Duality of Orbifoldized Elliptic Genera

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We discuss duality and mirror symmetry phenomena of Landau-Ginzburg orbifolds considering their elliptic genera. Under the duality (or mirror) transform performed by orbifoldizing the Landau-Ginzburg model via some discrete group of the superpotential we observe that the roles of the untwisted and twisted sectors are exchanged. As explicit evidence detailed orbifold data are presented for $N=2$ minimal models, Arnold's exceptional singularities, $K3$ surfaces constructed from Arnold's singularities and Fermat hypersurfaces.

§ 1. Introduction

Landau-Ginzburg field theory approach to two-dimensional critical phenomena uncovers qualitative physical properties behind the exact algebraic description in terms of conformal field theories.[1)~3)] When $N=2$ supersymmetry is considered even quantitative results can be deduced in the framework of the Landau-Ginzburg models.[4)~6)] The non-renormalization theorem for the superpotential of $N=2$ models is believed to be responsible for this miracle. $N=2$ Landau-Ginzburg descriptions have also proved to be efficient in constructing superstring vacua through orbifoldizing the Landau-Ginzburg models.[7),8)] This fact is somewhat mysterious since the Landau-Ginzburg models do not a priori possess the target space interpretation while more conventional sigma models have. There have been some arguments attempting to clarify the connection between the Landau-Ginzburg models and the sigma models with Calabi-Yau target spaces.[9)~11)]

Recently a novel scheme has been proposed to understand the Landau-Ginzburg /Calabi-Yau correspondence.[12)] One considers a $U(1)$ gauged Landau-Giuzburg model with the Fayet-Iliopoulos D-term and the theta term. The model contains several chiral superfields, one of which, say P, plays the role of an order parameter. The coefficient r of the Fayet-Iliopoulos term combined with that of the theta term, θ turns out to be a complex variable $t=\sqrt{-1}r+(\theta/2\pi)$ parametrizing the complexified Kähler cone. By tuning t one finds two extremum regimes. One regime ($r\ll 0$) represents the Landau-Ginzburg phase where p, bosonic component of P, acquires the vacuum expectation value $\langle p\rangle\neq 0$. The $U(1)$ symmetry then breaks down to some discrete group. This discrete group is employed to orbifoldize the Landau-Ginzburg model. In the other regime ($r\gg 0$) we have $\langle p\rangle=0$ and the bosonic

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components of the chiral superfields other than P are constrained to take values on a hypersurface in a weighted projective space. Hence this is the regime of the sigma model. Furthermore it is argued that one can make an analytic continuation from the Landau-Ginzburg to Calabi-Yau regimes and vice versa on complex t-plane. This picture was also confirmed from another independent point of view[13] and has been extended to the (0, 2) case.[12),14)]

One more evidence of the above scheme can be obtained by considering the elliptic genus,[15)~17)] i.e., the index of the right-moving supercharge. It is expected that the elliptic genus is independent of the parameter t because of its topological nature and thus should coincide in both the Calabi-Yau and Landau-Ginzburg orbifold phases. Calculations of the elliptic genera of the Landau-Ginzburg models were initiated in Ref. 18) where the A-type $N=2$ minimal model is taken to explain the essential idea. This was soon followed by several groups who have extended the original idea in order to incorporate various $N=2$ models.[18)~27)] In Ref. 20) it was confirmed that the elliptic genus of an appropriate Landau-Ginzburg orbifold takes the same form as that of the sigma model with a Calabi-Yau target manifold thus with a good agreement with the above expectation.

The orbifoldized elliptic genus is also an interesting arena to consider the mirror symmetry of Calabi-Yau manifolds[28] and similar phenomena. In fact the investigation of mirror symmetry via the elliptic genus has already been taken up in Ref. 24). A well-known procedure to construct mirror pairs is to orbifoldize the Landau-Ginzburg model via various symmetry groups of the superpotential.[29] To compute the elliptic genera of the resulting Landau-Ginzburg orbifolds necessitates a slightly more involved formula than that for the most frequently studied case. Thus after introducing *generic* Landau-Ginzburg orbifolds in § 2 we briefly summarize the formulas for their elliptic genera in § 3.

In § 4, which is the main part of this contribution, we study mirror phenomena and their cousins by employing these formulas. By picking up typical examples we present detailed data which have accumulated during our series of analyses of the elliptic genus. Although these data may be a sort of objects usually to be suppressed in the literature it is quite impressive to experience duality or mirror phenomena through explicit data. We thus think it worth publishing these data in a comprehensible manner. We also believe that our presentation is in accordance with the editorial spirit of these proceedings.

§ 2. $N=2$ Landau-Ginzburg model and its orbifolds

We consider the Landau-Ginzburg model whose Lagrangian density is given by

$$\int d^2\theta d^2\bar{\theta}\sum_{i=1}^{N} X_i\bar{X}_i+\int d^2\theta W(X_i)+\int d^2\bar{\theta}\bar{W}(\bar{X}_i)\,, \qquad (2\cdot 1)$$

where the superpotential W is a weighted homogeneous polynomial of N chiral superfields $X_1, \cdots, X_N$ with weights $\omega_1, \cdots, \omega_N$,

$$\lambda W(X_1, \cdots, X_N)=W(\lambda^{\omega_1}X_1, \cdots, \lambda^{\omega_N}X_N)\,. \qquad (2\cdot 2)$$

We assume that W has an isolated critical point at the origin and the ω_i's are strictly positive rational numbers such that $\omega_1, \cdots, \omega_N \leq 1/2$. The infrared fixed point theory is believed to be described by an $N=2$ superconformal field theory with

$$\hat{c}=\sum_{i=1}^{N}(1-2\omega_i)\,. \tag{2·3}$$

In general W is invariant under some discrete group $G\subset GL(N, \boldsymbol{C})$ acting on ${}^t(X_1, \cdots, X_N)$ and one can consider the orbifold theory with respect to G. The resulting theory is called the Landau-Ginzburg orbifold and will be denoted symbolically as $W /\!/ G$ in the following. We shall restrict ourselves to the case where G is abelian and their elements take the form $\mathrm{diag}(\boldsymbol{e}[a_1\omega_1], \cdots, \boldsymbol{e}[a_N\omega_N])$ where $a_i\in\boldsymbol{Z}$ and $\boldsymbol{e}[*]=\exp(2\pi\sqrt{-1}*)$. One distinguished example of such discrete groups that always exists for any W is the one generated by $\mathrm{diag}(\boldsymbol{e}[\omega_1], \cdots, \boldsymbol{e}[\omega_N])$. We shall call this group the *principal discrete group* and denote it by G_0. The Landau-Ginzburg orbifold $W /\!/ G_0$ is a fundamental and the most frequently studied case.

In a favorable situation (i.e. $\hat{c}\in\boldsymbol{Z}$ and $G\supseteq G_0$) the Landau-Ginzburg orbifold $W /\!/ G$ can be interpreted as an 'analytic continuation' of some $N=2$ sigma model with its target space smoothed.

§ 3. Elliptic genus

Topological properties of a supersymmetric theory in two space-time dimensions can be succinctly summarized by the elliptic genus.[15)~17)] This quantity was recently refined so as to incorporate $N=2$ theories and up until now various examples have been computed.[18)~27)]

The definition of the $N=2$ elliptic genus is[*),18)]

$$Z(\tau, z)=\mathrm{Tr}(-1)^F y^{(J^L)_0} q^{\mathcal{H}^L} \bar{q}^{\mathcal{H}^R}\,, \quad y=\boldsymbol{e}[z]\,, \quad q=\boldsymbol{e}[\tau]\,, \quad (\mathrm{Im}\,\tau>0) \tag{3·1}$$

where $(J^{L,R})_0$ are the left, right $U(1)$ charge operators and $\mathcal{H}^{L,R}$ are the left, right Hamiltonians. We have set $(-1)^F=\exp[-\pi\sqrt{-1}\{(J^L)_0-(J^R)_0\}]$. As usual, due to the right supersymmetry $Z(\tau, z)$ is $\bar{q}$ independent.

The basic properties of the elliptic genus[20)] are the modular invariance up to a prefactor

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)=\boldsymbol{e}\left[\frac{\hat{c}}{2}\frac{cz^2}{c\tau+d}\right]Z(\tau, z)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, \boldsymbol{Z})\,, \tag{3·2}$$

and the double quasi-periodicity

$$Z(\tau, z+\lambda\tau+\mu)=(-1)^{\hat{c}(\lambda+\mu)}\boldsymbol{e}\left[-\frac{\hat{c}}{2}(\lambda^2\tau+2\lambda z)\right]Z(\tau, z)\,, \quad \lambda, \mu\in h\boldsymbol{Z}\,, \tag{3·3}$$

where h is the least positive integer such that the $U(1)$ charge of any chiral ring element multiplied by h is an integer. These two properties together with the 'χ_y-genus'[30)] determined by

*) Throughout this paper we shall consider (2, 2) theories only.

$$\chi_y = y^{(\hat{c}/2)} \lim_{\tau \to i\infty} Z(\tau, z)\,, \tag{3·4}$$

characterize the elliptic genus.

The elliptic genus of the Landau-Ginzburg model can be computed as

$$Z[W](\tau, z) = \prod_{i=1}^{N} \frac{\vartheta_1(\tau, (1-\omega_i)z)}{\vartheta_1(\tau, \omega_i z)}\,, \tag{3·5}$$

where

$$\vartheta_1(\tau, z) = \sqrt{-1} \sum_{n \in \mathbf{Z}} (-1)^n q^{(1/2)(n-(1/2))^2} y^{n-(1/2)}$$

$$= \sqrt{-1} q^{1/8} y^{-1/2} \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1}y)(1-q^n y^{-1})\,, \tag{3·6}$$

is one of the Jacobi theta functions. Using the theta function formulae it is easy to check that $Z[W](\tau, z)$ obeys (3·2) and (3·3) with $\hat{c}$ given by (2·3) and h being the smallest positive integer such that $\omega_i h \in \mathbf{Z}$ for all $1 \le i \le N$.

The elliptic genus of the Landau-Ginzburg orbifold $W /\!/ G$ is given by[20),21),24)]

$$Z[W /\!/ G](\tau, z) = \frac{1}{|G|} \sum_{\alpha, \beta \in G} \epsilon(\alpha, \beta) {}_{\beta}\Box_{\alpha}(\tau, z)\,, \tag{3·7}$$

where

$$\epsilon(\alpha, \beta) = \prod_{i=1}^{N} (-1)^{\alpha_i + \beta_i + \alpha_i \beta_i}\,, \tag{3·8}$$

and

$${}_{\beta}\Box_{\alpha}(\tau, z) = \prod_{i=1}^{N} \boldsymbol{e}\left[\frac{1-2\omega_i}{2} \alpha_i \beta_i\right] \boldsymbol{e}\left[\frac{1-2\omega_i}{2} (\alpha_i^2 \tau + 2\alpha_i z)\right]$$

$$\times \frac{\vartheta_1(\tau, (1-\omega_i)(z + \alpha_i \tau + \beta_i))}{\vartheta_1(\tau, \omega_i(z + \alpha_i \tau + \beta_i))}\,. \tag{3·9}$$

Here we have made $\alpha = (\alpha_1, \cdots, \alpha_N)$ represent for the element diag$(\boldsymbol{e}[\alpha_1 \omega_1], \cdots, \boldsymbol{e}[\alpha_N \omega_N])$ of G. Correspondingly the χ_y-genus takes the form

$$\chi_y[W /\!/ G] = y^{(\hat{c}/2)} \frac{(-1)^N}{|G|} \sum_{\alpha, \beta \in G} \prod_{\substack{i \\ \omega_i \alpha_i \notin \mathbf{Z}}} y^{-((\omega_i \alpha_i))} \prod_{\substack{i \\ \omega_i \alpha_i \in \mathbf{Z}}} \frac{\sin \pi \{(\omega_i - 1)z + \omega_i \beta_i\}}{\sin \pi (\omega_i z + \omega_i \beta_i)}\,, \tag{3·10}$$

where $((*)) = * - [*] - (1/2)$.

