimes \bar{\theta}\}$, where $\{\bar{\theta}_t\}$ is the canonical action, (translation), of $\mathbf{R}$ on $L^\infty(\mathbf{R})$. Hence $\{\mathcal{Z}_0, \tilde{\theta}\}$ is isomorphic to $\{L^\infty(\mathbf{R}), \tilde{\theta}\}$. But this is excluded by the assumption.

Conversely, suppose that $\{\mathcal{Z}_0, \tilde{\theta}\}$ is isomorphic to $\{L^\infty(\mathbf{R}), \theta\}$. By Theorem 10.5, the covariant system $\{\mathcal{M}_0, \theta\}$ is induced from another covariant system $\{\mathcal{N}_0, \varrho\}$. But in this case, being an action of the trivial group $\{0\}$, ϱ must be the identity automorphism ι. Hence $\{\mathcal{M}_0, \theta\}$ is isomorphic to $\{\mathcal{N}_0 \otimes L^\infty(\mathbf{R}), \iota\otimes\tilde{\theta}\}$. Hence we have

$$\mathcal{M} \cong \mathcal{N}_0 \otimes \mathcal{L}(L^2(\mathbf{R})) \cong \mathcal{N}_0.$$

Thus $\mathcal{M}$ is semifinite.

We close this section with discussion of the relation between Theorem 10.6 and the structure theorem in the previous paper, [45]. In [45], we showed that a von Neumann algebra $\mathcal{M}$ equipped with a homogeneous periodic state φ may be written as the crossed product of the centralzier $\mathcal{M}_0^\varphi$ of φ by an endomorphism θ which is an isomorphism of $\mathcal{M}_0^\varphi$ onto the reduced algebra $e\mathcal{M}_0^\varphi e$. Let T be the period of φ. Then this θ is given by an isometry u in $\mathcal{M}$ with $\sigma_t^\varphi(u) = e^{2\pi i t/T}u$ in such a way that $\theta(x) = uxu^*$, $x \in \mathcal{M}_0^\varphi$ and $e = uu^*$. Let $\bar{\mathcal{M}}$ be the tensor product $\mathcal{M} \otimes \mathcal{L}(\mathfrak{H}_\infty)$ of $\mathcal{M}$ and the factor of type I_∞, and $\bar{\varphi} = \varphi \otimes \mathrm{Tr}$. Then $\bar{\varphi}$ is a faithful semifinite normal weight on $\bar{\mathcal{M}}$. Since $e\otimes 1$ is a properly infinite projection of the II_∞-von Neumann algebra $\mathcal{M}_0^\varphi \otimes \mathcal{L}(\mathfrak{H}_\infty) = \bar{\mathcal{M}}_0$, there exists a co-isometry w in $\bar{\mathcal{M}}_0$ such that $w^*w = e\otimes 1$. Put $\bar{u} = w(u\otimes 1)$. We get a unitary $\bar{u}$ in $\bar{\mathcal{M}}$ such that $\sigma_t^{\bar{\varphi}}(\bar{u}) = e^{2\pi i t/T}\bar{u}$ and $\bar{u}\bar{\mathcal{M}}_0\bar{u}^* = \bar{\mathcal{M}}_0$. Putting $\bar{\theta}(x) = \bar{u}x\bar{u}^*$, we obtain an automorphism $\bar{\theta}$ of $\bar{\mathcal{M}}_0$. It is now straightforward that $\bar{\mathcal{M}} \cong \mathcal{R}(\bar{\mathcal{M}}_0; \bar{\theta})$. Since $\mathcal{M}$ is of type III, we have $\mathcal{M} \cong \mathcal{R}(\bar{\mathcal{M}}_0; \bar{\theta})$. Hence $\mathcal{M}$ is the crossed product of a II_∞-von Neumann algebra $\bar{\mathcal{M}}_0$ by an automorphism $\bar{\theta}$. Since θ transforms the restriction φ_0 of φ to $\mathcal{M}_0^\varphi$, (which is a faithful normal trace), in such a way that $\varphi_0 \circ \theta = \lambda\varphi_0$ with $\lambda = e^{-2\pi/T}$, we have $\bar{\varphi}_0 \circ \bar{\theta} = \lambda\bar{\varphi}_0$, where $\bar{\varphi}_0$ is the restriction of $\bar{\varphi}$ to $\bar{\mathcal{M}}_0$, because $\bar{\varphi}_0(wxw^*) = \bar{\varphi}_0(x)$, $x \in \bar{\mathcal{M}}_0$. Therefore, the structure theorem in [45] coincides essentially with Theorem 10.6 in this case.

11. Example

By the Araki-Woods classification theory, [3] and [10], a type III_1 factor $\mathcal{M}$ which is an infinite tensor product of finite factors of type I is unique. In this section, we shall examine this factor $\mathcal{M}$.

Let $\{\mathcal{M}_n\}$ be an increasing sequence of subfactors of type I_{m_n} such that $\mathcal{M}$ is generated by $\bigcup_{n=1}^\infty \mathcal{M}_n$. Let φ be a faithful normal state whose modular automorphism $\{\sigma_t^\varphi\}$ leaves each $\mathcal{M}_n$ invariant. Let $\mathcal{M}_0$ be the crossed product $\mathcal{R}(\mathcal{M}; \sigma^\varphi)$. Since $\mathcal{M}$ is of type III_1, we know, by Corollary 9.7, that $\mathcal{M}_0$ is a factor of type II_∞. We denote by $v(t)$ the regular representation $\lambda(t)$ of $\mathbf{R}$ which together with the canonical image of $\mathcal{M}$ generates $\mathcal{M}_0$.

