TEL-AVIV

WP 7-85

THE FOERDER INSTITUTE FOR ECONOMIC RESEARCH

TEL-AVIV UNIVERSITY

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DUALITY IN NON-ADDITIVE EXPECTED UTILITY THEORY

By

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Working Paper No.7-85

April, 1985

Financial support from the Josef Buchmann Fund and typing facilities of the Foerder Institute for Economic Research are gratefully acknowledged.

I wish to thank Professor David Schmeidler for his guidance.

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ABSTRACT

Some duality problems in expected utility theory, raised by the introduction of non-additive probabilities, are examined. Characterization of the probability measures, for which these problems do not arise, leads to an argument in favor of addivity.

INTRODUCTION

For various reasons and in different contexts, expected utility theory was recently extended in some ways to include non-additive probability measures. This was done explicitly by Schmeidler (1982, 1984a and 1984b) and Gilboa (1985), and implicitly by Quiggin (1982), Yaari (1984) and others.

Although all these models are generalizations of the traditional ones (with Yaari's being an exception), and the conclusions of the former resemble those of the latter, non-additive probabilities pose a few new questions and allow some intriguing anomalies.

One of those is the fact that utility maximizing and disutility minimizing are not necessarily the same phenomenon. That is to say, an individual having a utility u and subjective probability measure v who maximizes $\int u dv$ may behave differently from another individual, with the very same u and v, who minimizes $\int (-u) dv$.

Another problem is the definition of the integration operation itself. In all the models under discussion, it is defined in one way, but it may also be defined in a symmetric way, inducing a different preference order. Similarly, every set of axioms of the above models has a dual set of axioms, equally reasonable, which leads to the dual-integral theory rather than the original one. It is not clear why, if at all, the latter should be preferred to the former.

Yet another problem is that any non-additive measure has a natural "dual" measure, and the integral-defined preference order again fails to exhibit invariance with respect to these two measures, while we have no convincing reason to prefer one of them to the other.

This paper studies these problems. In Section 2 their scope is examined, and it is shown that only two (rather than eight) different preference relations are induced by the pairs of utilities, measures and integrals. Section 3 is devoted to the characterization of preference relations which are immune to the discussed anomalies. Section 4 uses this characterization to raise an argument in favor of additivity.

1. PRELIMINARIES

The framework we will use is the following:

S – a nonempty set of states of nature, subsets of which are called events.

X – a nonempty set of consequences.

F $= \{f: S \to X\}$ – the set of acts.

$\geq \subset F \times F$ – a preference order over the acts.

A real function over X is called a utility.

A set function $v : 2^S \to [0,1]$ which satisfies:

(i) $E \subset F \Rightarrow v(E) \leq v(F)$

(ii) $v(\phi) = 0$; $v(S) = 1$

is called a measure.

For any measure v, the dual measure $\bar{v}$ is defined by $\bar{v}(A) = 1-v(A^c)$ for $A \subset S$. It is easily seen that $\bar{v}$ is indeed a measure on S and that $\bar{\bar{v}} = v$. A measure v is called symmetric iff $v = \bar{v}$, i.e. $v(A) + v(A^c) = 1$ for all $A \subset S$.

The integration operation that is usually used is the Choquet integral, defined in Choquet (1955) and further discussed in schmeidler (1984b). Here it will be called the upper Choquet integral, and it is defined as follows: Let w be a real function over S, and v a measure on S. The upper integral of w w.r.t. (with respect to) v is

$$\int^{*} w dv = \int_{0}^{\infty} v(w \geq t)dt - \int_{-\infty}^{0} [1 - v(w \geq t)]dt$$

(The integrals on the right hand side are Riemann's). Similarly, the lower (Choquet) integral of w w.r.t. v is

$$\int_{*} w dv = \int_{0}^{\infty} [1-v(w \leq t)]dt - \int_{-\infty}^{0} v(w \leq t)dt$$

A measure v is said to be locally convex valued iff for all $A \subset B \subset S$ and $\alpha \in [0,1]$ there is an event C such that $A \subset C \subset B$ and

$$v(C) = \alpha v(A) + (1-\alpha)v(B).$$

In this context, Gilboa (1985) provides an axiomatization of $\geq$ for which there are a bounded utility u and a locally convex valued measure v such that

$$f \geq g \iff \int^{*} u(f)dv \geq \int^{*} u(g)dv \quad \forall \; f,g \in F.$$

This axiomatization also assures the uniqueness of v and of u (up to a positive linear transformation). For brevity's sake we will not repeat the axioms here. However, when the need arises we will refer to them by their original names which are (for historical reasons) P1, P2*, P3*, P5*, P6*, P6**, P7*. (P_i^* is a variant of Savage's P_i. [See Savage (1954)].)

