between such functions as u and $\tilde{u}$ (the representative function agreeing with u on μ_δ and approaching u in the norm) the proof runs as follows:

From (23), there is, for any $\epsilon > 0$, a δ_0 such that, for $\delta < \delta_0$, $\|w - S_\delta^{-1}u\| < \epsilon$ where we write $Au = w$, $Sw = u$. On multiplying $w' = (w - \tilde{S}_\delta^{-1}u)$ by $\tilde{S}_\delta$ one has, from (24) $\|\tilde{S}_\delta w - u\| = \|S_\delta w'\| \leqq M'\|w'\| < M'\epsilon$ which gives (25) and the scheme is convergent.

As in Part I, the converse follows from the Principle of Uniform Boundedness.

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DUALITY IN HOMOGENEOUS PROGRAMMING[1]

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The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an $m \times n$ matrix A and vectors $a \in R^n$, $b \in R^m$.[2] The feasibility sets X and Y are defined by: $X = R_+^m \cap \{x \mid xA \leqq a\}$, $Y = R_+^n \cap \{y \mid Ay \geqq b\}$. Since $xA \leqq a$ if and only if $xAy \leqq ay$ for all $y \in R_+^n$ (and similarly for $Ay \geqq b$), we may write:

$$(1)\qquad \begin{aligned} X &= R_+^m \cap \{x \mid xAy \leqq \psi(y) \quad \text{all } y \in R_+^n\} \\ Y &= R_+^n \cap \{y \mid xAy \geqq \phi(x) \quad \text{all } x \in R_+^m\} \end{aligned}$$

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[2] R^m denotes the set of all real m-tuples. If $u, v \in R^m$ then $u \leqq v$ means that the inequality holds for each component. In particular, $R_+^m = R^m \cap \{x \mid x \geqq 0\}$. If M is a $p \times q$ matrix and N is a $q \times t$ matrix then MN represents the usual matrix product. To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.

where $\psi(y)=ay$ and $\phi(x)=bx$.

A fundamental theorem of linear programming (see, e.g., [**3**; **2**]) states that if X and Y are both nonempty then

$$\text{(2)} \qquad \max_{x\in X}\phi(x),\ \min_{y\in Y}\psi(y) \text{ exist and are equal.}$$

We propose to demonstrate that (2) holds for another class of triples (A, ϕ, ψ).

ASSUMPTION A_1. Let $\phi: R_+^m\to R$, $\psi: R_+^n\to R$ be positively homogeneous,[3] continuous, concave and convex respectively.

Let us first show that A_1 does not guarantee that (2) holds when X and Y are nonempty. If $m=2$, $n=1$ and

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$\phi(x)=\phi(\xi,\ \eta)=\xi\eta/(\xi+\eta)$ $(\phi(0)=0)$, $\psi(y)=y$ then A_1 is satisfied and $X=R_+^2\cap\{(\xi,\ \eta)\mid\eta\leqq 1\}$, $Y=R_+\cap\{y\mid y\geqq 1\}$ are nonempty. Thus $\min_{y\in Y}\psi(y)=1$, but if $\eta\leqq 1$ then $\phi(\xi,\eta)<1$, although $\sup_{x\in X}\phi(x)=1$, hence $\max_{x\in X}\phi(x)$ does not exist.

The situation just illustrated cannot occur if the following holds:

ASSUMPTION A_2.

(i) If $x\in R_+^m$, $xA\leqq 0$, $\phi(x)\geqq 0$ then $x=0$.

(ii) If $x\in R_+^n$, $Ay\geqq 0$, $\psi(y)\leqq 0$ then $y=0$.

One sees immediately that (i) is violated in the preceding example, for let $x=(1,\ 0)$ then $xA=0$ and $\phi(x)=0$.

Before proving our main result, that if A_1 and A_2 hold then so does (2), we require the following lemma which specializes to homogeneous functions the well-known fact that a concave function is the infimum of its supports. The proof is presented here for the sake of completeness.

