# Duality relations for overlaps of integrable boundary states in AdS/dCFT

**Charlotte Kristjansen,**[a] **Dennis Müller**[a] **and Konstantin Zarembo**[a,b,1]

[a]*Niels Bohr Institute, Copenhagen University,*
*Blegdamsvej 17, 2100 Copenhagen, Denmark*

[b]*Nordita, KTH Royal Institute of Technology and Stockholm University,*
*Hannes Alfvéns väg 12, SE-106 91 Stockholm, Sweden*

*E-mail:* kristjan@nbi.dk, dennis.muller@nbi.ku.dk, zarembo@nordita.org

ABSTRACT: The encoding of all possible sets of Bethe equations for a spin chain with $\mathrm{SU}(N|M)$ symmetry into a $QQ$-system calls for an expression of spin chain overlaps entirely in terms of $Q$-functions. We take a significant step towards deriving such a universal formula in the case of overlaps between Bethe eigenstates and integrable boundary states, of relevance for AdS/dCFT, by determining the transformation properties of the overlaps under fermionic as well as bosonic dualities which allows us to move between any two descriptions of the spin chain encoded in the $QQ$-system. An important part of our analysis involves introducing a suitable regularization for singular Bethe root configurations.

---

[1]Also at ITEP, Moscow, Russia.

## Contents

## 1  Introduction

Duality relations certainly interweave the AdS/CFT correspondence as well as its various defect versions at the macroscopical level [1, 2]. We will address a class of duality relations which rather pertain to the microscopical level. These are dualities which characterize the integrable super spin chain underlying the AdS/CFT correspondence [3–6] and its associated integrable boundary states describing the D3-D5 defect set-up [7–13].

The spectral problem of $\mathcal{N} = 4$ SYM as well as a number of other problems, some of which can be related to the free energy of the integrable spin chain [14–16], have been effectively dealt with exploiting these dualities that take the form of a set of so-called QQ-relations [17] (for reviews, see [18–20]) which for these particular problems can be elevated to a quantum spectral curve [21, 22]. The QQ relations, a set of relations between certain Q-functions, encode the various ways that one can choose the vacuum of the spin chain and the sequence of excitations at the various levels of nesting.

Addressing correlation functions of $\mathcal{N} = 4$ SYM amounts to studying wave functions and norms of spin chain eigenstates and the quantum spectral curve approach is typically not directly applicable. A possible strategy is the separation of variables method [23] which have been applied to certain sub-classes of correlation functions [24, 25]. Here we shall address the simplest possible correlation functions of $\mathcal{N} = 4$ SYM namely one-point

functions of the integrable D3-D5 probe brane set-up. These one-point functions can be expressed as overlaps between Bethe eigenstates and integrable boundary states in the form of either matrix product states or valence bond states [7, 8, 12].

Closed expressions found for these overlaps [7, 8, 12, 26, 27] have lead to a derivation of an asymptotic all-loop formula for the one-point functions in question [9–13], as well as for three-point functions involving a single trace operator and two giant gravitons both in $\mathcal{N} = 4$ SYM [28] and in ABJM theory [29, 30], and have sparked novel developments in statistical physics [31–34] where the overlaps are of relevance for the study of quantum quenches. The overlap formulas contain a universal part expressed in terms of the super determinant of the Gaudin matrix [35] of the Bethe state in question and a dynamical factor depending on the boundary state and being expressible as a product of Q-functions.

As a first step towards finding an expression for the one-point functions entirely in terms of Q-functions and thus making them amenable to the quantum spectral curve approach we complete our earlier initiated study of the behavior of the super determinant of the Gaudin matrix under the duality transformations encoded in the QQ-system.[1] The duality transformations consist of two classes of transformations, bosonic and fermionic. Both types correspond to exchanging the role of a vacuum configuration and an excitation at some nesting level. A fermionic duality implies a change of statistics for certain excitations, and corresponds to a change of Dynkin diagram for the underlying super Lie algebra, whereas a bosonic duality does not involve any change of statistics or of Dynkin diagram. In the present paper we focus on the bosonic dualities.

Our paper is organized as follows: we begin in section 2 by discussing known overlap formulas for the Heisenberg spin chain and complete the existing picture by extending the results to the case of singular Bethe root configurations. While interesting in its own right this extension is also mandatory for the overarching goal of our work as duality transformations of non-singular Bethe root configurations are known to introduce singular roots [41]. In section 3 we give a short review of the QQ-relations for integrable spin chains of SU($N|M$) type, illustrating the idea with the simplest possible example of the spin chain based on the super Lie algebra SU(2|1). Subsequently, in section 4, we turn to determining the transformation properties of the super determinant of the Gaudin matrix under bosonic dualities, starting with the simplest case of the Heisenberg spin chain, moving on the SU(2|1) spin chain and finishing with the most general case. Finally, section 5 contains our conclusion.

## 2 Integrable boundary states and overlaps

### 2.1 Boundary states

We start with boundary states in the Heisenberg model, which we review in quite some detail. The model is defined by the Hamiltonian

$$H = \sum_{l=1}^{L} (1 - P_{l,l+1}), \tag{2.1}$$

---

[1]At a full non-perturbative level the one-point correlator maps to a worldsheet g-function [36], potentially calculable in any integrable field theory with boundary [37]. Amenable to a combinatorial analysis [38, 39], the g-function has recently been connected with the quantum spectral curve formalism [40].

where $P_{l,l'}$ is the permutation operator, and is solved by Bethe ansatz

$$-1 = \frac{Q_\theta^-(u_j)}{Q_\theta^+(u_j)} \frac{\mathcal{Q}^{++}(u_j)}{\mathcal{Q}^{--}(u_j)} \equiv \mathrm{e}^{\,i\chi_j}, \tag{2.2}$$

written here in terms of the Q-functions

$$\mathcal{Q}(u) = \prod_{j=1}^{M} \left(u - u_j\right). \tag{2.3}$$

The trivial Q-function

$$Q_\theta(u) = u^L, \tag{2.4}$$

acts as a source in the Bethe equations. We use the standard notations for rapidity shifts:

$$f^{[\pm q]}(u) = f\left(u \pm \frac{iq}{2}\right), \qquad f^\pm \equiv f^{[\pm 1]}, \qquad f^{++} \equiv f^{[\pm 2]}. \tag{2.5}$$

The Bethe eigenstates $|\{u_j\}\rangle$ are $\mathfrak{su}(2)$ highest weights:

$$S^+ |\{u_j\}\rangle = 0, \qquad S^3 |\{u_j\}\rangle = \left(\frac{L}{2} - M\right) |\{u_j\}\rangle, \tag{2.6}$$

with the total spin normalized as

$$S^i = \sum_{l=1}^{L} \frac{\sigma_l^i}{2}. \tag{2.7}$$

Other members of the multiplet are generated by repeated application of $S^-$.

The commuting charges of the integrable hierarchy have definite parity under reversal of the spin chain's orientation, typically chosen to alternate with $n$: $P^{-1}Q_n P = (-1)^n Q_n$. Parity interchanges $u_j$ with $-u_j$, and an eigenvalue of $Q_n$ is an even function of Bethe roots if $n$ is even and an odd function if $n$ is odd.

An integrable boundary state is a state annihilated by all parity-odd charges: $Q_{2n+1}|B\rangle = 0$ [31, 42]. The ensuing selection rule imposes parity invariance on the set of rapidities $\{u_j\} = \{-u_j\}$ so long as the overlap $\langle B|\{u_j\}\rangle$ is non-zero. The particle content of a boundary state thus consists of momentum-conjugate pairs $(p, -p)$. Crossing (were it well-defined) would map each pair to a single particle reflecting off a spatial boundary [42]. The paired structure of the boundary state guarantees that reflection is elastic and proceeds without particle production. Originally proposed for relativistic systems [42], where crossing is well defined, the definition is extended to spin chains by analogy [31].

While integrable boundary states have never been completely classified, known examples fall into two broad categories. One type of boundary states is obtained by associating to each allowed spin state, $s$, a $k \times k$ matrix, $\sigma^s$, and tracing the product of $L$ such matrices ($L$ is the length of the spin chain) over the $k$-dimensional auxiliary space [7, 31]:

$$\langle \mathrm{MPS}| = \sum_{\{s_l\}} \mathrm{tr}\, \sigma^{s_1} \ldots \sigma^{s_L} \langle s_1 \ldots s_L|. \tag{2.8}$$

Not any choice of matrices $\sigma^s$ gives an integrable boundary state, but many examples where it does are known. For instance, an MPS generated by $\sigma^\uparrow = \sigma^1$, $\sigma^\downarrow = \sigma^2$ in the Heisenberg model is integrable.

