JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 4, Pages 943–973
S 0894-0347(05)00496-0
Article electronically published on July 12, 2005

DUALITY OF HARDY AND BMO SPACES ASSOCIATED WITH OPERATORS WITH HEAT KERNEL BOUNDS

XUAN THINH DUONG AND LIXIN YAN

1. Introduction

The introduction and development of Hardy and BMO spaces on Euclidean spaces $\mathbb{R}^n$ in the 1960s and 1970s played an important role in modern harmonic analysis and applications in partial differential equations. These spaces were studied extensively in [32], [22], [18], [19], [31] and many others.

An L^1 function f on $\mathbb{R}^n$ is in the Hardy space $H^1(\mathbb{R}^n)$ if the area integral function of the Poisson integral $e^{-t\sqrt{\triangle}}f$ satisfies

$$(1.1)\quad \mathcal{S}(f)(x) = \Big(\int_0^\infty\int_{|y-x|<t}\Big|\frac{\partial}{\partial t}e^{-t\sqrt{\triangle}}f(y)\Big|^2\, t^{1-n}dy\,dt\Big)^{1/2} \in L^1(\mathbb{R}^n).$$

There are a number of equivalent characterizations of functions in the H^1 space, including the all-important atomic decomposition (see [21], [31]).

A locally integrable function f defined on $\mathbb{R}^n$ is said to be in BMO, the space of functions of bounded mean oscillation, if

$$(1.2)\qquad \|f\|_{\rm BMO} = \sup_B \frac{1}{|B|}\int_B |f(y)-f_B|dy < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, and f_B stands for the mean of f over B, i.e.,

$$f_B = |B|^{-1}\int_B f(y)dy.$$

In [19], Fefferman and Stein showed that the space BMO is the dual space of the Hardy space H^1. They also obtained a characterization of the BMO space in terms of the Carleson measure, the H^1-H^1 boundedness of convolution operators which satisfy the Hörmander condition, and an interpolation theorem between L^p spaces and the BMO space. From the viewpoint of Calderón-Zygmund operator theory, H^1 and BMO spaces are natural substitutes for L^1 and L^∞ spaces, respectively.

Recently, Auscher, McIntosh and the first-named author introduced a class of Hardy spaces H^1_L associated with an operator L by means of the L^1 area integral

Received by the editors August 3, 2004.

2000 *Mathematics Subject Classification.* Primary 42B30, 42B35, 47F05.

Key words and phrases. Hardy space, BMO, semigroup, holomorphic functional calculi, tent space, Carleson measure, second-order elliptic operator, Schrödinger operator.

Both authors are supported by a grant from the Australia Research Council. The second author is also supported by NNSF of China (Grant No. 10371134) and the Foundation of Advanced Research Center, Zhongshan University.

functions in (1.1) in which the Poisson semigroup $e^{-t\sqrt{\triangle}}$ was replaced by the semigroup e^{-tL} ([4]). They then obtained an L-molecular characterization for H^1_L by using the theory of tent spaces developed by Coifman, Meyer and Stein ([7], [8] and [4]). See also Sections 3.2.1 and 4.1 below. In [16], we introduced and developed a new function space BMO_L associated with an operator L by using a maximal function introduced by Martell in [25]. Roughly speaking, if L is the infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t\geq 0}$ on L^2 with kernel $p_t(x,y)$ (which decays fast enough), we can view $P_tf = e^{-tL}f$ as an average version of f (at the scale t) and use the quantity

$$P_{t_B}f(x) = \int_{\mathbb{R}^n} p_{t_B}(x,y)f(y)dy \tag{1.3}$$

to replace the mean value f_B in the definition (1.2) of the classical BMO space, where t_B is scaled to the radius of the ball B. We then say that a function f (with suitable bounds on growth) is in BMO_L if

$$\sup_B \frac{1}{|B|}\int_B |f(x) - P_{t_B}f(x)|dx < \infty.$$

See Section 3.2.2 below. We also studied and established a number of important features of the BMO_L space such as the John-Nirenberg inequality and complex interpolation ([16], Section 3). Note that the spaces $H^1_{\sqrt{\triangle}}$ and $\mathrm{BMO}_{\sqrt{\triangle}}$ coincide with the classical Hardy and BMO spaces, respectively ([16], Section 2).

The main purpose of this paper is to prove a generalization of Fefferman and Stein's result on the duality of H^1 and BMO spaces. We will show that if L has a bounded holomorphic functional calculus on L^2 and the kernel $p_t(x,y)$ of the operator P_t in (1.3) satisfies an upper bound of Poisson type, then the space BMO_{L^*} is the dual space of the Hardy space H^1_L in which L^* denotes the adjoint operator of L. We also obtain a characterization of functions in BMO_L in terms of the Carleson measure. See Theorems 3.1 and 3.2 below.

We note that a valid choice of P_t in (1.3) is the Poisson integral $P_tf = e^{-t\sqrt{\triangle}}f$, which is defined by

$$P_tf(x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)dy,\ t>0, \quad \text{where} \ \ p_t(x) = \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}.$$

For this choice of P_t, Theorems 3.1 and 3.2 of this article give the classical results of Theorem 2 and the equivalence (i)$\Leftrightarrow$ (iii) of Theorem 3 of [19], respectively. See also Chapter IV of [31].

Note that in our main result, Theorem 3.1, we assume only an upper bound on the kernel $p_t(x,y)$ of P_t in (1.3) and no regularities on the space variables x or y. Another feature of our result is that we do not assume the conservation property of the semigroup $P_t(1) = 1$ for $t > 0$. This allows our method to be applicable to a large class of operators L.

The paper is organised as follows. In Section 2 we will give some preliminaries on holomorphic functional calculi of operators and on integral operators P_t with kernels $p_t(x,y)$ satisfying upper bounds of Poisson type. In Section 3 we introduce and describe the assumptions of the operator L in this paper, and recall the definitions of H^1_L and BMO_L spaces as in [4] and [16]. We then state our main result, Theorem 3.1, which says that the dual space of H^1_L is BMO_{L^*}. In Section 4 we prove a number of important estimates for functions in H^1_L and BMO_L spaces. We then prove Theorem 3.1 in Section 5 by combining the key estimates of Section 4 with certain estimates using the theory of tent spaces and Carleson measures. In Section 6, we study the dimensions of the kernel spaces $\mathcal{K}_L$ of BMO_L when L is a second-order elliptic operator of divergence form and when L is a Schrödinger operator. We conclude this article with a study of inclusion between the classical BMO space and BMO_L spaces associated with some differential operators, including a sufficient condition for the classical BMO and BMO_L spaces to coincide.

Throughout this paper, the letter "c" will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

We first give some preliminary definitions of holomorphic functional calculi as introduced by McIntosh [26].

Let $0 \le \omega < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$ by

$$S_\omega = \{z \in \mathbb{C} : |\mathrm{arg} z| \le \omega\} \cup \{0\}$$

and denote the interior of S_ω by S_ω^0.

We employ the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0:

$$H_\infty(S_\nu^0) = \{b \in H(S_\nu^0) : \ ||b||_\infty < \infty\},$$

where $||b||_\infty = \sup\{|b(z)| : z \in S_\nu^0\}$, and

$$\Psi(S_\nu^0) = \{\psi \in H(S_\nu^0) : \exists\, s > 0,\ |\psi(z)| \le c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Let $0 \le \omega < \pi$. A closed operator L in $L^2(\mathbb{R}^n)$ is said to be of type ω if $\sigma(L) \subset S_\omega$, and for each $\nu > \omega$, there exists a constant c_ν such that

$$\|(L - \lambda\mathcal{I})^{-1}\| \le c_\nu|\lambda|^{-1}, \qquad \lambda \notin S_\nu.$$

If L is of type ω and $\psi \in \Psi(S_\nu^0)$, we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\psi(L) = \frac{1}{2\pi i}\int_\Gamma (L - \lambda\mathcal{I})^{-1}\psi(\lambda)d\lambda, \tag{2.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \ge 0\}$ parametrized clockwise around S_ω, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$. If, in addition, L is one-one and has dense range and if $b \in H_\infty(S_\nu^0)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L),$$

where $\psi(z) = z(1+z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$. We say that L has a bounded H_∞ calculus on L^2 if there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_\infty(S_\nu^0)$,

$$||b(L)|| \le c_{\nu,2}||b||_\infty.$$

For a detailed study of operators which have holomorphic functional calculi, see [6].

In this paper, we will work with a class of integral operators $\{P_t\}_{t>0}$, which plays the role of generalized approximations to the identity. We assume that for each $t > 0$, the operator P_t is defined by its kernel $p_t(x, y)$ in the sense that

$$P_t f(x) = \int_{\mathbb{R}^n} p_t(x, y) f(y) dy$$

for every function f which satisfies the growth condition (3.3) in Section 3.1 below.

We also assume that the kernel $p_t(x,y)$ of P_t satisfies a Poisson bound of order $m>0$:

$$|p_t(x,y)| \le h_t(x,y) = t^{-n/m} s\Big(\frac{|x-y|}{t^{1/m}}\Big), \tag{2.2}$$

in which s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon} s(r) = 0 \tag{2.3}$$

for some $\epsilon > 0$.

It is easy to check that there exists a constant $c>0$ such that $h_t(x,y)$ satisfies

$$c^{-1} \le \int_{\mathbb{R}^n} h_t(x,y)dx \le c \quad \text{and} \quad c^{-1} \le \int_{\mathbb{R}^n} h_t(y,x)dx \le c$$

uniformly in $y \in \mathbb{R}^n$, $t>0$. See Section 2 of [14].

We recall that the Hardy-Littlewood maximal operator Mf is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy,$$

where the *sup* is taken over all balls containing x. It is well known that the Hardy-Littlewood maximal operator is bounded on L^r for all $r\in(1,\infty]$. Because of the decay of the kernel $p_t(x,y)$ in (2.2) and (2.3), one has

Proposition 2.1. *There exists a constant $c>0$ such that for any $f\in L^r$, $1\le r\le\infty$, we have*

$$|P_t f(x)| \le \int_{\mathbb{R}^n} h_t(x,y)|f(y)|dy \le cMf(x)$$

for all $t>0$.

Proof. This is a consequence of the conditions (2.2), (2.3) and the definition of Mf. See [15], Proposition 2.4. □

3. Duality between H^1_L and BMO_{L^*} spaces

In this section, we will give the framework and the main result of this paper.

3.1. Assumptions and notation.

Let L be a linear operator of type ω on $L^2(\mathbb{R}^n)$ with $\omega<\pi/2$; hence L generates a holomorphic semigroup e^{-zL}, $0\le|\mathrm{Arg}(z)|<\pi/2-\omega$. Assume the following two conditions.

Assumption (a). The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}(z)|<\pi/2-\omega$, is represented by the kernel $p_z(x,y)$ which satisfies the upper bound

$$|p_z(x,y)| \le c_\theta h_{|z|}(x,y)$$

for $x,y\in\mathbb{R}^n$, $|\mathrm{Arg}(z)|<\pi/2-\theta$ for $\theta>\omega$, and h_t is defined on $\mathbb{R}^n\times\mathbb{R}^n$ by (2.2).

Assumption (b). The operator L has a bounded H_∞-calculus on $L^2(\mathbb{R}^n)$. That is, there exists $c_{\nu,2}>0$ such that $b(L)\in\mathcal{L}(L^2,L^2)$, and for $b\in H_\infty(S^0_\nu)$:

$$||b(L)f||_2 \le c_{\nu,2}||b||_\infty \|f\|_2$$

for any $f\in L^2(\mathbb{R}^n)$.

We now give some consequences of assumptions (a) and (b) which will be useful in the sequel.

(i) If $\{e^{-tL}\}_{t\geq 0}$ is a bounded analytic semigroup on $L^2(\mathbb{R}^n)$ whose kernel $p_t(x,y)$ satisfies the estimate (2.2), then for all $k\in\mathbb{N}$, the time derivatives of p_t satisfy

$$\Big|\frac{\partial^k p_t}{\partial t^k}(x,y)\Big| \leq ct^{-\frac{n+km}{m}} s\Big(\frac{|x-y|}{t^{1/m}}\Big) \tag{3.1}$$

for all $t>0$ and almost all $x,y\in\mathbb{R}^n$. For each $k\in\mathbb{N}$, the function s might depend on k but it always satisfies (2.3). See Lemma 2.5 of [5].

(ii) L has a bounded H_∞-calculus on $L^2(\mathbb{R}^n)$ if and only if for any non-zero function $\psi\in\Psi(S^0_\nu)$, L satisfies the square function estimate and its reverse

$$c_1\|f\|_2 \leq \Big(\int_0^\infty \|\psi_t(L)f\|_2^2\frac{dt}{t}\Big)^{1/2} \leq c_2\|f\|_2 \tag{3.2}$$

for some $0<c_1\leq c_2<\infty$, where $\psi_t(\xi)=\psi(t\xi)$. Note that different choices of $\nu>\omega$ and $\psi\in\Psi(S^0_\nu)$ lead to equivalent quadratic norms of f. See [26].

As noted in [26], positive self-adjoint operators satisfy the quadratic estimate (3.2), as do normal operators with spectra in a sector, and maximal accretive operators. For definitions of these classes of operators, we refer the reader to [36].

(iii) Under the assumptions (a) and (b), it was proved in Theorem 3.1 of [15] and Theorem 6 of [14] that the operator L has a bounded holomorphic functional calculus on $L^p(\mathbb{R}^n)$, $1<p<\infty$; that is, there exists $c_{\nu,p}>0$ such that $b(L)\in\mathcal{L}(L^p,L^p)$, and for $b\in H_\infty(S^0_\nu)$:

$$\|b(L)f\|_p \leq c_{\nu,p}\|b\|_\infty\|f\|_p$$

for any $f\in L^p(\mathbb{R}^n)$. For $p=1$, the operator $b(L)$ is of weak-type $(1,1)$. In [16], it was proved that for $p=\infty$, the operator $b(L)$ is bounded from L^∞ into BMO_L.