2. The equivalence of dualities

We need a few lemmas:

Lemma 1: Let w and v be a real function and a measure on S, respectively. Then

$$\int^* w d\bar{v} = \int_* w dv.$$

Proof: Since v is monotone, $v(w \geq t)$ is a monotone function of t. Therefore

$\{t \,/\, v(w > t) < v(w \geq t)\}$ is countable, whence the inequalities in the definitions of the Choquet integrals may be strict:

$$\int^* w d\bar{v} = \int_0^\infty \bar{v}(w > t)dt - \int_{-\infty}^0 [1 - \bar{v}(w > t)]dt =$$

$$= \int_0^\infty [1 - v(w \leq t)]dt - \int_{-\infty}^0 v(w \leq t)dt = \int_* w dv. //$$

Lemma 2: Let $w : S \to [0,1]$ and let v be a measure on S. Then $\int^* w dv = 1 - \int_*(1-w)dv$.

Proof: All that is needed is the following calculation:

$$\int^* w dv = \int_0^1 v(w \geq t)dt = \int_0^1 v(w \geq 1-t)dt =$$

$$= 1 - \int_0^1 [1 - v(1 - w \leq t)]dt = 1 - \int_*(1 - w)dv.$$

Now let there be given a bounded utility u and a measure v on S. Suppose w.l.o.g. (without loss of generality) that $\sup u = 1$ and $\inf u = 0$. We define eight preference orders on F as follows:

$v \setminus \int$	$\int^*$	$\int_*$
v	$\geq_1$	$\geq_2$
$\bar{v}$	$\geq_4$	$\geq_3$

u maximizers

$v \setminus \int$	$\int^*$	$\int_*$
v	$\geq_6$	$\geq_5$
$\bar{v}$	$\geq_7$	$\geq_8$

(-u) minimizers

(e.g.: $f \geq_1 g \iff \int^* u(f)dv \geq \int^* u(g)dv$;

$f \geq_8 g \iff \int_* -u(f)d\bar{v} \leq \int_* -u(g)d\bar{v}$ etc.)

<u>Theorem 1</u>: In the above conditions, all the preference orders with odd indices are the same, and so are all those with even indices.

<u>Proof</u>: Lemma 1 proves that $\geq_1 = \geq_3$; $\geq_2 = \geq_4$; $\geq_6 = \geq_8$; $\geq_5 = \geq_7$.

By Lemma 2,

$$\int^* u(f)dv \geq \int^* u(g)dv \iff \int_*(1-u(f))dv \leq \int_*(1-u(g))dv$$

– whence $\geq_1 = \geq_5$ and similarly $\geq_2 = \geq_6$, and the proof is complete.//

3. Characterization of symmetric measures

The previous section leads to

Lemma 3: Let there be given a bounded utility u and a locally convex valued measure v on S.

The following statements are equivalent:

(i) v is symmetric;

(ii) the maximization of $\int^* u dv$ and the minimization of $\int^*(-u)dv$ are equivalent;

(iii) the maximization of $\int^* u dv$ is equivalent to that of $\int_* u dv$.

Proof: (ii) means that for all $f, g \in F$,

$$\int^* u(f)dv \geq \int^* u(g)dv \iff \int^*(-u)(f)dv \leq \int^*(-u)(g)dv,$$

or $\geq_1 = \geq_6$. Similarly, (iii) means $\geq_1 = \geq_2$, whence (ii) and (iii) are equivalent.