The orbifoldized elliptic genus $Z[W /\!/ G]$ obeys the same modular transformation property and double quasi-periodicity as those of $Z[W]$. In addition, if $\hat{c}$ is an integer, the conditions for the double quasi-periodicity λ, $\mu \in h\mathbf{Z}$ are relaxed to λ, $\mu \in \mathbf{Z}$ and the orbifold theory has a chance to have correspondence with an $N=2$ sigma model.

The Witten index $Z[W /\!/ G](\tau, 0) = \chi_{y=1}[W /\!/ G]$ reads

$$\frac{(-1)^N}{|G|}\sum_{\alpha,\beta\in G}\prod_{\substack{i\\ \omega_i\alpha_i\in Z \text{ and } \omega_i\beta_i\in Z}}\left(1-\frac{1}{\omega_i}\right), \tag{3·11}$$

and this reproduces the result of Roan[31] which is in turn the extension of Vafa's formula.[*),7]

§ 4. Self-duality, strange duality and mirror symmetry

In this section we restrict ourselves to Landau-Ginzburg orbifolds $W /\!/ G$ with either $G=\{id\}$ or $G\supseteq G_0$. We now wish to present a variety of computational results for the following phenomena:

There are some cases in which the Landau-Ginzburg orbifold $W /\!/ G$ has a partner $W^* /\!/ G^*$ such that

- *$W^* /\!/ G^*$ has the same central charge $\hat{c}$ as $W /\!/ G$.*
- $$Z[W /\!/ G](\tau, z)=\pm Z[W^* /\!/ G^*](\tau, z), \tag{4·1}$$

  or equivalently

  $$\chi_y[W /\!/ G]=\pm\chi_y[W^* /\!/ G^*]. \tag{4·2}$$
- *By going from $W /\!/ G$ to $W^* /\!/ G^*$ the roles of the untwisted sectors and twisted sectors are interchanged.*

Here we have to explain what we mean by untwisted and twisted sectors. If $G=\{id\}$ we have only untwisted sectors and no twisted sectors. If $G\supseteq G_0$, by untwisted sectors we mean the ones with respect to the subgroup G_0. Thus the number of the untwisted sectors is equal to $|G|/|G_0|$.

Conceptual understanding of these observations is still lacking and it is certainly true that a mere consideration of elliptic genus does not suffice and perhaps we have to view things from a broader perspective (see § 5 for discussion). Nevertheless we hope that a relative ease of computations and their explicitness make these results worth presenting.

4.1. *Self-duality of minimal models*

Our first example is the well-known self-duality of $N=2$ minimal model. The $N=2$ minimal model is in one to one correspondence with the Landau-Ginzburg model with its superpotential given by one of ADE potentials and its central charge is given by $\hat{c}=1-(2/h)$ where h is the Coxeter number of ADE. If we take $W^*=W$ and choose $G=\{id\}$ and $G^*=G_0$ then the above situation is realized as we now see. As mentioned the Landau-Ginzburg model $W=W /\!/ \{id\}$ has no twisted sectors and its χ_y-genus is, as well-known, given by

*) Note that in the formula of $Z[W /\!/ G_0](\tau, z)$ given in Ref. 20) we took $\epsilon(\alpha, \beta)=(-1)^{D(\alpha+\beta+\alpha\beta)}$ where D is an integer such that $Dh\equiv\hat{c}h$ (mod 2). Equation (3·8) corresponds to the choice $D=N$ which is possible since $\hat{c}h=Nh-2\sum_i\omega_i h\equiv Nh$ (mod 2). If $\hat{c}$ is an integer we can instead take $D=\hat{c}$ which leads to the original Vafa's formula[7] of the Euler characteristic.

$$\chi_y[W]=\sum_{i=1}^{l}t^{m_i-1}\,, \tag{4·3}$$

where $t=y^{1/h}$ and $m_1,\cdots,m_l$ are the exponents of ADE. The χ_y-genus of the Landau-Ginzburg orbifold $W /\!/ G_0$ can be decomposed as

$$\chi_y[W /\!/ G_0]=\sum_{\alpha=0}^{h-1}\chi_y^\alpha[W /\!/ G_0]\,, \tag{4·4}$$

and each contribution is given as follows. For A_l we have

$$\chi_y^\alpha[W /\!/ G_0]=\begin{cases}0\,, & \alpha=0\,,\\ -t^{l-\alpha}\,, & \alpha=1,\cdots,l\,.\end{cases} \tag{4·5}$$

For D_l and if l is even, we have

$$\chi_y^\alpha[W /\!/ G_0]=\begin{cases}0\,, & \alpha=\text{even}\,,\\ -t^{2l-3-\alpha}\,, & \alpha=\text{odd}\,,\quad \neq l-1\,,\\ -2t^{l-2}\,, & \alpha=l-1\,,\end{cases} \tag{4·6}$$

while if l is odd

$$\chi_y^\alpha[W /\!/ G_0]=\begin{cases}0 & \alpha=\text{even}\,,\quad \neq l-1\,,\\ -t^{2l-3-\alpha} & \alpha=\text{odd}\,,\\ -t^{l-2}\,, & \alpha=l-1\,.\end{cases} \tag{4·7}$$

For the remaining cases we have

	$\{-\chi_y^0[W /\!/ G_0], -\chi_y^1[W /\!/ G_0],\cdots,-\chi_y^{h-1}[W /\!/ G_0]\}$
E_6	$\{0, t^{10}, 0, 0, t^7, t^6, 0, t^4, t^3, 0, 0, 1\}$
E_7	$\{0, t^{16}, 0, 0, 0, t^{12}, 0, t^{10}, 0, t^8, 0, t^6, 0, t^4, 0, 0, 0, 1\}$
E_8	$\{0, t^{28}, 0, 0, 0, 0, 0, t^{22}, 0, 0, 0, t^{18}, 0, t^{16}, 0, 0, 0, t^{12}, 0, t^{10}, 0, 0, 0, t^6, 0, 0, 0, 0, 0, 1\}$

To summarize we found that

$$\chi_y^\alpha[W /\!/ G_0]=-\text{mult}(\alpha)t^{h-1-\alpha}\,, \tag{4·8}$$

where mult(α) is the multiplicity of α appearing in the set of exponents $\{m_1,\cdots,m_l\}$. Hence it follows that

$$\chi_y[W]=-\chi_y[W /\!/ G_0]\,, \tag{4·9}$$

and the twisted and the untwisted sectors are interchanged (with minus signs) between W and $W /\!/ G_0$.

4.2. *Arnold's strange duality in terms of Landau-Ginzburg orbifolds*

Let W be the potential corresponding to one of Arnold's 14 exceptional singularities and let W^* denote its dual in the sense of strange duality. (See Table I.) W and W^* share the same Coxeter number h and hence $\hat{c}=1+2/h$. Take $G=\{id\}$ and $G^*=G_0^*$ where G_0^* is the principal discrete group of W^*. Comparing Tables I and II we find that

Table I. Exceptional singularities: χ_y-genera.

(h, d_1, d_2, d_3)	$W(z_1, z_2, z_3)$	$\chi_y[W] \quad (t=y^{1/h})$
(12, 4, 4, 3)	$z_1^3+z_2^3+z_3^4$	$1+t^3+2t^4+t^6+2t^7+t^8+2t^{10}+t^{11}+t^{14}$
(13, 4, 3, 5)	$z_1^2z_3+z_2z_3^2+z_1z_2^3$	$1+t^3+t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{15}$
(15, 6, 5, 3)	$z_1^2z_3+z_2^3+z_2^5$	$1+t^3+t^5+2t^6+t^8+t^9+2t^{11}+t^{12}+t^{14}+t^{17}$
(16, 5, 4, 6)	$z_1^2z_3+z_2z_3^2+z_2^4$	$1+t^4+t^5+t^6+t^8+t^9+t^{10}+t^{12}+t^{13}+t^{14}+t^{18}$
(16, 4, 3, 8)	$z_1^4+z_1z_2^4+z_3^2$	$1+t^3+t^4+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{14}+t^{15}+t^{18}$
(18, 7, 6, 4)	$z_1^2z_3+z_2^3+z_2z_3^3$	$1+t^4+t^6+t^7+t^8+t^{10}+t^{12}+t^{13}+t^{14}+t^{16}+t^{20}$
(18, 5, 3, 9)	$z_1^3z_2+z_2^6+z_3^2$	$1+t^3+t^5+t^6+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{14}+t^{15}+t^{17}+t^{20}$
(20, 5, 4, 10)	$z_1^4+z_2^5+z_3^2$	$1+t^4+t^5+t^8+t^9+t^{10}+t^{12}+t^{13}+t^{14}+t^{17}+t^{18}+t^{22}$
(22, 6, 4, 11)	$z_1^3z_2+z_1z_2^4+z_3^2$	$1+t^4+t^6+t^8+t^{10}+2t^{12}+t^{14}+t^{16}+t^{18}+t^{20}+t^{24}$
(24, 9, 8, 6)	$z_1^2z_3+z_2^3+z_3^4$	$1+t^6+t^8+t^9+t^{12}+t^{14}+t^{17}+t^{18}+t^{20}+t^{26}$
(24, 8, 3, 12)	$z_1^3+z_2^8+z_3^2$	$1+t^3+t^6+t^8+t^9+t^{11}+t^{12}+t^{14}+t^{15}+t^{17}+t^{18}+t^{20}+t^{23}+t^{26}$
(30, 8, 6, 15)	$z_1^3z_2+z_2^5+z_3^2$	$1+t^6+t^8+t^{12}+t^{14}+t^{16}+t^{18}+t^{20}+t^{24}+t^{26}+t^{32}$
(30, 10, 4, 15)	$z_1^3+z_1z_2^5+z_3^2$	$1+t^4+t^8+t^{10}+t^{12}+t^{14}+t^{16}+t^{18}+t^{20}+t^{22}+t^{24}+t^{28}+t^{32}$
(42, 14, 6, 21)	$z_1^3+z_2^7+z_3^2$	$1+t^6+t^{12}+t^{14}+t^{18}+t^{20}+t^{24}+t^{26}+t^{30}+t^{32}+t^{38}+t^{44}$

Table II. Exceptional singularities: the untwisted and twisted sector contributions to the Landau-Ginzburg orbifold χ_y-genera.

(h, d_1, d_2, d_3)	$\{-\chi_y^0[W /\!/ G_0], -\chi_y^1[W /\!/ G_0], \cdots, -\chi_y^{h-1}[W /\!/ G_0]\}$
(12, 4, 4, 3)	$\{0, t^{14}, t^3, 2t^4, 0, t^6, 2t^7, t^8, 0, 2t^{10}, t^{11}, 1\}$
(13, 4, 3, 5)	$\{0, t^{15}, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}, 1\}$
(15, 6, 5, 3)	$\{0, t^{17}, t^3, 0, t^5, 2t^6, 0, t^8, t^9, 0, 2t^{11}, t^{12}, 0, t^{14}, 1\}$
(16, 5, 4, 6)	$\{0, t^{18}, t^3, t^4, 0, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}, 0, t^{14}, t^{15}, 1\}$
(16, 4, 3, 8)	$\{0, t^{18}, 0, t^4, t^5, t^6, 0, t^8, t^9, t^{10}, 0, t^{12}, t^{13}, t^{14}, 0, 1\}$
(18, 7, 6, 4)	$\{0, t^{20}, t^3, 0, t^5, t^6, 0, t^8, t^9, t^{10}, t^{11}, t^{12}, 0, t^{14}, t^{15}, 0, t^{17}, 1\}$
(18, 5, 3, 9)	$\{0, t^{20}, 0, t^4, 0, t^6, t^7, t^8, 0, t^{10}, 0, t^{12}, t^{13}, t^{14}, 0, t^{16}, 0, 1\}$
(20, 5, 4, 10)	$\{0, t^{22}, 0, t^4, t^5, 0, 0, t^8, t^9, t^{10}, 0, t^{12}, t^{13}, t^{14}, 0, 0, t^{17}, t^{18}, 0, 1\}$
(22, 6, 4, 11)	$\{0, t^{24}, 0, t^4, 0, t^6, 0, t^8, 0, t^{10}, 0, 2t^{12}, 0, t^{14}, 0, t^{16}, 0, t^{18}, 0, t^{20}, 0, 1\}$
(24, 9, 8, 6)	$\{0, t^{26}, t^3, 0, 0, t^6, 0, t^8, t^9, 0, t^{11}, t^{12}, 0, t^{14}, t^{15}, 0, t^{17}, t^{18}, 0, t^{20}, 0, 0, t^{23}, 1\}$
(24, 8, 3, 12)	$\{0, t^{26}, 0, 0, 0, t^6, 0, t^8, t^9, 0, 0, t^{12}, 0, t^{14}, 0, 0, t^{17}, t^{18}, 0, t^{20}, 0, 0, 0, 1\}$
(30, 8, 6, 15)	$\{0, t^{32}, 0, t^4, 0, 0, 0, t^8, 0, t^{10}, 0, t^{12}, 0, t^{14}, 0, t^{16}, 0, t^{18}, 0, t^{20}, 0, t^{22}, 0, t^{24}, 0, 0, 0, t^{28}, 0, 1\}$
(30, 10, 4, 15)	$\{0, t^{32}, 0, 0, 0, t^6, 0, t^8, 0, 0, 0, t^{12}, 0, t^{14}, 0, t^{16}, 0, t^{18}, 0, t^{20}, 0, 0, 0, t^{24}, 0, t^{26}, 0, 0, 0, 1\}$
(42, 14, 6, 21)	$\{0, t^{44}, 0, 0, 0, t^6, 0, 0, 0, 0, 0, t^{12}, 0, t^{14}, 0, 0, 0, t^{18}, 0, t^{20}, 0, 0, 0, t^{24}, 0, t^{26}, 0, 0, 0, t^{30}, 0, t^{32}, 0, 0, 0, 0, 0, t^{38}, 0, 0, 0, 1\}$