We now define the class of functions that the operators P_t act upon. For any $\beta>0$, a function $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be a function of β-type if f satisfies

$$\Big(\int_{\mathbb{R}^n}\frac{|f(x)|^2}{1+|x|^{n+\beta}}dx\Big)^{1/2}\leq c<\infty. \tag{3.3}$$

We denote by $\mathcal{M}_\beta$ the collection of all functions of β-type. If $f\in\mathcal{M}_\beta$, the norm of f in $\mathcal{M}_\beta$ is denoted by

$$\|f\|_{\mathcal{M}_\beta}=\inf\Big\{c\geq 0:\ (3.3)\text{ holds}\Big\}.$$

It is easy to see that $\mathcal{M}_\beta$ is a Banach space under the norm $\|f\|_{\mathcal{M}_\beta}$. Note that we use $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ instead of the space $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as in [19] and [16] since this gives the appropriate setting for the duality between H^1_L and BMO_L. For any given operator L, we let $\Theta(L)=\sup\big\{\epsilon>0 : (2.3)\text{ holds }\big\}$, and define

$$\mathcal{M}=\begin{cases}\mathcal{M}_{\Theta(L)} & \text{if } \Theta(L)<\infty;\\[2ex] \displaystyle\bigcup_{\beta:\ 0<\beta<\infty}\mathcal{M}_\beta & \text{if } \Theta(L)=\infty.\end{cases}$$

Note that if L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then $\Theta(\triangle)=\infty$. When $L=\sqrt{\triangle}$, we have $\Theta(\sqrt{\triangle})=1$.

For any $(x,t)\in\mathbb{R}^n\times(0,+\infty)$ and $f\in\mathcal{M}$, we define

$$P_tf(x)=e^{-tL}f(x)=\int_{\mathbb{R}^n}p_t(x,y)f(y)dy \tag{3.4}$$

and

$$Q_t f(x) = tLe^{-tL}f(x) = \int_{\mathbb{R}^n} -t\Big(\frac{d}{dt}p_t(x,y)\Big)f(y)dy. \tag{3.5}$$

It follows from the estimate (3.1) that the operators $P_t f$ and $Q_t f$ are well-defined. Moreover, the operator Q_t has the following properties:

(i) for any $t_1, t_2 > 0$ and almost all $x \in \mathbb{R}^n$,

$$Q_{t_1}Q_{t_2}f(x) = t_1t_2\Big(\frac{d^2P_t}{dt^2}\big|_{t=t_1+t_2}f\Big)(x);$$

(ii) the kernel $q_{t^m}(x,y)$ of Q_{t^m} satisfies

$$|q_{t^m}(x,y)| \le ct^{-n}s\Big(\frac{|x-y|}{t}\Big), \tag{3.6}$$

where the function s satisfies the condition (2.3). This property is the same as the estimate (3.1).

3.2. **Hardy spaces and BMO spaces associated with operators.**

3.2.1. *Hardy space* H^1_L. We assume that L is an operator which satisfies the assumptions of Section 3.1. $\mathbb{R}^{n+1}_+$ will denote the usual upper half-space in $\mathbb{R}^{n+1}$. The notation $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ denotes the standard cone (of aperture 1) with vertex $x \in \mathbb{R}^n$. For any closed subset $F \subset \mathbb{R}^n$, $\mathcal{R}(F)$ will be the union of all cones with vertices in F, i.e., $\mathcal{R}(F) = \bigcup_{x\in F}\Gamma(x)$. If O is an open subset of $\mathbb{R}^n$, then the "tent" over O, denoted by $\widehat{O}$, is given as $\widehat{O} = [\mathcal{R}(O^c)]^c$.

Given a function $f \in L^1(\mathbb{R}^n)$, the area integral function $\mathcal{S}_L(f)$ associated with an operator L is defined by

$$\mathcal{S}_L(f)(x) = \Big(\int_{\Gamma(x)} |Q_{t^m}f(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\Big)^{1/2}. \tag{3.7}$$

It follows from the assumption (b) of L that the area integral function $\mathcal{S}_L(f)$ is bounded on $L^2(\mathbb{R}^n)$ ([26]). It then follows from the assumption (a) of L that $\mathcal{S}_L(f)$ is bounded on L^p, $1 < p < \infty$. See Theorem 6 of [4]. More specifically, there exist constants c_1, c_2 such that $0 < c_1 \le c_2 < \infty$ and

$$c_1\|f\|_p \le \|\mathcal{S}_L(f)\|_p \le c_2\|f\|_p \tag{3.8}$$

for all $f \in L^p, 1 < p < \infty$. See also [35].

By duality, the operator $S_{L^*}(f)$ also satisfies the estimate (3.8), where L^* is the adjoint operator of L.

The following definition was introduced in [4]. We say that $f \in L^1$ belongs to a Hardy space associated with an operator L, denoted by H^1_L, if $S_L(f) \in L^1$. We define its H^1_L norm by

$$\|f\|_{H^1_L} = \|\mathcal{S}_L(f)\|_{L^1}.$$

Note that if L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then it follows from the area integral characterization of a Hardy space by using convolution that the classical space $H^1(\mathbb{R}^n)$ coincides with the spaces $H^1_\triangle(\mathbb{R}^n)$ and $H^1_{\sqrt{\triangle}}(\mathbb{R}^n)$ and their norms are equivalent. See [19] and [31].

3.2.2. *The function space* BMO_L. Following [16], we say that $f \in \mathcal{M}$ is of bounded mean oscillation associated with an operator L (abbreviated as BMO_L) if

$$\sup_B \frac{1}{|B|}\int_B |f(x) - P_{r_B^m} f(x)|dx = \|f\|_{\mathrm{BMO}_L} < \infty, \tag{3.9}$$

where the *sup* is taken over all balls in $\mathbb{R}^n$, and r_B is the radius of the ball B. The class of functions of BMO_L, modulo $\mathcal{K}_L$, where

$$\mathcal{K}_L = \Big\{ f \in \mathcal{M} : P_t f(x) = f(x) \text{ for almost all } \ x \in \mathbb{R}^n \text{ and all } t > 0 \Big\}, \tag{3.10}$$

is a Banach space with the norm $\|f\|_{\mathrm{BMO}_L}$ defined as in (3.9). We refer to Corollary 5.2 in Section 5 for completeness of the space BMO_L. See also Section 6.1 for a discussion of the kernel space $\mathcal{K}_L$.

We now give the following list of a number of important properties of the spaces BMO_L. For the proofs, we refer the reader to Sections 2 and 3 of [16].

(i) If a function f is in the classical space BMO, then it follows from the John-Nirenberg inequality that $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $f \in \mathcal{M}$. See [22]. Under the extra condition that L satisfies the conservation property of the semigroup $P_t(1) = 1$ for every $t > 0$, it can be verified that BMO is a subspace of BMO_L. Moreover, the spaces BMO, $\mathrm{BMO}_{\triangle}$ and $\mathrm{BMO}_{\sqrt{\triangle}}$ coincide and their norms are equivalent. See also Theorem 6.10 in Section 6.

(ii) If $f \in\mathrm{BMO}_L$, then for every $t > 0$ and every $K > 1$, there exists a constant $c > 0$ such that for almost all $x \in \mathbb{R}^n$, we have

$$|P_t f(x) - P_{Kt} f(x)| \le c(1 + \log K)\|f\|_{\mathrm{BMO}_L}. \tag{3.11}$$

(iii) If $f \in\mathrm{BMO}_L$, then for any $\delta > 0$ and any $x_0 \in \mathbb{R}^n$, there exists a constant c_δ which depends on δ such that

$$\int_{\mathbb{R}^n} \frac{|f(x) - P_t f(x)|}{(t^{1/m} + |x - x_0|)^{n+\delta}} dx \le \frac{c_\delta}{t^{\delta/m}}\|f\|_{\mathrm{BMO}_L}. \tag{3.12}$$

(iv) A variant of the John-Nirenberg inequality holds for functions in BMO_L. That is, there exist positive constants c_1 and c_2 such that for every ball B and $\alpha > 0$,

$$|\{x \in B : |f(x) - P_{r_B^m} f(x)| > \alpha\}| \le c_1|B| \exp\Big\{-\frac{c_2\alpha}{\|f\|_{\mathrm{BMO}_L}}\Big\}.$$

This and (3.9) imply that for any $f \in\mathrm{BMO}_L$ and $1 \le p < \infty$, the norms

$$\|f\|_{p,\mathrm{BMO}_L} = \sup_B \Big(\frac{1}{|B|}\int_B |f(x) - P_{r_B^m} f(x)|^p dx\Big)^{1/p} \tag{3.13}$$

with different choices of p are all equivalent.

3.3. **Main theorems.** We now state the main result of this paper.

Theorem 3.1. *Assume that the operator L satisfies the assumptions* (a) *and* (b) *in Section* 3.1. *Denote by L^* the adjoint operator of L. Then, the dual space of the H^1_L space is the* BMO_{L^*} *space, in the following sense.*

(i) *Suppose $f \in\mathrm{BMO}_{L^*}$. Then the linear functional ℓ given by*

$$\ell(g) = \int_{\mathbb{R}^n} f(x)g(x)dx, \tag{3.14}$$

initially defined on the dense subspace $H^1_L \cap L^2$, has a unique extension to H^1_L.

(ii) *Conversely, every continuous linear functional* ℓ *on the* H^1_L *space can be realized as above; i.e., there exists* $f \in \mathrm{BMO}_{L^*}$ *such that* (3.14) *holds and* $\|f\|_{\mathrm{BMO}_{L^*}} \le c\|\ell\|$.

To state the next theorem, we recall that a measure μ defined on $\mathbb{R}^{n+1}_+$ is said to be a Carleson measure if there is a positive constant c such that for each ball B on $\mathbb{R}^n$,

$$\mu(\widehat{B}) \le c|B|, \tag{3.15}$$

where $\widehat{B}$ is the tent over B. The smallest bound c in (3.15) is defined to be the norm of μ and is denoted by $|||\mu|||_c$.

The Carleson measure is closely related to the classical BMO space. We note that for every $f \in \mathrm{BMO}$,

$$\mu_f(x,t) = \big|t\frac{\partial}{\partial t}e^{-t\sqrt{\triangle}}f(x)\big|^2\frac{dxdt}{t}$$

is a Carleson measure on $\mathbb{R}^{n+1}_+$. See [19] and Chapter 4 of [21].

For the space BMO_L, we have the following characterization of BMO_L functions in terms of the Carleson measure.

Theorem 3.2. *Assume that the operator* L *satisfies the assumptions* (a) *and* (b) *in Section* 3.1. *The following conditions are equivalent:*

(i) f *is a function in* $\mathrm{BMO}_L(\mathbb{R}^n)$*;*

(ii) $f \in \mathcal{M}$, *and* $\mu_f(x,t) = |Q_{t^m}(\mathcal{I} - P_{t^m})f(x)|^2\frac{dxdt}{t}$ *is a Carleson measure, with* $|||\mu_f|||_c \sim \|f\|^2_{\mathrm{BMO}_L}$.

The proofs of Theorem 3.1 and the implication (ii) $\Rightarrow$ (i) of Theorem 3.2 will be given in Section 5. For the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 3.2, we refer to Lemma 4.6 of Section 4.

Remark. Using Theorems 3.1 and 3.2, we can obtain more information about the Hardy spaces H^1_L and the BMO_L spaces. We will discuss the inclusion between the classical BMO space and the BMO_L spaces associated with some differential operators. See Section 6.

4. Properties of H^1_L and BMO_L spaces

In [7], [8], Coifman, Meyer and Stein introduced and studied a new family of function spaces, the so-called "tent spaces". These spaces are useful for the study of a variety of problems in harmonic analysis. In particular, we note that the tent spaces give a natural and simple approach to the atomic decomposition of functions in the classical Hardy space by using the area integral functions and the connection with the theory of Carleson measure. In this paper, we will adopt the same approach of tent spaces.

4.1. **Tent spaces and applications.** For any function $f(y,t)$ defined on $\mathbb{R}^{n+1}_+$ we will denote

$$\mathcal{A}(f)(x) = \Big(\int_{\Gamma(x)}|f(y,t)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \tag{4.1}$$

and

$$\mathcal{C}(f)(x) = \sup_{x\in B}\Big(\frac{1}{|B|}\int_{\widehat{B}}|f(y,t)|^2\frac{dydt}{t}\Big)^{1/2}. \tag{4.2}$$

As in [8], the "tent space" T_2^p is defined as the space of functions f such that $\mathcal{A}(f) \in L^p(\mathbb{R}^n)$, when $p < \infty$. The resulting equivalence classes are then equipped with the norm $|||f|||_{T_2^p} = \|\mathcal{A}(f)\|_p$. When $p = \infty$, the space T_2^∞ is the class of functions f for which $\mathcal{C}(f) \in L^\infty(\mathbb{R}^n)$ and the norm $|||f|||_{T_2^\infty} = \|\mathcal{C}(f)\|_\infty$. Thus, $f \in H_L^1$ if and only if $Q_{t^m} f \in T_2^1$, i.e., $\mathcal{A}(Q_{t^m} f) = \mathcal{S}_L(f) \in L^1$.

Next, a function $a(t,x)$ is called a T_2^1-atom if

(i) the function $a(t,x)$ is supported in $\widehat{B}$ (for some ball $B \subset \mathbb{R}^n$);

(ii) $\displaystyle\int_{\widehat{B}} |a(t,x)|^2 \frac{dxdt}{t} \leq |B|^{-1}$.

The following proposition on duality and atomic decomposition for functions in T_2^1 was proved in [8].