We identify $\mathcal{M}$ with the canonical image of $\mathcal{M}$ in $\mathcal{M}_0$. Hence we have $\sigma_t^\varphi(x) = v(t)xv(t)^*$, $x \in \mathcal{M}$, $t \in \mathbf{R}$. Let $\bar{\mathcal{M}}_n$ be the von Neumann subalgebra of $\mathcal{M}_0$ generated by $\mathcal{M}_n$ and $\{v(t)\}$. It is obvious that $\bar{\mathcal{M}}_n = \mathcal{R}(\mathcal{M}_n; \sigma^\varphi)$, and $\mathcal{M}_0$ is generated by $\bigcup_{n=1}^\infty \bar{\mathcal{M}}_n$. Let h_n be the Radon-Nikodym derivative of $\varphi|_{\mathcal{M}_n}$ with respect to the trace of $\mathcal{M}_n$. Then we have

$$\sigma_t^\varphi(x) = h_n^{it} x h_n^{-it}, \quad x \in \mathcal{M}_n.$$

Hence $\bar{\mathcal{M}}_n \cong \mathcal{M}_n \otimes L^\infty(\mathbf{R})$, so that each $\bar{\mathcal{M}}_n$ is homogeneous von Neumann algebra of type I_{m_n}. Let $\mathcal{A}$ denote the abelian von Neumann subalgebra generated by $\{v(t)\}$, which is isomorphic to $L^\infty(\mathbf{R})$. Let $\tilde{\varphi}$ be the weight on $\mathcal{M}_0$ dual to φ.

LEMMA 11.1. *The restriction $\tilde{\varphi}|_{\mathcal{A}}$ of $\tilde{\varphi}$ to $\mathcal{A}$ is semifinite.*

Proof. Since the Tomita algebra $\mathfrak{A}_0$ based on φ contains an identity, say ξ_0, the Tomita algebra $\mathcal{K}(\mathfrak{A}_0; \mathbf{R})$ constructed in § 5 contains the convolution algebra $\mathcal{K}(\mathbf{R})$. For each $\xi \in \mathcal{K}(\mathbf{R})$, we have $\pi_l(\xi) = \int_{\mathbf{R}} \xi(t) v(t) dt$, so that $\{\pi_l(\xi) : \xi \in \mathcal{K}(\mathbf{R})\}$ generates $\mathcal{A}$; hence $\tilde{\varphi}|_{\mathcal{A}}$ is semifinite. Q.E.D.

Identifying $\mathcal{A}$ with $L^\infty(\mathbf{R})$, we see by construction that the dual action $\{\theta_t\}$ on $\mathcal{A}$ nothing but translation. Since $\tilde{\varphi}|_{\mathcal{A}}$ is semifinite and θ-invariant, $\tilde{\varphi}|_{\mathcal{A}}$ is nothing but integration with respect to Lebesgue measure. Namely, we have

$$\tilde{\varphi}(f) = \int_{-\infty}^{\infty} f(s)\, ds, \quad f \in L^\infty(\mathbf{R}), f \geq 0.$$

The unitary group $\{v(s)\}$ is given by

$$v(t)(s) = e^{ist}, \quad s, t \in \mathbf{R}.$$

The action θ of $\mathbf{R}$ turns out to be:

$$\theta_t(f)(s) = f(s-t), \quad f \in L^\infty(\mathbf{R}).$$

Define

$$h(s) = e^s, \quad s \in \mathbf{R}.$$

Then we have $v(t) = h^{it}$. Hence the restriction $\tau|_{\mathcal{A}}$ of the trace on $\mathcal{M}_0$ onto $\mathcal{A}$ is given by

$$\tau(f) = \int_{-\infty}^{\infty} e^{-s} f(s)\, ds, \quad f \in L^\infty(\mathbf{R}).$$

Hence the projection p in $\mathcal{A}$ given by the characteristic function $[0, \infty)$ is finite because

$$\tau(p) = \int_0^\infty e^{-s}\, ds = 1.$$

The projection $\theta_t(p)$ corresponds to the interval $[t, \infty)$. Hence $\theta_t(p) \leqslant p$ for $t \geqslant 0$. Since $\mathcal{A}$ is contained in $\bar{\mathfrak{M}}_n$, $n=1, 2, \ldots$ and p belongs to every $\bar{\mathfrak{M}}_n$. Therefore, the reduced factor $p\mathfrak{M}_0 p$ of type II_1 is generated by the union $\bigcup_{n=1}^{\infty} p\bar{\mathfrak{M}}_n p$ of reduced finite type I von Neumann algebras. Since each $p\bar{\mathfrak{M}}_n p$ is approximated by finite dimensional subalgebras, $p\mathfrak{M}_0 p$ is approximated by finite dimensional subalgebras. Hence by Murray–von Neumann's Theorem [26; Theorem X II] $p\mathfrak{M}_0 p$ is a hyperfinite factor of type II_1. Thus we have obtained the following conclusion by the unicity of a hyperfinite factor of type II_1.

THEOREM 11.2. *Let $\mathfrak{F}$ be a hyperfinite factor of type II_1. There exists a decreasing one parameter family $\{p_t\}_{t\geqslant 0}$ of projections in $\mathfrak{F}$ and a continuous one parameter semigroup $\{\theta_t; t\to 0\}$ of endomorphisms such that $\theta_t(p_s)=p_{s+t}$, $p_0=1$, and $\tau(p_s)=e^{-s}$, $s, t \in \mathbf{R}_+$, where τ means, of course, the canonical trace on $\mathfrak{F}$.*

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