Another common type of boundary states is built from entangled two-site blocks:

$$\langle \text{VBS}| = \langle D|^{\otimes \frac{L}{2}}, \qquad \langle D| = \sum_{s\,s'} K_{ss'} \langle ss'|, \tag{2.9}$$

and can be called Valence Bond States (VBS). In the Heisenberg model any VBS is integrable, independently of the coefficients $K_{ss'}$ [43].

All known boundary states in the Heisenberg model actually descend from the generalized dimer:

$$\langle D_\nu| = (1 + \nu)\langle\uparrow\downarrow| + (1 - \nu)\langle\downarrow\uparrow|. \tag{2.10}$$

Applying an SU(2) rotation one gets:

$$\begin{aligned}
\langle D_\nu|\,\mathrm{e}^{i\beta S^1}\,\mathrm{e}^{i\alpha S^3} = (\cos\beta + \nu)\langle\uparrow\downarrow| + (\cos\beta - \nu)\langle\downarrow\uparrow| \\
+ i\,\mathrm{e}^{i\alpha}\sin\beta\,\langle\uparrow\uparrow| + i\,\mathrm{e}^{-i\alpha}\sin\beta\,\langle\downarrow\downarrow|,
\end{aligned} \tag{2.11}$$

a generic two-site state. One can also show that [31]

$$\langle \text{MPS}| = i^{-\frac{L}{2}}\left(\langle \text{VBS}_1| + \langle \text{VBS}_{-1}|\right)\mathrm{e}^{\frac{i\pi S^1}{2}}, \tag{2.12}$$

where the subscripts on VBS refer to the subscripts on the associated D, cf. eqs. (2.9) and (2.10) Both types of boundary states admit higher-rank generalizations, albeit at higher rank not all VBS are integrable. Likewise, higher-rank MPS are related to VBS by symmetry transformations only in particular cases [44] but not in general. With rather few exceptions (see section 4.2 of [44]) boundary states are defined on a spin chain of even length. We shall always take $L$ to be even.

## 2.2 Overlap formulas

Integrable boundary states have a number of remarkable properties, in particular their overlaps with on-shell Bethe states admit a concise determinant representation:

$$\langle B|\{u_j\}\rangle \propto \prod_j v(u_j)\sqrt{\mathrm{Sdet}\,G}. \tag{2.13}$$

The function $v(u)$ depends on the boundary state at hand, while the determinant factor is universal and depends only on the Gaudin matrix $G$, the Jacobian of the Bethe equations (2.2):

$$G_{jk} = \frac{\partial \chi_j}{\partial u_k} = \delta_{jk}\left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_m K_{jm}\right) + K_{jk}, \qquad K_{jk} = \frac{2}{(u_j - u_k)^2 + 1}. \tag{2.14}$$

The overlap formula in this form was first derived for the Néel state ($|\text{VBS}_1\rangle$) in the Heisenberg model [45–47] by exploiting connections [48] to the six-vertex partition function with specific boundary conditions [49]. As for higher-rank spin chains, rigorous derivations
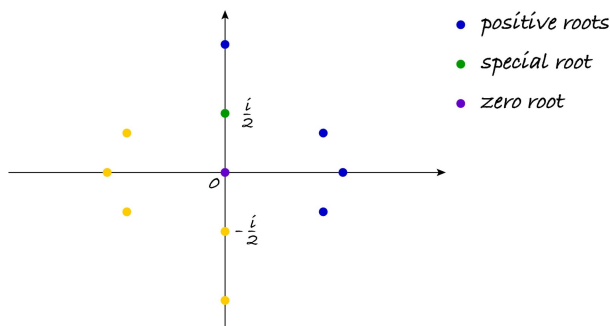
**Figure 1**. A symmetric configuration of Bethe roots.

are seldom available, but many overlap formulas were conjectured [12, 26, 27, 30] or deduced from scattering theory [10, 13, 36], coordinate Bethe ansatz [50] and TBA [44].

As discussed above, the Bethe roots in $\langle B|\{u_j\}\rangle$ must be invariant under reflection: $\{u_j\} = \{-u_j\}$. A typical symmetric configuration is illustrated in figure 1. The roots mostly form pairs $(u_j, -u_j)$ but a solitary root at zero is also allowed. We pick one root from each pair, it does not matter which one, for instance the root with a positive real part, and call this root positive. Reflection acts as a $\mathbb{Z}_2$ permutation (represented by a matrix $\Omega$) on rows and columns of the Gaudin matrix, and it is this $\mathbb{Z}_2$ action that defines the superdeterminant:

$$\mathrm{Sdet}\, G = \mathrm{e}^{\,\mathrm{tr}\,\Omega \ln G}. \tag{2.15}$$

Additional complications arise for singular solutions of the Bethe ansatz equations with roots at $\pm i/2$ [51–55]. The Bethe-ansatz equations require regularization in this case. The cleanest way to proceed is via Baxter equations or the QQ-relations, not dealing with Bethe roots and circumventing the problem altogether [56]. Some entries of the Gaudin matrix become singular at $u_j = \pm i/2$, compelling us to treat singular solutions separately.

We parameterize a symmetric Bethe state in terms of positive roots $\mathbf{u} = \{u_j\}_{j=1\ldots M'}$ and two discrete parameters $\delta_z$ and $\delta_s$ indicating additional roots at zero and at $\pm i/2$:

$$|\mathbf{u}\rangle = |\{u_j, -u_j\}\rangle \qquad\qquad \delta_{\mathbf{z}} = 0, \qquad\qquad \delta_{\mathbf{s}} = 0 \tag{2.16}$$

$$|\mathbf{u}\rangle = |\{u_j, -u_j, 0\}\rangle \qquad\qquad \delta_{\mathbf{z}} = 1, \qquad\qquad \delta_{\mathbf{s}} = 0 \tag{2.17}$$

$$|\mathbf{u}\rangle = |\{u_j, -u_j, i/2, -i/2\}\rangle \qquad \delta_{\mathbf{z}} = 0, \qquad\qquad \delta_{\mathbf{s}} = 1 \tag{2.18}$$

$$|\mathbf{u}\rangle = |\{u_j, -u_j, 0, i/2, -i/2\}\rangle \qquad \delta_{\mathbf{z}} = 1, \qquad\qquad \delta_{\mathbf{s}} = 1, \tag{2.19}$$

and introduce the abridged Q-function

$$Q(u) = \prod_{j=1}^{M'} \left(u^2 - u_j^2\right), \tag{2.20}$$

with the product over positive roots only. The total number of roots is $M = 2M' + 2\delta_{\mathbf{s}} + \delta_{\mathbf{z}}$, while the total momentum is

$$\mathrm{e}^{iP} = (-1)^{\delta_{\mathbf{z}} + \delta_{\mathbf{s}}}. \tag{2.21}$$

The master overlap formula for the SU(2) spin chain can be written in the following way (see [43] or [57]):[2]

$$\frac{\langle \text{VBS}_\nu | \, e^{i\beta S^1} \, |\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{\frac{1}{2}}} = 2^{\delta_\mathbf{s}} i^{(L+1)\delta_\mathbf{z}} \, (i \sin \beta)^{\frac{L}{2}-M} \, \frac{\mathcal{Q}\left(\frac{i\nu}{2}\right)}{\sqrt{Q(0)Q\left(\frac{i}{2}\right)}} \sqrt{\text{Sdet}\, G} \,. \tag{2.22}$$

The prefactor contains both normal and abridged Q-functions, the latter evaluated at the argument where the normal Q-function would be zero for special states.

This formula holds for highest weights, for descendants one can use the identity

$$e^{i\beta S^1} = e^{i \tan \frac{\beta}{2} S^-} \left( \cos \frac{\beta}{2} \right)^{2S^3} e^{i \tan \frac{\beta}{2} S^+}, \tag{2.23}$$

take into account (2.6), expand in $\tan \beta/2$, and normalize by

$$\langle \mathbf{u}| \left( S^+ \right)^n \left( S^- \right)^n |\mathbf{u}\rangle = \frac{n!(L-2M)!}{(L-2M-n)!} \, \langle \mathbf{u}|\mathbf{u}\rangle \,. \tag{2.24}$$

Because the dependence on $\beta$ in the master formula is so simple, the $\mathfrak{su}(2)$ quantum numbers enter only through a combinatorial prefactor.