Proposition 4.1. (a) *The following inequality holds, whenever $f \in T_2^1$ and $g \in T_2^\infty$:*

$$\int_{\mathbb{R}_+^{n+1}} \big|f(x,t)g(x,t)\big| \frac{dxdt}{t} \leq c \int_{\mathbb{R}^n} \mathcal{A}(f)(x)\mathcal{C}(g)(x)dx. \tag{4.3}$$

(b) *The pairing*

$$\langle f, g\rangle \to \int_{\mathbb{R}_+^{n+1}} f(x,t)g(x,t)\frac{dxdt}{t}$$

realizes T_2^∞ as equivalent to the Banach space dual of T_2^1.

(c) *Every element $f \in T_2^1$ can be written as $f = \sum \lambda_j a_j$, where the a_j are T_2^1 atoms, $\lambda_j \in \mathbb{C}$, and $\sum |\lambda_j| \leq c|||f|||_{T_2^1}$.*

Proof. For the proof of Proposition 4.1, we refer to Theorem 1 of [8]. See also Theorem 1 of [11] for a proof of (a). $\square$

Proposition 4.1 gives a quick proof of the atomic decomposition for the classical Hardy space H^1. Let $L = \sqrt{\triangle}$. For any $f \in H^1$, we denote by $P_t f(x)$ the Poisson integral $P_t f = e^{-t\sqrt{\triangle}} f$ and set $F = Q_t f(x) = -t\frac{d}{dt}P_t f \in T_2^1$. The atomic decomposition of F in T_2^1 leads to the atomic decomposition of f in H^1 by using the following identity on H^1:

$$f(x) = \pi_\phi(F)(x) = \int_0^\infty F(x,t) * \phi_t \frac{dt}{t}, \tag{4.4}$$

where $\phi_t = t^{-n}\phi(\cdot/t)$ for all $t > 0$, the function ϕ is radial and in C_0^∞ with $\int \phi(x)dx = 0$, and $-2\pi \int_0^\infty \hat{\phi}(\xi t)|\xi| e^{-2\pi|\xi|t} dt = 1$ for all $\xi \neq 0$. Note that instead of the condition $\phi \in C_0^\infty$, we may assume that $|\phi(x)| + |\nabla\phi(x)| \leq M(1+|x|)^{-n-1}$ for some $M > 0$. Then, the operator π_ϕ maps T_2^1 atoms to appropriate "molecules". See Lemma 7 of [7].

We now give a short discussion of the Hardy space H_L^1. For more details, see [4]. First, we need a variant of formula (4.4), which is inspired from the H_∞-calculus for L. We start from the identity:

$$\frac{1}{4m} = \int_0^\infty (t^m z e^{-t^m z})(t^m z e^{-t^m z})\frac{dt}{t},$$

which is valid for all $z \neq 0$ in a sector S_μ^0 with $\mu \in (\omega, \pi)$. As a consequence, one has

$$\mathrm{Id} = 4m \int_0^\infty Q_{t^m} Q_{t^m} \frac{dt}{t}, \tag{4.5}$$

where the integral converges strongly in L^2. See [26]. For any $f \in H_L^1$, we let $F(x,t) = \big(Q_{t^m} f\big)(x)$. We then have the following identity for all $f \in H_L^1 \cap L^2$:

$$f(x) = \pi_L(F)(x) = 4m \int_0^\infty Q_{t^m}\big(Q_{t^m} f\big)(x)\frac{dt}{t}. \tag{4.6}$$

Recall that in [4], a function $\alpha(x)$ is called an L-molecule if

$$\alpha(x) = \int_0^\infty Q_{t^m}(a(t,\cdot))(x)\frac{dt}{t}, \tag{4.7}$$

where $a(t,x)$ is a T_2^1-atom supported in the tent $\widehat{B}$ of some ball $B \subset \mathbb{R}^n$, and $a(t,x)$ satisfies the condition $\int_{\widehat{B}} |a(t,x)|^2 dxdt/t \le |B|^{-1}$. By using the identity (4.6) in place of (4.4), an L-molecule decomposition of f in the space H_L^1 is obtained in Theorem 7 of [4] as follows.

Proposition 4.2. *Let $f \in H_L^1 \cap L^2$. There exist L-molecules $\alpha_k(x)$ and numbers λ_k for $k = 0, 1, 2, \cdots$ such that*

$$f(x) = \sum_k \lambda_k \alpha_k(x). \tag{4.8}$$

The sequence λ_k satisfies $\sum\limits_k |\lambda_k| \le c\|f\|_{H_L^1}$. Conversely, the decomposition (4.8) *satisfies*

$$\|f\|_{H_L^1} \le c \sum_k |\lambda_k|.$$

Proof. The proof of Proposition 4.2 follows from an argument using certain estimates on area integrals and tent spaces. For the details, we refer the reader to Theorem 7 of [4]. □

4.2. **Properties for H_L^1 and BMO_L spaces.** Let $T_{2,c}^p$ be the set of all $f \in T_2^p$ with compact support in $\mathbb{R}_+^{n+1}$. Consider the operator π_L of (4.6) initially defined on $T_{2,c}^p$ by

$$\pi_L(f)(x) = 4m \int_0^\infty Q_{t^m}(f(\cdot,t))(x)\frac{dt}{t}. \tag{4.9}$$

Note that for any compact set K in $\mathbb{R}_+^{n+1}$,

$$\int_K |f(x,t)|^2 dxdt \le c(K,p)\|\mathcal{A}(f)\|_p^2.$$

This and the estimate (3.2) imply that the integral (4.9) is well-defined, and $\pi_L(f) \in L^2$ for $f \in T_{2,c}^p$.

Lemma 4.3. *The operator π_L, initially defined on $T_{2,c}^p$, extends to a bounded linear operator from*

(a) T_2^p *to* L^p, *if* $1 < p < \infty$;
(b) T_2^1 *to* H_L^1;
(c) T_2^∞ *to* BMO_L.

Proof. The property (b) is contained in the second part of Proposition 4.2. The property (c) will be shown in Section 5.2 as it is a direct result of Theorem 3.1 and the duality of H_L^1 and BMO_{L^*} spaces.

We now verify (a). By using (5.1) of [8], we have

$$\int_{\mathbb{R}^{n+1}_+} |f(x,t)h(x,t)|\frac{dxdt}{t} \leq \int_{\mathbb{R}^n} \mathcal{A}(f)(x)\mathcal{A}(h)(x)dx.$$

This, together with (4.9) and the estimate (3.8), yield

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} \pi_L(f)(x)g(x)dx\Big| &\leq c\Big|\int_{\mathbb{R}^{n+1}_+} f(x,t)Q^*_{t^m}g(x)\frac{dxdt}{t}\Big| \\
&\leq c\Big|\int_{\mathbb{R}^n} \mathcal{A}(f)(x)\mathcal{A}(Q^*_{t^m}g)(x)dx\Big| \\
&\leq c\|\mathcal{A}(f)\|_p\|\mathcal{A}(Q^*_{t^m}g)\|_{p'} \\
&\leq c|||f|||_{T^p_2}\|\mathcal{S}_{L^*}g\|_{p'} \\
&\leq c|||f|||_{T^p_2}\|g\|_{p'}
\end{aligned}$$

for any $g \in L^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$. Hence, we obtain $\|\pi_L(f)\|_p \leq c|||f|||_{T^p_2}$. □

As a consequence of Lemma 4.3, we have the following corollary.

Corollary 4.4. *The space $H^1_L \cap L^2$ is dense in H^1_L.*

Proof. For any $f \in H^1_L$, by the definition of H^1_L we have $Q_{t^m}f \in T^1_2$. Define $\tilde{O}_k = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x| \leq k,\ k^{-1} < t \leq k\}$, and let

$$f_k(x) = 4m\int_0^\infty Q_{t^m}\Big([Q_{t^m}f]\chi_{\tilde{O}_k}\Big)(x)\frac{dt}{t}$$

for all $k \in \mathbb{N}$. This family of functions $\{f_k\}_{k\in\mathbb{N}}$ satisfies

(i) $f_k \in L^2 \cap H^1_L$;

(ii) $\|f - f_k\|_{H^1_L} \to 0$ as $k \to \infty$.

By (a) and (b) of Lemma 4.3, the estimate (i) is straightforward since for each $k \in \mathbb{N}$, $[Q_{t^m}f]\chi_{\tilde{O}_k} \in T^1_2 \cap T^2_2$. Moreover, by (b) of Lemma 4.3,

$$\begin{aligned}
\|f - f_k\|_{H^1_L} &\leq c|||Q_{t^m}f(x) - (Q_{t^m}f)\chi_{\tilde{O}_k}(x)|||_{T^1_2} \\
&\leq c|||(Q_{t^m}f)\chi_{(\tilde{O}_k)^c}(x)|||_{T^1_2} \\
&\to 0
\end{aligned}$$

as $k \to \infty$. This proves property (ii) and completes the proof of Corollary 4.4. □

Remark. From Corollary 4.4, it follows from a standard argument that for any $f \in H^1_L$, f has an L-molecular decomposition (4.8). See, for example, Chapter III of [31].

We next prove the following H^1_L-estimate for functions in the space H^1_L, which will be useful in proving our Theorems 3.1 and 3.2 in Section 5.

Lemma 4.5. *For any L^2-function f supported on a ball B with radius r_B, there exists a positive constant c such that*

$$\big\|(\mathcal{I} - P_{r^m_B})f\big\|_{H^1_L} \leq c|B|^{1/2}\|f\|_{L^2}. \tag{4.10}$$

Proof. Assume that $B = B(z_0, r_B)$ is a ball of radius r_B and centered at z_0. One writes

$$\begin{aligned}\|\mathcal{S}_L(\mathcal{I} - P_{r_B^m})f\|_{L^1} &= \int_{4B} \mathcal{S}_L(\mathcal{I} - P_{r_B^m})f(x)dx + \int_{(4B)^c} \mathcal{S}_L(\mathcal{I} - P_{r_B^m})f(x)dx \\ &= \mathrm{I} + \mathrm{II}, \quad \text{respectively.}\end{aligned}$$

Note that $\|P_t f\|_{L^2} \leq c\|f\|_{L^2}$ for any $t > 0$. Using Hölder's inequality and the fact that the area integral function $\mathcal{S}_L$ is bounded on L^2, one obtains

$$\begin{aligned}\int_{4B} \mathcal{S}_L(\mathcal{I} - P_{r_B^m})f(x)dx &\leq c|B|^{\frac{1}{2}}\|\mathcal{S}_L(\mathcal{I} - P_{r_B^m})f\|_{L^2} \\ &\leq c|B|^{\frac{1}{2}}\|(\mathcal{I} - P_{r_B^m})f\|_{L^2} \\ &\leq c|B|^{1/2}\|f\|_{L^2}.\end{aligned}$$

We now estimate the term II. First, we will show that there exists a constant $c > 0$ such that for any $x \notin 4B$,

$$\Big(\mathcal{S}_L(\mathcal{I} - e^{-r_B^m L})f\Big)^2(x) \leq c r_B^{2\epsilon}\|f\|_{L^1}^2 |x - z_0|^{-2(n+\epsilon)}. \tag{4.11}$$

Let us verify (4.11). Let

$$\Psi_{t,s}(L)f(x) = (t^m + s^m)^2\Big(\frac{d^2 P_r}{dr^2}\big|_{r=t^m+s^m} f\Big)(x)$$

and $h(x) = mx^m(1 + x^m)^{-2}$. Since $(\mathcal{I} - P_{r_{\frac{B}{m}}}) = m\int_0^{r_B} Q_{s^m}\frac{ds}{s}$, we obtain

$$Q_{t^m}(\mathcal{I} - P_{r_B^m}) = m\int_0^{r_B} Q_{t^m}Q_{s^m}\frac{ds}{s} = \int_0^{r_B} h(\frac{s}{t})\Psi_{t,s}(L)\frac{ds}{s}.$$

It follows from the estimate (3.1) that the kernel $\Psi_{t,s}(L)(y, z)$ of the operator $\Psi_{t,s}(L)$ satisfies

$$|\Psi_{t,s}(L)(y, z)| \leq c\frac{(t + s)^\epsilon}{(t + s + |y - z|)^{n+\epsilon}},$$

where ϵ is the positive constant in (2.3). Therefore,

$$\begin{aligned}&\Big(\mathcal{S}_L(\mathcal{I} - P_{r_B^m})f\Big)^2(x) \\ &\leq \int_0^\infty \int_{|y-x|\leq t}\Big[\int_0^{r_B} h(\frac{s}{t})\Psi_{t,s}(L)f(y)\frac{ds}{s}\Big]^2\frac{dydt}{t^{n+1}} \\ &\leq c\Big(\int_0^{r_B} + \int_{r_B}^\infty\Big)\int_{|y-x|\leq t}\Big[\int_0^{r_B} h(\frac{s}{t})\int_B \frac{(t + s)^\epsilon}{(t + s + |y - z|)^{n+\epsilon}}|f(z)|\frac{dzds}{s}\Big]^2\frac{dydt}{t^{n+1}} \\ &= \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

We only consider the term II_2 since the estimate of the term II_1 is even simpler. For $x \notin 4B$ and $t \geq r_B$, we set $B = B_1 \cup B_2$, where $B_1 = B \cap \{z : |y - z| \leq \frac{|x-z_0|}{2}\}$. For any $z \in B_1$ and $|y - x| < t$, we have

$$|x - z_0| \leq |y - x| + |y - z| + |z - z_0| \leq t + \frac{|x - z_0|}{2} + r_B \leq 2t + \frac{|x - z_0|}{2},$$

which implies $t \geq |x - z_0|/4$; hence $(t + s + |y - z|) \geq |x - z_0|/4$. Obviously, for any $z \in B_2$ and $|y - x| < t$, we also have $(t + s + |y - z|) \geq |x - z_0|/2$. Note that

$$(t + s)^\epsilon h(\frac{s}{t}) \leq c(t + s)^\epsilon(ts)^m(t^m + s^m)^{-2} \leq ct^{-\epsilon/2}s^{3\epsilon/2}.$$

It follows from elementary integration that

$$\begin{aligned}
\mathrm{II}_2 &\le c\int_{r_B}^{\infty}\int_{|y-x|\le t}\Big[\int_0^{r_B}(t+s)^{\epsilon}h(\frac{s}{t})\frac{ds}{s}\Big]^2\frac{dydt}{t^{n+1}}\|f\|_{L^1}^2|x-z_0|^{-2(n+\epsilon)}\\
&\le c\Big(\int_{r_B}^{\infty}\Big[\int_0^{r_B}t^{-\epsilon/2}s^{3\epsilon/2}\frac{ds}{s}\Big]^2\frac{dt}{t}\Big)\|f\|_{L^1}^2|x-z_0|^{-2(n+\epsilon)}\\
&\le cr_B^{2\epsilon}\|f\|_{L^1}^2|x-z_0|^{-2(n+\epsilon)}.
\end{aligned}$$

The estimate (4.11) then follows readily. Therefore,

$$\begin{aligned}
\int_{(4B)^c}\mathcal{S}_L(\mathcal{I}-P_{r_B^m})f(x)dx &\le cr_B^{\epsilon}\|f\|_{L^1}\int_{(4B)^c}|x-z_0|^{-(n+\epsilon)}dx\\
&\le c\|f\|_{L^1}\\
&\le c|B|^{1/2}\|f\|_{L^2}.
\end{aligned}$$

Combining the estimates of the terms I and II, we obtain that $\|\mathcal{S}_L(\mathcal{I}-P_{r_B^m})f\|_{L^1}\le c|B|^{1/2}\|f\|_{L^2}$. The proof of Lemma 4.5 is complete. □

We now follow Theorem 2.14 of [16] to prove the implication (i) ⇒ (ii) of Theorem 3.2. For the implication (ii) ⇒ (i) of Theorem 3.2, we will present its proof in Section 5.3.