$$\chi_y[W]=-\chi_y[W^* /\!/ G_0^*]\,, \tag{4·10}$$

and the twisted and the untwisted sectors are interchanged (with minus signs) between W and $W^* /\!/ G_0^*$.

4.3. *Landau-Ginzburg orbifolds corresponding to K3 constructed from exceptional singularities*

Let $\tilde{W}(z_1, z_2, z_3)$ be the potential corresponding to one of Arnold's 14 exceptional singularities and let $\tilde{W}^*(z_1, z_2, z_3)$ denote its dual in the sense of strange duality. Set $W=W(z_1, z_2, z_3, z_4)=\tilde{W}(z_1, z_2, z_3)+z_4^h$ and similarly for W^*. Then it is known that one can construct the $K3$ surface as the resolution of

$$\{(z_1, \cdots, z_4)\in \boldsymbol{WP}^3_{\{d_1,d_2,d_3,1\}} \mid W(z_1, z_2, z_3, z_4)=0\}\,. \tag{4·11}$$

The Landau-Ginzburg orbifold $W /\!/ G_0$ describes the analytic continuation of the $N=2$

Table III. The $K3$ associated with exceptional singularities: the untwisted and twisted sector contributions to the Landau-Ginzburg orbifold χ_y-genera.

(h, d_1, d_2, d_3)	$\{\chi_y^0[W /\!/ G_0], \chi_y^1[W /\!/ G_0], \cdots, \chi_y^{h-1}[W /\!/ G_0]\}$
(12, 4, 4, 3)	$\{1+10y+y^2, y^2, y, 2y, 0, y, 2y, y, 0, 2y, y, 1\}$
(13, 4, 3, 5)	$\{1+10y+y^2, y^2, y, y, y, y, y, y, y, y, y, y, 1\}$
(15, 6, 5, 3)	$\{1+10y+y^2, y^2, y, 0, y, 2y, 0, y, y, 0, 2y, y, 0, y, 1\}$
(16, 5, 4, 6)	$\{1+9y+y^2, y^2, y, y, 0, y, y, y, y, y, y, y, 0, y, y, 1\}$
(16, 4, 3, 8)	$\{1+11y+y^2, y^2, 0, y, y, y, 0, y, y, y, 0, y, y, y, 0, 1\}$
(18, 7, 6, 4)	$\{1+9y+y^2, y^2, y, 0, y, y, 0, y, y, y, y, y, 0, y, y, 0, y, 1\}$
(18, 5, 3, 9)	$\{1+11y+y^2, y^2, 0, y, 0, y, y, y, 0, y, 0, y, y, y, 0, y, 0, 1\}$
(20, 5, 4, 10)	$\{1+10y+y^2, y^2, 0, y, y, 0, 0, y, y, y, 0, y, y, y, 0, 0, y, y, 0, 1\}$
(22, 6, 4, 11)	$\{1+10y+y^2, y^2, 0, y, 0, y, 0, y, 0, y, 0, 2y, 0, y, 0, y, 0, y, 0, y, 0, 1\}$
(24, 9, 8, 6)	$\{1+8y+y^2, y^2, y, 0, 0, y, 0, y, y, 0, y, y, 0, y, y, 0, y, y, 0, y, 0, 0, y, 1\}$
(24, 8, 3, 12)	$\{1+12y+y^2, y^2, 0, 0, 0, y, 0, y, y, 0, 0, y, 0, y, 0, 0, y, y, 0, y, 0, 0, 0, 1\}$
(30, 8, 6, 15)	$\{1+9y+y^2, y^2, 0, y, 0, 0, 0, y, 0, y, 0, y, 0, y, 0, y, 0, y, 0, y, 0, y, 0, y, 0, 0, 0, y, 0, 1\}$
(30, 10, 4, 15)	$\{1+11y+y^2, y^2, 0, 0, 0, y, 0, y, 0, 0, 0, y, 0, y, 0, y, 0, y, 0, y, 0, 0, 0, y, 0, y, 0, 0, 0, 1\}$
(42, 14, 6, 21)	$\{1+10y+y^2, y^2, 0, 0, 0, y, 0, 0, 0, 0, 0, y, 0, y, 0, 0, 0, y, 0, y, 0, 0, 0, y, 0, y, 0, 0, 0, y, 0, y, 0, 0, 0, 0, 0, y, 0, 0, 0, 1\}$

sigma model whose target space is the $K3$ surface. The χ_y-genus of the $K3$ surface is

$$\chi_y(K3)=2+20y+2y^2\,, \tag{4·12}$$

and we find $\chi_y[W /\!/ G_0]=\chi_y(K3)=\chi_y[W^* /\!/ G_0^*]$. Let us denote the αth twisted sector contribution to the $\chi_y[W /\!/ G_0]$ by $\chi_y^\alpha[W /\!/ G_0]$. Table III shows that the contributions from the untwisted sector and those from the twisted sectors are interchanged between $W /\!/ G_0$ and $W^* /\!/ G_0^*$. To put differently,

$$\chi_y^0[W /\!/ G_0]+\chi_y^0[W^* /\!/ G_0^*]=\chi_y(K3)\,. \tag{4·13}$$

Thus we have seen that the partner of $W /\!/ G_0$ is given by $W^* /\!/ G_0^*$.

We remark that subjects related to what has been presented in the previous and the present subsections were earlier discussed by Martinec.[4)]

4.4. *Mirror symmetry*

Our last example is mirror symmetry considered by Greene and Plesser.[29)] We consider the family of superpotentials given by

$$W=z_1^d+\cdots+z_d^d\,, \quad d=3, 4, 5 \tag{4·14}$$

and take $W=W^*$. We choose G to satisfy

$$\boldsymbol{Z}_d\simeq G_0\subseteq G\subseteq(\boldsymbol{Z}_d)^{d-1}\,. \tag{4·15}$$

Apparently the number of such G's is 2^{d-2} and they are given by

$d=3$		
G	generators	$\chi(\mathscr{M}_G)$
$G_0\simeq\boldsymbol{Z}_3$	(1, 1, 1)	0
$G_1\simeq(\boldsymbol{Z}_3)^2$	(1, 1, 1), (0, 1, 2)	0

$d=4$		
G	generators	$\chi(\mathcal{M}_G)$
$G_0\simeq \boldsymbol{Z}_4$	(1, 1, 1, 1)	24
$G_1\simeq (\boldsymbol{Z}_4)^2$	(1, 1, 1, 1), (0, 0, 1, 3)	24
$G_2\simeq (\boldsymbol{Z}_4)^2$	(1, 1, 1, 1), (0, 1, 1, 2)	24
$G_3\simeq (\boldsymbol{Z}_4)^3$	(1, 1, 1, 1), (0, 0, 1, 3), (0, 1, 1, 2)	24

$d=5$		
G	generators	$\chi(\mathcal{M}_G)$
$G_0\simeq \boldsymbol{Z}_5$	(1, 1, 1, 1, 1)	−200
$G_1\simeq (\boldsymbol{Z}_5)^2$	(1, 1, 1, 1, 1), (0, 0, 0, 1, 4)	−88
$G_2\simeq (\boldsymbol{Z}_5)^2$	(1, 1, 1, 1, 1), (0, 1, 2, 3, 4)	−40
$G_3\simeq (\boldsymbol{Z}_5)^2$	(1, 1, 1, 1, 1), (0, 1, 1, 4, 4)	8
$G_4\simeq (\boldsymbol{Z}_5)^3$	(1, 1, 1, 1, 1), (0, 1, 1, 4, 4), (0, 1, 2, 3, 4)	−8
$G_5\simeq (\boldsymbol{Z}_5)^3$	(1, 1, 1, 1, 1), (0, 1, 3, 1, 0), (0, 1, 1, 0, 3)	40
$G_6\simeq (\boldsymbol{Z}_5)^3$	(1, 1, 1, 1, 1), (0, 1, 4, 0, 0), (0, 3, 0, 1, 1)	88
$G_7\simeq (\boldsymbol{Z}_5)^4$	(1, 1, 1, 1, 1), (0, 1, 2, 3, 4), (0, 1, 1, 4, 4), (0, 0, 0, 1, 4)	200

If $G=G_k$ then we take $G^*=G_{2^{d-2}-1-k}$. Note that $|G||G^*|=d^d$. The Landau-Ginzburg orbifold $W/\!/G$ corresponds to the sigma model on $\mathcal{M}_G$ which is a resolution of the orbifold

$$\mathcal{M}_G=\{(z_1, \cdots, z_d)\in \boldsymbol{CP}^{d-1}:\ W(z_1, \cdots, z_d)=0\}/(G/G_0)\,. \tag{4·16}$$

The Euler characteristic of $\mathcal{M}_G$ is related to the χ_y-genus by

$$\chi(\mathcal{M}_G)=(-1)^d\chi_{y=1}[W/\!/G]\,. \tag{4·17}$$

By examining the data presented below we can confirm that the asserted situation indeed occurs. However before seeing this let us explain how to look at tables below. The elements of G_i are ordered from left to right then from top to bottom in their tabulations. Note that the elements of G_i corresponding to the untwisted sectors take the form $(0, *, \cdots, *)$. The $\chi_y^g[W/\!/G_i]$ are arrayed in the same order as for the elements of G_i and should again be read from left to right then from top to bottom in their tabulations. Thus for example the first and second rows of the table of $d=5$, G_4 correspond respectively to $1+5y+5y^2+y^3$, 0, 0, 0, 0 and 0, 0, $2y+2y^2$, 0, $2y+2y^2$ in the table of $\chi_y^g[W/\!/G_4]$.

Now let us consider, as an illustration, the pair of $W/\!/G_1$ and $W/\!/G_2$ for $d=4$. Both theories have 4 untwisted sectors and 12 twisted sectors. The total twisted contribution to $\chi_y[W/\!/G_1]$ reads $y^2+0+y+0+y+y+y+y+1+0+y+0=1+6y+y^2$ while the total untwisted contributions to $\chi_y[W/\!/G_2]$ reads $(1+5y+y^2)+0+y+0=1+6y+y^2$. As another example let us take the pair of $W/\!/G_0$ and $W/\!/G_7$ for $d=5$. The total twisted contribution to $\chi_y[W/\!/G_0]$ reads $-y^3-y^2-y-1$. The theory $W/\!/G_7$ has $5^4/5=125$ untwisted sectors. The first one makes a contribution of $1+y+y^2+y^3$ while each of the remaining 124 ones of 0.

The other cases can be checked similarly. Though we have not worked out, it is also likely that similar results can be obtained for a class of mirror pairs considered in Ref. 32).