Now, if (i) holds, i.e. $\bar{v} = v$, then $\geq_1 = \geq_4$, and (ii) follows. Conversely, if (ii) holds, and consequently $\geq_1 = \geq_4$, v must equal $\bar{v}$ by the uniqueness of the measure under these conditions.//

This lemma shows that a preference order which is invariant w.r.t. one of the dualities is invariant w.r.t. the other two as well. However, we would like to have a characterization of these preference orders in terms of the primitives of the model. To this end we must introduce some new definitions and notations.

For $f \in F$, $x \in X$, $A \subset S$, $f / {x \atop A}$ is the act $h \in F$ satisfying $h(s) = f(s)$ $s \in A^c$; $h(s) = x$ $s \in A$. $f, g \in F$ are <u>comonotonic</u> iff there do not exist $s, t \in S$ such that $f(s) > f(t)$ and $g(s) < g(t)$. For $A \subset B \subset S$ and $C \subset D \subset S$ we write $(A,B) \geq' (C,D)$ iff the following is true:

There are $f, g \in F$, $x, y \in X$, $y > x$, such that

(i) $f / {x \atop B-A}$ and $f / {y \atop B-A}$ are comonotonic, and so are $g / {x \atop D-C}$ and $g / {y \atop D-C}$;

(ii) $\{s \; / \; f / {x \atop B-A}(s) \geq y\} = A$; $\{s \; / \; g / {x \atop D-C}(s) \geq y\} = C$;

(iii) $f / {x \atop B-A} \sim g / {x \atop D-C}$ but $f / {y \atop B-A} \geq g / {y \atop D-C}$.

The meaning of $(A,B) \geq' (C,D)$ is that the transition from A to B is "bigger" (or more weighty) than that from C to D. Axiom P2* assures that $\geq$ is a weak order on $\{(A,B) \; / \; A \subset B \subset S\}$.

We may now formulate

<u>Lemma 4</u>: Let $\geq$ satisfy P1-P7*, and let u and v be the utility and measure attached to it, respectively. Then v is symmetric iff for all $A \subset B$, $C \subset D$.

$(*)$ $(A,B) \geq' (C,D) \iff (B^c, A^c) \geq' (D^c, C^c)$.

<u>Proof</u>: We observe, first of all, that, given an integral representation of $\geq$ on F, $(A,B) \geq' (C,D)$ iff

$$v(B) - v(A) \geq v(D) - v(C).$$

Similarly, $(B^c, A^c) \geq' (D^c, C^c)$ iff

$v(A^c) - v(B^c) \geq v(C^c) - v(D^c)$ or $\bar{v}(B) - \bar{v}(A) \geq \bar{v}(D) - \bar{v}(C)$.

Hence it is evident that if $v = \bar{v}$, (*) holds. To prove the converse, assume (*) to be true. Taking $A = C = \phi$, one gets $v(B) \geq v(D)$ iff $\bar{v}(B) \geq \bar{v}(D)$, whence there is a strictly increasing $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$, $\psi(1) = 1$, such that $\bar{v}(\cdot) = \psi(v(\cdot))$. Furthermore, since v is locally convex valued, for any B with $V(B) > 0$ there is an event $A \subset B$ satisfying $v(A)=1/2v(B)$. Letting $D = A$, $C = \phi$, we have $\bar{v}(A) = 1/2\ \bar{v}(B)$, whence ψ is additive over the dyadic rationals and consequently $\bar{v} = v$.//

The two preceding lemmas are summarized in

<u>Theorem 2</u>: Suppose $\geq$ satisfies P1–P7*, and u and v are the associated utility and measure, respectively. Then the following statements are equivalent:

(i) for all $A \subset B$, $C \subset D$, $(A,B) \geq' (C,D) \iff (B^c,A^c) \geq' (D^c,C^c)$;

(ii) v is symmetric;

(iii) the maximization of $\int^* u\,dv$ is equivalent to that of $\int_* u\,dv$;

(iv) the maximization of $\int^* u\,dv$ is equivalent to the minimization of $\int^*(-u)dv$.

Before concluding this section we should mention and characterize another property the measure may possess.

