Duality Results for Block-Structured Transition Matrices

Yiqiang Q. Zhao[1], Wei Li[1,2] and Attahiru Sule Alfa[3]

[1] *Department of Mathematics and Statistics*
University of Winnipeg
Winnipeg, Manitoba Canada R3B 2E9
E-mail: zhao@uwinnipeg.ca
E-mail: wli@uwinnipeg.ca

[2] *Institute of Applied Mathematics*
Chinese Academy of Sciences
Beijing, P.R. China

[3] *Department of Mechanical and Industrial Engineering*
University of Manitoba
Winnipeg, Manitoba Canada R3T 5V6
E-mail: alfa@cc.umanitoba.ca

In this paper, we consider a certain class of Markov renewal processes where the matrix of the transition kernel governing the Markov renewal process possesses some block-structured property, including repeating rows. Duality conditions and properties are obtained on two probabilistic measures which often play a key role in the analysis and computations of such a block-structured process. The method used here unifies two different concepts of duality. Applications of duality are also provided, including a characteristic theorem concerning recurrence and transience of a transition matrix with repeating rows and a batch arrival queueing model.

Subject classification: AMS(MOS)

Keywords: Markov renewal processes, duality, block-structures.

1   Introduction

For any Markov chain with transition probability matrix P either stochastic or substochastic, if there exists a positive left superregular vector $\alpha = (\alpha_0, \alpha_1, \ldots)$, that is, $\alpha \geq \alpha P$ with $\alpha > 0$, then we can define a dual Markov chain by

$$\hat{P} = \operatorname{diag}^{-1}(\alpha) P' \operatorname{diag}(\alpha), \tag{1}$$

where $\operatorname{diag}(\alpha)$ is the diagonal matrix with diagonal entries α_i and $'$ is the transpose of a matrix. The dual process can be probabilistically interpreted in terms of time reversibility of the original Markov chain and finds many important applications, for example, in proving the uniqueness of regular vectors of a recurrent Markov chain and in potential theory or Brownian motions [3]. The concept of duality also applies to queueing theory. For example, for a single arrival and single server queue, the dual queue can be defined by interchanging the arrival process and the service process. The duality or time reversibility properties can be well demonstrated using the $GI/M/1$ queue and the $M/G/1$ queue. However, for more complex queueing models (say a batch arrival or a batch service model), obtaining duality results can be very challenging and interchanging the arrival process and the service process may not lead to any duality or time reversibility result. To distinguish the dual defined by (1) from other definitions, we refer to it as the classical dual.

Recently, Ramaswami [6] obtained a duality theorem for the Markov renewal processes arising in the two-dimensional analogues of the classical $GI/M/1$ and $M/G/1$ queues. Later on, Asmussen and Ramaswami [2] provided a probabilistic proof for this duality theorem in terms of time reversal of sample paths of the Markov renewal processes. The probabilistic measure or matrix $R(x)$ for a Markov renewal process of $GI/M/1$ type and the probabilistic measure or matrix $G(x)$ for a Markov renewal process of $M/G/1$ type are key measures to analyze the models similar to the role played by matrices R and G for $GI/M/1$ and $M/G/1$ paradigms ([4, 5]). Ramaswami showed that given a Markov renewal process of $M/G/1$ type with building blocks $A_i(x)$ of the transition kernel and the invariant probability vector π of $A = \sum A_i(\infty)$, if each entry $D_i(x)$ (in block form) of a Markov renewal process of $GI/M/1$ type is constructed as the dual matrix defined by

$$D_i(x) = \operatorname{diag}^{-1}(\pi) A_i'(x) \operatorname{diag}(\pi), \tag{2}$$

then the $G(x)$ matrix for the $M/G/1$ paradigm is the dual of the $R(x)$ matrix for the $GI/M/1$ paradigm in the sense of

$$R(x) = \operatorname{diag}^{-1}(\pi) G'(x) \operatorname{diag}(\pi). \tag{3}$$

It is not difficult to see that in general, the matrix of the dual transition kernel defined by (2) is not the dual defined by (1), since a superregular vector for the submatrix excluding the boundaries may no longer be a superregular vector for the whole matrix. However, we can see a similarity between the classical duality and the duality defined by Ramaswami due to the probabilistic interpretation provided by Asmussen and Ramaswami. Roughly speaking, the definition in (2) induces the duality for the building blocks, not for boundaries. This seems reasonable because both the R and G-measures are independent of the boundaries. This also tells us why the dual defined by (2) cannot be the classical dual in general. From the above discussion, we conclude that the

dual defined by (2) can be actually encompassed by the classical duality if the transition matrix P is allowed to be substochastic. By allowing the transition matrix to be substochastic, some important properties of the dual process will no longer hold. For example, the fact that the classical dual and the original process have the same property of recurrence or transience will not be true any more in general if the original transition matrix is substochastic. In this case, new characterization theorems are sought.