(i) $d = 3$

G_0		
(0, 0, 0)	(1, 1, 1)	(2, 2, 2)

G_1		
(0, 0, 0)	(0, 1, 2)	(0, 2, 4)
(1, 1, 1)	(1, 2, 3)	(1, 3, 5)
(2, 2, 2)	(2, 3, 4)	(2, 4, 6)

$\chi_y^\alpha[W//G_0]$
$1+y, -y, -1$

$\chi_y^\alpha[W//G_1]$		
$1+y, 0, 0,$	$-y, 0, 0,$	$-1, 0, 0$

(ii) $d = 4$

G_0			
(0, 0, 0, 0)	(1, 1, 1, 1)	(2, 2, 2, 2)	(3, 3, 3, 3)

G_1			
(0, 0, 0, 0)	(0, 0, 1, 3)	(0, 0, 2, 6)	(0, 0, 3, 9)
(1, 1, 1, 1)	(1, 1, 2, 4)	(1, 1, 3, 7)	(1, 1, 4, 10)
(2, 2, 2, 2)	(2, 2, 3, 5)	(2, 2, 4, 8)	(2, 2, 5, 11)
(3, 3, 3, 3)	(3, 3, 4, 6)	(3, 3, 5, 9)	(3, 3, 6, 12)

G_2			
(0, 0, 0, 0)	(0, 1, 1, 2)	(0, 2, 2, 4)	(0, 3, 3, 6)
(1, 1, 1, 1)	(1, 2, 2, 3)	(1, 3, 3, 5)	(1, 4, 4, 7)
(2, 2, 2, 2)	(2, 3, 3, 4)	(2, 4, 4, 6)	(2, 5, 5, 8)
(3, 3, 3, 3)	(3, 4, 4, 5)	(3, 5, 5, 7)	(3, 6, 6, 9)

G_3			
(0, 0, 0, 0)	(0, 1, 1, 2)	(0, 2, 2, 4)	(0, 3, 3, 6)
(0, 0, 1, 3)	(0, 1, 2, 5)	(0, 2, 3, 7)	(0, 3, 4, 9)
(0, 0, 2, 6)	(0, 1, 3, 8)	(0, 2, 4, 10)	(0, 3, 5, 12)
(0, 0, 3, 9)	(0, 1, 4, 11)	(0, 2, 5, 13)	(0, 3, 6, 15)
(1, 1, 1, 1)	(1, 2, 2, 3)	(1, 3, 3, 5)	(1, 4, 4, 7)
(1, 1, 2, 4)	(1, 2, 3, 6)	(1, 3, 4, 8)	(1, 4, 5, 10)
(1, 1, 3, 7)	(1, 2, 4, 9)	(1, 3, 5, 11)	(1, 4, 6, 13)
(1, 1, 4, 10)	(1, 2, 5, 12)	(1, 3, 6, 14)	(1, 4, 7, 16)
(2, 2, 2, 2)	(2, 3, 3, 4)	(2, 4, 4, 6)	(2, 5, 5, 8)
(2, 2, 3, 5)	(2, 3, 4, 7)	(2, 4, 5, 9)	(2, 5, 6, 11)
(2, 2, 4, 8)	(2, 3, 5, 10)	(2, 4, 6, 12)	(2, 5, 7, 14)
(2, 2, 5, 11)	(2, 3, 6, 13)	(2, 4, 7, 15)	(2, 5, 8, 17)
(3, 3, 3, 3)	(3, 4, 4, 5)	(3, 5, 5, 7)	(3, 6, 6, 9)
(3, 3, 4, 6)	(3, 4, 5, 8)	(3, 5, 6, 10)	(3, 6, 7, 12)
(3, 3, 5, 9)	(3, 4, 6, 11)	(3, 5, 7, 13)	(3, 6, 8, 15)
(3, 3, 6, 12)	(3, 4, 7, 14)	(3, 5, 8, 16)	(3, 6, 9, 18)

$\chi_y^{\alpha}[W//G_0]$
$1+19y+y^2, y^2, y, 1$

$\chi_y^{\alpha}[W//G_1]$			
$1+5y+y^2, 3y, 3y, 3y,$	$y^2, 0, y, 0,$	$y, y, y, y,$	$1, 0, y, 0$

$\chi_y^{\alpha}[W//G_2]$			
$1+5y+y^2, 0, y, 0,$	$y^2, y, y, 3y,$	$y, 0, 3y, 0,$	$1, 3y, y, y$

$\chi_y^\alpha[W//G_3]$			
$1+y+y^2,0,0,0,$	$0,0,0,0,$	$0,0,0,0,$	$0,0,0,0,$
$y^2,y,y,0,$	$0,y,0,0,$	$y,0,y,0,$	$0,0,y,0,$
$y,0,0,0,$	$y,0,0,y,$	$0,y,0,y,$	$y,y,0,0,$
$1,0,y,y,$	$0,0,y,0,$	$y,0,y,0,$	$0,0,0,y$

(iii) $d=5$

G_0				
(0, 0, 0, 0, 0)	(1, 1, 1, 1, 1)	(2, 2, 2, 2, 2)	(3, 3, 3, 3, 3)	(4, 4, 4, 4, 4)

G_1				
(0, 0, 0, 0, 0)	(0, 0, 0, 1, 4)	(0, 0, 0, 2, 8)	(0, 0, 0, 3, 12)	(0, 0, 0, 4, 16)
(1, 1, 1, 1, 1)	(1, 1, 1, 2, 5)	(1, 1, 1, 3, 9)	(1, 1, 1, 4, 13)	(1, 1, 1, 5, 17)
(2, 2, 2, 2, 2)	(2, 2, 2, 3, 6)	(2, 2, 2, 4, 10)	(2, 2, 2, 5, 14)	(2, 2, 2, 6, 18)
(3, 3, 3, 3, 3)	(3, 3, 3, 4, 7)	(3, 3, 3, 5, 11)	(3, 3, 3, 6, 15)	(3, 3, 3, 7, 19)
(4, 4, 4, 4, 4)	(4, 4, 4, 5, 8)	(4, 4, 4, 6, 12)	(4, 4, 4, 7, 16)	(4, 4, 4, 8, 20)

G_2				
(0, 0, 0, 0, 0)	(0, 1, 2, 3, 4)	(0, 2, 4, 6, 8)	(0, 3, 6, 9, 12)	(0, 4, 8, 12, 16)
(1, 1, 1, 1, 1)	(1, 2, 3, 4, 5)	(1, 3, 5, 7, 9)	(1, 4, 7, 10, 13)	(1, 5, 9, 13, 17)
(2, 2, 2, 2, 2)	(2, 3, 4, 5, 6)	(2, 4, 6, 8, 10)	(2, 5, 8, 11, 14)	(2, 6, 10, 14, 18)
(3, 3, 3, 3, 3)	(3, 4, 5, 6, 7)	(3, 5, 7, 9, 11)	(3, 6, 9, 12, 15)	(3, 7, 11, 15, 19)
(4, 4, 4, 4, 4)	(4, 5, 6, 7, 8)	(4, 6, 8, 10, 12)	(4, 7, 10, 13, 16)	(4, 8, 12, 16, 20)

G_3				
(0, 0, 0, 0, 0)	(0, 1, 1, 4, 4)	(0, 2, 2, 8, 8)	(0, 3, 3, 12, 12)	(0, 4, 4, 16, 16)
(1, 1, 1, 1, 1)	(1, 2, 2, 5, 5)	(1, 3, 3, 9, 9)	(1, 4, 4, 13, 13)	(1, 5, 5, 17, 17)
(2, 2, 2, 2, 2)	(2, 3, 3, 6, 6)	(2, 4, 4, 10, 10)	(2, 5, 5, 14, 14)	(2, 6, 6, 18, 18)
(3, 3, 3, 3, 3)	(3, 4, 4, 7, 7)	(3, 5, 5, 11, 11)	(3, 6, 6, 15, 15)	(3, 7, 7, 19, 19)
(4, 4, 4, 4, 4)	(4, 5, 5, 8, 8)	(4, 6, 6, 12, 12)	(4, 7, 7, 16, 16)	(4, 8, 8, 20, 20)

G_4				
(0, 0, 0, 0, 0)	(0, 1, 2, 3, 4)	(0, 2, 4, 6, 8)	(0, 3, 6, 9, 12)	(0, 4, 8, 12, 16)
(0, 1, 1, 4, 4)	(0, 2, 3, 7, 8)	(0, 3, 5, 10, 12)	(0, 4, 7, 13, 16)	(0, 5, 9, 16, 20)
(0, 2, 2, 8, 8)	(0, 3, 4, 11, 12)	(0, 4, 6, 14, 16)	(0, 5, 8, 17, 20)	(0, 6, 10, 20, 24)
(0, 3, 3, 12, 12)	(0, 4, 5, 15, 16)	(0, 5, 7, 18, 20)	(0, 6, 9, 21, 24)	(0, 7, 11, 24, 28)
(0, 4, 4, 16, 16)	(0, 5, 6, 19, 20)	(0, 6, 8, 22, 24)	(0, 7, 10, 25, 28)	(0, 8, 12, 28, 32)
(1, 1, 1, 1, 1)	(1, 2, 3, 4, 5)	(1, 3, 5, 7, 9)	(1, 4, 7, 10, 13)	(1, 5, 9, 13, 17)
(1, 2, 2, 5, 5)	(1, 3, 4, 8, 9)	(1, 4, 6, 11, 13)	(1, 5, 8, 14, 17)	(1, 6, 10, 17, 21)
(1, 3, 3, 9, 9)	(1, 4, 5, 12, 13)	(1, 5, 7, 15, 17)	(1, 6, 9, 18, 21)	(1, 7, 11, 21, 25)
(1, 4, 4, 13, 13)	(1, 5, 6, 16, 17)	(1, 6, 8, 19, 21)	(1, 7, 10, 22, 25)	(1, 8, 12, 25, 29)
(1, 5, 5, 17, 17)	(1, 6, 7, 20, 21)	(1, 7, 9, 23, 25)	(1, 8, 11, 26, 29)	(1, 9, 13, 29, 33)
(2, 2, 2, 2, 2)	(2, 3, 4, 5, 6)	(2, 4, 6, 8, 10)	(2, 5, 8, 11, 14)	(2, 6, 10, 14, 18)
(2, 3, 3, 6, 6)	(2, 4, 5, 9, 10)	(2, 5, 7, 12, 14)	(2, 6, 9, 15, 18)	(2, 7, 11, 18, 22)
(2, 4, 4, 10, 10)	(2, 5, 6, 13, 14)	(2, 6, 8, 16, 18)	(2, 7, 10, 19, 22)	(2, 8, 12, 22, 26)
(2, 5, 5, 14, 14)	(2, 6, 7, 17, 18)	(2, 7, 9, 20, 22)	(2, 8, 11, 23, 26)	(2, 9, 13, 26, 30)
(2, 6, 6, 18, 18)	(2, 7, 8, 21, 22)	(2, 8, 10, 24, 26)	(2, 9, 12, 27, 30)	(2, 10, 14, 30, 34)
(3, 3, 3, 3, 3)	(3, 4, 5, 6, 7)	(3, 5, 7, 9, 11)	(3, 6, 9, 12, 15)	(3, 7, 11, 15, 19)
(3, 4, 4, 7, 7)	(3, 5, 6, 10, 11)	(3, 6, 8, 13, 15)	(3, 7, 10, 16, 19)	(3, 8, 12, 19, 23)
(3, 5, 5, 11, 11)	(3, 6, 7, 14, 15)	(3, 7, 9, 17, 19)	(3, 8, 11, 20, 23)	(3, 9, 13, 23, 27)
(3, 6, 6, 15, 15)	(3, 7, 8, 18, 19)	(3, 8, 10, 21, 23)	(3, 9, 12, 24, 27)	(3, 10, 14, 27, 31)
(3, 7, 7, 19, 19)	(3, 8, 9, 22, 23)	(3, 9, 11, 25, 27)	(3, 10, 13, 28, 31)	(3, 11, 15, 31, 35)
(4, 4, 4, 4, 4)	(4, 5, 6, 7, 8)	(4, 6, 8, 10, 12)	(4, 7, 10, 13, 16)	(4, 8, 12, 16, 20)
(4, 5, 5, 8, 8)	(4, 6, 7, 11, 12)	(4, 7, 9, 14, 16)	(4, 8, 11, 17, 20)	(4, 9, 13, 20, 24)
(4, 6, 6, 12, 12)	(4, 7, 8, 15, 16)	(4, 8, 10, 18, 20)	(4, 9, 12, 21, 24)	(4, 10, 14, 24, 28)
(4, 7, 7, 16, 16)	(4, 8, 9, 19, 20)	(4, 9, 11, 22, 24)	(4, 10, 13, 25, 28)	(4, 11, 15, 28, 32)
(4, 8, 8, 20, 20)	(4, 9, 10, 23, 24)	(4, 10, 12, 26, 28)	(4, 11, 14, 29, 32)	(4, 12, 16, 32, 36)