In Lemma 4 we have, in fact, proved that two locally convex valued measures v_1 and v_2 are equal iff

$$v_1(B)-v_1(A) \geq v_1(D)-v_1(C) \iff v_2(B)-v_2(A) \geq v_2(D)-v_2(C)$$

for all $A \subset B$ and $C \subset D$.

However, a weaker property of two measures is sometimes of interest: v_1 agrees with v_2 iff $v_1(A) \geq v_1(B) \iff v_2(A) \geq v_2(B) \quad \forall\, A,B \subset S$.

In this context we would like to know when is it true that v agrees with $\bar{v}$. (We will henceforth call such a measure semi-symmetric.) Again we need an order on events. This time we define $A \geq^* B$ iff $(\phi,A) \geq' (\phi,B)$. Obviously $A \geq^* B$ iff $v(A) \geq v(B)$.

For $\bar{v}$ one may also define

$$A \geq_* B \text{ iff } (A^c,S) \geq' (B^c,S),$$

and $A \geq_* B$ iff $\bar{v}(A) \geq \bar{v}(B)$, or, equivalently,

$$A^c \;{}^*\!\leq B^c.$$

Therefore the following is obvious:

Observation v is semi-symmetric iff

$$A \geq^* B \iff A \geq_* B \quad \forall\, A,B \subset S.$$

Another characterization is given by

Theorem 3: If P1-P7* hold, and v is the measure induced by $\geq$, the following are equivalent:

(i) If $\{s \,/\, f(s) \leq x\} \sim^* \{s \,/\, g(s) \leq x\}$ for all $x \in X$, then $f \sim g$;

(ii) If $\{s \,/\, f(s) \geq x\} \sim_* \{s \,/\, g(s) \geq x\}$ for all $x \in X$, then $f \sim g$;

(iii) v is semi-symmetric.

<u>Proof</u>: First suppose (i) holds. Evidently, $A \sim^* B$ iff $A^c \sim^* B^c$, which is equivalent to $A \sim_* B$. In light of the monotonicity of both v and $\bar{v}$, (iii) is proved.

Now assume (iii). Let $f, g \in F$ satisfy the condition of (i), whence $v(f \leq x) = v(g \leq x)$ for all $x \in X$. By the semi-symmetry, $\bar{v}(f \leq x) = \bar{v}(g \leq x)$ or $v(f > x) = v(g > x)$ for all $x \in X$, and hence $\int^* u(f)dv = \int^* u(g)dv$ where u is the utility for $\geq$.

The proof that (ii) is equivalent to (iii) is (totally) symmetric.//

4. <u>An argument for additivity</u>

The previous section, which provides an axiomatic characterization of symmetric measures, may be interpreted as a normative argument for symmetry. That is to say, it is "more rational" to compute expectation w.r.t. a symmetric measure than w.r.t. a non-symmetric one.

In this section we proceed to raise another rationality argument for additivity. We will consider the concept of conditional probability measure, and eventually we will see that, given a symmetric measure, one may define for it a conditional measure satisfying some traditional conditions only if the original measure is additive.

So we begin with

<u>Definition</u>: Suppose v is a measure on S. Denote $N = \{A \subset S \,/\, v(A) = 0\}$. A two-argument set function $w : 2^S \times (2^S - N) \rightarrow [0,1]$ is called a <u>conditional probability measure</u> for v iff the following conditions are satisfied:

(i) For all $A \in 2^S - N$, $w(\cdot / A)$ restricted to 2^A is a measure on A;

(ii) If $v(A) = 1$, then $w(\cdot / A) = v(\cdot)$;

(iii) For all $A \in 2^S - N$, $B, C \subset S$,

$w(B / A) \geq w(C / A) \Leftrightarrow v(B \cap A) \geq v(C \cap A)$;

(iv) For all $A_1, A_2 \in 2^S - N$, $A_1 \cap A_2 = \phi$,

and all $B \subset S$, if $w(B / A_1) \geq w(B / A_2)$,

then $w(B / A_1) \geq w(B / A_1 \cup A_2) \geq w(B / A_2)$.