It is well-known that properties of matrix G can be used to characterize a Markov chain of $M/G/1$ type. One such theorem is that the Markov chain is transient if and only if G is substochastic; or $Ge < e$ where e is a column vector of ones. Properties of the matrix R can be used to characterize a Markov chain of $GI/M/1$ type, usually in terms of the spectrum and the generalized traffic intensity. One of the theorems is that the Markov chain is positive recurrent if and only if $\pi\beta^* > 1$ where π is the probability invariant vector of A and β^* is defined as in (1.3.2) of Neuts (1980). It is noticed that the second theorem is not a dual result of the first one. Since the R and G-measures are considered dual to each other, it is of interest to know what is the dual result to the first theorem. One of the obvious differences between R and G is that G is always stochastic or substochastic, while R does not have such a property. The use of the duality result in (3) enable us to discover a dual result of the first theorem mentioned above and to obtain a complete characterization, in terms of both R and G, of the Markov chain with block-repeating transition entries $P = P(\infty)$ defined in (16). We prove that P is positive recurrent if and only if $\pi R < \pi$; P is transient if and only if $Ge < e$; and P is null recurrent if and only if $\pi R = \pi$ and $Ge = e$.

The main contributions in this paper include: 1. A generalization of the duality concept and result in (2) and (3) by Ramaswami to an arbitrary Markov renewal process whose transition kernel is partitioned into blocks; 2. A characteristic theorem for recurrence and transience of the dual process in terms of the behaviour of the original process; and 3. Two applications: the first uses duality results to develop new necessary and sufficient conditions for a Markov chain with block-repeating structure to be positive recurrent, null recurrent and transient; the second shows duality properties of a batch arrival queueing model.

The rest of the paper consists of four sections. In Section 2, for an arbitrary Markov renewal process, we define two probabilistic measures: the R and G-measures and define the dual Markov renewal process. A dual relationship of the R and G-measures between the original and the dual processes is given. Section 3 contains: i) A generalization of the duality theorem obtained by Ramaswami to a Markov renewal kernel with repeating rows; ii) The relationship concerning recurrence and transience between the original and the dual processes; and iii) An application of duality results, which characterizes the recurrence and transience of a Markov chain with repeating property in terms of the R and G-measures. In Section 4, a batch arrival queueing model is considered as another application. The final section concludes the paper.

2 Duality of Markov renewal processes with block-partitioned transition kernels

Consider a Markov renewal process on the state space $\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, m\}$ defined by the transition kernel

$$
P(x) = \begin{bmatrix}
P_{00}(x) & P_{01}(x) & P_{02}(x) & \cdots & \cdots \\
P_{10}(x) & P_{11}(x) & P_{12}(x) & \cdots & \cdots \\
P_{20}(x) & P_{21}(x) & P_{22}(x) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\tag{4}
$$

where all $P_{ij}(x)$ are $m \times m$ matrices. The above kernel corresponds to a partition of the states into *levels* $L = 0, 1, 2, \ldots$ and *phases* $J = 1, 2, \ldots, m$. A state in level i and phase u is written as (i, u). Therefore, $P_{ij}(x)$ represents a matrix of the transition probabilities from level i to level j in the time interval $[0, x]$. Let X_n and Y_n be the level and the phase after the nth renewal, let t_n be the inter-renewal times between the $(n-1)$st and nth renewals of the Markov renewal process, and let $T_n = \sum_{k=1}^{n} t_k$ be the total time up to the nth renewal. A general entry $P_{(i,u)(j,v)}(x)$ in $P(x)$ is the following probability:

$$
P_{(i,u)(j,v)}(x) = P\{(X_{n+1}, Y_{n+1}) = (j, v), t_{n+1} \leq x | (X_n, Y_n) = (i, u)\}.
$$

For the Markov renewal process defined above, define two probabilistic measures, called the R-measure and G-measure respectively as follows. Let $L_k = \{(k, 1), (k, 2), \ldots, (k, m)\}$ of states consisting of all states in level k and let $L_{\leq n} = \cup_{k=0}^{n} L_k$ be the set of states consisting of all states in from level 0 to level n. For $i < j$, $R_{ij}(n, x)$ is a $m \times m$ matrix whose (u, v) entry $R_{(i,u)(j,v)}(n, x)$ is the probability that starting in state (i, u) at time 0, the Markov renewal process makes its nth transition in the interval $[0, x]$ and such a transition is a visit into state (j, v) without visiting any states in $L_{\leq j-1}$ in intermediate steps; or

$$
\begin{aligned}
R_{(i,u)(j,v)}(n, x) &= P\{(X_n, Y_n) = (j, v), (X_k, Y_k) \notin L_{\leq j-1} \text{ for } k = 1, 2, \ldots, n-1, \\
&\quad T_n \leq x | (X_0, Y_0) = (i, u)\}.
\end{aligned}
$$