Lemma 4.6. *If $f\in$BMO$_L$, then $\mu_f(x,t)=|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)|^2\frac{dxdt}{t}$ is a Carleson measure with $|||\mu_f|||_c\sim\|f\|_{\mathrm{BMO}_L}^2$.*

Proof. We will prove that there exists a positive constant $c>0$ such that for any ball $B=B(x_B,r_B)$ on $\mathbb{R}^n$,

$$\iint_{\widehat{B}}\big|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)\big|^2\frac{dxdt}{t}\le c|B|\|f\|_{\mathrm{BMO}_L}^2. \tag{4.12}$$

Note that

$$Q_{t^m}(\mathcal{I}-P_{t^m})=Q_{t^m}(\mathcal{I}-P_{t^m})(\mathcal{I}-P_{r_B^m})+Q_{t^m}(I-P_{t^m})P_{r_B^m}.$$

Hence, (4.12) follows from the following estimates (4.13) and (4.14):

$$\iint_{\widehat{B}}|Q_{t^m}(\mathcal{I}-P_{t^m})(\mathcal{I}-P_{r_B^m})f(x)|^2\frac{dxdt}{t}\le c|B|\|f\|_{\mathrm{BMO}_L}^2 \tag{4.13}$$

and

$$\iint_{\widehat{B}}|Q_{t^m}(\mathcal{I}-P_{t^m})P_{r_B^m}f(x)|^2\frac{dxdt}{t}\le c|B|\|f\|_{\mathrm{BMO}_L}^2. \tag{4.14}$$

We will prove these two estimates by adapting the argument in pp. 85-86 of [21]. To prove (4.13), let us consider the square function $\mathcal{G}f$ given by

$$\mathcal{G}(f)(x)=\Big(\int_0^{\infty}|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)|^2\frac{dt}{t}\Big)^{1/2}.$$

From (3.2), the function $\mathcal{G}(f)$ is bounded on L^2. Let $b_1 = (\mathcal{I} - P_{r_B^m})f\chi_{2B}$ and $b_2 = (\mathcal{I} - P_{r_B^m})f\chi_{(2B)^c}$. Using the properties (3.13) and (3.11), we obtain

$$\begin{aligned}
&\iint_{\widehat{B}} |Q_{t^m}(\mathcal{I} - P_{t^m})b_1(x)|^2 \frac{dxdt}{t} \\
&\quad\leq \iint_{\mathbb{R}^{n+1}_+} |Q_{t^m}(\mathcal{I} - P_{t^m})b_1(x)|^2 \frac{dxdt}{t} \\
&\quad\leq c\|b_1\|^2_{L^2(\mathbb{R}^n)} \\
&\quad= c\int_{2B} |(\mathcal{I} - P_{r_B^m})f(x)|^2 dx \\
&\quad\leq c\Big(\int_{2B} |(\mathcal{I} - P_{r_{2B}^m})f(x)|^2 dx + |B| \cdot \sup_{x\in 2B} |P_{r_B^m}f(x) - P_{r_{2B}^m}f(x)|^2\Big) \\
&\quad\leq c|B|\|f\|^2_{2,\mathrm{BMO}_L} + c|B|\|f\|^2_{\mathrm{BMO}_L} \quad \text{(using (3.13) and (3.11))} \\
&\quad\leq c|B|\|f\|^2_{\mathrm{BMO}_L} \quad \text{(using the equivalence of } p \text{ norms in (3.13)).}
\end{aligned} \tag{4.15}$$

On the other hand, for any $x \in B$ and $y \in (2B)^c$, one has $|x - y| \geq r_B$. By (3.6) and the property (3.12),

$$\begin{aligned}
|Q_{t^m}(\mathcal{I} - P_{t^m})b_2(x)| &\leq c\int_{\mathbb{R}^n\setminus 2B} \frac{t^\epsilon}{(t + |x-y|)^{n+\epsilon}} |(\mathcal{I} - P_{r_B^m})f(y)|dy \\
&\leq c\Big(\frac{t}{r_B}\Big)^\epsilon \int_{\mathbb{R}^n} \frac{r_B^\epsilon}{(r_B + |x-y|)^{n+\epsilon}} |(\mathcal{I} - P_{r_B^m})f(y)|dy \\
&\leq c\Big(\frac{t}{r_B}\Big)^\epsilon \|f\|_{\mathrm{BMO}_L}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint_{\widehat{B}} |Q_{t^m}(\mathcal{I} - P_{t^m})b_2(x)|^2 \frac{dxdt}{t} &\leq \frac{c}{r_B^{2\epsilon}} \iint_{\hat{B}} t^{2\epsilon} \ \frac{dxdt}{t} \|f\|^2_{\mathrm{BMO}_L} \\
&\leq c|B|\|f\|^2_{\mathrm{BMO}_L}.
\end{aligned}$$

This, together with (4.15), give the estimate (4.13).

Let us prove (4.14). Noting that for $0 < t < r_B$, it follows from the property (3.11) that for any $x \in \mathbb{R}^n$,

$$|P_{\frac{1}{2}r_B^m} f(x) - P_{(t^m + \frac{1}{2}r_B^m)} f(x)| \leq c\|f\|_{\mathrm{BMO}_L}.$$

By (3.6), the kernel $k_{t,r_B}(x, y)$ of the operator $Q_{t^m}P_{\frac{1}{2}r_B^m} = \frac{t^m}{t^m + \frac{1}{2}r_B^m} Q_{(t^m + \frac{1}{2}r_B^m)}$ satisfies

$$|k_{t,r_B}(x, y)| \leq c\Big(\frac{t}{r_B}\Big)^m \frac{r_B^\epsilon}{(r_B + |x - y|)^{n+\epsilon}}.$$

Using the commutative property of the semigroup $\{P_t\}_{t>0}$ and the estimate (3.6), we then obtain

$$\begin{aligned}
&|Q_{t^m}(I - P_{t^m})P_{r_B^m} f(x)| = |Q_{t^m}P_{\frac{1}{2}r_B^m}(P_{\frac{1}{2}r_B^m} - P_{(t^m + \frac{1}{2}r_B^m)})f(x)| \\
&\quad\leq c\Big(\frac{t}{r_B}\Big)^m \int_{\mathbb{R}^n} \frac{r_B^\epsilon}{(r_B + |x-y|)^{n+\epsilon}} |(P_{\frac{1}{2}r_B^m} - P_{(t^m + \frac{1}{2}r_B^m)})f(y)|dy \\
&\quad\leq c\Big(\frac{t}{r_B}\Big)^m \|f\|_{\mathrm{BMO}_L}.
\end{aligned}$$

Hence,

$$\begin{aligned}\iint_{\widehat{B}} |Q_{t^m}(I-P_{t^m})P_{r_B^m})f(x)|^2 \frac{dxdt}{t} &\leq \frac{c}{r_B^m}\iint_{\hat{B}} t^{m-1}\, dxdt\|f\|^2_{\mathrm{BMO}_L}\\ &\leq c|B|\|f\|^2_{\mathrm{BMO}_L},\end{aligned}$$

which gives the estimate (4.14). Hence, the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 3.2 is complete. ☐

This lemma, together with the estimate (3.12), give the following result. We leave the details of the proof to the reader.

Corollary 4.7. *Assume that* $2 \leq q < \infty$. *For any* $f \in \mathrm{BMO}_L$,

$$\mu_f(x,t) = \big|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)\big|^q \frac{dxdt}{t}$$

is a Carleson measure on $\mathbb{R}^{n+1}_+$ *with* $\||\mu_f\||_c \sim \|f\|^q_{\mathrm{BMO}_L}$.

5. Proofs of Theorems 3.1 and 3.2

5.1. **An identity related to Carleson measures.** Suppose that f is a function in $\mathcal{M}$ such that $\mu_f(x,t) = |Q^*_{t^m}(\mathcal{I}-P^*_{t^m})f(x)|^2 \frac{dxdt}{t}$ is a Carleson measure and g is an L-molecule of H^1_L. Let

$$(5.1) \qquad F(x,t) = Q^*_{t^m}(\mathcal{I}-P^*_{t^m})f(x) \quad \text{and} \quad G(x,t) = Q_{t^m}g(x), \quad (x,t)\in\mathbb{R}^{n+1}_+.$$

We first establish the following identity, which will play an important role in the proof of Theorems 3.1 and 3.2.

Proposition 5.1. *For any functions* F, G *defined as in* (5.1), *we have the following identity with constant* $b_m = \frac{36}{5}m$:

$$(5.2) \qquad \int_{\mathbb{R}^n} f(x)g(x)dx = b_m \int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dxdt}{t}.$$

As a consequence, for any $f \in \mathrm{BMO}_{L^*}$ *and* $g \in H^1_L \cap L^2$, *the above identity* (5.2) *holds.*

Proof. For any L-molecule g of H^1_L, we first observe that $\mathcal{A}(G)(x) = \mathcal{A}(Q_{t^m}g)(x) \in L^1$, where the mapping $\mathcal{A}$ is given in (4.1). Since $\mu_f(x,t) = |Q^*_{t^m}(\mathcal{I}-P^*_{t^m})f(x)|^2 \frac{dxdt}{t}$ is a Carleson measure, then by (a) of Proposition 4.1 and the dominated convergence theorem, the following integral converges absolutely and satisfies

$$\int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dxdt}{t} = \lim_{\delta\to 0}\lim_{N\to\infty}\int_\delta^N \int_{\mathbb{R}^n} F(x,t)G(x,t)\frac{dxdt}{t}.$$

Next, by Fubini's theorem, together with the commutative property of the semi-group $\{P_t\}_{t>0}$, we have

$$\int_{\mathbb{R}^n} Q^*_{t^m}(\mathcal{I}-P^*_{t^m})f(x)Q_{t^m}g(x)dx = \int_{\mathbb{R}^n} f(y)Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)dy, \qquad \forall t>0.$$

Without loss of generality, we assume that $g(y) = \int_0^\infty Q_{t^m}(a(t,\cdot))(y)\frac{dt}{t}$ where $a(t,z)$ is a T^1_2-atom supported in $\widehat{B}$, and the ball $B = B(z_0, r_B)$ is centered at z_0 and of radius r_B. We have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} & F(x,t)G(x,t)\frac{dxdt}{t} \\ &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_\delta^N \Big[\int_{\mathbb{R}^n} f(y)Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)dy\Big]\frac{dt}{t} \\ &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f(y)\Big[\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big]dy \\ &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_1(y)\Big[\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big]dy \\ &\quad + \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_2(y)\Big[\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big]dy \\ &= \mathrm{I}+\mathrm{II}, \end{aligned} \tag{5.3}$$

where $f_1 = f\chi_{4B}$ and $f_2 = f\chi_{(4B)^c}$.