G_5				
(0, 0, 0, 0, 0)	(0, 1, 1, 0, 3)	(0, 2, 2, 0, 6)	(0, 3, 3, 0, 9)	(0, 4, 4, 0, 12)
(0, 1, 3, 1, 0)	(0, 2, 4, 1, 3)	(0, 3, 5, 1, 6)	(0, 4, 6, 1, 9)	(0, 5, 7, 1, 12)
(0, 2, 6, 2, 0)	(0, 3, 7, 2, 3)	(0, 4, 8, 2, 6)	(0, 5, 9, 2, 9)	(0, 6, 10, 2, 12)
(0, 3, 9, 3, 0)	(0, 4, 10, 3, 3)	(0, 5, 11, 3, 6)	(0, 6, 12, 3, 9)	(0, 7, 13, 3, 12)
(0, 4, 12, 4, 0)	(0, 5, 13, 4, 3)	(0, 6, 14, 4, 6)	(0, 7, 15, 4, 9)	(0, 8, 16, 4, 12)
(1, 1, 1, 1, 1)	(1, 2, 2, 1, 4)	(1, 3, 3, 1, 7)	(1, 4, 4, 1, 10)	(1, 5, 5, 1, 13)
(1, 2, 4, 2, 1)	(1, 3, 5, 2, 4)	(1, 4, 6, 2, 7)	(1, 5, 7, 2, 10)	(1, 6, 8, 2, 13)
(1, 3, 7, 3, 1)	(1, 4, 8, 3, 4)	(1, 5, 9, 3, 7)	(1, 6, 10, 3, 10)	(1, 7, 11, 3, 13)
(1, 4, 10, 4, 1)	(1, 5, 11, 4, 4)	(1, 6, 12, 4, 7)	(1, 7, 13, 4, 10)	(1, 8, 14, 4, 13)
(1, 5, 13, 5, 1)	(1, 6, 14, 5, 4)	(1, 7, 15, 5, 7)	(1, 8, 16, 5, 10)	(1, 9, 17, 5, 13)
(2, 2, 2, 2, 2)	(2, 3, 3, 2, 5)	(2, 4, 4, 2, 8)	(2, 5, 5, 2, 11)	(2, 6, 6, 2, 14)
(2, 3, 5, 3, 2)	(2, 4, 6, 3, 5)	(2, 5, 7, 3, 8)	(2, 6, 8, 3, 11)	(2, 7, 9, 3, 14)
(2, 4, 8, 4, 2)	(2, 5, 9, 4, 5)	(2, 6, 10, 4, 8)	(2, 7, 11, 4, 11)	(2, 8, 12, 4, 14)
(2, 5, 11, 5, 2)	(2, 6, 12, 5, 5)	(2, 7, 13, 5, 8)	(2, 8, 14, 5, 11)	(2, 9, 15, 5, 14)
(2, 6, 14, 6, 2)	(2, 7, 15, 6, 5)	(2, 8, 16, 6, 8)	(2, 9, 17, 6, 11)	(2, 10, 18, 6, 14)
(3, 3, 3, 3, 3)	(3, 4, 4, 3, 6)	(3, 5, 5, 3, 9)	(3, 6, 6, 3, 12)	(3, 7, 7, 3, 15)
(3, 4, 6, 4, 3)	(3, 5, 7, 4, 6)	(3, 6, 8, 4, 9)	(3, 7, 9, 4, 12)	(3, 8, 10, 4, 15)
(3, 5, 9, 5, 3)	(3, 6, 10, 5, 6)	(3, 7, 11, 5, 9)	(3, 8, 12, 5, 12)	(3, 9, 13, 5, 15)
(3, 6, 12, 6, 3)	(3, 7, 13, 6, 6)	(3, 8, 14, 6, 9)	(3, 9, 15, 6, 12)	(3, 10, 16, 6, 15)
(3, 7, 15, 7, 3)	(3, 8, 16, 7, 6)	(3, 9, 17, 7, 9)	(3, 10, 18, 7, 12)	(3, 11, 19, 7, 15)
(4, 4, 4, 4, 4)	(4, 5, 5, 4, 7)	(4, 6, 6, 4, 10)	(4, 7, 7, 4, 13)	(4, 8, 8, 4, 16)
(4, 5, 7, 5, 4)	(4, 6, 8, 5, 7)	(4, 7, 9, 5, 10)	(4, 8, 10, 5, 13)	(4, 9, 11, 5, 16)
(4, 6, 10, 6, 4)	(4, 7, 11, 6, 7)	(4, 8, 12, 6, 10)	(4, 9, 13, 6, 13)	(4, 10, 14, 6, 16)
(4, 7, 13, 7, 4)	(4, 8, 14, 7, 7)	(4, 9, 15, 7, 10)	(4, 10, 16, 7, 13)	(4, 11, 17, 7, 16)
(4, 8, 16, 8, 4)	(4, 9, 17, 8, 7)	(4, 10, 18, 8, 10)	(4, 11, 19, 8, 13)	(4, 12, 20, 8, 16)

G_6				
(0, 0, 0, 0, 0)	(0, 3, 0, 1, 1)	(0, 6, 0, 2, 2)	(0, 9, 0, 3, 3)	(0, 12, 0, 4, 4)
(0, 1, 4, 0, 0)	(0, 4, 4, 1, 1)	(0, 7, 4, 2, 2)	(0, 10, 4, 3, 3)	(0, 13, 4, 4, 4)
(0, 2, 8, 0, 0)	(0, 5, 8, 1, 1)	(0, 8, 8, 2, 2)	(0, 11, 8, 3, 3)	(0, 14, 8, 4, 4)
(0, 3, 12, 0, 0)	(0, 6, 12, 1, 1)	(0, 9, 12, 2, 2)	(0, 12, 12, 3, 3)	(0, 15, 12, 4, 4)
(0, 4, 16, 0, 0)	(0, 7, 16, 1, 1)	(0, 10, 16, 2, 2)	(0, 13, 16, 3, 3)	(0, 16, 16, 4, 4)
(1, 1, 1, 1, 1)	(1, 4, 1, 2, 2)	(1, 7, 1, 3, 3)	(1, 10, 1, 4, 4)	(1, 13, 1, 5, 5)
(1, 2, 5, 1, 1)	(1, 5, 5, 2, 2)	(1, 8, 5, 3, 3)	(1, 11, 5, 4, 4)	(1, 14, 5, 5, 5)
(1, 3, 9, 1, 1)	(1, 6, 9, 2, 2)	(1, 9, 9, 3, 3)	(1, 12, 9, 4, 4)	(1, 15, 9, 5, 5)
(1, 4, 13, 1, 1)	(1, 7, 13, 2, 2)	(1, 10, 13, 3, 3)	(1, 13, 13, 4, 4)	(1, 16, 13, 5, 5)
(1, 5, 17, 1, 1)	(1, 8, 17, 2, 2)	(1, 11, 17, 3, 3)	(1, 14, 17, 4, 4)	(1, 17, 17, 5, 5)
(2, 2, 2, 2, 2)	(2, 5, 2, 3, 3)	(2, 8, 2, 4, 4)	(2, 11, 2, 5, 5)	(2, 14, 2, 6, 6)
(2, 3, 6, 2, 2)	(2, 6, 6, 3, 3)	(2, 9, 6, 4, 4)	(2, 12, 6, 5, 5)	(2, 15, 6, 6, 6)
(2, 4, 10, 2, 2)	(2, 7, 10, 3, 3)	(2, 10, 10, 4, 4)	(2, 13, 10, 5, 5)	(2, 16, 10, 6, 6)
(2, 5, 14, 2, 2)	(2, 8, 14, 3, 3)	(2, 11, 14, 4, 4)	(2, 14, 14, 5, 5)	(2, 17, 14, 6, 6)
(2, 6, 18, 2, 2)	(2, 9, 18, 3, 3)	(2, 12, 18, 4, 4)	(2, 15, 18, 5, 5)	(2, 18, 18, 6, 6)
(3, 3, 3, 3, 3)	(3, 6, 3, 4, 4)	(3, 9, 3, 5, 5)	(3, 12, 3, 6, 6)	(3, 15, 3, 7, 7)
(3, 4, 7, 3, 3)	(3, 7, 7, 4, 4)	(3, 10, 7, 5, 5)	(3, 13, 7, 6, 6)	(3, 16, 7, 7, 7)
(3, 5, 11, 3, 3)	(3, 8, 11, 4, 4)	(3, 11, 11, 5, 5)	(3, 14, 11, 6, 6)	(3, 17, 11, 7, 7)
(3, 6, 15, 3, 3)	(3, 9, 15, 4, 4)	(3, 12, 15, 5, 5)	(3, 15, 15, 6, 6)	(3, 18, 15, 7, 7)
(3, 7, 19, 3, 3)	(3, 10, 19, 4, 4)	(3, 13, 19, 5, 5)	(3, 16, 19, 6, 6)	(3, 19, 19, 7, 7)
(4, 4, 4, 4, 4)	(4, 7, 4, 5, 5)	(4, 10, 4, 6, 6)	(4, 13, 4, 7, 7)	(4, 16, 4, 8, 8)
(4, 5, 8, 4, 4)	(4, 8, 8, 5, 5)	(4, 11, 8, 6, 6)	(4, 14, 8, 7, 7)	(4, 17, 8, 8, 8)
(4, 6, 12, 4, 4)	(4, 9, 12, 5, 5)	(4, 12, 12, 6, 6)	(4, 15, 12, 7, 7)	(4, 18, 12, 8, 8)
(4, 7, 16, 4, 4)	(4, 10, 16, 5, 5)	(4, 13, 16, 6, 6)	(4, 16, 16, 7, 7)	(4, 19, 16, 8, 8)
(4, 8, 20, 4, 4)	(4, 11, 20, 5, 5)	(4, 14, 20, 6, 6)	(4, 17, 20, 7, 7)	(4, 20, 20, 8, 8)