Note that all these conditions are satisfied by $w(B / A) = P(A \cap B)/P(B)$ in case P is additive. However, (i)-(iv) are supposed to be justified on intuitive grounds as well: (i) simply states that, when informed that A has occurred, w enables the decision maker to use the same decision rules as before. (ii) and (iii) connect the conditional measure and the original one: (iii) is a qualitative condition, stating that, given A, B is more likely than C iff $B \cap A$ was originally considered more likely than $C \cap A$. (ii) is a quantitative condition, fixing $w(\cdot / A)$ at $v(\cdot)$ for the events A which are certain.

The last condition means that taking a partition of A (into A_1 and A_2), one may not find conditional probabilities $(w(B / A_1), w(B / A_2))$ which are all above or all below the original one $(w(B / A))$. Of course, the additivity argument will eventually emerge out of this condition.

Besides formulating axiomatic conditions on a conditional measure, one would like to have an algorithm for computing $w(\cdot / \cdot)$, given $v(\cdot)$. In view of (iii), it is evident that should $w(\cdot / \cdot)$ be a conditional measure for v, there must exist $\{f_A\}_{A \in 2^S - N}$ strictly monotone functions from the unit interval into itself, such that

$$w(B / A) = f_A(v(A \cap B) / v(A)).$$

Of special interest is the case where f_A is the identity function for all A. For this case we have

<u>Theorem 4</u>: Let v be a symmetric locally convex valued measure. Then $w(A / B) \equiv v(A \cap B)/v(B)$ is a conditional measure for v iff v is additive.

<u>Proof</u>: If v is indeed additive, w is the usual conditional measure, and it surely satisfies (i)-(iv).

Suppose, then, that w is a conditional measure for v, and let $A_1 \dot{\cup} A_2 = A$.

First suppose that $v(A_1) = 0$.

Consider $w(A / A_1^c) = v(A \cap A_1^c) / v(A_1^c) = v(A_2)/v(A_1^c)$.

But v is symmetric, so that $v(A_1^c) = 1$, hence, by (ii), $w(A / A_1^c) = v(A)$, and we have shown that $w(A / A_1^c) = v(A_2)$, whence $v(A) = v(A_1) + v(A_2)$.

Since the case $v(A_2) = 0$ is dealt with symmetrically, assume $v(A_1), v(A_2) > 0$.

We would like to prove the existence of an event B such that $B, B^c \in 2^S - N$; $B \cap A = A_1$; $B^c \cap A = A_2$; $w(A_1 / B) = w(A_2 / B^c)$.

Suppose such an event was found. By (iii), $w(A / B) = w(A \cap B/B) = w(A_1 / B)$ and $w(A / B^c) = w(A_2/B^c)$. Using (iv), $w(A / B) = w(A / B^c) = w(A / S) = v(A)$. Now write:

$$
\begin{aligned}
v(A) &= v(A)v(B) + v(A)(1 - v(B)) = \\
&= v(A)v(B) + v(A)v(B^c) = \\
&= w(A / B)v(B) + w(A / B^c)v(B^c) = \\
&= v(A \cap B) + v(A \cap B^c) = v(A_1) + v(A_2).
\end{aligned}
$$

so v is additive.

To find the required B, let $\alpha = v(A_1)$; $\beta = v(A_1 \cap A^c)$, so that $1 > \beta \geq \alpha > 0$.

Note that $\frac{\alpha}{1-\alpha} \leq \frac{\alpha}{1-\beta} \leq \frac{\beta}{1-\beta}$ whence there is a number $\gamma \in [\alpha,\beta]$ such that $\frac{\gamma}{1-\gamma} = \frac{\alpha}{1-\beta}$, or $\frac{\alpha}{\gamma} = \frac{1-\beta}{1-\gamma}$. v is locally convex valued, so that there is an event B, $A_1 \subset B \subset A_1 \cup A^c$, such that $v(B) = \gamma$ (and $v(B^c) = 1 - \gamma$). It is easily seen that

$$
w(A_1 / B) = \frac{v(A_1)}{v(B)} = \frac{\alpha}{\gamma} = \frac{1-\beta}{1-\gamma} = \frac{v(A_2)}{v(B^c)} = w(A_2 / B^c),
$$

and B is indeed the event we are looking for, so the proof is complete.//

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