For example, the MPS overlaps following from (2.12) are given by

$$\frac{\langle \text{MPS}| \, (S^-)^{2n} \, |\mathbf{u}\rangle}{\langle \mathbf{u}| \, (S^+)^{2n} \, (S^-)^{2n} \, |\mathbf{u}\rangle^{\frac{1}{2}}} = 2i^{-M} \mathbb{C}_{nML} \sqrt{\frac{Q\left(\frac{i}{2}\right)}{Q(0)} \, \text{Sdet}\, G} \,, \tag{2.25}$$

where

$$\mathbb{C}_{nML} = \frac{\left(\frac{L}{2}-M\right)!}{n! \left(\frac{L}{2}-M-n\right)!} \sqrt{\frac{(2n)! \, (L-2M-2n)!}{(L-2M)!}} \,, \tag{2.26}$$

in agreement with [59]. For all special states the MPS overlaps vanish, and it is easy to understand why: states with $\delta_\mathbf{z}+\delta_\mathbf{s} = 1$ are orthogonal to MPS by momentum conservation. The states with $\delta_\mathbf{z} = \delta_\mathbf{s} = 1$ carry zero momentum but contain an odd number of roots while the number of flipped spins in MPS is necessarily even.

Diagonalizing $\Omega$ converts the superdeterminant into a ratio of ordinary determinants of rank $(M' + \delta_\mathbf{z} + \delta_\mathbf{s}) \times (M' + \delta_\mathbf{z} + \delta_\mathbf{s})$ and $M' \times M'$ [45]:

$$\text{Sdet}\, G = \frac{\det G^+}{\det G^-} \,, \tag{2.27}$$

whereas the Gaudin determinant factorizes as

$$\det G = \det G^+ \det G^- \,. \tag{2.28}$$

---

[2]The phase of the overlap is not really well-defined, and depends on the phase convention for the Bethe eigenstates. Our convention corresponds to the Algebraic Bethe Ansatz as described in [58].

The matrix elements of the factors are

$$G_{jk}^{\pm} = \delta_{jk}\left(\frac{L-\delta_{\mathbf{s}}}{u_j^2+\frac{1}{4}} - \frac{3\delta_{\mathbf{s}}}{u_j^2+\frac{9}{4}} - \frac{2\delta_{\mathbf{z}}}{u_j^2+1} - \sum_m K_{jm}^+\right) + K_{jk}^{\pm}$$

$$G_{j\mathbf{z}}^{+} = \frac{2\sqrt{2}}{u_j^2+1} = G_{\mathbf{z}j}^{+}$$

$$G_{j\mathbf{s}}^{+} = \frac{1}{u_j^2+\frac{1}{4}} + \frac{3}{u_j^2+\frac{9}{4}} = G_{\mathbf{s}j}^{+}$$

$$G_{\mathbf{z}\mathbf{z}}^{+} = 4L - \frac{16\delta_{\mathbf{s}}}{3} - \sum_m \frac{4}{u_m^2+1}$$

$$G_{\mathbf{z}\mathbf{s}}^{+} = \frac{8\sqrt{2}}{3} = G_{\mathbf{s}\mathbf{z}}^{+}$$

$$G_{\mathbf{s}\mathbf{s}}^{+} = L - \frac{8\delta_{\mathbf{z}}}{3} - \sum_m \left(\frac{1}{u_m^2+\frac{1}{4}} + \frac{3}{u_m^2+\frac{9}{4}}\right), \tag{2.29}$$

where

$$K_{jk}^{\pm} = \frac{2}{(u_j-u_k)^2+1} \pm \frac{2}{(u_j+u_k)^2+1}. \tag{2.30}$$

The factorization formula that takes into account states with a root at zero was derived in [47]. Extension to singular states with roots at $\pm i/2$ is new, we believe, and can be justified by theta-regularization discussed in appendix A.

## 2.3 Nested Bethe Ansatz

Given a Dynkin diagram for a super Lie algebra of type $\mathrm{SU}(N|M)$ and an associated set of Dynkin labels one can write down a set of Bethe equations for an integrable nearest neighbor spin chain where each spin chain site carries the representation defined by the Dynkin labels [60]. The Bethe equations can be compactly expressed in terms of $Q$-functions, that in turn depend on a number of Bethe roots, and read

$$(-1)^{F_a+1} = \frac{Q_\theta^{[-q_a]}(u_{aj})}{Q_\theta^{[+q_a]}(u_{aj})} \prod_b \frac{\mathcal{Q}_b^{[+M_{ab}]}(u_{aj})}{\mathcal{Q}_b^{[-M_{ab}]}(u_{aj})} \equiv e^{i\chi_{aj}}. \tag{2.31}$$

Here $M_{ab}$, $a,b \in \{1,\ldots,N+M-1\}$ are elements of the (tridiagonal) Cartan matrix corresponding to the chosen Dynkin diagram and $q_a$, $a \in \{1,\ldots,N+M-1\}$ are the Dynkin labels, and $F_a$ is the fermionic parity of the $a$th node: $F_a = 0$ for $M_{aa} \neq 0$; $F_a = 1$ for $M_{aa} = 0$. There is one $Q$-function, $\mathcal{Q}_b$, $b \in \{1,\ldots,N+M-1\}$ associated with each node in the Dynkin diagram and in addition there is a trivial Q-function $Q_\theta$ defined in (2.4). The functions $\mathcal{Q}_a$ are polynomials whose zeros are the Bethe roots:

$$\mathcal{Q}_a(u) = \prod_{j=1}^{\mathcal{K}_a} (u - u_{aj}), \tag{2.32}$$

where the $\mathcal{K}_a$ are excitation numbers that characterize the spin chain eigenstate.

The norm of a Bethe eigenstate is encoded in the determinant of the Gaudin matrix [35, 61]

$$G_{aj,bk} = \frac{\partial \chi_{aj}}{\partial u_{bk}}. \tag{2.33}$$

Bethe eigenstates that have non-trivial overlaps with integrable boundary states of VBS or MPS nature are characterized by the Bethe roots being paired as $\{u_{aj}, -u_{aj}\}$, possibly up a single root at zero.[3] For such root configurations the determinant of the Gaudin matrix factorizes as in eq. (2.28) and the overlaps are expressed in terms of the super determinant of $G$, defined in eq. (2.27), in combination with a number of $\mathcal{Q}$-functions.

The factors are given by the obvious generalization of (2.29)[4]:

$$G^{\pm}_{aj,bk} = \delta_{ab}\delta_{jk}\left\{ \frac{Lq_a}{u_{aj}^2 + \frac{q_a^2}{4}} - \sum_c \left[ \frac{\delta_{c\mathbf{s}}\,(M_{ac}+q_c)}{u_{aj}^2 + \frac{(M_{ac}+q_c)^2}{4}} + \frac{\delta_{c\mathbf{s}}\,(M_{ac}-q_c)}{u_{aj}^2 + \frac{(M_{ac}-q_c)^2}{4}} + \frac{\delta_{c\mathbf{z}}M_{ac}}{u_{aj}^2 + \frac{M_{ac}^2}{4}} \right] \right.$$
$$\left. - \sum_{cl} K^+_{aj,cl} \right\} + K^{\pm}_{aj,bk}$$

$$G^+_{aj,b\mathbf{z}} = \frac{\sqrt{2}\,M_{ab}}{u_{aj}^2 + \frac{M_{ab}^2}{4}} = G^+_{b\mathbf{z},aj}$$

$$G^+_{aj,b\mathbf{s}} = \frac{M_{ab}+q_b}{u_{aj}^2 + \frac{(M_{ab}+q_b)^2}{4}} + \frac{M_{ab}-q_b}{u_{aj}^2 + \frac{(M_{ab}-q_b)^2}{4}} = G^+_{b\mathbf{s},aj}$$

$$G^+_{a\mathbf{z},b\mathbf{z}} = \delta_{ab}\left[ \frac{4L}{q_a} - \sum_c \left( \frac{8\delta_{c\mathbf{s}}M_{ac}}{M_{ac}^2 - q_c^2} + \frac{4\delta_{c\mathbf{z}}}{M_{ac}} \right) - \sum_{cl} \frac{2M_{ac}}{u_{cl}^2 + \frac{M_{ac}^2}{4}} \right] + \frac{4}{M_{ab}}$$

$$G^+_{a\mathbf{z},b\mathbf{s}} = \frac{4\sqrt{2}\,M_{ab}}{M_{ab}^2 - q_b^2} = G^+_{b\mathbf{s},a\mathbf{z}}$$

$$G^+_{a\mathbf{s},b\mathbf{s}} = \delta_{ab}\left\{ \frac{L}{q_a} - \sum_c \left[ \frac{4\delta_{c\mathbf{s}}M_{ac}}{M_{ac}^2 - (q_a-q_c)^2} + \frac{4\delta_{c\mathbf{s}}M_{ac}}{M_{ac}^2 - (q_a+q_c)^2} + \frac{4\delta_{c\mathbf{z}}M_{ac}}{M_{ac}^2 - q_a^2} \right] \right.$$
$$\left. - \sum_{cl} \left[ \frac{M_{ac}+q_a}{u_{cl}^2 + \frac{(M_{ac}+q_a)^2}{4}} + \frac{M_{ac}-q_a}{u_{cl}^2 + \frac{(M_{ac}-q_a)^2}{4}} \right] \right\}$$
$$+ \frac{4M_{ab}}{M_{ab}^2 - (q_a-q_b)^2} + \frac{4M_{ab}}{M_{ab}^2 - (q_a+q_b)^2} \tag{2.34}$$

with

$$K^{\pm}_{aj,bk} = \frac{M_{ab}}{(u_{aj}-u_{bk})^2 + \frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj}+u_{bk})^2 + \frac{M_{ab}^2}{4}}. \tag{2.35}$$

The Gaudin superdeterminant will be the main focus of our study.