G_7				
(0, 0, 0, 0, 0)	(0, 0, 0, 1, 4)	(0, 0, 0, 2, 8)	(0, 0, 0, 3, 12)	(0, 0, 0, 4, 16)
(0, 1, 1, 4, 4)	(0, 1, 1, 5, 8)	(0, 1, 1, 6, 12)	(0, 1, 1, 7, 16)	(0, 1, 1, 8, 20)
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(0, 4, 4, 16, 16)	(0, 4, 4, 17, 20)	(0, 4, 4, 18, 24)	(0, 4, 4, 19, 28)	(0, 4, 4, 20, 32)
(0, 1, 2, 3, 4)	(0, 1, 2, 4, 8)	(0, 1, 2, 5, 12)	(0, 1, 2, 6, 16)	(0, 1, 2, 7, 20)
(0, 2, 3, 7, 8)	(0, 2, 3, 8, 12)	(0, 2, 3, 9, 16)	(0, 2, 3, 10, 20)	(0, 2, 3, 11, 24)
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(0, 5, 6, 19, 20)	(0, 5, 6, 20, 24)	(0, 5, 6, 21, 28)	(0, 5, 6, 22, 32)	(0, 5, 6, 23, 36)
(0, 2, 4, 6, 8)	(0, 2, 4, 7, 12)	(0, 2, 4, 8, 16)	(0, 2, 4, 9, 20)	(0, 2, 4, 10, 24)
(0, 3, 5, 10, 12)	(0, 3, 5, 11, 16)	(0, 3, 5, 12, 20)	(0, 3, 5, 13, 24)	(0, 3, 5, 14, 28)
(0, 4, 6, 14, 16)	(0, 4, 6, 15, 20)	(0, 4, 6, 16, 24)	(0, 4, 6, 17, 28)	(0, 4, 6, 18, 32)
(0, 5, 7, 18, 20)	(0, 5, 7, 19, 24)	(0, 5, 7, 20, 28)	(0, 5, 7, 21, 32)	(0, 5, 7, 22, 36)
(0, 6, 8, 22, 24)	(0, 6, 8, 23, 28)	(0, 6, 8, 24, 32)	(0, 6, 8, 25, 36)	(0, 6, 8, 26, 40)
(0, 3, 6, 9, 12)	(0, 3, 6, 10, 16)	(0, 3, 6, 11, 20)	(0, 3, 6, 12, 24)	(0, 3, 6, 13, 28)
(0, 4, 7, 13, 16)	(0, 4, 7, 14, 20)	(0, 4, 7, 15, 24)	(0, 4, 7, 16, 28)	(0, 4, 7, 17, 32)
(0, 5, 8, 17, 20)	(0, 5, 8, 18, 24)	(0, 5, 8, 19, 28)	(0, 5, 8, 20, 32)	(0, 5, 8, 21, 36)
(0, 6, 9, 21, 24)	(0, 6, 9, 22, 28)	(0, 6, 9, 23, 32)	(0, 6, 9, 24, 36)	(0, 6, 9, 25, 40)
(0, 7, 10, 25, 28)	(0, 7, 10, 26, 32)	(0, 7, 10, 27, 36)	(0, 7, 10, 28, 40)	(0, 7, 10, 29, 44)
(0, 4, 8, 12, 16)	(0, 4, 8, 13, 20)	(0, 4, 8, 14, 24)	(0, 4, 8, 15, 28)	(0, 4, 8, 16, 32)
(0, 5, 9, 16, 20)	(0, 5, 9, 17, 24)	(0, 5, 9, 18, 28)	(0, 5, 9, 19, 32)	(0, 5, 9, 20, 36)
(0, 6, 10, 20, 24)	(0, 6, 10, 21, 28)	(0, 6, 10, 22, 32)	(0, 6, 10, 23, 36)	(0, 6, 10, 24, 40)

(continued)

(0, 7, 11, 24, 28)	(0, 7, 11, 25, 32)	(0, 7, 11, 26, 36)	(0, 7, 11, 27, 40)	(0, 7, 11, 28, 44)
(0, 8, 12, 28, 32)	(0, 8, 12, 29, 36)	(0, 8, 12, 30, 40)	(0, 8, 12, 31, 44)	(0, 8, 12, 32, 48)
(1, 1, 1, 1, 1)	(1, 1, 1, 2, 5)	(1, 1, 1, 3, 9)	(1, 1, 1, 4, 13)	(1, 1, 1, 5, 17)
(1, 2, 2, 5, 5)	(1, 2, 2, 6, 9)	(1, 2, 2, 7, 13)	(1, 2, 2, 8, 17)	(1, 2, 2, 9, 21)
(1, 3, 3, 9, 9)	(1, 3, 3, 10, 13)	(1, 3, 3, 11, 17)	(1, 3, 3, 12, 21)	(1, 3, 3, 13, 25)
(1, 4, 4, 13, 13)	(1, 4, 4, 14, 17)	(1, 4, 4, 15, 21)	(1, 4, 4, 16, 25)	(1, 4, 4, 17, 29)
(1, 5, 5, 17, 17)	(1, 5, 5, 18, 21)	(1, 5, 5, 19, 25)	(1, 5, 5, 20, 29)	(1, 5, 5, 21, 33)
(1, 2, 3, 4, 5)	(1, 2, 3, 5, 9)	(1, 2, 3, 6, 13)	(1, 2, 3, 7, 17)	(1, 2, 3, 8, 21)
(1, 3, 4, 8, 9)	(1, 3, 4, 9, 13)	(1, 3, 4, 10, 17)	(1, 3, 4, 11, 21)	(1, 3, 4, 12, 25)
(1, 4, 5, 12, 13)	(1, 4, 5, 13, 17)	(1, 4, 5, 14, 21)	(1, 4, 5, 15, 25)	(1, 4, 5, 16, 29)
(1, 5, 6, 16, 17)	(1, 5, 6, 17, 21)	(1, 5, 6, 18, 25)	(1, 5, 6, 19, 29)	(1, 5, 6, 20, 33)
(1, 6, 7, 20, 21)	(1, 6, 7, 21, 25)	(1, 6, 7, 22, 29)	(1, 6, 7, 23, 33)	(1, 6, 7, 24, 37)
(1, 3, 5, 7, 9)	(1, 3, 5, 8, 13)	(1, 3, 5, 9, 17)	(1, 3, 5, 10, 21)	(1, 3, 5, 11, 25)
(1, 4, 6, 11, 13)	(1, 4, 6, 12, 17)	(1, 4, 6, 13, 21)	(1, 4, 6, 14, 25)	(1, 4, 6, 15, 29)
(1, 5, 7, 15, 17)	(1, 5, 7, 16, 21)	(1, 5, 7, 17, 25)	(1, 5, 7, 18, 29)	(1, 5, 7, 19, 33)
(1, 6, 8, 19, 21)	(1, 6, 8, 20, 25)	(1, 6, 8, 21, 29)	(1, 6, 8, 22, 33)	(1, 6, 8, 23, 37)
(1, 7, 9, 23, 25)	(1, 7, 9, 24, 29)	(1, 7, 9, 25, 33)	(1, 7, 9, 26, 37)	(1, 7, 9, 27, 41)
(1, 4, 7, 10, 13)	(1, 4, 7, 11, 17)	(1, 4, 7, 12, 21)	(1, 4, 7, 13, 25)	(1, 4, 7, 14, 29)
(1, 5, 8, 14, 17)	(1, 5, 8, 15, 21)	(1, 5, 8, 16, 25)	(1, 5, 8, 17, 29)	(1, 5, 8, 18, 33)
(1, 6, 9, 18, 21)	(1, 6, 9, 19, 25)	(1, 6, 9, 20, 29)	(1, 6, 9, 21, 33)	(1, 6, 9, 22, 37)
(1, 7, 10, 22, 25)	(1, 7, 10, 23, 29)	(1, 7, 10, 24, 33)	(1, 7, 10, 25, 37)	(1, 7, 10, 26, 41)
(1, 8, 11, 26, 29)	(1, 8, 11, 27, 33)	(1, 8, 11, 28, 37)	(1, 8, 11, 29, 41)	(1, 8, 11, 30, 45)
(1, 5, 9, 13, 17)	(1, 5, 9, 14, 21)	(1, 5, 9, 15, 25)	(1, 5, 9, 16, 29)	(1, 5, 9, 17, 33)
(1, 6, 10, 17, 21)	(1, 6, 10, 18, 25)	(1, 6, 10, 19, 29)	(1, 6, 10, 20, 33)	(1, 6, 10, 21, 37)
(1, 7, 11, 21, 25)	(1, 7, 11, 22, 29)	(1, 7, 11, 23, 33)	(1, 7, 11, 24, 37)	(1, 7, 11, 25, 41)
(1, 8, 12, 25, 29)	(1, 8, 12, 26, 33)	(1, 8, 12, 27, 37)	(1, 8, 12, 28, 41)	(1, 8, 12, 29, 45)
(1, 9, 13, 29, 33)	(1, 9, 13, 30, 37)	(1, 9, 13, 31, 41)	(1, 9, 13, 32, 45)	(1, 9, 13, 33, 49)
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(2, 3, 3, 6, 6)	(2, 3, 3, 7, 10)	(2, 3, 3, 8, 14)	(2, 3, 3, 9, 18)	(2, 3, 3, 10, 22)
(2, 4, 4, 10, 10)	(2, 4, 4, 11, 14)	(2, 4, 4, 12, 18)	(2, 4, 4, 13, 22)	(2, 4, 4, 14, 26)
(2, 5, 5, 14, 14)	(2, 5, 5, 15, 18)	(2, 5, 5, 16, 22)	(2, 5, 5, 17, 26)	(2, 5, 5, 18, 30)
(2, 6, 6, 18, 18)	(2, 6, 6, 19, 22)	(2, 6, 6, 20, 26)	(2, 6, 6, 21, 30)	(2, 6, 6, 22, 34)
(2, 3, 4, 5, 6)	(2, 3, 4, 6, 10)	(2, 3, 4, 7, 14)	(2, 3, 4, 8, 18)	(2, 3, 4, 9, 22)
(2, 4, 5, 9, 10)	(2, 4, 5, 10, 14)	(2, 4, 5, 11, 18)	(2, 4, 5, 12, 22)	(2, 4, 5, 13, 26)
(2, 5, 6, 13, 14)	(2, 5, 6, 14, 18)	(2, 5, 6, 15, 22)	(2, 5, 6, 16, 26)	(2, 5, 6, 17, 30)
(2, 6, 7, 17, 18)	(2, 6, 7, 18, 22)	(2, 6, 7, 19, 26)	(2, 6, 7, 20, 30)	(2, 6, 7, 21, 34)
(2, 7, 8, 21, 22)	(2, 7, 8, 22, 26)	(2, 7, 8, 23, 30)	(2, 7, 8, 24, 34)	(2, 7, 8, 25, 38)
(2, 4, 6, 8, 10)	(2, 4, 6, 9, 14)	(2, 4, 6, 10, 18)	(2, 4, 6, 11, 22)	(2, 4, 6, 12, 26)
(2, 5, 7, 12, 14)	(2, 5, 7, 13, 18)	(2, 5, 7, 14, 22)	(2, 5, 7, 15, 26)	(2, 5, 7, 16, 30)
(2, 6, 8, 16, 18)	(2, 6, 8, 17, 22)	(2, 6, 8, 18, 26)	(2, 6, 8, 19, 30)	(2, 6, 8, 20, 34)
(2, 7, 9, 20, 22)	(2, 7, 9, 21, 26)	(2, 7, 9, 22, 30)	(2, 7, 9, 23, 34)	(2, 7, 9, 24, 38)
(2, 8, 10, 24, 26)	(2, 8, 10, 25, 30)	(2, 8, 10, 26, 34)	(2, 8, 10, 27, 38)	(2, 8, 10, 28, 42)
(2, 5, 8, 11, 14)	(2, 5, 8, 12, 18)	(2, 5, 8, 13, 22)	(2, 5, 8, 14, 26)	(2, 5, 8, 15, 30)
(2, 6, 9, 15, 18)	(2, 6, 9, 16, 22)	(2, 6, 9, 17, 26)	(2, 6, 9, 18, 30)	(2, 6, 9, 19, 34)
(2, 7, 10, 19, 22)	(2, 7, 10, 20, 26)	(2, 7, 10, 21, 30)	(2, 7, 10, 22, 34)	(2, 7, 10, 23, 38)
(2, 8, 11, 23, 26)	(2, 8, 11, 24, 30)	(2, 8, 11, 25, 34)	(2, 8, 11, 26, 38)	(2, 8, 11, 27, 42)
(2, 9, 12, 27, 30)	(2, 9, 12, 28, 34)	(2, 9, 12, 29, 38)	(2, 9, 12, 30, 42)	(2, 9, 12, 31, 46)
(2, 6, 10, 14, 18)	(2, 6, 10, 15, 22)	(2, 6, 10, 16, 26)	(2, 6, 10, 17, 30)	(2, 6, 10, 18, 34)
(2, 7, 11, 18, 22)	(2, 7, 11, 19, 26)	(2, 7, 11, 20, 30)	(2, 7, 11, 21, 34)	(2, 7, 11, 22, 38)
(2, 8, 12, 22, 26)	(2, 8, 12, 23, 30)	(2, 8, 12, 24, 34)	(2, 8, 12, 25, 38)	(2, 8, 12, 26, 42)
(2, 9, 13, 26, 30)	(2, 9, 13, 27, 34)	(2, 9, 13, 28, 38)	(2, 9, 13, 29, 42)	(2, 9, 13, 30, 46)
(2, 10, 14, 30, 34)	(2, 10, 14, 31, 38)	(2, 10, 14, 32, 42)	(2, 10, 14, 33, 46)	(2, 10, 14, 34, 50)
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(3, 4, 4, 7, 7)	(3, 4, 4, 8, 11)	(3, 4, 4, 9, 15)	(3, 4, 4, 10, 19)	(3, 4, 4, 11, 23)