We first consider the term I. From (a) of Lemma 4.3, the function $g \in L^2$. Since L has a bounded H_∞-calculus on L^2, we obtain

$$g = \lim_{\delta\to 0}\lim_{N\to\infty} b_m\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})(g)\frac{dt}{t}$$

in L^2, where $b_m = \frac{36}{5}m$ is the constant such that $1 = b_m\int_0^\infty t^{2m}e^{-2t^m}(1-e^{-t^m})\frac{dt}{t}$. See [26]. Since $f \in \mathcal{M}$, (3.3) ensures that $f_1 \in L^2$. Hence

$$\begin{aligned}\mathrm{I} &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_1(y)\Big[\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})(g)(y)\frac{dt}{t}\Big]dy \\ &= b_m^{-1}\int_{\mathbb{R}^n} f_1(y)g(y)dy.\end{aligned}$$

In order to estimate the term II, we need to show that for all $y \notin 4B$, there exists a constant $c = c(a, L)$ such that

$$\sup_{\delta>0,\ N>0}\Big|\int_\delta^N Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big| \le c(1+|y-z_0|)^{-(n+\epsilon)}. \tag{5.4}$$

Let us verify (5.4). Let

$$\Psi_{t,s}(L)g(y) = (2t^m+s^m)^3\Big(\frac{d^3P_r}{dr^3}\Big|_{r=2t^m+s^m}(\mathcal{I}-P_{t^m})g\Big)(y).$$

By (3.1), we have

$$\begin{aligned}\Big|\int_\delta^N & Q^2_{t^m}(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big| \\ &= \Big|\int_\delta^N\int_0^\infty Q^2_{t^m}Q_{s^m}(\mathcal{I}-P_{t^m})a(s,\cdot)(y)\frac{ds}{s}\frac{dt}{t}\Big| \\ &\le c\int_\delta^N\int_0^{r_B}\frac{t^{2m}s^m}{(t^m+s^m)^3}|\Psi_{t,s}(L)a(s,\cdot)(y)|\frac{ds}{s}\frac{dt}{t} \\ &\le c\int_\delta^N\int_0^{r_B}\int_{B(x_0,r_B)}\frac{t^{2m}s^m}{(t^m+s^m)^3}\frac{(t+s)^\epsilon}{(t+s+|y-z|)^{n+\epsilon}}|a(s,z)|\frac{dzds}{s}\frac{dt}{t}.\end{aligned}$$

Note that for $y \notin 4B$, we have $|y-z| \geq |y-z_0|/2$. Using the inequality

$$\frac{t^{2m}s^m(t+s)^\epsilon}{(t^m+s^m)^3} \leq c\min\Big((ts)^{\epsilon/2}, t^{-\epsilon/2}s^{3\epsilon/2}\Big),$$

together with Hölder's inequality and elementary integration, it can be verified that there exists a positive constant c independent of $\delta, N > 0$ such that for all $y \notin 4B$,

$$\begin{aligned} \Big|\int_\delta^N Q_{t^m}^2(\mathcal{I}-P_{t^m})g(y)\frac{dt}{t}\Big| &\leq cr_B^{\epsilon+\frac{n}{2}}\Big(\int_{\widehat{B}}|a(s,z)|^2\frac{dzds}{s}\Big)^{1/2}|y-z_0|^{-(n+\epsilon)} \\ &\leq cr_B^\epsilon|y-z_0|^{-(n+\epsilon)}. \end{aligned}$$

Estimate (5.4) then follows readily.

We now estimate the term II. For $f \in \mathcal{M}$, it follows from (3.3) that the function $f_2 \in L^2((1+|x|)^{-(n+\epsilon)}dx)$. The estimate (5.4) implies that there exists a constant $c > 0$ such that

$$\sup_{\delta>0,\ N>0}\int_{\mathbb{R}^n}\Big|f_2(y)\int_\delta^N Q_{t^m}^2(\mathcal{I}-P_{t^m})(g)\frac{dt}{t}(y)\Big|dy \leq c.$$

This allows us to pass the limit inside the integral of II. Hence

$$\begin{aligned} \mathrm{II} &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_2(y)\Big[\int_\delta^N Q_{t^m}^2(\mathcal{I}-P_{t^m})(g)(y)\frac{dt}{t}\Big]dy \\ &= \int_{\mathbb{R}^n} f_2(y)\lim_{\delta\to 0}\lim_{N\to\infty}\Big[\int_\delta^N Q_{t^m}^2(\mathcal{I}-P_{t^m})(g)(y)\frac{dt}{t}\Big]dy \\ &= b_m^{-1}\int_{\mathbb{R}^n} f_2(y)g(y)dy. \end{aligned}$$

Combining the estimates of I and II, we obtain the identity (5.2). The proof of Proposition 5.1 is complete. □

5.2. **Proof of Theorem 3.1.** First, we prove (i) of Theorem 3.1. Note that for any $g \in H^1_{\mathrm{L}} \cap L^2$ and $f \in \mathrm{BMO}_{L^*}$, the assumptions of Proposition 5.1 are satisfied since we have

$$\mathcal{A}\big(Q_{t^m}g\big)(x) = \mathcal{S}_L(g)(x) \in L^1$$

and by Lemma 4.6,

$$\mathcal{C}\big(Q_{t^m}^*(\mathcal{I}-P_{t^m}^*)f\big)(x) \in L^\infty$$

with $\|\mathcal{C}\big(Q_{t^m}^*(\mathcal{I}-P_{t^m}^*)f\big)(x)\|_{L^\infty} \leq c|||\mu_f|||_c^{1/2}$.

Let $b_m = \frac{36}{5}m$ be the constant in Proposition 5.1. Applying the identity (5.2), together with (a) of Proposition 4.1, we obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} f(x)g(x)dx\Big| &= b_m\Big|\int_{\mathbb{R}^{n+1}_+}\big(Q_{t^m}^*(\mathcal{I}-P_{t^m}^*)f\big)(x)\big(Q_{t^m}g\big)(x)\frac{dxdt}{t}\Big| \\ &\leq c\int_{\mathbb{R}^n}\mathcal{C}\big(Q_{t^m}^*(\mathcal{I}-P_{t^m}^*)f\big)(x)\mathcal{A}\big(Q_{t^m}g\big)(x)dx \\ &\leq c|||\mu_f|||_c^{1/2}\int_{\mathbb{R}^n}\mathcal{S}_L(g)(x)dx \\ &\leq c\|f\|_{\mathrm{BMO}_{L^*}}\|g\|_{H^1_L}, \end{aligned}$$

and thus $\mathrm{BMO}_{L^*} \subset \big(H^1_L \cap L^2\big)'$. Since $H^1_L \cap L^2$ is dense in H^1, (i) of Theorem 3.1 follows from a standard density argument.

We now prove (ii) of Theorem 3.1. We define

$$\Omega_L = \Big\{ h(x,t) : h(x,t) = Q_t g(x) \text{ for some } g \in H^1_L \Big\}.$$

By the definition of H^1_L, we have that $\Omega_L \subset T^1_2$, where T^1_2 is the standard tent space. See Section 4.1. Note that by (b) of Lemma 4.3,

$$\mathcal{R}(h)(x) = 4m \int_0^\infty Q_{t^m}(h_t)(x) \frac{dt}{t} \in H^1_L$$

for every $h_t(x) \in T^1_2$.

On the other hand, from (4.6) we have that for any $g \in H^1_L \cap L^2$,

$$g(x) = 4m \int_0^\infty Q_{t^m} Q_{t^m} g(x) \frac{dt}{t}.$$

Therefore, for each continuous linear functional ℓ on H^1_L, we obtain

$$\ell(g) = \ell \circ \mathcal{R} \circ Q_{t^m}(g) \tag{5.5}$$

for all $g \in H^1_L \cap L^2$. Furthermore, $\ell \circ \mathcal{R}$ is a continuous linear functional on Ω_L which satisfies

$$\|\ell \circ \mathcal{R}\|_{T^1_2 \to \mathbb{C}} \le \|\ell\|_{(H^1_L)'} \cdot \|\mathcal{R}\|_{T^1_2 \to H^1_L} \le c < \infty.$$

Applying the Hahn-Banach theorem, we can extend $\ell \circ \mathcal{R}$ to a continuous linear functional on T^1_2. Note that by (b) of Proposition 4.1, the dual of T^1_2 is equivalent to T^∞_2. By restricting attention to Ω_L, we can conclude that if ℓ is a continuous linear functional on H^1_L, then it follows from (5.5) that there exists a $w_t(x) \in T^\infty_2$ such that

$$\begin{aligned} \ell(g) &= l \circ \mathcal{R} \circ Q_{t^m}(g) \\ &= \int_{\mathbb{R}^{n+1}_+} w_t(x) Q_{t^m} g(x) \frac{dxdt}{t} \\ &= \int_{\mathbb{R}^n} \Big(\int_0^\infty Q^*_{t^m} w_t(x) \frac{dt}{t} \Big) g(x) dx \\ &\overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x) g(x) dx, \end{aligned} \tag{5.6}$$

where $f(x) = \int_0^\infty Q^*_{t^m} w_t(x) \frac{dt}{t}$.

We now prove that $f \in \mathrm{BMO}_{L^*}$. For any ball $B = B(x_B, r_B)$, it follows from (5.6) and Lemma 4.5 that

$$\begin{aligned} \Big(\int_B |f - P^*_{r_B^m} f|^2 dx \Big)^{1/2} &= \sup_{\|g\|_{L^2(B)} \le 1} \Big| \int_{\mathbb{R}^n} \big(f(x) - P^*_{r_B^m} f(x) \big) g(x) dx \Big| \\ &= \sup_{\|g\|_{L^2(B)} \le 1} \Big| \int_{\mathbb{R}^n} f(x) (\mathcal{I} - P_{r_B^m}) g(x) dx \Big| \\ &\le \sup_{\|g\|_{L^2(B)} \le 1} \Big| \ell\big((\mathcal{I} - P_{r_B^m}) g \big) \Big| \\ &\le \|\ell\| \sup_{\|g\|_{L^2(B)} \le 1} \|(\mathcal{I} - P_{r_B^m}) g\|_{H^1_L} \\ &\le c \|\ell\| |B|^{1/2}. \end{aligned}$$

This proves that $f \in \mathrm{BMO}_{L^*}$ with $\|f\|_{\mathrm{BMO}_{L^*}} \leq c\|\ell\|$. Hence, the proof of (ii) of Theorem 3.1 is complete.

Proof of (c) of Lemma 4.3. We now use Theorem 3.1 to prove property (c) of Lemma 4.3. As in Definition (4.9), we consider the operator π_{L^*} associated with L^* defined on T_2^p by

$$\pi_{L^*}(f)(x) = 4m \int_0^\infty Q^*_{t^m}(f(\cdot,t))(x)\frac{dt}{t}. \tag{5.7}$$

In order to prove (c) of Lemma 4.3, it suffices to prove that π_{L^*} is bounded from T_2^∞ to BMO_{L^*}. Note that for any $f \in T_2^\infty$ and $g \in H^1_L$,

$$\begin{aligned}
\big|\langle \pi_{L^*}(f), g\rangle\big| &= \Big|\int_{\mathbb{R}^{n+1}_+} f(x,t)Q_{t^m}g(x)\frac{dxdt}{t}\Big| \\
&\leq c\int_{\mathbb{R}^n} \mathcal{C}(f)(x)\mathcal{A}(Q_{t^m}g)(x)dx \\
&\leq c\|\mathcal{C}(f)\|_\infty \int_{\mathbb{R}^n} \mathcal{S}_L(g)(x)dx \\
&\leq c|||f|||_{T_2^\infty}\|g\|_{H^1_L}.
\end{aligned}$$

Since Theorem 3.1 shows that the predual space of BMO_{L^*} is the Hardy space H^1_L, property (c) of Lemma 4.3 follows readily. □

Corollary 5.2. *The spaces* BMO_L *and* BMO_{L^*} *are Banach spaces.*

Proof. Note that $H^1_{L^*}$ is a normed linear space. It follows from Theorem 3.1 and a standard argument of functional analysis that $\mathrm{BMO}_L = \big(H^1_{L^*}\big)'$ is a Banach space. See, for example, page 111 of [36]. The same argument holds for the space BMO_{L^*}. Hence, the proof of Corollary 5.2 is complete. □

5.3. **Proof of Theorem 3.2.** In Lemma 4.6, we proved the implication (i) $\Rightarrow$ (ii) of Theorem 3.2. We now prove the implication (ii) $\Rightarrow$ (i). Suppose that $f \in \mathcal{M}$ such that $\mu_f(x,t) = |Q_{t^m}(\mathcal{I} - P_{t^m})f(x)|^2\frac{dxdt}{t}$ is a Carleson measure. For any $g \in H^1_{L^*} \cap L^2$, using the identity (5.2) with L^* in place of L, we obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} g(x)f(x)dx\Big| &= b_m\Big|\int_{\mathbb{R}^{n+1}_+} \big(Q_{t^m}(\mathcal{I} - P_{t^m})f\big)(x)\big(Q^*_{t^m}g\big)(x)\frac{dxdt}{t}\Big| \\
&\leq c\int_{\mathbb{R}^n} \mathcal{C}\big(Q_{t^m}(\mathcal{I} - P_{t^m})f\big)(x)\mathcal{A}(Q^*_{t^m}g)(x)dx \\
&\leq c|||\mu_f|||_c^{1/2}\int_{\mathbb{R}^n} \mathcal{A}(Q^*_{t^m}g)(x)dx \\
&\leq c|||\mu_f|||_c^{1/2}\int_{\mathbb{R}^n} \mathcal{S}_{L^*}(g)(x)dx \\
&\leq c|||\mu_f|||_c^{1/2}\|g\|_{H^1_{L^*}},
\end{aligned}$$

which gives $f \in \big(H^1_{L^*} \cap L^2\big)'$ and thus $f \in \mathrm{BMO}_L$ with $\|f\|_{\mathrm{BMO}_L} \leq c|||\mu_f|||_c^{1/2}$. The proof of Theorem 3.2 is complete.

6. The H^1_L and BMO_L spaces associated with some differential operators

In this section, we conduct further study on the Hardy and BMO spaces associated with some differential operators such as the divergence form operators and the Schrödinger operators on $\mathbb{R}^n$ (Section 6.1). We will also discuss the inclusion between the classical BMO space and the BMO_L spaces associated with operators (Section 6.2).

Note first that smooth functions with compact support do not necessarily belong to $H^1_{L^*}$ in general. The reason is that $(\mathrm{BMO}_L, \|\cdot\|_{\mathrm{BMO}_L})$ is a Banach space, with the norm vanishing on the kernel space $\mathcal{K}_L$ of (3.10) defined by

$$\mathcal{K}_L = \Big\{ f \in \mathcal{M} : P_t f(x) = f(x) \text{ for almost all } \ x \in \mathbb{R}^n \text{ and all } t > 0 \Big\}; \tag{6.1}$$

hence if $g \in H^1_{L^*}$, then g satisfies the cancellation condition

$$\int_{\mathbb{R}^n} g(x) f(x) dx = 0$$

for all $f \in \mathcal{K}_L$.

6.1. **Kernel spaces $\mathcal{K}_L$ of some differential operators.** We first note that the classical BMO space is a Banach space modulo the constant functions. In this section, we will study the kernel spaces $\mathcal{K}_L$ of BMO_L spaces associated with second-order uniformly elliptic operators of divergence form and with Schrödinger operators with certain potentials.