---

[3]Parity $\Omega$ may act non-trivially on the nodes of the Dynkin diagram: $u_{aj} \to -u_{\sigma(a)j}$, involving a transposition $\sigma$. We will not consider this possibility, but examples can be found in [30, 36].

[4]The markers $\delta_{a\mathbf{z}}$, $\delta_{a\mathbf{s}}$ indicate if the node $a$ carries zero and special roots (special roots sit at $\pm iq_a/2$). By convention, $1/q_a$, $1/M_{ab}$ are understood to be zero if $q_a$, $M_{ab}$ are zero.

## 3  Duality transformations

For simplicity, let us specialize to the simplest possible super Lie algebra SU(2|1) and let us choose the Cartan matrix and the Dynkin labels to be

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{3.1}$$

which in particular implies picking the Dynkin diagram ○—⊗. The Bethe equations then take the following form

$$-1 = \frac{Q_\theta^-(u_{1j})}{Q_\theta^+(u_{1j})} \frac{\mathcal{Q}_1^{++}(u_{1j})}{\mathcal{Q}_1^{--}(u_{1j})} \frac{\mathcal{Q}_2^-(u_{1j})}{\mathcal{Q}_2^+(u_{1j})}, \tag{3.2}$$

$$1 = \frac{\mathcal{Q}_1^-(u_{2j})}{\mathcal{Q}_1^+(u_{2j})}. \tag{3.3}$$

We note that the roots associated with the fermionic node, $\{u_{2j}\}$, do not have any self-interactions. Fermionic duality relies precisely on this property of the Bethe equations and amounts to a change of variables where the fermionic roots $\{u_{2j}\}_{j=1}^{\mathcal{K}_2}$ are eliminated in favor of a set of dual roots $\{\tilde{u}_{2k}\}_{k=1}^{\widetilde{\mathcal{K}}_2}$ which are the roots of the polynomial $\widetilde{Q}_2$ defined via the relation

$$\mathcal{Q}_1^+(u) - \mathcal{Q}_1^-(u) = \mathcal{Q}_2(u)\widetilde{\mathcal{Q}}_2(u). \tag{3.4}$$

We note that the degree of the polynomial $\widetilde{\mathcal{Q}}_2$ is given by

$$\widetilde{\mathcal{K}}_2 = \mathcal{K}_1 - \mathcal{K}_2 - 1. \tag{3.5}$$

After the duality transformation the Bethe equations take the form

$$1 = \frac{Q_\theta^-(u_{1j})}{Q_\theta^+(u_{1j})} \frac{\widetilde{\mathcal{Q}}_2^-(u_{1j})}{\widetilde{\mathcal{Q}}_2^+(u_{1j})}, \tag{3.6}$$

$$1 = \frac{\mathcal{Q}_1^-(\tilde{u}_{2j})}{\mathcal{Q}_1^+(\tilde{u}_{2j})}. \tag{3.7}$$

This simple example illustrates the general phenomenon that a fermionic duality transformation after a given node leaves the Bethe equations associated with that node unchanged but changes the nature of the roots at the neighboring nodes from bosonic to fermionic or vice versa, thus effectively changing the Dynkin diagram underlying the description. Furthermore, in the case where a non-zero Dynkin label is associated with the fermionic node, the Dynkin label of the node and its neighboring nodes will change (see [41] for further discussion focussed on overlap formulas). Due to the tri-diagonal nature of the Cartan matrix, in general only the Bethe equations corresponding to the node itself and its two neighbors can be affected by the duality transformation. Similarly, only the part of the Gaudin matrix that refers to the phases $\chi_{aj}$ of the dualized node, $a$, and its two neighbors, $a-1$ and $a+1$, will change.

Under a general fermionic duality transformation the Gaudin super determinant transforms in an astonishingly simple way. More precisely, starting from a regular root configuration with all roots paired and dualizing after the node $a$ one finds the following transformation formula [41]

$$S\det \tilde{G} \propto \frac{Q_a(0)\widetilde{Q}_a(0)}{Q_{a+1}(\frac{i}{2})Q_{a-1}(\frac{i}{2})} \, S\det G, \qquad (3.8)$$

where the constant of proportionality depends on the excitation numbers as well as the local Dynkin labels and Cartan matrix elements. In the $Q$-functions appearing in (3.8) the roots at $\pm i/2$ and at zero are left out. The relation holds semi-off-shell meaning that the Bethe roots $u_{aj}$, $\tilde{u}_{aj}$ must fulfill a duality relation à la eq. (3.4) but the remaining roots can be chosen arbitrarily. With this transformation rule for the super determinant the overlaps between Bethe eigenstates of the $PSU(2,2|4)$ spin chain and the integrable boundary states of relevance for domain wall versions of $\mathcal{N}=4$ SYM transform covariantly [41]. We note that by means of fermionic duality transformations we can move between all the possible Dynkin diagrams of the underlying super Lie algebra.

The fermionic dualities only constitute a sub-set of the possible duality transformations of the Bethe equations. The various possible forms of the Bethe equations reflect the different possible choices of vacuum state and of excitations at the various levels of nesting. All duality transformations are conveniently expressed in terms of a set of $QQ$-relations which in turn can be encoded in a Hasse diagram [17], see also [18–20]. For a Lie-algebra of type $SU(N|M)$ there is a total of $2^{M+N}$ $Q$-functions and each of them is associated with a node in the corresponding Hasse diagram.

Again, for simplicity, let us specialize to $SU(2|1)$ for which we show the Hasse diagram in figure 2. Each path from the bottom node to the top node corresponds to a particular version of the Bethe equations and a fermionic duality transformation of the Bethe equations corresponds to flipping two segments of the path vertically across a plaquette.

Our original Bethe equations (3.2) and (3.3) correspond to the blue path whereas the Bethe equations after dualization, i.e. eqs. (3.6) and (3.7) correspond the red path with the following identification of $\mathcal{Q}$-functions

$$\mathcal{Q}_{\emptyset|\emptyset} = Q_\theta, \qquad\qquad \mathcal{Q}_{1|\emptyset} = \mathcal{Q}_1, \qquad\qquad \mathcal{Q}_{12|\emptyset} = \mathcal{Q}_2, \qquad (3.9)$$

$$\mathcal{Q}_{12|1} = 1, \qquad\qquad \mathcal{Q}_{1|1} = \widetilde{\mathcal{Q}}_2. \qquad\qquad\qquad\qquad (3.10)$$

In addition to the fermionic duality already considered one can perform a bosonic duality transformation on the first node on the Dynkin diagram $\bigcirc\!\!-\!\!\otimes$. This amounts to a change of variables from the roots $u_{1j}$ to a set of dual roots $\tilde{u}_{1k}$, where the dual variables are the roots of the polynomial $\widetilde{\mathcal{Q}}_1$, which is to be identified with $\mathcal{Q}_{2|\emptyset}$, and which is defined via

$$\mathcal{Q}_1^+(u)\widetilde{\mathcal{Q}}_1^-(u) - \mathcal{Q}_1^-(u)\widetilde{\mathcal{Q}}_1^+(u) = Q_\theta(u)\mathcal{Q}_2(u). \qquad (3.11)$$

A bosonic duality transformation corresponds to flipping the path in the Hasse diagram horizontally across a plaquette and in the present case transforms the blue path into the
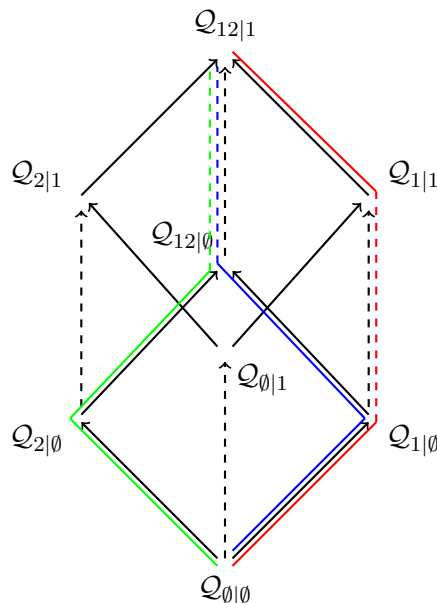
**Figure 2**. The Hasse diagram of SU(2|1). Each path from the bottom to the top node corresponds to a particular form of the Bethe equations. The blue path turns into the red path under a fermionic duality transformation and into the green path under a bosonic duality transformation.