LEMMA. *Let ϕ be as in assumption A_1, consider*

$$T = R^m \cap \{t \mid tx \geqq \phi(x)\ \text{all}\ x \in R_+^m\},$$

then T is nonempty, and $\phi(x)=\inf_{t\in T} tx$, for all $x\in R_+^m$.

PROOF. Let $C=\{(x,\lambda)\mid x\in R_+^m,\ \lambda\leqq\phi(x)\}$ then C is a closed convex cone. Now if $x_0\in R_+^m$, $\epsilon>0$, then $(x_0,\ \epsilon+\phi(x_0))\notin C$, whence (see [**2**, Theorem 1]) there exist $t\in R^m$ and $\alpha\in R$ such that $tx_0-\alpha[\epsilon+\phi(x_0)]<0\leqq tx-\alpha\lambda$ all $(x,\ \lambda)\in C$.

[3] A function f: $C\to R^q$, where $C\subset R^p$ is a cone, is positively homogeneous providing $f(\lambda x)=\lambda f(x)$ for all $x\in C$ and $\lambda\in R_+$.

It then follows that $\alpha>0$, so that (dividing by α) we may assume $\alpha=1$, but then $t\in T$. Reiterating, if $x_0\in R_+^m$, $\epsilon>0$ then $\exists t\in T$ such that:

$$tx_0-\epsilon\leqq\phi(x_0)\leqq tx_0$$

giving the desired result. We are now able to prove:

THEOREM 1. *If assumptions* A_1 *and* A_2 *hold then* (2) *holds.*

PROOF. Let

$$S=R^n\cap\{s\mid sy\leqq\psi(y)\quad\text{all } y\in R_+^n\},$$
$$T=R^m\cap\{t\mid tx\geqq\phi(x)\quad\text{all } x\in R_+^m\}. \tag{3}$$

Then S and T are convex sets; now consider the system of inequalities:

$$\begin{gathered} x\in R_+^m,\ y\in R_+^n,\ s\in S,\ t\in T,\\ s-xA>0,\\ -t+Ay>0,\\ \phi(x)-\psi(y)>0. \end{gathered} \tag{4}$$

If (4) has a solution x, y, s, t then

$$\psi(y)<\phi(x)\leqq tx\leqq xAy\leqq sy\leqq\psi(y)$$

which is a contradiction. Thus (see [1, Theorem 1]) there exist $x_0\in R_+^m$, $y_0\in R_+^n$, $\lambda\in R_+$, not all zero and such that $(s-xA)y_0+x_0(Ay-t)+\lambda[\phi(x)-\psi(y)]\leqq 0$ for all $x\in R_+^m$, $y\in R_+^n$, $s\in S$, $t\in T$. From the homogeneity and continuity of ϕ and ψ it then follows that:

$$\begin{aligned} xAy_0&\geqq\lambda\phi(x) &&\text{all } x\in R_+^m,\\ x_0Ay&\leqq\lambda\psi(y) &&\text{all } y\in R_+^n,\\ sy_0&\leqq tx_0 &&\text{all } s\in S, t\in T. \end{aligned} \tag{5}$$

The last condition together with our lemma imply:

$$\psi(y_0)\leqq\phi(x_0).$$

Now if $\lambda=0$ then either $x_0\neq 0$ or $y_0\neq 0$ and $Ay_0\geqq 0$, $x_0A\leqq 0$. Suppose $x_0\neq 0$, then by A_2 (i) we have $\phi(x_0)<0$, whence $\psi(y_0)<0$ and $y_0\neq 0$, contradicting A_2 (ii). Thus $\lambda>0$ and, dividing all inequalities by λ, we may assume $\lambda=1$. This tells us that $x_0\in X$, $y_0\in Y$ and $\phi(x_0)\leqq x_0Ay_0\leqq\psi(y_0)\leqq\phi(x_0)$. So that if $x\in X$, $y\in Y$ then

$$\phi(x) \leqq xAy_0 \leqq \psi(y_0) = \phi(x_0),$$
$$\psi(y) \geqq x_0Ay \geqq \phi(x_0) = \psi(y_0)$$

proving the theorem.