6.1.1. *Second-order elliptic operators of divergence form.* Let $A = A(x)$ be an $n \times n$ matrix of bounded complex coefficients defined on $\mathbb{R}^n$ which satisfies the ellipticity (or "accretivity") condition

$$\lambda |\xi|^2 \le \mathrm{Re}\, A\xi \cdot \bar{\xi} \equiv \mathrm{Re} \sum_{i,j} a_{ij}(x) \xi_j \bar{\xi}_i, \qquad \|A\|_\infty \le \Lambda \tag{6.2}$$

for $\xi \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \le \Lambda < \infty$. We define the second-order divergence form operator

$$Lf = -\mathrm{div}(A \nabla f) \tag{6.3}$$

on $L^2(\mathbb{R}^n)$, which we interpret in the weak sense via a sesquilinear form. See [3].

Since L is maximal accretive, it has a bounded H_∞-calculus on $L^2(\mathbb{R}^n)$ ([1], [3]); i.e., L satisfies assumption (b) of Section 3.1. Note that when A has real entries, or when the dimension $n = 1$ or 2 in the case of complex entries, the operator L generates an analytic semigroup e^{-tL} on $L^2(\mathbb{R}^n)$ with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound; that is,

$$|p_t(x, y)| \le \frac{C}{t^{n/2}} e^{-c \frac{|x-y|^2}{t}} \tag{6.4}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$. In this case, L satisfies assumption (a) of Section 3.1. For dimensions 5 and higher, it is known that the Gausssian bounds (6.4) may fail. See [2] and Chapter 1 of [3].

Recall that $f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is said to be L-harmonic if it is a weak solution of the equation $Lf = 0$, i.e., for any $\varphi \in C^1_0(\mathbb{R}^n)$,

$$\langle Lf, \varphi\rangle = \int_{\mathbb{R}^n} A\nabla f \cdot \overline{\nabla\varphi}\, dx = 0.$$

For any real number $d \geq 0$, one denotes

$$\mathcal{H}_d(L) = \left\{ f \in W^{1,2}_{\rm loc}(\mathbb{R}^n) : Lf = 0 \text{ and } \ |f(x)| = O(|x|^d) \ \text{ as } |x| \to \infty \right\}, \tag{6.5}$$

which is the space of all polynomial growth L-harmonic functions of degree at most d. See [23] and [24].

For second-order uniformly elliptic operators with real measurable coefficients, De Giorgi-Nash-Moser theory asserts that any weak solution f must be C^α for some $0 < \alpha < 1$. A global version of this theory implies that there exists $0 < \alpha < 1$ such that any L-harmonic function f satisfying the growth condition

$$|f(x)| = O(|x|^\alpha)$$

as $|x| \to \infty$ must be a constant function. This means that for all $0 \leq d \leq \alpha < 1$, the dimension of $\mathcal{H}_d(L)$ is 1. In [23] and [24], P. Li and J.P. Wang proved that for each real number $d \geq 1$, the space $\mathcal{H}_d(L)$ is of finite dimension. More specifically, there exists a constant c depending only on n, λ and Λ in (6.2) such that the dimension $h_d(L)$ of $\mathcal{H}_d(L)$ satisfies

$$h_d(L) \leq c d^{n-1}.$$

For any fixed constant $\epsilon > 0$ in (2.3), we let

$$\mathcal{H}_{\epsilon,L} = \bigcup_{d:\ 0 \leq d \leq [\frac{n+\epsilon}{2}]+1} \mathcal{H}_d(L).$$

Proposition 6.1. *Let L be the divergence form operator as in* (6.3)*. Assume that the operator L satisfies assumption* (a) *in Section* 3.1 *for $m = 2$ and some $\epsilon > 0$ as in* (2.3)*. Then*

(i) *The results of Theorems* 3.1 *and* 3.2 *hold for the operator L.*

(ii) *The following inclusion between the kernel space $\mathcal{K}_L$ and the space $\mathcal{H}_d(L)$ holds:*

(ii)$_1$ $(\mathcal{K}_L \cap \mathcal{M}_\epsilon) \subset \mathcal{H}_{\epsilon,L}$*;*

(ii)$_2$ *Conversely, we have that $\mathcal{H}_d(L) \subset (\mathcal{K}_L \cap \mathcal{M}_{2\epsilon})$ for any $0 \leq d < \epsilon$.*

(iii) *If the semigroup e^{-tL} has a kernel $p_t(x, y)$ satisfying the Gaussian upper bound* (6.4)*, then*

$$\mathcal{K}_L = \bigcup_{d:\ 0 \leq d < \infty} \mathcal{H}_d(L).$$

(iv) *In the case that L has real coefficients, then for each $\epsilon > 0$, the kernel space $(\mathcal{K}_L \cap \mathcal{M}_\epsilon)$ has finite dimension.*

In order to prove Proposition 6.1, we need the following Lemmas 6.2 and 6.3. For any two closed sets E and F of $\mathbb{R}^n$, we denote the distance between E and F by $\mathrm{dist}(E, F)$. We first have

Lemma 6.2. *Let L be the divergence form operator as in* (6.3) *with ellipticity constants λ and Λ as in* (6.2)*. For any two closed sets E and F of $\mathbb{R}^n$, the following*

L^2 off-diagonal estimate of Gaffney type holds:

$$\int_F |t^{\frac{1}{2}}\nabla e^{-tL}f(x)|^2dx \leq Ce^{-\frac{\mathrm{dist}(E,F)^2}{ct}}\int_E |f(x)|^2dx, \quad \mathrm{supp} f \subset E, \tag{6.6}$$

where $c > 0$ depends only on λ, Λ, and C depends on n, λ, Λ.

Proof. For the proof, we refer to Lemma 2.1 of [20]. See also Lemma 2.1 of [1].

Lemma 6.3. *Let L be the divergence form operator as in* (6.3). *Assume that the operator L satisfies the assumption* (a) *in Section* 3.1 *for $m = 2$ and some $\epsilon > 0$ as in* (2.3). *Then for any $f \in \mathcal{M}_\epsilon$,*

(i) *for any $t > 0$, there exists a constant c_t which depends on t such that*

$$|e^{-tL}f(x)| \leq c_t(1 + |x|)^{(n+\epsilon)/2}\|f\|_{\mathcal{M}_\epsilon}$$

for almost all $x \in \mathbb{R}^n$.

(ii) *For almost all $x \in \mathbb{R}^n$,*

$$\lim_{t\to 0^+} e^{-tL}f(x) = f(x);$$

(iii) *For any $t > 0$, $e^{-tL}f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. The proof of (i) is a simple consequence of direct integration using the decay of heat kernels (2.2), (2.3) and the triangle inequality. We omit the details.

We now prove (ii). We fix a ball B of radius r_B and set ourselves the task of showing that $\lim\limits_{t\to 0^+} e^{-tL}f(x) = f(x)$ for almost every $x \in B$. Let B_1 be the ball with the same centre as B and with radius $r_B + 1$. Let $f_1(x) = f(x)$ for $x \in B_1$ and 0 for $x \notin B_1$; and let $f = f_1 + f_2$. Then $f_1 \in L^2(B_1)$. Note that under the conditions (2.2) and (2.3), L satisfies the conservation property of the semigroup $e^{-tL}(1) = 1$ for all $t > 0$. See page 55 of [3]. By a standard argument using the heat kernel bounds, for example, Section 2, Chapter 3 of [30] for the case of convolution operators, we have that $\lim\limits_{t\to 0^+} e^{-tL}f_1(x) = f_1(x)$ for almost every $x \in B$. However for any $x \in B$ and $y \in (B_1)^c$, we have $|x - y| \geq 1$, and then by the conditions (2.2) and (2.3),

$$\begin{aligned} |e^{-tL}f_2(x)| &\leq c\int_{|x-y|\geq 1} h_t(x, y)|f(y)|dy \\ &\leq c_x t^{\epsilon/4}\|f\|_{\mathcal{M}_\epsilon} \to 0 \end{aligned}$$

as $t \to 0^+$. Hence, $\lim_{t\to 0^+} e^{-tL}f_2(x) = 0$ for almost all $x \in B$. Thus (ii) is proved.

For the proof of (iii), it suffices to prove that for any ball $B = B(0, r_B)$ with its center at the origin and of radius r_B, there exists a constant $c = c(t, r_B)$ which depends on t and r_B such that

$$\||\nabla e^{-tL}f|\|_{L^2(B)} \leq c\|f\|_{\mathcal{M}_\epsilon}. \tag{6.7}$$

Let us prove (6.7). For any integer $l \geq 0$, we denote by 2^lB the ball with center at the origin and of radius 2^lr_B, except that the notation $2^{-1}B$ means the empty set $\emptyset$. We define $f_l(x) = f\chi_{2^lB\backslash 2^{l-1}B}(x)$ for any $l \geq 0$, and write $f(x) = \sum\limits_{l=0}^{\infty} f_l(x)$. Since $f \in \mathcal{M}_\epsilon$, we have that for $l \geq 0$, $\|f\|_{L^2(2^lB\backslash 2^{l-1}B)} \leq c(1+2^lr_B)^{(n+\epsilon)/2}\|f\|_{\mathcal{M}_\epsilon}$.

Using Lemma 6.2, one has

$$
\begin{aligned}
&\||\nabla e^{-tL} f|\|_{L^2(B)} \\
&\quad \le \sum_{l=0}^{\infty} \||\nabla e^{-tL} f_l|\|_{L^2(B)} \\
&\quad \le ct^{-1/2}(1+r_B)^{(n+\epsilon)/2}\|f\|_{\mathcal{M}_\epsilon} + ct^{-1/2}\sum_{l=2}^{\infty} e^{-\frac{(2^{l-2}r_B)^2}{ct}}\|f\|_{L^2(2^lB\setminus 2^{l-1}B)} \\
&\quad \le c\|f\|_{\mathcal{M}_\epsilon} + ct^{-1/2}\sum_{l=2}^{\infty} e^{-\frac{(2^l r_B)^2}{16ct}}(2^l r_B)^{(n+\epsilon)/2}\|f\|_{\mathcal{M}_\epsilon} \\
&\quad \le c\|f\|_{\mathcal{M}_\epsilon} + ct^{-1/2} r_B^{(n+\epsilon)/2}\sum_{l=2}^{\infty} e^{-c_{t,B}2^{2l}} 2^{(n+\epsilon)l/2}\|f\|_{\mathcal{M}_\epsilon} \quad (\text{where } c_{t,B} = \frac{r_B^2}{16ct}) \\
&\quad \le c\|f\|_{\mathcal{M}_\epsilon} < \infty.
\end{aligned}
$$

This shows (6.7), and hence $e^{-tL}f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$. The proof of Lemma 6.3 is complete. $\square$

Remark 6.4. Property (iii) of Lemma 6.3 holds for any differential operator which satisfies the Gaffney estimate (6.6) and assumption (a) in Section 3.1 for $m = 2$. This will be used in the proof of Proposition 6.5 below.

Proof of Proposition 6.1. For the proof of (i), it is straightforward that L satisfies the assumptions (a) and (b) of Section 3.1; hence Theorems 3.1 and 3.2 hold.

We now prove (ii)$_1$. If $f \in (\mathcal{K}_L \cap \mathcal{M}_\epsilon)$, then $f = e^{-tL}f$ for any $t > 0$ and $f \in \mathcal{M}_\epsilon$. It follows from (i) of Lemma 6.3 that $f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ and $|f(x)| = O(|x|^{(n+\epsilon)/2})$. Because of the growth of f, we use a standard approximation argument through a sequence f_k as follows. For any $k \in \mathbb{N}$, we denote by η_k a standard C^∞ cut-off function which is 1 inside the ball $B(0,k)$, zero outside $B(0,k+1)$, and let $f_k = f\eta_k \in W^{1,2}(\mathbb{R}^n)$. Since $f = e^{-tL}f$, we have that for any $\varphi \in C^1_0(\mathbb{R}^n)$,

$$
\begin{aligned}
\langle Lf, \varphi\rangle = \langle Le^{-tL}f, \varphi\rangle &= \lim_{k\to\infty}\langle Le^{-tL}f_k, \varphi\rangle \\
&= -\lim_{k\to\infty}\langle \frac{d}{dt}e^{-tL}f_k, \varphi\rangle = -\langle \frac{d}{dt}e^{-tL}f, \varphi\rangle \\
&= -\langle \frac{d}{dt}f, \varphi\rangle = 0,
\end{aligned}
$$

which proves that $f \in \mathcal{H}_{\epsilon,L}$.

Next, we prove (ii)$_2$. Since $f \in \mathcal{H}_d(L)$ for $0 \le d < \epsilon$, we have $Lf = 0$ and $|f(x)| = O(|x|^d)$. This gives that $f \in \mathcal{M}_{2\epsilon}$. Hence (ii) of Lemma 6.3 holds. Since $Lf = 0$, we have that for any $\varphi \in C^1_0(\mathbb{R}^n)$,

$$
\begin{aligned}
\langle \frac{d}{dt}e^{-tL}f, \varphi\rangle = \lim_{k\to\infty}\langle \frac{d}{dt}e^{-tL}f_k, \varphi\rangle &= -\lim_{k\to\infty}\langle e^{-tL}Lf_k, \varphi\rangle \\
&= -\langle e^{-tL}Lf, \varphi\rangle = 0.
\end{aligned}
$$

This gives $\frac{d}{dt}e^{-tL}f = 0$ a.e; hence $e^{-tL}f(x) = \lim_{t\to 0} e^{-tL}f(x) = f(x)$, a.e. This proves that $f \in (\mathcal{K}_L \cap \mathcal{M}_{2\epsilon})$, and (ii)$_2$ is proved.