green path. Under this change of variables the Bethe equations turn into

$$-1 = \frac{Q_\theta^-(\tilde{u}_{1j})}{Q_\theta^+(\tilde{u}_{1j})} \frac{\widetilde{\mathcal{Q}}_1^{++}(\tilde{u}_{1j})}{\widetilde{\mathcal{Q}}_1^{--}(\tilde{u}_{1j})} \frac{\mathcal{Q}_2^-(\tilde{u}_{1j})}{\mathcal{Q}_2^+(\tilde{u}_{1j})}, \tag{3.12}$$

$$1 = \frac{\widetilde{\mathcal{Q}}_1^-(u_{2j})}{\widetilde{\mathcal{Q}}_1^+(u_{2j})}. \tag{3.13}$$

This simple example illustrates the fact that a bosonic duality transformation as opposed to a fermionic one does not change the nature of the Bethe equations and thus does not change the Dynkin diagram. Likewise the Dynkin labels on all nodes are invariant under the transformation. Finally, as before, due to the tri-diagonal form of the Cartan matrix only the part of the Gaudin matrix that refers to the phases $\chi_{aj}$ of the dualized node, $a$, and its two neighbors, $a-1$ and $a+1$, will change.

In the following we will investigate the transformation properties of the super determinant of the Gaudin matrix under bosonic dualities with the aim of determining whether a bosonic equivalent of the fermionic transformation formula (3.8) exists. We will start by considering the simplest possible case of SU(2), where the bosonic node is necessarily momentum carrying, from there proceed to SU(2|1), where the bosonic mode can be either momentum carrying or not, and finally consider the case of the most general three-node Dynkin diagram having a bosonic node in the middle. As we shall see, the transformation rule is the same in all three cases and it is natural to expect that it holds for any Dynkin diagram of $\mathfrak{su}(N|M)$ type.

# 4 Bosonic dualities

In this section we study the behavior of the super determinant of the Gaudin matrix under bosonic dualities. We start from the SU(2) spin chain and from there move on to more general cases.

## 4.1 SU(2): Q-functions and regularization

The Bethe equations of the SU(2) Heisenberg spin chain are given by eq. (2.2) which we note constitute a special case of the Bethe equations (3.2) and (3.3) with $\mathcal{Q}_2 = 1$ (and thus no roots of type $u_{2j}$). Accordingly, the bosonic duality transformation is given by eq. (3.11). With the appropriate simplifications after a slight change in normalization the latter becomes:

$$\mathcal{Q}^+ \widetilde{\mathcal{Q}}^- - \mathcal{Q}^- \widetilde{\mathcal{Q}}^+ = i(2M - L - 1)u^L. \tag{4.1}$$

The roots of $\widetilde{\mathcal{Q}}$ lie "beyond the equator" [62]: the initial number of excitations $M$ is limited by the highest-weight condition (2.6) to be $M \leqslant L/2$, while after dualizing the number of roots becomes

$$\widetilde{M} = L - M + 1, \tag{4.2}$$

and exceeds $L/2$.

The dual roots are ambiguous: a linear combination $\widetilde{\mathcal{Q}} + \alpha \mathcal{Q}$ solves the QQ-equation equally well as $\widetilde{\mathcal{Q}}$. This could have caused unnecessary complications, but Bethe eigenstates of interest are reflection invariant and that saves the day. Assuming all roots are paired, the original Baxter polynomial is an even function. It is then natural to require the dual Q-function to have an opposite parity:

$$\mathcal{Q}(-u) = \mathcal{Q}(u), \qquad \widetilde{\mathcal{Q}}(-u) = -\widetilde{\mathcal{Q}}(u). \tag{4.3}$$

This choice makes the "large" solution unique.

It is true in general that

$$\widetilde{\delta}_{\mathbf{z}} = 1 - \delta_{\mathbf{z}}, \tag{4.4}$$

and, somewhat less obviously, that

$$\widetilde{\delta}_{\mathbf{s}} = 1 - \delta_{\mathbf{s}}. \tag{4.5}$$

In other words, if the state $|\{u_j\}\rangle$ is not special then $\{\widetilde{u}_j\}$ contain roots at $\pm i/2$, and vice versa. Same for the root at zero.

Degeneracy of the Gaudin matrix poses a more serious problem: it always happens that $\det \widetilde{G}_+ = 0$.[5] The dual superdeterminant is consequently ill defined and has to be regularized to make any meaningful statement. The regularization we use singles out the

---

[5]We have no analytic proof of this statement, but have checked on a huge number of examples that $\det G_+ = 0$ so long as $M > L/2$.

root at zero:[6] consider an $(\widetilde{M}' + \widetilde{\delta}_{\mathbf{s}}) \times (\widetilde{M}' + \widetilde{\delta}_{\mathbf{s}})$ matrix $\widetilde{G}'_+$ given by (2.29) with $u_j$ replaced by $\widetilde{u}_j$ and with the $\widetilde{G}^+_{\mathbf{z}j}$ row and $\widetilde{G}^+_{j\mathbf{z}}$ column dropped. We then define

$$\mathrm{Sdet}' \widetilde{G} = \frac{\det \widetilde{G}'_+}{\det \widetilde{G}_-} \,. \tag{4.6}$$

With this regularization we find the following relation between the original and the dual superdeterminant

$$\mathrm{Sdet}' \widetilde{G} = 2^{2-4\delta_{\mathbf{s}}} \mathbb{A}_{L/2-M} \frac{Q(0)\widetilde{Q}(i/2)}{Q(i/2)\widetilde{Q}(0)} \, \mathrm{Sdet}\, G, \tag{4.7}$$

where we introduced a shorthand notation for the combinatorial prefactor:

$$\mathbb{A}_n = \frac{(2^n n!)^4}{(2n)!(2n+1)!} \,. \tag{4.8}$$

This relation, which was found by numerical investigations, has several features in common with the transformation law found for fermionic dualities [41]. Incidentally, it involves the same ratio of Baxter polynomials that appears in the overlap of the VBS$_0$ with the on-shell Bethe states. The latter is thus duality-invariant having precisely the same form in terms of original and dual roots, up to a somewhat involved pre-factor that resembles the one appearing in the overlaps with descendants, cf. [59] or eq. (2.26). The overlaps with VBS$_0$ are precisely those that appear in the Bethe ansatz description of the one-point functions in the simplest domain-wall dCFT [12].

## 4.2 From SU(2) to SU(2|1)

As the next to simplest case, let us move on to SU(2|1). We pick the Cartan matrix and the Dynkin labels as in the example treated in section 3, i.e. as given in eq. (3.1). Hence, we consider the Dynkin diagram ○—⊗ and take the bosonic node to be momentum carrying. The bosonic duality transformation corresponds to moving from the blue path to the green path in figure 2 and amounts to invoking the QQ-relation given in eq. (3.11). The number of bosonic roots before and after the dualization are related by

$$\widetilde{\mathcal{K}}_1 = L + \mathcal{K}_2 - \mathcal{K}_1 + 1. \tag{4.9}$$

For simplicity, let us consider the case where the number of original bosonic roots, $\mathcal{K}_1$, as well as the number of fermionic roots, $\mathcal{K}_2$, are even. Then there will be an odd number of dual bosonic roots, $\widetilde{\mathcal{K}}_1$, and hence a single dual root at zero. Under these assumptions the QQ-relation is adapted to

$$\mathcal{Q}_1^+ \widetilde{\mathcal{Q}}_1^- - \mathcal{Q}_1^- \widetilde{\mathcal{Q}}_1^+ = i(2\mathcal{K}_1 - L - \mathcal{K}_2 - 1)u^L \mathcal{Q}_2. \tag{4.10}$$

---

[6]Other possibilities include turning on the twist [57], removing the zero eigenvector from $\widetilde{G}_+$ [25] or dropping the special roots. We did not find any simple dualization formula in the first two cases, the last one is discussed in appendix B and leads to results very similar to regularization adopted in the main text.

Using the same regularization procedure as above we find from numerical investigations a transformation formula for the super determinant which generalizes the SU(2) result to the nested Bethe ansatz, namely

$$\text{Sdet}' \, \widetilde{G} = 2^{2-4\delta_{1s}} \mathbb{A}_{(L+\mathcal{K}_2)/2-\mathcal{K}_1} \, \frac{Q_1(0)\widetilde{Q}_1(i/2)}{Q_1(i/2)\widetilde{Q}_1(0)} \, \text{Sdet} \, G. \tag{4.11}$$

The effect of the extra fermionic node is to replace $L$ by $L+\mathcal{K}_2$. The same formula applies, without any change, for SU(3) where the auxiliary node is bosonic: $\bigcirc\!\!-\!\!\bigcirc$. Regularization by omitting roots at $\pm i/2$ leads to a formula with the Jacobian inverted, the precise transformation rule is spelled out in the appendix B.