(continued)

(3, 5, 5, 11, 11)	(3, 5, 5, 12, 15)	(3, 5, 5, 13, 19)	(3, 5, 5, 14, 23)	(3, 5, 5, 15, 27)
(3, 6, 6, 15, 15)	(3, 6, 6, 16, 19)	(3, 6, 6, 17, 23)	(3, 6, 6, 18, 27)	(3, 6, 6, 19, 31)
(3, 7, 7, 19, 19)	(3, 7, 7, 20, 23)	(3, 7, 7, 21, 27)	(3, 7, 7, 22, 31)	(3, 7, 7, 23, 35)
(3, 4, 5, 6, 7)	(3, 4, 5, 7, 11)	(3, 4, 5, 8, 15)	(3, 4, 5, 9, 19)	(3, 4, 5, 10, 23)
(3, 5, 6, 10, 11)	(3, 5, 6, 11, 15)	(3, 5, 6, 12, 19)	(3, 5, 6, 13, 23)	(3, 5, 6, 14, 27)
(3, 6, 7, 14, 15)	(3, 6, 7, 15, 19)	(3, 6, 7, 16, 23)	(3, 6, 7, 17, 27)	(3, 6, 7, 18, 31)
(3, 7, 8, 18, 19)	(3, 7, 8, 19, 23)	(3, 7, 8, 20, 27)	(3, 7, 8, 21, 31)	(3, 7, 8, 22, 35)
(3, 8, 9, 22, 23)	(3, 8, 9, 23, 27)	(3, 8, 9, 24, 31)	(3, 8, 9, 25, 35)	(3, 8, 9, 26, 39)
(3, 5, 7, 9, 11)	(3, 5, 7, 10, 15)	(3, 5, 7, 11, 19)	(3, 5, 7, 12, 23)	(3, 5, 7, 13, 27)
(3, 6, 8, 13, 15)	(3, 6, 8, 14, 19)	(3, 6, 8, 15, 23)	(3, 6, 8, 16, 27)	(3, 6, 8, 17, 31)
(3, 7, 9, 17, 19)	(3, 7, 9, 18, 23)	(3, 7, 9, 19, 27)	(3, 7, 9, 20, 31)	(3, 7, 9, 21, 35)
(3, 8, 10, 21, 23)	(3, 8, 10, 22, 27)	(3, 8, 10, 23, 31)	(3, 8, 10, 24, 35)	(3, 8, 10, 25, 39)
(3, 9, 11, 25, 27)	(3, 9, 11, 26, 31)	(3, 9, 11, 27, 35)	(3, 9, 11, 28, 39)	(3, 9, 11, 29, 43)
(3, 6, 9, 12, 15)	(3, 6, 9, 13, 19)	(3, 6, 9, 14, 23)	(3, 6, 9, 15, 27)	(3, 6, 9, 16, 31)
(3, 7, 10, 16, 19)	(3, 7, 10, 17, 23)	(3, 7, 10, 18, 27)	(3, 7, 10, 19, 31)	(3, 7, 10, 20, 35)
(3, 8, 11, 20, 23)	(3, 8, 11, 21, 27)	(3, 8, 11, 22, 31)	(3, 8, 11, 23, 35)	(3, 8, 11, 24, 39)
(3, 9, 12, 24, 27)	(3, 9, 12, 25, 31)	(3, 9, 12, 26, 35)	(3, 9, 12, 27, 39)	(3, 9, 12, 28, 43)
(3, 10, 13, 28, 31)	(3, 10, 13, 29, 35)	(3, 10, 13, 30, 39)	(3, 10, 13, 31, 43)	(3, 10, 13, 32, 47)
(3, 7, 11, 15, 19)	(3, 7, 11, 16, 23)	(3, 7, 11, 17, 27)	(3, 7, 11, 18, 31)	(3, 7, 11, 19, 35)
(3, 8, 12, 19, 23)	(3, 8, 12, 20, 27)	(3, 8, 12, 21, 31)	(3, 8, 12, 22, 35)	(3, 8, 12, 23, 39)
(3, 9, 13, 23, 27)	(3, 9, 13, 24, 31)	(3, 9, 13, 25, 35)	(3, 9, 13, 26, 39)	(3, 9, 13, 27, 43)
(3, 10, 14, 27, 31)	(3, 10, 14, 28, 35)	(3, 10, 14, 29, 39)	(3, 10, 14, 30, 43)	(3, 10, 14, 31, 47)
(3, 11, 15, 31, 35)	(3, 11, 15, 32, 39)	(3, 11, 15, 33, 43)	(3, 11, 15, 34, 47)	(3, 11, 15, 35, 51)
(4, 4, 4, 4, 4)	(4, 4, 4, 5, 8)	(4, 4, 4, 6, 12)	(4, 4, 4, 7, 16)	(4, 4, 4, 8, 20)
(4, 5, 5, 8, 8)	(4, 5, 5, 9, 12)	(4, 5, 5, 10, 16)	(4, 5, 5, 11, 20)	(4, 5, 5, 12, 24)
(4, 6, 6, 12, 12)	(4, 6, 6, 13, 16)	(4, 6, 6, 14, 20)	(4, 6, 6, 15, 24)	(4, 6, 6, 16, 28)
(4, 7, 7, 16, 16)	(4, 7, 7, 17, 20)	(4, 7, 7, 18, 24)	(4, 7, 7, 19, 28)	(4, 7, 7, 20, 32)
(4, 8, 8, 20, 20)	(4, 8, 8, 21, 24)	(4, 8, 8, 22, 28)	(4, 8, 8, 23, 32)	(4, 8, 8, 24, 36)
(4, 5, 6, 7, 8)	(4, 5, 6, 8, 12)	(4, 5, 6, 9, 16)	(4, 5, 6, 10, 20)	(4, 5, 6, 11, 24)
(4, 6, 7, 11, 12)	(4, 6, 7, 12, 16)	(4, 6, 7, 13, 20)	(4, 6, 7, 14, 24)	(4, 6, 7, 15, 28)
(4, 7, 8, 15, 16)	(4, 7, 8, 16, 20)	(4, 7, 8, 17, 24)	(4, 7, 8, 18, 28)	(4, 7, 8, 19, 32)
(4, 8, 9, 19, 20)	(4, 8, 9, 20, 24)	(4, 8, 9, 21, 28)	(4, 8, 9, 22, 32)	(4, 8, 9, 23, 36)
(4, 9, 10, 23, 24)	(4, 9, 10, 24, 28)	(4, 9, 10, 25, 32)	(4, 9, 10, 26, 36)	(4, 9, 10, 27, 40)
(4, 6, 8, 10, 12)	(4, 6, 8, 11, 16)	(4, 6, 8, 12, 20)	(4, 6, 8, 13, 24)	(4, 6, 8, 14, 28)
(4, 7, 9, 14, 16)	(4, 7, 9, 15, 20)	(4, 7, 9, 16, 24)	(4, 7, 9, 17, 28)	(4, 7, 9, 18, 32)
(4, 8, 10, 18, 20)	(4, 8, 10, 19, 24)	(4, 8, 10, 20, 28)	(4, 8, 10, 21, 32)	(4, 8, 10, 22, 36)
(4, 9, 11, 22, 24)	(4, 9, 11, 23, 28)	(4, 9, 11, 24, 32)	(4, 9, 11, 25, 36)	(4, 9, 11, 26, 40)
(4, 10, 12, 26, 28)	(4, 10, 12, 27, 32)	(4, 10, 12, 28, 36)	(4, 10, 12, 29, 40)	(4, 10, 12, 30, 44)
(4, 7, 10, 13, 16)	(4, 7, 10, 14, 20)	(4, 7, 10, 15, 24)	(4, 7, 10, 16, 28)	(4, 7, 10, 17, 32)
(4, 8, 11, 17, 20)	(4, 8, 11, 18, 24)	(4, 8, 11, 19, 28)	(4, 8, 11, 20, 32)	(4, 8, 11, 21, 36)
(4, 9, 12, 21, 24)	(4, 9, 12, 22, 28)	(4, 9, 12, 23, 32)	(4, 9, 12, 24, 36)	(4, 9, 12, 25, 40)
(4, 10, 13, 25, 28)	(4, 10, 13, 26, 32)	(4, 10, 13, 27, 36)	(4, 10, 13, 28, 40)	(4, 10, 13, 29, 44)
(4, 11, 14, 29, 32)	(4, 11, 14, 30, 36)	(4, 11, 14, 31, 40)	(4, 11, 14, 32, 44)	(4, 11, 14, 33, 48)
(4, 8, 12, 16, 20)	(4, 8, 12, 17, 24)	(4, 8, 12, 18, 28)	(4, 8, 12, 19, 32)	(4, 8, 12, 20, 36)
(4, 9, 13, 20, 24)	(4, 9, 13, 21, 28)	(4, 9, 13, 22, 32)	(4, 9, 13, 23, 36)	(4, 9, 13, 24, 40)
(4, 10, 14, 24, 28)	(4, 10, 14, 25, 32)	(4, 10, 14, 26, 36)	(4, 10, 14, 27, 40)	(4, 10, 14, 28, 44)
(4, 11, 15, 28, 32)	(4, 11, 15, 29, 36)	(4, 11, 15, 30, 40)	(4, 11, 15, 31, 44)	(4, 11, 15, 32, 48)
(4, 12, 16, 32, 36)	(4, 12, 16, 33, 40)	(4, 12, 16, 34, 44)	(4, 12, 16, 35, 48)	(4, 12, 16, 36, 52)

$\chi_y^\alpha[W//G_0]$
$1+101y+101y^2+y^3, -y^3, -y^2, -y, -1$

$\chi_y^\alpha[W//G_1]$
$1+25y+25y^2+y^3, 6y+6y^2, 6y+6y^2, 6y+6y^2, 6y+6y^2,$
$-y^3, 0, -y^2, -y^2, 0,$
$-y^2, -y^2, 0, 0, -y^2,$
$-y, -y, 0, 0, -y,$
$-1, 0, -y, -y, 0$

$\chi_y^\alpha[W//G_2]$
$1+21y+21y^2+y^3, 0, 0, 0, 0,$
$-y^3, 0, 0, 0, 0,$
$-y^2, 0, 0, 0, 0,$
$-y, 0, 0, 0, 0,$
$-1, 0, 0, 0, 0$

$\chi_y^\alpha[W//G_3]$
$1+17y+17y^2+y^3, 0, 0, 0, 0,$
$-y^3, -4y^2, -y, -y, -4y^2,$
$-y^2, -y^2, -4y, -4y, -y^2,$
$-y, -y, -4y^2, -4y^2, -y,$
$-1, -4y, -y^2, -y^2, -4y$

$\chi_y^\alpha[W//G_4]$		
$1+5y+5y^2+y^3, 0, 0, 0, 0,$	$0, 0, 2y+2y^2, 0, 2y+2y^2,$	$0, 0, 0, 2y+2y^2, 2y+2y^2,$
$0, 2y+2y^2, 2y+2y^2, 0, 0,$	$0, 2y+2y^2, 0, 2y+2y^2, 0,$	$-y^3, 0, 0, 0, 0,$
$0, -y, -y^2, 0, 0,$	$-y, 0, 0, -y^2, 0,$	$-y, 0, -y^2, 0, 0,$
$0, 0, 0, -y^2, -y,$	$-y^2, 0, 0, 0, 0,$	$-y^2, 0, 0, 0, -y^2,$
$0, 0, -y^2, 0, -y^2,$	$0, -y^2, 0, -y^2, 0,$	$-y^2, -y^2, 0, 0, 0,$
$-y, 0, 0, 0, 0,$	$-y, 0, 0, 0, -y,$	$0, 0, -y, 0, -y,$
$0, -y, 0, -y, 0,$	$-y, -y, 0, 0, 0,$	$-1, 0, 0, 0, 0,$
$0, -y^2, -y, 0, 0,$	$-y^2, 0, 0, -y, 0,$	$-y^2, 0, -y, 0, 0,$
$0, 0, 0, -y, -y^2$		