For (iii), that $\mathcal{K}_L = \bigcup_{d:\ 0\le d<\infty}\mathcal{H}_d(L)$ is a consequence of (ii). For (iv), it follows from (ii), [23] and [24] that for each $\epsilon > 0$, the kernel space $(\mathcal{K}_L \cap \mathcal{M}_\epsilon)$ has a finite dimension. The proof of Proposition 6.1 is complete. $\square$

6.1.2. *Schrödinger operators.* Let $V \in L^2_{\rm loc}(\mathbb{R}^n)$ be a nonnegative function on $\mathbb{R}^n$. The Schrödinger operator with potential V is defined by

$$L = -\triangle + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3. \tag{6.8}$$

The operator L is a self-adjoint positive definite operator; hence it has a bounded H_∞-calculus on $L^2(\mathbb{R}^n)$ ([26]). From the Feynman-Kac formula, it is well known that the kernel $p_t(x,y)$ of the semigroup e^{-tL} satisfies the estimate

$$0 \leq p_t(x,y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}. \tag{6.9}$$

However, unless V satisfies additional conditions, the heat kernel can be a discontinuous function of the space variables and the Hölder continuity estimates may fail to hold. See, for example, [10].

As in [29], a function $f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is said to be a weak solution of $Lf = 0$ in $\mathbb{R}^n$ if for any $\varphi \in C^1_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla f \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^n} V f \cdot \varphi \, dx = 0.$$

For any $d \geq 0$, one writes

$$\mathcal{H}_d(L) = \Big\{ f \in W^{1,2}_{\rm loc}(\mathbb{R}^n) : Lf = 0 \text{ and } \ |f(x)| = O(|x|^d) \ \text{ as } |x| \to \infty \Big\}$$

and

$$\mathcal{H}_L = \bigcup_{d:\ 0 \leq d < \infty} \mathcal{H}_d(L).$$

Recall that a nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to the reverse Hölder class B_q with $1 < q < \infty$ if there exists a constant $c > 0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|} \int_B V^q dx \Big)^{1/q} \leq c \Big(\frac{1}{|B|} \int_B V dx \Big) \tag{6.10}$$

holds for every ball B in $\mathbb{R}^n$.

Note that if V is a nonnegative polynomial, then $V \in B_q$ for all q, $1 < q < \infty$. If $V \in B_q$ for some $q \geq n/2$, then the fundamental solution decays faster than any power of $\frac{1}{|x|}$. See page 517 of [28]. It follows from Corollary 2.8 of [28] that $(-\triangle + V)u = 0$ in $\mathbb{R}^n$ has a unique weak solution $u = 0$ in $\mathcal{H}_L$. Hence for any $d \geq 0$,

$$\mathcal{H}_L = \mathcal{H}_d(L) = \big\{ 0 \big\}. \tag{6.11}$$

See also Proposition 2.3 of [29].

Proposition 6.5. *Let L be the Schrödinger operator as in* (6.8). *Then,*

(i) *the results of Theorems* 3.1 *and* 3.2 *hold for the operator L;*

(ii) *for any $\epsilon > 0$, we have that $(\mathcal{K}_L \cap \mathcal{M}_\epsilon) \subset \mathcal{H}_L$.*

As a consequence, if $V \in B_q$ for some $q \geq n/2$, then $\mathcal{K}_L = \{0\}$.

Proof. For the proof of (i), it is straightforward that L satisfies the assumptions (a) and (b) of Section 3.1; hence Theorems 3.1 and 3.2 hold.

We now prove (ii). Assume that $f \in (\mathcal{K}_L \cap \mathcal{M}_\epsilon)$. Let us prove that $f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$. First, for any two closed sets E and F of $\mathbb{R}^n$, we observe that L satisfies the following L^2 off-diagonal estimate of Gaffney type:

$$\int_F |t^{\frac{1}{2}} \nabla e^{-tL} f(x)|^2 dx \le c e^{-\frac{{\rm dist}(E,F))^2}{ct}} \int_E |f(x)|^2 dx, \quad {\rm supp} f \subset E.$$

The proof of this estimate for the Schrödinger operator L is similar to that of the case when L is a divergence form operator. See, for examples, Lemma 2.1 of [20] and Lemma 2.1 of [1]. Then it follows from the Gaffney estimate and Remark 6.4 that $f \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$.

Note that if $f \in (\mathcal{K}_L \cap \mathcal{M}_\epsilon)$, then $f = e^{-tL} f$ for any $t > 0$. For any $k \in \mathbb{N}$, we denote by η_k a standard C^∞ cut-off function which is 1 inside the ball $B(0,k)$, zero outside $B(0,k+1)$, and let $f_k = f\eta_k \in W^{1,2}(\mathbb{R}^n)$. Since $f = e^{-tL} f$, we have that for any $\varphi \in C^1_0(\mathbb{R}^n)$,

$$\begin{aligned}\langle Lf, \varphi\rangle = \langle Le^{-tL} f, \varphi\rangle &= \lim_{k\to\infty} \langle Le^{-tL} f_k, \varphi\rangle \\ &= -\lim_{k\to\infty} \langle \frac{d}{dt} e^{-tL} f_k, \varphi\rangle = -\langle \frac{d}{dt} e^{-tL} f, \varphi\rangle \\ &= -\langle \frac{d}{dt} f, \varphi\rangle = 0,\end{aligned}$$

which proves that $f \in \mathcal{H}_L$. The proof of Proposition 6.5 is complete. □

6.2. **Inclusion between the classical** BMO **space and** BMO_L **spaces associated with operators.** An important application of the BMO_L space is the following interpolation result of operators.

Proposition 6.6. *Assume that T is a sublinear operator which is bounded on $L^q(\mathbb{R}^n)$ for some $1 \le q < \infty$, and for any $f \in L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\|Tf\|_{\mathrm{BMO}_L} \le c\|f\|_{L^\infty}$. Then, T is bounded on $L^p(\mathbb{R}^n)$ for all $q < p < \infty$.*

Proof. For the proof, we refer to Theorem 5.2 of [16]. □

Because of this interpolation result, we would like to compare the classical BMO space with the spaces BMO_L associated with operators.

6.2.1. *A necessary and sufficient condition for* $\mathrm{BMO} \subseteq \mathrm{BMO}_L$. The following proposition is essentially Proposition 3.1 of [25].

Proposition 6.7. *Suppose L is an operator which generates a semigroup e^{-tL} with the heat kernel bounds* (2.2) *and* (2.3). *A necessary and sufficient condition for the classical space* $\mathrm{BMO} \subseteq \mathrm{BMO}_L$ *with*

$$\|f\|_{\mathrm{BMO}_L} \le c\|f\|_{\mathrm{BMO}} \tag{6.12}$$

is that for every $t > 0$, $e^{-tL}(1) = 1$ almost everywhere, that is, $\int_{\mathbb{R}^n} p_t(x,y)dy = 1$ for almost all $x \in \mathbb{R}^n$.

Proof. Assume that for every $t > 0$, $e^{-tL}(1) = 1$ almost everywhere. By Proposition 3.1 of [25], we have that $\mathrm{BMO} \subseteq \mathrm{BMO}_L$ and the estimate (6.12) holds. See also Proposition 2.5 of [16]. We now show that the condition $e^{-tL}(1) = 1$ a.e. is necessary for $\mathrm{BMO} \subseteq \mathrm{BMO}_L$. Indeed, let us consider $f(x) = 1$. Then, (6.12) implies that $\|1\|_{\mathrm{BMO}_L} = 0$, and thus for every $t > 0$, $e^{-tL}(1) = 1$ almost everywhere. □

We now give an example of $\mathrm{BMO} \subsetneqq \mathrm{BMO}_L$.

Proposition 6.8. *There exists an operator L which satisfies the assumptions* (a) *and* (b) *of Section* 3.1 *such that*

$$H^1_L \subsetneqq H^1 \quad \text{and} \quad \mathrm{BMO} \subsetneqq \mathrm{BMO}_L.$$

Proof. We recall that $\mathbb{R}^n_+$ denotes the upper-half space of $\mathbb{R}^n$, i.e.,

$$\mathbb{R}^n_+ = \Big\{(x', x_n) \in \mathbb{R}^n : \ x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, \ x_n > 0\Big\}.$$

Similarly, $\mathbb{R}^n_-$ denotes the lower-half space in $\mathbb{R}^n$.

By $\triangle_{N_+}$ (resp. $\triangle_{N_-}$) we denote the Neumann Laplacian on $\mathbb{R}^n_+$ (resp. on $\mathbb{R}^n_-$). See page 57 of [33]. The Neumann Laplacians are self-adjoint and positive definite operators. Using the spectral theory one can define the semigroup $\{\exp(-t\triangle_{N_+})\}_{t\geq 0}$ (resp. $\{\exp(-t\triangle_{N_-})\}_{t\geq 0}$) generated by the operator $\triangle_{N_+}$ (resp. $\triangle_{N_-}$). For any f defined on $\mathbb{R}^n$, we set

$$f_- = f|_{\mathbb{R}^n_-} \ \text{ and } \ f_+ = f|_{\mathbb{R}^n_+},$$

where $f|_{\mathbb{R}^n_+}$ and $f|_{\mathbb{R}^n_-}$ are restrictions of the function f to $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, respectively. Let $\triangle_N$ be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that

$$(\triangle_N f)_+ = \triangle_{N_+} f_+ \quad \text{and} \quad (\triangle_N f)_- = \triangle_{N_-} f_-$$

for all $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $f_+ \in W^{1,2}(\mathbb{R}^n_+)$ and $f_- \in W^{1,2}(\mathbb{R}^n_-)$.

Then, $\triangle_N$ generates the conservative semigroup $e^{-t\triangle_N}$ for every $t > 0$, which satisfies the assumptions (a) and (b) of Section 3.1. Moreover, it can be proved that this operator $\triangle_N$ generates the spaces $H^1_{\triangle_N}$ and $\mathrm{BMO}_{\triangle_N}$ such that $H^1_{\triangle_N} \subsetneqq H^1$ and $\mathrm{BMO} \subsetneqq \mathrm{BMO}_{\triangle_N}$. For the details, we refer the reader to [12]. $\square$

6.2.2. *A sufficient condition for* BMO_L *spaces to coincide with the classical* BMO *space.* Assume that L is a linear operator of type ω on $L^2(\mathbb{R}^n)$ with $\omega < \pi/2$; hence L generates an analytic semigroup $e^{-zL}, 0 \leq |\mathrm{Arg}(z)| < \pi/2 - \omega$. We assume that for each $t > 0$, the kernel $p_t(x, y)$ of e^{-tL} is Hölder continuous in both variables x, y and there exist positive constants m, $\beta > 0$ and $0 < \gamma \leq 1$ such that for all $t > 0$, and $x, y, h \in \mathbb{R}^n$,

$$|p_t(x, y)| \leq c\frac{t^{\beta/m}}{(t^{1/m} + |x - y|)^{n+\beta}}, \tag{6.13}$$

$$|p_t(x+h, y)-p_t(x, y)|+|p_t(x, y+h) - p_t(x, y)| \leq c|h|^\gamma \frac{t^{\beta/m}}{(t^{1/m}+|x-y|)^{n+\beta+\gamma}} \tag{6.14}$$

whenever $2|h| \leq t^{1/m} + |x - y|$; and

$$\int_{\mathbb{R}^n} p_t(x, y)dx = \int_{\mathbb{R}^n} p_t(x, y)dy = 1, \qquad \forall t > 0. \tag{6.15}$$

We have the following lemma.

Lemma 6.9. *Assume that L satisfies* (6.13) *and* (6.14)*. Then the kernel of the operator $t\frac{d}{dt}e^{-tL}$ also satisfies* (6.13) *and* (6.14) *in which the constants β and γ are replaced by some constants $0 < \beta_1 < \beta$ and $0 < \gamma_1 < \gamma$, respectively. Moreover, for any $0 < \mu < \pi/2 - \omega$ there exist constants c, $0 < \gamma_2 < \gamma$ and $0 < \beta_2 < \beta$ such that for all z with $|\mathrm{arg} z| \leq \mu$,*

$$|p_z(x, y)| \leq c\frac{|z|^{\beta_2/m}}{(|z|^{1/m} + |x - y|)^{n+\beta_2}} \tag{6.16}$$

and

$$|p_z(x+h,y)-p_z(x,y)|+|p_z(x,y+h)-p_z(x,y)| \le c|h|^{\gamma_2}\frac{|z|^{\beta_2/m}}{(|z|^{1/m}+|x-y|)^{n+\beta_2+\gamma_2}} \tag{6.17}$$

whenever $2|h| \le |z|^{1/m}+|x-y|$.

Proof. The proof of Lemma 6.9 is standard. We give a brief argument of this proof for completeness and the convenience of the reader.

Assume that the statement on $p_z(x,y)$ is proved. Then, using the Cauchy formula applied to the holomorphic function $z \to p_z(x,y)$, we obtain the desired estimates for the kernel of $t\frac{d}{dt}e^{-tL}$. See, for example, Lemma 2.5 of [5].

It remains to prove the statement on $p_z(x,y)$. An argument of Davies, as adapted in Proposition 3.3 of [15], enables one to obtain (6.16). See also Lemma 2.4 of [5].