## 4.3 General momentum carrying case

Let us consider the general case of a three-node Dynkin diagram with a momentum carrying bosonic node in the middle, more precisely

$$M = \begin{bmatrix} \eta_1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \eta_2 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \eta_1, \eta_2 \in \{0, 2\}. \tag{4.12}$$

In this case the relevant QQ-relation reads

$$\mathcal{Q}_m^+ \widetilde{\mathcal{Q}}_m^- - \mathcal{Q}_m^- \widetilde{\mathcal{Q}}_m^+ = i(2\mathcal{K}_m - L - \mathcal{K}_l - \mathcal{K}_r - 1)u^L \mathcal{Q}_l \, \mathcal{Q}_r, \tag{4.13}$$

where $l, r$ and $m$ stand for left, right and middle. We assume that the original roots are all paired, then as before there will be a single zero root at the middle node after the dualization. We find a transformation formula which generalizes the two cases considered before, namely

$$\text{Sdet}' \, \widetilde{G} = 2^{2-4\delta_{ms}} \mathbb{A}_{(L+\mathcal{K}_r+\mathcal{K}_l)/2-\mathcal{K}_m} \, \frac{Q_m(0)\widetilde{Q}_m(i/2)}{Q_m(i/2)\widetilde{Q}_m(0)} \, \text{Sdet} \, G. \tag{4.14}$$

The neighboring nodes have resulted in $L$ being replaced by $L + \mathcal{K}_l + \mathcal{K}_r$ but otherwise play no role. In particular, the result is independent of whether these nodes are bosonic or fermionic. The transformation rule for the alternative regularization is displayed in the appendix B.

## 4.4 Dualizing a non-momentum carrying node

Let us go back to SU(2|1) but consider a different grading with the Dynkin diagram $\otimes\!\!-\!\!\bigcirc$ where now the bosonic node does not carry any momentum, more precisely

$$M = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.15}$$

The associated Bethe equations read

$$1 = \frac{Q_\theta^-(u_{1j})}{Q_\theta^+(u_{1j})} \frac{\mathcal{Q}_2^-(u_{1j})}{\mathcal{Q}_2^+(u_{1j})}, \tag{4.16}$$

$$-1 = \frac{\mathcal{Q}_1^{++}(u_{2j})}{\mathcal{Q}_1^{--}(u_{2j})} \frac{\mathcal{Q}_1^-(u_{2j})}{\mathcal{Q}_1^+(u_{2j})}, \tag{4.17}$$
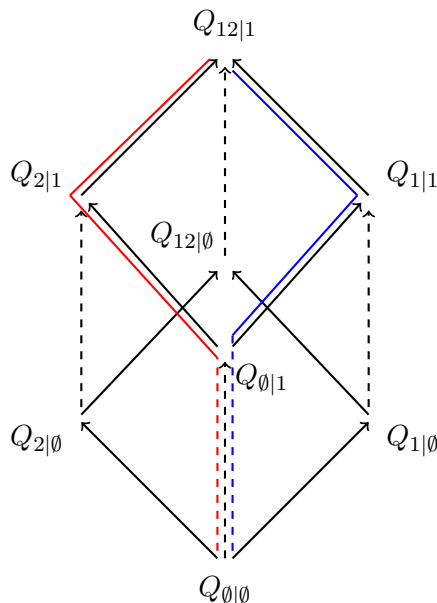
**Figure 3**. The QQ-relation (4.19) encodes the change of path from blue to red on the Hasse diagram.

and correspond to the blue path on the Hasse diagram in figure 3 with the identifications

$$\mathcal{Q}_{\emptyset|\emptyset} = Q_\theta, \qquad \mathcal{Q}_{\emptyset|1} = \mathcal{Q}_1, \qquad \mathcal{Q}_{1|1} = \mathcal{Q}_2, \qquad \mathcal{Q}_{12|1} = 1. \qquad (4.18)$$

The bosonic duality relation reads

$$\mathcal{Q}_2^+ \widetilde{\mathcal{Q}}_2^- - \mathcal{Q}_2^- \widetilde{\mathcal{Q}}_2^+ = i(2\mathcal{K}_2 - \mathcal{K}_1 - 1)\mathcal{Q}_1, \qquad (4.19)$$

and the implied change of variables corresponds to flipping two links of the blue path horizontally across a plaquette to arrive at the red path with the understanding that $\widetilde{\mathcal{Q}}_2$ should be identified with $Q_{2|1}$. Assuming again the original roots to be paired, we delete the row and the column corresponding to the zero dual root. This leads to a closed transformation formula which reads

$$\text{Sdet}' \, \widetilde{G} = \mathbb{A}_{\mathcal{K}_1/2 - \mathcal{K}_2} \frac{\widetilde{Q}_2(i/2)Q_2(0)}{\widetilde{Q}_2(0)Q_2(i/2)} \, \text{Sdet} \, G. \qquad (4.20)$$

The relative factor of $2^{2-4\delta_{\mathbf{s}}}$ is missing because neither $\mathcal{Q}_2$ nor $\widetilde{\mathcal{Q}}_2$ have special roots. For the same reason we cannot use the alternative regularization. The roots at $\pm i/2$ cannot appear on the auxiliary node because there is no momentum term in the Bethe equations to compensate for the pole in the scattering amplitude.

One can repeat the experiment for a non-momentum carrying node in SU(3), i.e. considering the Dynkin diagram $\bigcirc$—$\bigcirc$ and

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad (4.21)$$

where one dualizes after the second node. The QQ-relation takes the same form as (4.19) and the transformation formula is again the same.

## 5 Conclusion

We have found that a combination of Q-functions and the super determinant of the Gaudin matrix for an integrable spin chain of SU($N|M$) type transforms covariantly under the bosonic dualities encoded in the QQ-relations of the spin chain. Such combinations are characteristic of overlaps between Bethe eigenstates and integrable boundary states, which appear in the study of one-point functions in AdS/dCFT. We have earlier reported on similar observations concerning the fermionic duality relations and pointed out that the overlap formulas of relevance for the D3-D5 brane induced domain wall version of $\mathcal{N}=4$ SYM were singled out by transforming covariantly under fermionic dualities [41]. The overlaps do not have equally appealing transformation properties under bosonic dualities, except when restricted to the scalar SO(6) sector. With the present work we have analyzed all available duality relations and it is our hope that our findings could provide an important clue towards a completely universal formula for overlaps in terms of $Q$-functions thereby making the one-point function problem amenable to the quantum spectral curve approach and allowing one to pursue these observables beyond the asymptotic regime which has so far only been possible for protected operators [11]. The full non-perturbative answer is expected to take the form of a g-function [37], with the Gaudin matrix superseded by a ratio of Fredholm determinants [36]. It would be interesting to understand how those transform under the symmetries of the Quantum Spectral Curve.

## Acknowledgments

## A   Theta regularization

The Algebraic Bethe ansatz admits impurities. If those are paired: $\{\theta_l, -\theta_l\}$, $l = 1 \ldots L/2$, reflection symmetry is preserved and one can identify integrable boundary states as those preserving parity-odd charges of the hierarchy.

Impurities present, the Bethe ansatz equations become

$$\prod_{l=1}^{\frac{L}{2}} \frac{\left(u_j + \frac{i}{2}\right)^2 - \theta_l^2}{\left(u_j - \frac{i}{2}\right)^2 - \theta_l^2} = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{A.1}$$

The Gaudin matrix and its factors change accordingly:

$$G_{jk}^{\pm} = \left\{ \sum_{l=1}^{\frac{L}{2}} \left[ \frac{1}{(u_j - \theta_l)^2 + \frac{1}{4}} + \frac{1}{(u_j + \theta_l)^2 + \frac{1}{4}} \right] - \frac{2\delta_{\mathbf{z}}}{u_j^2 + 1} - \sum_m K_{jm}^+ \right\} \delta_{jk} + K_{jk}^{\pm}$$

$$G_{j\mathbf{z}}^+ = \frac{2\sqrt{2}}{u_j^2 + 1} = G_{\mathbf{z}j}^+ \qquad G_{\mathbf{zz}}^+ = \sum_{l=1}^{\frac{L}{2}} \frac{2}{\theta_l^2 + \frac{1}{4}} - \sum_m \frac{4}{u_m^2 + 1} \tag{A.2}$$

with the same $K_{jk}^{\pm}$ as in (2.30).