$\chi_y^\alpha[W//G_5]$		
$1+y+y^2+y^3,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$-y^3,-y^2,-y^2,0,0,$
$-y^2,0,-y^2,0,-y^2,$	$-y^2,-y,0,0,-y^2,$	$0,0,-y^2,0,-y,$
$0,0,0,0,0,$	$-y^2,0,-y,0,-y^2,$	$0,0,0,-y^2,-y,$
$-y,0,0,-y^2,-y,$	$0,0,0,0,0,$	$-y^2,0,-y^2,-y^2,0,$
$-y,-y,0,-y^2,0,$	$-y,0,-y,-y,0,$	$0,0,0,0,0,$
$-y^2,-y^2,-y,0,0,$	$0,-y^2,-y,0,0,$	$-1,0,0,-y,-y,$
$0,0,0,0,0,$	$0,-y^2,0,-y,0,$	$-y,-y,0,0,-y^2,$
$-y,-y,0,-y,0$		

$\chi_y^\alpha[W//G_6]$		
$1+5y+5y^2+y^3,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$-y^3,-y^2,-y^2,0,-4y^2,$
$0,0,0,0,0,$	$-y^2,-y^2,-y,-y,0,$	$-y^2,-y^2,0,-y,-4y^2,$
$0,-y^2,-y^2,-y,-4y^2,$	$-y^2,0,-y,-4y^2,-y^2,$	$-y^2,-y^2,-y,-4y^2,0,$
$0,0,0,0,0,$	$0,-y,-y,-4y,-y^2,$	$-y^2,-y,-y,0,-y^2,$
$-y,-y,-4y,-y^2,0,$	$-y,-y,0,-y^2,-y^2,$	$0,-y,-4y^2,-y^2,-y^2,$
$0,0,0,0,0,$	$-y,0,-4y,-y^2,-y,$	$-1,-4y,0,-y,-y,$
$0,-4y,-y^2,-y,-y,$	$-y,-4y,-y^2,0,-y,$	$-y,0,-y^2,-y^2,-y,$
$0,0,0,0,0$		

$\chi_y^\alpha[W//G_7]$		
$1+y+y^2+y^3,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$-y^3,0,-y^2,-y^2,0,$	$0,-y^2,-y^2,-y^2,-y^2,$
$-y,0,-y^2,-y^2,0,$	$-y,-y,0,0,-y,$	$0,0,0,0,0,$
$0,0,-y^2,-y^2,-y^2,$	$-y,-y,0,-y^2,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,-y^2,-y^2,-y^2,$	$0,0,0,0,0,$
$-y^2,-y^2,-y^2,0,0,$	$0,0,0,0,0,$	$-y^2,0,-y^2,-y^2,-y^2,$
$0,-y,0,-y^2,-y^2,$	$0,-y^2,-y^2,0,-y,$	$0,0,0,0,0,$
$-y^2,0,0,-y^2,-y^2,$	$0,0,0,0,0,$	$-y^2,-y^2,-y^2,-y^2,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,-y^2,-y^2,-y^2,0,$
$0,-y^2,-y^2,-y^2,0,$	$-y,0,-y^2,0,-y,$	$-y^2,-y^2,0,0,-y^2,$
$-y^2,0,-y,-y,0,$	$0,-y,-y,-y,-y,$	$0,0,0,0,0,$
$-y^2,-y^2,0,0,-y^2,$	$0,0,-y,-y,-y,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$-y^2,-y^2,-y^2,0,-y^2,$	$-y^2,-y^2,0,-y,0,$
$0,-y,0,-y^2,-y^2,$	$0,0,0,0,0,$	$-y^2,-y^2,-y^2,0,0,$
$0,-y^2,0,-y,-y,$	$0,0,0,0,0,$	$0,0,0,0,0,$
$0,-y^2,-y^2,0,-y,$	$0,0,0,0,0,$	$-y^2,0,0,-y^2,-y^2,$
$0,-y,-y,0,-y^2,$	$0,0,0,0,0,$	$-y^2,-y^2,0,-y^2,-y^2,$
$-y^2,0,-y,0,-y^2,$	$0,-y,-y,-y,0,$	$0,0,0,0,0,$

(continued)

$-y,-y,0,0,-y,$	$-y,-y,0,0,-y,$	$0,0,0,0,0,$
$0,-y^2,-y^2,-y^2,-y^2,$	$-y,0,-y^2,-y^2,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$0,0,-y^2,-y^2,-y^2,$	$-y,-y,0,-y^2,0,$
$-y,-y,-y,0,-y,$	$0,0,0,0,0,$	$0,-y,0,-y^2,-y^2,$
$-y,-y,-y,0,0,$	$0,0,0,0,0,$	$0,-y^2,0,-y,-y,$
$0,-y,-y,0,-y^2,$	$0,0,0,0,0,$	$0,-y^2,-y^2,0,-y,$
$-y,0,0,-y,-y,$	$0,0,0,0,0,$	$0,-y^2,-y^2,-y^2,0,$
$-y,0,-y^2,0,-y,$	$-y,-y,0,-y,-y,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$-1,0,-y,-y,0,$	$0,0,0,0,0,$
$-y^2,-y^2,0,0,-y^2,$	$-y^2,0,-y,-y,0,$	$0,-y,-y,-y,-y,$
$0,0,0,0,0,$	$-y^2,-y^2,0,-y,0,$	$0,0,-y,-y,-y,$
$0,0,-y,-y,-y,$	$0,0,0,0,0,$	$0,-y^2,0,-y,-y,$
$-y,0,-y,-y,-y,$	$0,0,0,0,0,$	$-y,-y,-y,0,0,$
$0,0,0,0,0,$	$0,0,0,0,0,$	$0,-y,-y,0,-y^2,$
$-y,-y,-y,-y,0,$	$0,0,0,0,0,$	$-y,0,0,-y,-y,$
$0,-y,-y,-y,0,$	$0,-y,-y,-y,0,$	$0,0,0,0,0,$
$0,0,0,0,0,$	$-y^2,0,-y,0,-y^2$	

§ 5. Discussion

We have pointed out in the previous section that there exist instances in which the elliptic genera for a pair of Landau-Ginzburg orbifolds obey the relation (4·1) (or equivalently (4·2)) and moreover the roles of the untwisted and twisted sectors are exchanged. To consider this phenomenon a little bit further we shall now concentrate on the case where two Landau-Ginzburg orbifolds are in correspondence with sigma models as investigated in §§ 4.3 and 4.4.

Let us first extend the χ_y-genus to describe the full $U(1)\times U(1)$ charge spectrum for the (c, c) ring. For a Calabi-Yau $\hat{c}$-fold $\mathcal{M}$ we define

$$\chi[\mathcal{M}](y, \bar{y})=\sum_{q_L,q_R=0}^{\hat{c}}(-1)^{q_L+q_R}h_{q_L,\hat{c}-q_R}y^{q_L}\bar{y}^{q_R}\,, \tag{5·1}$$

where $h_{q_L,\hat{c}-q_R}$ is the number of states with charge (q_L, q_R) and is also equal to the Hodge number, $\dim H^{q_L,\hat{c}-q_R}(\mathcal{M})$. Suppose that a pair of Calabi-Yau $\hat{c}$-folds $(\mathcal{M}, \tilde{\mathcal{M}})$ consist of a mirror pair, then we have

$$\tilde{h}_{p,q}=h_{p,\hat{c}-q}\,, \tag{5·2}$$

where $\tilde{h}_{p,q}$ are the Hodge numbers for $\tilde{\mathcal{M}}$. Hence the extended genus (5·1) for $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are related through

$$\chi[\mathcal{M}](y, \bar{y})=(-1)^{\hat{c}}\bar{y}^{\hat{c}}\chi[\tilde{\mathcal{M}}](y, 1/\bar{y})\,. \tag{5·3}$$

Setting $\bar{y}=1$ yields the relation for χ_y-genera

$$\chi_y[\mathcal{M}]=(-1)^{\hat{c}}\chi_y[\tilde{\mathcal{M}}]\,. \tag{5·4}$$

When the $\hat{c}$-fold $\mathcal{M}$ has a correspondence with a Landau-Ginzburg orbifold $W/\!/G$ one can write down $\chi[\mathcal{M}](y, \bar{y})=\pm\chi[W/\!/G](y, \bar{y})$ explicitly. Following the reasoning in Refs. 33) and 8) we find from (3·10) that

$$\chi[W /\!/ G](y, \bar{y}) = \frac{(-1)^N}{|G|} \sum_{\alpha,\beta\in G} \sum_{\substack{i \\ \omega_i\alpha_i\notin Z}} (y\bar{y})^{(1-2\omega_i)/2}(y/\bar{y})^{-((\omega_i\alpha_i))}$$

$$\times \prod_{\substack{i \\ \omega_i\alpha_i\in Z}} e\left[\omega_i\beta_i+\frac{1}{2}\right]\frac{1-e[(1-\omega_i)\beta_i](y\bar{y})^{1-\omega_i}}{1-e[\omega_i\beta_i](y\bar{y})^{\omega_i}}. \quad (5\cdot5)$$

We now consider a 3-fold $\mathcal{M}=\tilde{\mathcal{M}}_G$ where $\mathcal{M}_G$ is given by (4·16) with $d=5$. Inspecting the tables for G_k ($k=0, 1, \cdots, 7$) we see that $\sum_{\omega_i\alpha_i\notin Z}((\omega_i\alpha_i))=0$ occurs only in the untwisted sectors.*) Thus it is clear from (5·5) that the states with $(q_L, q_R)=(1, 1)$ corresponding to $H^{1,2}(\mathcal{M})$, come from the untwisted sectors while the states with $(q_L, q_R)=(1, 2)$ corresponding to $H^{1,1}(\mathcal{M})$, from the twisted sectors. Similarly, for its mirror partner $\tilde{\mathcal{M}}$ described as $W /\!/ G^*$, the elements of $H^{1,2}(\tilde{\mathcal{M}})$ (or $H^{1,1}(\tilde{\mathcal{M}})$) arise from the untwisted (or twisted) sectors. To see more explicitly let us evaluate (5·5) for $G=G_2$ and $G^*=G_5$. We obtain

$$\chi[W /\!/ G](y, \bar{y}) = \chi_u[W /\!/ G](y, \bar{y}) + \chi_t[W /\!/ G](y, \bar{y}), \quad (5\cdot6)$$

where χ_u(or χ_t) stands for the contribution from the untwisted (or twisted) sectors:

$$\chi_u[W /\!/ G](y, \bar{y}) = 1+21y\bar{y}+21y^2\bar{y}^2+y^3\bar{y}^3, \quad (5\cdot7)$$

$$\chi_t[W /\!/ G](y, \bar{y}) = -y^3-y^2\bar{y}-y\bar{y}^2-\bar{y}^3. \quad (5\cdot8)$$

For $W /\!/ G^*$ we get

$$\chi_u[W /\!/ G^*](y, \bar{y}) = 1+y\bar{y}+y^2\bar{y}^2+y^3\bar{y}^3, \quad (5\cdot9)$$

$$\chi_t[W /\!/ G^*](y, \bar{y}) = -y^3-21y^2\bar{y}-21y\bar{y}^2-\bar{y}^3. \quad (5\cdot10)$$

Hence

$$\chi_u[W /\!/ G](y, \bar{y}) = -\bar{y}^3\chi_t[W /\!/ G^*](y, 1/\bar{y}), \quad (5\cdot11)$$

$$\chi_t[W /\!/ G](y, \bar{y}) = -\bar{y}^3\chi_u[W /\!/ G^*](y, 1/\bar{y}). \quad (5\cdot12)$$

Upon setting $\bar{y}=1$ this reduces to the relation for χ_y-genera we found in § 4.4. We have checked using (5·5) that similar results hold for all the examples given in § 4.4. Therefore what we have observed seems to be natural whenever a mirror pair has a corresponding pair of Landau-Ginzburg orbifolds.

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