In case ϕ and ψ are linear-homogeneous then it is true that $\max_{x\in X} \phi(x)$ exists if and only if $\min_{y\in Y} \psi(y)$ exists, in which case they are equal. As above, this statement is not true under assumption A_1; however, we show:

THEOREM 2. (I) *If* A_1 *and* A_2 (ii) *hold and* $\max_{x\in X} \phi(x)$ *exists then* $\min_{y\in Y} \psi(y)$ *exists and the two are equal.*

(II) *If* A_1 *and* A_2 (i) *hold and* $\min_{y\in Y} \psi(y)$ *exists then* $\max_{x\in X} \phi(x)$ *exists and the two are equal.*

We prove (I), the proof of (II) is similar. Suppose that $x_0 \in X$ and $\phi(x_0) = \max_{x\in X} \phi(x)$ then the system:

$$\begin{aligned} x \in R_+^m,\ s \in S \\ s - xA > 0, \\ \phi(x) - \phi(x_0) > 0, \end{aligned} \tag{6}$$

has no solution. Thus (see [1, Theorem 1]) there exist $y_0 \in R_+^n$, $\lambda \in R_+$, not both zero and such that

$$sy_0 - xAy_0 + \lambda[\phi(x) - \phi(x_0)] \leqq 0 \qquad \text{for all } x \in R_+^m,\ s \in S.$$

From the homogeneity of ϕ and our lemma it then follows that

$$\begin{aligned} xAy_0 &\geqq \lambda\phi(x) \qquad \text{for all } x \in R_+^m, \\ \psi(y_0) &\leqq \lambda\phi(x_0). \end{aligned} \tag{7}$$

Now if $\lambda = 0$ then $y_0 \neq 0$ and $Ay_0 \geqq 0$, $\psi(y_0) \leqq 0$ contradicting A_2(ii). It may then be assumed that $\lambda > 0$ and, in fact, that $\lambda = 1$ (replacing y_0 by λy_0). Thus, from (7), $y_0 \in Y$ and for any $y \in Y$ we have:

$$\psi(y_0) \leqq \phi(x_0) \leqq x_0Ay \leqq \psi(y),$$

i.e.,

$$\psi(y_0) = \min_{y\in Y} \psi(y) = \phi(x_0).$$

It should be remarked that if A_1 holds then (i) and (ii) of assumption A_2 are equivalent to (i)′ and (ii)′ respectively of:

ASSUMPTION A_2'.

(i)′ $\exists y_0 \in R_+^n \ni xAy_0 > \phi(x)$ all $x \in R_+^m$, $x \neq 0$.

(ii)′ $\exists x_0 \in R_+^m \ni x_0Ay < \psi(y)$ all $y \in R_+^n$, $y \neq 0_s$.

These in turn are equivalent to the familiar conditions that X, Y have nonempty interiors. To see, for instance, that (i) and (i)′ are equivalent it suffices to show that (i) implies (i)′ since the implication in the other direction is trivial. Assuming (i)′ false, the system

$$y \in R_+^n,\ t \in T, \qquad Ay - t > 0 \tag{8}$$

has no solution, whence (see [1, Theorem 1]) there is an $x \in R_+^m$, $x \neq 0$, and such that $xAy \leqq tx$ for all $y \in R_+^n$ and $t \in T$. Thus $xA \leqq 0$ and (using our lemma) $\phi(x) \geqq 0$, contradicting (i). To return to our remark about maximizing a concave homogeneous and continuous function $\phi: R_+^m \rightarrow R$, subject to the inequalities $x \geqq 0$ and $xA \leqq a$, the dual is then: minimize ay subject to $y \in Y$. Conditions (i) and (ii)′ become:

$$x \in R_+^m,\ x \neq 0,\ xA \leqq 0,\ \phi(x) \geqq 0 \text{ has no solution; and}$$
$$x \in R_+^m,\ xA < a \text{ has a solution; respectively.}$$

Also, since $y \in Y$ providing $y \geqq 0$ and $Ay \geqq t$ for some support t of ϕ, we may characterize Y by means of the gradient of ϕ.

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