The singular states are removed or, better say, regularized by the impurities. If $\theta_l$ are very small, the solution for singular roots is

$$u_{1,2} = \pm\frac{i}{2} \pm \varepsilon \tag{A.3}$$

with

$$\varepsilon \approx i \prod_{k>2} \frac{u_k - \frac{3i}{3}}{u_k + \frac{i}{2}} \prod_{l=1}^{\frac{L}{2}} \theta_l^2. \tag{A.4}$$

In the limit $\theta_l \to 0$ two apparent singularities occur in the first equation in (A.2). The diagonal matrix elements $G_{11}^{\pm}$ corresponding to the special root, those that arise from the kinetic term in the Bethe equations, hit the pole. In addition, the scattering elements $K_{11}^{\pm}$ diverge as $\pm 1/[(i/2 + i/2)^2 + 1]$. This divergence only affects $G_{11}^-$, while in $G_{11}^+$ it cancels. The divergence in the kinetic term is actually fictitious, because:[7]

$$\lim_{\theta \to 0} \left[ \frac{1}{\left(\frac{i}{2} - \theta\right)^2 + \frac{1}{4}} + \frac{1}{\left(\frac{i}{2} + \theta\right)^2 + \frac{1}{4}} \right] = 2. \tag{A.5}$$

As a consequence, $G_{11}^+$ remains finite in the limit:

$$\lim_{\theta_l \to 0} G_{11}^+ = L - \frac{8\delta_{\mathbf{z}}}{3} - \sum_m \left( \frac{1}{u_m^2 + \frac{1}{4}} + \frac{3}{u_m^2 + \frac{9}{4}} \right). \tag{A.6}$$

The scattering singularity in $G_{11}^-$, on the contrary, is not fake:

$$G_{11}^- = \frac{i}{\varepsilon} + \text{finite}, \tag{A.7}$$

and $\det G^-$ diverges as

$$\det G^- = \frac{i}{\varepsilon} \det G_{\text{reg}}^-, \tag{A.8}$$

where $G_{\text{reg}}^-$ is the matrix with the first row and first column dropped. The divergence is compensated by the zero of the Q-function:

$$Q(i/2) = -i\varepsilon Q_{\text{reg}}(i/2), \tag{A.9}$$

the product staying finite in the limit:

$$\lim_{\theta_l \to 0} Q(i/2) \det G^- = Q_{\text{reg}}(i/2) \det G_{\text{reg}}^-. \tag{A.10}$$

This leads to the regularization prescription in the main text. An extra factor of $2^{\delta_{\mathbf{s}}}$ in (2.22) arises from removing the $i/2$ root from $Q(0)$:

$$\lim_{\theta_l \to 0} Q(0) = \frac{1}{4} Q_{\text{reg}}(0). \tag{A.11}$$

---

[7]Here we are taking into account that $\varepsilon \ll \theta$.

# B Alternative regularization

In the main text we regularized the dual Gaudin matrix by dropping the row and column of the zero root. One can alternatively remove the roots at $\pm i/2$. This regularization works if the initial state was not special, the singular roots are then necessarily present in the dual Baxter function. Shifting the singular dual roots away from $\pm\frac{i}{2}$ according to (A.3) yields a dual Gaudin matrix $\widetilde{G}$ with $\varepsilon$-poles in the upper left $2 \times 2$ corner but with otherwise finite matrix elements, and we define the reduced Gaudin matrix as the matrix which is obtained by crossing out the first two rows and columns in $\widetilde{G}$:

$$\widetilde{G}^{\text{red}} = \widetilde{G}_{ij}\Big|_{i,j=3,\dots,L-M+1}. \tag{B.1}$$

Since there are no divergences in $\widetilde{G}^{\text{red}}$, all Bethe roots and in particular $u_1$ and $u_2$ can directly be set to their face value. The reduced Gaudin matrix enjoys parity invariance and can be factorized much in the same way as the Gaudin matrix without the singular roots. We define the reduced Gaudin super determinant as the ration of factors:

$$\text{Sdet}_{\text{r}} \widetilde{G} = \frac{\det \widetilde{G}^{\text{red}}_+}{\det \widetilde{G}^{\text{red}}_-} \tag{B.2}$$

By experimenting with numerics we found that the reduced superdeterminant follows a simple transformation law:

$$\text{Sdet}_{\text{r}} \widetilde{G} = 2\mathbb{A}_{L/2-M} \frac{Q(i/2)\widetilde{Q}(0)}{Q(0)\widetilde{Q}(i/2)} \text{Sdet}\, G, \tag{B.3}$$

with the numerical prefactor defined in (4.8) and the Jacobian inverted compared to (4.7). Interestingly, the overlaps which transform covariantly in this case are those between the Bethe eigenstates and the Néel state, or the MPS overlaps which prominently feature in the $\mathcal{N} = 4$ dCFT [7, 8].

Adding extra nodes to the Dynkin diagram only changes the numerical prefactor in the transformation law. For the general case considered in section 4.3 the transformation formula is

$$\text{Sdet}_{\text{r}} \widetilde{G} = 2\mathbb{A}_{(L+\mathcal{K}_l+\mathcal{K}_r)/2-\mathcal{K}_m} \frac{Q_m(i/2)\widetilde{Q}_m(0)}{Q_m(0)\widetilde{Q}_m(i/2)} \text{Sdet}\, G. \tag{B.4}$$

The two-node case from section 4.2 is obtained by setting $\mathcal{K}_l = 0$.

## References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [INSPIRE].

[2] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *JHEP* **06** (2001) 063 [hep-th/0105132] [INSPIRE].

[3] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super-Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [INSPIRE].

[4] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of conformal $N = 4$ super-Yang-Mills theory*, *Nucl. Phys. B* **664** (2003) 131 [hep-th/0303060] [INSPIRE].

[5] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [hep-th/0307042] [INSPIRE].

[6] N. Beisert and M. Staudacher, *Long-range $PSU(2,2|4)$ Bethe ansatze for gauge theory and strings*, *Nucl. Phys. B* **727** (2005) 1 [hep-th/0504190] [INSPIRE].

[7] M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point functions in defect CFT and integrability*, *JHEP* **08** (2015) 098 [arXiv:1506.06958] [INSPIRE].

[8] I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point functions in AdS/dCFT from matrix product states*, *JHEP* **02** (2016) 052 [arXiv:1512.02532] [INSPIRE].

[9] I. Buhl-Mortensen, M. de Leeuw, A.C. Ipsen, C. Kristjansen and M. Wilhelm, *Asymptotic one-point functions in gauge-string duality with defects*, *Phys. Rev. Lett.* **119** (2017) 261604 [arXiv:1704.07386] [INSPIRE].

[10] T. Gombor and Z. Bajnok, *Boundary states, overlaps, nesting and bootstrapping AdS/dCFT*, *JHEP* **10** (2020) 123 [arXiv:2004.11329] [INSPIRE].

[11] S. Komatsu and Y. Wang, *Non-perturbative defect one-point functions in planar $\mathcal{N} = 4$ super-Yang-Mills*, *Nucl. Phys. B* **958** (2020) 115120 [arXiv:2004.09514].

[12] C. Kristjansen, D. Müller and K. Zarembo, *Integrable boundary states in D3-D5 dCFT: beyond scalars*, *JHEP* **08** (2020) 103 [arXiv:2005.01392] [INSPIRE].

[13] T. Gombor and Z. Bajnok, *Boundary state bootstrap and asymptotic overlaps in AdS/dCFT*, *JHEP* **03** (2021) 222 [arXiv:2006.16151] [INSPIRE].

[14] N. Gromov and F. Levkovich-Maslyuk, *Quantum spectral curve for a cusped Wilson line in $\mathcal{N} = 4$ SYM*, *JHEP* **04** (2016) 134 [arXiv:1510.02098] [INSPIRE].

[15] T. Harmark and M. Wilhelm, *Hagedorn temperature of $AdS_5/CFT_4$ via integrability*, *Phys. Rev. Lett.* **120** (2018) 071605 [arXiv:1706.03074] [INSPIRE].

[16] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Quantum spectral curve and structure constants in $\mathcal{N} = 4$ SYM: cusps in the ladder limit*, *JHEP* **10** (2018) 060 [arXiv:1802.04237] [INSPIRE].

[17] Z. Tsuboi, *Analytic Bethe ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r + 1|s + 1)$*, *Physica A* **252** (1998) 565 [arXiv:0911.5387] [INSPIRE].

[18] N. Gromov, *Introduction to the spectrum of $N = 4$ SYM and the quantum spectral curve*, arXiv:1708.03648 [INSPIRE].

[19] V. Kazakov, *Quantum spectral curve of $\gamma$-twisted $\mathcal{N} = 4$ SYM theory and fishnet CFT*, arXiv:1802.02160 [INSPIRE].

[20] F. Levkovich-Maslyuk, *A review of the AdS/CFT quantum spectral curve*, *J. Phys. A* **53** (2020) 283004 [arXiv:1911.13065] [INSPIRE].