We now prove (6.17). We only consider the part $|p_z(x+h,y)-p_z(x,y)|$ since the proof of $|p_z(x,y+h)-p_z(x,y)|$ is similar. It can be verified that it is equivalent to the following: there exist constants c and $\nu > 0$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,

$$|h|^{-\nu}\big|p_z(x+h,y)-p_z(x,y)\big| \le c|z|^{-(n+\nu)/m}. \tag{6.18}$$

Let us prove (6.18). By Lemma 17 of Chapter 1 of [3], this inequality is equivalent to the boundedness of e^{-zL} from L^1 to the homogeneous space $\dot{C}^\nu$ with the right-hand side of (6.18) being its operator norm. For $1 \le p \le q$, we denote by $\|T\|_{p,q}$ the operator norm of T from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. We deduce from (6.13) and Lemma 17 of Chapter 1 of [3] that $\|e^{-tL}\|_{1,\infty} \le ct^{-n/m}$, $\|e^{-tL}\|_{1,1} \le c$ and $\|e^{-tL}\|_{\infty,\infty} \le c$. Hence, by interpolation,

$$\|e^{-tL}\|_{p,q} \le ct^{(\frac{1}{q}-\frac{1}{p})\frac{n}{m}}, \quad 1 \le p \le q \le \infty.$$

On the other hand, it follows from (6.14) that

$$\|e^{-tL}f\|_{\dot{C}^\nu} \le ct^{-(n+\nu)/m}\|f\|_1$$

and

$$\|e^{-tL}f\|_{\dot{C}^\nu} \le ct^{-\nu/m}\|f\|_\infty.$$

Hence, by interpolation,

$$\|e^{-tL}f\|_{\dot{C}^\nu} \le ct^{-(\frac{n}{p}+\nu)/m}\|f\|_p, \quad 1 \le p \le \infty.$$

One writes $z = t+t+\xi$ where $t > 0$, $|\arg\xi| < \pi/2-\omega$ and $|z| \sim t \sim |\xi|$. Then using the semigroup property $e^{-zL} = e^{-tL}e^{-\xi L}e^{-tL}$, we have

$$\begin{aligned}\|e^{-zL}f\|_{\dot{C}^\nu} &\le c|z|^{-(\frac{n}{2}+\nu)/m}\|e^{-\xi L}e^{-tL}f\|_2\\ &\le c|z|^{-(\frac{n}{2}+\nu)/m}\|e^{-tL}f\|_2\\ &\le c|z|^{-(n+\nu)/m}\|f\|_1,\end{aligned}$$

which gives (6.18). This gives the desired estimate of $|p_z(x+h,y)-p_z(x,y)|$ in (6.17). Hence, Lemma 6.9 is proved. □

Using Lemma 6.9, we have the following equivalence between the classical BMO space and BMO_L spaces associated with differential operators.

Theorem 6.10. *Assume that L satisfies the assumptions* (6.13), (6.14) *and* (6.15). *Then, the* BMO *space (modulo constant functions) and the* BMO_L *space (modulo* $\mathcal{K}_L$*) coincide, and their norms are equivalent.*

Proof. We remark that for L satisfying (6.13), (6.14) and (6.15), our proof below shows that L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ because the area integral functions $\mathcal{S}_L$ and $\mathcal{S}_{L^*}$ are bounded on $L^2(\mathbb{R}^n)$ where L^* is the adjoint operator of L. Hence, Theorem 3.1 holds for the operators L and L^*.

This follows from Proposition 6.7 and the assumption (6.15) that $\mathrm{BMO} \subset \mathrm{BMO}_L$. We now prove $\mathrm{BMO}_L \subset \mathrm{BMO}$. From Theorem 3.1 and a duality argument, this reduces to proving that $H^1 \subset H^1_{L^*}$ with $\|f\|_{H^1_{L^*}} \leq c\|f\|_{H^1}$. Using the atomic decomposition of H^1, it suffices to prove that for any atom a, we have $\|a\|_{H^1_{L^*}} \leq c$, where c is a positive constant independent of a. See [31]. Denote by $q_t^*(x,y)$ the kernel of the operator $Q_t^* = t\frac{d}{dt}e^{-tL^*}$. By (6.15) we have $Q_t(1) = Q_t^*(1) = 0$. It follows from Lemma 6.9 that there exist constants $c > 0$, $0 < \gamma_1 < \gamma$ and $0 < \beta_1 < \beta$ such that

$$|q_t^*(x,y)| \leq c\frac{t^{\beta_1/m}}{(t^{1/m} + |x-y|)^{n+\beta_1}},$$

and whenever $2|h| \leq t^{1/m} + |x-y|$,

$$|q_t^*(x+h,y) - q_t^*(x,y)| + |q_t^*(x,y+h) - q_t^*(x,y)| \leq c|h|^{\gamma_1}\frac{t^{\beta_1/m}}{(t^{1/m} + |x-y|)^{n+\beta_1+\gamma_1}}.$$

From Theorem 3 of [27], the area integral function $\mathcal{S}_{L^*}(f)$ is bounded on $L^2(\mathbb{R}^n)$; hence $\|\mathcal{S}_{L^*}(a)\|_2 \leq c\|a\|_2$. It follows from a standard harmonic analysis argument that we have $\|a\|_{H^1_{L^*}} = \|S_{L^*}(a)\|_1 \leq c$. See, for example, Proposition 1.2, Chapter 14 of [34].

This proves that $H^1 \subset H^1_{L^*}$; hence $\mathrm{BMO}_L \subset \mathrm{BMO}$. The proof of Theorem 6.10 is complete. □

Remarks. (i) As noted in Section 6.1.1, the assumptions (6.13), (6.14) and (6.15) are satisfied for the divergence form operator L in (6.3) when L has real coefficients or when the dimension $n = 1$ or 2 in the case of complex coefficients. See Chapter 1 of [3] and [2].

(ii) The Laplacian $\triangle$ on $\mathbb{R}^n$ satisfies the assumptions of Theorem 6.10; hence the spaces $\mathrm{BMO}_{\triangle}$ and $\mathrm{BMO}_{\sqrt{\triangle}}$ coincide with the classical BMO space and Theorem 6.10 generalizes the results of Theorems 2.14 and 2.15 of [16].

6.2.3. *An example of* $\mathrm{BMO}_L \subsetneqq \mathrm{BMO}$. In [13], a space of BMO type associated with a Schrödinger operator was introduced as follows. Let $L = -\triangle + V(x)$ on $\mathbb{R}^n$, $n \geq 3$, where

$$V(x) = \sum_{\beta \leq \alpha} a_\beta x^\beta \tag{6.19}$$

is a nonnegative nonzero polynomial on $\mathbb{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$. Such a function V in (6.19) belongs to the reverse Hölder class B_q for all q, $1 < q < \infty$. See the condition (6.10) in Section 6.1.2.

Denote by $\rho(x) = \sup\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\}$. The space BMO_s associated with L was defined by

$$\mathrm{BMO}_s = \Big\{f \in \mathrm{BMO} : \frac{1}{|B|}\int_B |f(x)|dx \le c \text{ for all } B = B_R(x) : R > \rho(x)\Big\}.$$

It is obvious that $\mathrm{BMO}_s \subset \mathrm{BMO}$. It was observed in [13] that $\mathrm{BMO_s}$ is a proper subspace of the classical BMO space (for example, $\log|x| \notin \mathrm{BMO}_s$). In [13], they also proved that

$$(\widetilde{H^1_L})' = \mathrm{BMO}_s, \tag{6.20}$$

where the Hardy space $\widetilde{H^1_L}$ is defined by means of a maximal function associated with the semigroup $\{e^{-tL}\}_{t>0}$, i.e.,

$$\widetilde{H^1_L} = \Big\{f \in L^1 : \sup_{t>0}|e^{-tL}f(x)| \in L^1\Big\}.$$

See [17]. Note that by Theorem 3 of [37],

$$\widetilde{H^1_L} \equiv H^1_L = \Big\{f \in L^1 : S_L(f) \in L^1\Big\}. \tag{6.21}$$

Theorem 3.1, together with (6.20) and (6.21), give the following proposition.

Proposition 6.11. *Assume that $L = -\triangle + V(x)$, where V is a nonnegative nonzero polynomial* (6.19)*. Then, the spaces* BMO_L *and* BMO_s *coincide and their norms are equivalent.*

As a consequence, we have $\mathrm{BMO}_L \subsetneqq \mathrm{BMO}$*. That is,* BMO_L *is a proper subspace of the classical* BMO *space.*

Acknowledgments

The authors would like to thank the referee for helpful comments and suggestions. The authors thank A. McIntosh for helpful suggestions and P. Auscher as some of these ideas originate with him in [4]. The second-named author thanks C. Kenig and Z. Shen for useful discussions and references and D.G. Deng for all support, encouragement and guidance given over the years.

References

[1] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Annals of Math., **156** (2002), 633-654. MR1933726 (2004c:47096c)

[2] P. Auscher and P. Tchamitchian, *Calcul fonctionnel précisé pour des opérateurs elliptiques complexes en dimension un (et applications à certaines équations elliptiques complexes en dimension deux)*, Ann. Institut Fourier (Grenoble), **45** (1995), 721-778. MR1340951 (96f:35036)

[3] P. Auscher and P. Tchamitchian, *Square root problem for divergence operators and related topics*, Asterisque, **249**, Soc. Math. France, 1998. MR1651262 (2000c:47092)

[4] P. Auscher, X.T. Duong and A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*, preprint, 2004.

[5] T. Coulhon and X.T. Duong, *Maximal regularity and kernel bounds: Observations on a theorem by Hieber and Prüss*, Adv. Differential Equations, **5** (2000), 343-368. MR1734546 (2001d:34087)

[6] M. Cowling, I. Doust, A. McIntosh and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Austral. Math. Soc. Ser. A, **60** (1996), 51-89. MR1364554 (97d:47023)

[7] R.R. Coifman, Y. Meyer and E.M. Stein, *Un nouvel éspace adapté à l'étude des opérateurs définis par des intégrales singulières*, in "Proc. Conf. Harmonic Analysis, Cortona", Lecture Notes in Math., Vol. **992**, pp. 1-15, Springer-Verlag, Berlin/New York, 1983. MR0729344 (85j:42032)

[8] R.R. Coifman, Y. Meyer and E.M. Stein, *Some new functions and their applications to harmonic analysis*, J. Funct. Analysis, **62**(1985), 304-335. MR0791851 (86i:46029)

[9] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis,* Bull. Amer. Math. Soc., **83** (1977), 569–645. MR0447954 (56:6264)

[10] E.B. Davies, *Heat kernels and spectral theory,* Cambridge Univ. Press, 1989. MR0990239 (90e:35123)

[11] D.G. Deng, *On a generalized Carleson inequality*, Studia Math., **78** (1984), 245-251. MR0782661 (86h:42033)

[12] D.G. Deng, X.T. Duong, A. Sikora and L.X. Yan, *Comparison between the classical* BMO *and the* BMO *spaces associated with operators and applications*, preprint, (2005).

[13] J. Dziubański, G. Garrigós, T. Martínez, J. Torrea and J. Zienkiewicz, BMO *spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z., **249** (2005), 329-356. MR2115447.

[14] X.T. Duong and A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamericana, **15** (1999), 233-265. MR1715407 (2001e:42017a)

[15] X.T. Duong and D.W. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, J. Funct. Anal., **142** (1996), 89–128. MR1419418 (97j:47056)

[16] X.T. Duong and L.X. Yan, *New function spaces of* BMO *type, the John-Nirenberg inequality, interpolation and applications*, to appear, Comm. Pure Appl. Math., (2005).

[17] J. Dziubański and J. Zienkiewicz, *Hardy spaces associated with some Schrödinger operator*, Studia Math., **126** (1997), 149-160. MR1472695 (98k:42029)

[18] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc., **77** (1971), 587-588. MR0280994 (43:6713)

[19] C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–195. MR0447953 (56:6263)

[20] S. Hofmann, J.M. Martell, *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators*, Pub. Mat., **47** (2003), 497-515. MR2006497 (2004i:35067)

[21] J.L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*. Lecture Notes in Mathematics, 994. Springer, Berlin-New York, 1983. MR0706075 (85i:42021)

[22] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14**(1961), 415–426. MR0131498 (24:A1348)

[23] P. Li, *Harmonic sections of polynomial growth*, Math. Research Letters, **4** (1997), 35-44. MR1432808 (98i:53054)

[24] P. Li and J.P. Wang, *Counting dimensions of L-harmonic functions*, Ann. of Math., **152** (2000), 645-658. MR1804533 (2002c:58032)

[25] J.M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math., **161**(2004), 113-145. MR2033231 (2005b:42016)

[26] A. McIntosh, *Operators which have an H_∞-calculus*, Miniconference on operator theory and partial differential equations, (1986) Proc. Centre Math. Analysis, ANU, Canberra **14** (1986), 210-231. MR0912940 (88k:47019)

[27] S. Semmes, *Square function estimates and the $T(b)$ theorem*, Proc. Amer. Math. Soc., **110** (1990), 721-726. MR1028049 (91h:42018)

[28] Z. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Institut Fourier (Grenoble), **45** (1995), 513-546. MR1343560 (96h:35037)

[29] Z. Shen, *On fundamental solutions of generalized Schrödinger operators*, J. Funct. Anal., **167** (1999), 521-564. MR1716207 (2000j:35055)

[30] E.M. Stein, *Singular integral and differentiability properties of functions*, Princeton Univ. Press, **30**, (1970). MR0290095 (44:7280)

[31] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, (1993). MR1232192 (95c:42002)

[32] E.M. Stein and G. Weiss, *On the theory of harmonic functions of several variables I, The theory of H^p spaces*, Acta Math., **103** (1960), 25-62. MR0121579 (22:12315)

[33] W.A. Strauss, *Partial differential equations: An introduction.* John Wiley & Sons, Inc., New York, 1992. MR1159712 (92m:35001)

[34] A. Torchinsky, *Real-variable methods in harmonic analysis*, Pure and Applied Math., Vol 123, Academic Press, (1986). MR0869816 (88e:42001)

[35] L.X. Yan, *Littlewood-Paley functions associated to second order elliptic operators*, Math. Z., **246** (2004), 655-666. MR2045834 (2005a:42015)

[36] K. Yosida, *Functional Analysis* (fifth edition), Springer-Verlag, Berlin, 1978. MR0500055 (58:17765)

[37] Y.P. Zhu, *Area functions on Hardy spaces associated to Schrödinger operators*, Acta Math. Sci. Ser. B Engl. Ed., **23** (2003), 521–530. MR2032556 (2004k:42029)

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: duong@ics.mq.edu.au

Department of Mathematics, Macquarie University, NSW 2109, Australia and Department of Mathematics, Zhongshan University, Guangzhou, 510275, People's Republic of China
E-mail address: lixin@ics.mq.edu.au; mcsylx@zsu.edu.cn