[21] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for planar* $\mathcal{N} = 4$ *Super-Yang-Mills theory*, *Phys. Rev. Lett.* **112** (2014) 011602 [arXiv:1305.1939] [INSPIRE].

[22] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *Quantum spectral curve for arbitrary state/operator in* $AdS_5/CFT_4$, *JHEP* **09** (2015) 187 [arXiv:1405.4857] [INSPIRE].

[23] E.K. Sklyanin, *Separation of variables*, *Prog. Theor. Phys. Suppl.* **118** (1995) 3.

[24] N. Gromov, F. Levkovich-Maslyuk and P. Ryan, *Determinant form of correlators in high rank integrable spin chains via separation of variables*, *JHEP* **05** (2021) 169 [arXiv:2011.08229] [INSPIRE].

[25] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Separation of variables in AdS/CFT: functional approach for the fishnet CFT*, *JHEP* **06** (2021) 131 [arXiv:2103.15800] [INSPIRE].

[26] M. de Leeuw, C. Kristjansen and S. Mori, *AdS/dCFT one-point functions of the* SU(3) *sector*, *Phys. Lett. B* **763** (2016) 197 [arXiv:1607.03123] [INSPIRE].

[27] M. De Leeuw, C. Kristjansen and G. Linardopoulos, *Scalar one-point functions and matrix product states of AdS/dCFT*, *Phys. Lett. B* **781** (2018) 238 [arXiv:1802.01598] [INSPIRE].

[28] Y. Jiang, S. Komatsu and E. Vescovi, *Exact three-point functions of determinant operators in planar* $N = 4$ *supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **123** (2019) 191601 [arXiv:1907.11242] [INSPIRE].

[29] G. Chen, R. De Mello Koch, M. Kim and H.J.R. Van Zyl, *Structure constants of heavy operators in ABJM and ABJ theory*, *Phys. Rev. D* **100** (2019) 086019 [arXiv:1909.03215] [INSPIRE].

[30] P. Yang, Y. Jiang, S. Komatsu and J.-B. Wu, *Three-point functions in ABJM and Bethe ansatz*, arXiv:2103.15840 [INSPIRE].

[31] L. Piroli, B. Pozsgay and E. Vernier, *What is an integrable quench?*, *Nucl. Phys. B* **925** (2017) 362 [arXiv:1709.04796] [INSPIRE].

[32] L. Piroli, E. Vernier, P. Calabrese and B. Pozsgay, *Integrable quenches in nested spin chains II: fusion of boundary transfer matrices*, *J. Stat. Mech.* **1906** (2019) 063104 [arXiv:1812.05330] [INSPIRE].

[33] L. Piroli, E. Vernier, P. Calabrese and B. Pozsgay, *Integrable quenches in nested spin chains I: the exact steady states*, *J. Stat. Mech.* **1906** (2019) 063103 [arXiv:1811.00432] [INSPIRE].

[34] B. Pozsgay, L. Piroli and E. Vernier, *Integrable matrix product states from boundary integrability*, *SciPost Phys.* **6** (2019) 062 [arXiv:1812.11094] [INSPIRE].

[35] M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, *J. Phys. France* **37** (1976) 1087.

[36] Y. Jiang, S. Komatsu and E. Vescovi, *Structure constants in* $\mathcal{N} = 4$ *SYM at finite coupling as worldsheet g-function*, *JHEP* **07** (2020) 037 [arXiv:1906.07733] [INSPIRE].

[37] P. Dorey, D. Fioravanti, C. Rim and R. Tateo, *Integrable quantum field theory with boundaries: the exact g function*, *Nucl. Phys. B* **696** (2004) 445 [hep-th/0404014] [INSPIRE].

[38] I. Kostov, D. Serban and D.L. Vu, *Boundary TBA, trees and loops*, *Nucl. Phys. B* **949** (2019) 114817 [arXiv:1809.05705].

[39] D.-L. Vu, I. Kostov and D. Serban, *Boundary entropy of integrable perturbed SU (2)$_k$ WZNW*, *JHEP* **08** (2019) 154 [arXiv:1906.01909] [INSPIRE].

[40] J. Caetano and S. Komatsu, *Functional equations and separation of variables for exact g-function*, *JHEP* **09** (2020) 180 [arXiv:2004.05071] [INSPIRE].

[41] C. Kristjansen, D. Müller and K. Zarembo, *Overlaps and fermionic dualities for integrable super spin chains*, *JHEP* **03** (2021) 100 [arXiv:2011.12192] [INSPIRE].

[42] S. Ghoshal and A.B. Zamolodchikov, *Boundary S matrix and boundary state in two-dimensional integrable quantum field theory*, *Int. J. Mod. Phys. A* **9** (1994) 3841 [*Erratum ibid.* **9** (1994) 4353] [hep-th/9306002] [INSPIRE].

[43] B. Pozsgay, *Overlaps with arbitrary two-site states in the XXZ spin chain*, *J. Stat. Mech.* **1805** (2018) 053103 [arXiv:1801.03838] [INSPIRE].

[44] M. De Leeuw, T. Gombor, C. Kristjansen, G. Linardopoulos and B. Pozsgay, *Spin chain overlaps and the twisted Yangian*, *JHEP* **01** (2020) 176 [arXiv:1912.09338] [INSPIRE].

[45] M. Brockmann, J. De Nardis, B. Wouters and J.S. Caux, *A Gaudin-like determinant for overlaps of Néel and XXZ Bethe states*, *J. Phys. A* **47** (2014) 145003 [arXiv:1401.2877].

[46] M. Brockmann, *Overlaps of q-raised Néel states with XXZ Bethe states and their relation to the Lieb-Liniger Bose gas*, *J. Stat. Mech.* **5** (2014) P05006 [arXiv:1402.1471].

[47] M. Brockmann, J. De Nardis, B. Wouters and J.-S. Caux, *Néel-XXZ state overlaps: odd particle numbers and Lieb-Liniger scaling limit*, *J. Phys. A* **47** (2014) 345003 [arXiv:1403.7469].

[48] B. Pozsgay, *Overlaps between eigenstates of the XXZ spin-1/2 chain and a class of simple product states*, *J. Stati. Mech.* **6** (2014) P06011 [arXiv:1309.4593].

[49] O. Tsuchiya, *Determinant formula for the six-vertex model with reflecting end*, *J. Math. Phys.* **39** (1998) 135 [solv-int/9804010].

[50] Y. Jiang and B. Pozsgay, *On exact overlaps in integrable spin chains*, *JHEP* **06** (2020) 022 [arXiv:2002.12065] [INSPIRE].

[51] H. Bethe, *On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain*, *Z. Phys.* **71** (1931) 205 [INSPIRE].

[52] L.V. Avdeev and A.A. Vladimirov, *On exceptional solutions of the Bethe ansatz equations*, *Theor. Math. Phys.* **69** (1987) 1071 [INSPIRE].

[53] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139] [INSPIRE].

[54] R.I. Nepomechie and C. Wang, *Algebraic Bethe ansatz for singular solutions*, *J. Phys. A* **46** (2013) 325002 [arXiv:1304.7978] [INSPIRE].

[55] R.I. Nepomechie and C. Wang, *Twisting singular solutions of Bethe's equations*, *J. Phys. A* **47** (2014) 505004 [arXiv:1409.7382] [INSPIRE].

[56] C. Marboe and D. Volin, *Fast analytic solver of rational Bethe equations*, *J. Phys. A* **50** (2017) 204002 [arXiv:1608.06504] [INSPIRE].

[57] T. Gombor and B. Pozsgay, *On factorized overlaps: algebraic Bethe Ansatz, twists, and separation of variables*, *Nucl. Phys. B* **967** (2021) 115390 [arXiv:2101.10354].

[58] L.D. Faddeev, *How algebraic Bethe ansatz works for integrable model*, in *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*, pp. pp. 149–219, 5, 1996 [`hep-th/9605187`] [ɪɴSPIRE].

[59] M. de Leeuw, A.C. Ipsen, C. Kristjansen, K.E. Vardinghus and M. Wilhelm, *Two-point functions in AdS/dCFT and the boundary conformal bootstrap equations*, *JHEP* **08** (2017) 020 [`arXiv:1705.03898`] [ɪɴSPIRE].

[60] H. Saleur, *The continuum limit of sl(N/K) integrable superspin chains*, *Nucl. Phys. B* **578** (2000) 552 [`solv-int/9905007`] [ɪɴSPIRE].

[61] V.E. Korepin, *Calculation of norms of Bethe wave functions*, *Commun. Math. Phys.* **86** (1982) 391 [ɪɴSPIRE].

[62] G.P. Pronko and Y.G. Stroganov, *Bethe equations 'on the wrong side of equator'*, *J. Phys. A* **32** (1999) 2333 [`hep-th/9808153`] [ɪɴSPIRE].