For $i > j$, $G_{ij}(n, x)$ is a $m \times m$ matrix whose (u, v) entry $G_{(i,u)(j,v)}(n, x)$ is the probability that starting in state (i, u) at time 0, the Markov renewal process makes its nth transition in the interval $[0, x]$ and such a transition is a visit into state (j, v) without visiting any states in $L_{\leq i-1}$ in intermediate steps; or

$$
\begin{aligned}
G_{(i,u)(j,v)}(n, x) &= P\{(X_n, Y_n) = (j, v), (X_k, Y_k) \notin L_{\leq i-1} \text{ for } k = 1, 2, \ldots, n-1, \\
&\quad T_n \leq x | (X_0, Y_0) = (i, u)\}.
\end{aligned}
$$

Remark 1

(i)   *We will refer level 0 to boundary level and entries $P_{0j}(x)$ and $P_{i0}(x)$ to boundary probabilities.*

(ii)   *The R-measure and the G-measure are often key probabilistic measures in the analysis of the process.*

In the rest of this section, we prove a dual relationship between the R-measure for the original process and the G-measure for the dual process. To do this, we need the following lemma.

Lemma 1
Let $P_{ij}^{(n)}(x)$ be the matrix with its (u,v)th entry

$$P_{(i,u)(j,v)}^{(n)}(x) = P\{(X_n, Y_n) = (j,v), T_n \in [0,x] | (X_0, Y_0) = (i,u)\}, \tag{5}$$

being the n step transition probability with a total transition time less than or equal to x. Then for $i > j \geq 0$,

$$G_{ij}(n,x) = P_{ij}^{(n)}(x) - \sum_{k=0}^{i-1} \sum_{l=1}^{n-1} G_{ik}(l, \cdot) * P_{kj}^{(n-l)}(x); \tag{6}$$

and for $0 \leq i < j$,

$$R_{ij}(n,x) = P_{ij}^{(n)}(x) - \sum_{k=0}^{j-1} \sum_{l=1}^{n-1} P_{ik}^{(n-l)}(\cdot) * R_{kj}(l,x), \tag{7}$$

where for two matrix functions in x, $A(x) = (a_{ij}(x))$ and $B(x) = (b_{ij}(x))$, the entries $c_{ij}(x)$ of $A(\cdot) * B(x)$ are defined as

$$c_{ij}(x) = \sum_k a_{ik}(\cdot) * b_{kj}(x) = \sum_k \int_0^x a_{ik}(x-t) db_{kj}(t).$$

Proof
Based on conditioning, we have

$$P_{ij}^{(n)}(x) = G_{ij}(n,x) + \sum_{k=0}^{i-1} \sum_{l=1}^{n-1} G_{ik}(l, \cdot) * P_{kj}^{(n-l)}(x) \tag{8}$$

and

$$P_{ij}^{(n)}(x) = R_{ij}(n,x) + \sum_{k=0}^{j-1} \sum_{l=1}^{n-1} P_{ik}^{(n-l)}(\cdot) * R_{kj}(l,x). \tag{9}$$

$\square$

Remark 2
Since the boundary probabilities in $P(x)$ usually behave differently from the non-boundary blocks when $P(x)$ possesses some block structure, the boundary components (either $i = 0$ or $j = 0$) of the R-measure $R_{ij}(n,x)$ and G-measure $G_{i,j}(n,x)$ also often behave differently from the non-boundary components (both $i > 0$ and $j > 0$). It is important to express the non-boundary components of the R and G-measures using only non-boundary transition probabilities as will be done in the next section.

Let $_0P(x)$ be the submatrix of $P(x)$ by deleting the boundary entries:

$$_0P(x) = \begin{bmatrix} P_{11}(x) & P_{12}(x) & \cdots & \cdots \\ P_{21}(x) & P_{22}(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{10}$$

Let $_0P_{ij}^{(n)}(x)$ be the n step transition probabilities defined by (5) for the transition kernel $_0P_{ij}(x)$. A similar argument as in Lemma 1 leads to the following lemma.

Lemma 2
Let $G_{ij}(n,x)$ and $R_{ij}(n,x)$ be the G and R-measures for the original transition kernel $P(x)$. For $i > j \geq 1$,

$$G_{ij}(n,x) = {_0P_{ij}^{(n)}}(x) - \sum_{k=1}^{i-1}\sum_{l=1}^{n-1} G_{ik}(l,\cdot) * {_0P_{kj}^{(n-l)}}(x); \tag{11}$$

and for $1 \leq i < j$,

$$R_{ij}(n,x) = {_0P_{ij}^{(n)}}(x) - \sum_{k=1}^{j-1}\sum_{l=1}^{n-1} {_0P_{ik}^{(n-l)}}(\cdot) * R_{kj}(l,x). \tag{12}$$

For a Markov renewal process defined by (4), we define a dual Markov renewal process in the classical way, for example, see page 136 of [3]. Let α be any positive left superregular vector of $P = P(\infty)$; that is, $\alpha \geq \alpha P$ with $\alpha > 0$. It is clear that α is also a positive left superregular vector of $P(x)$ for any $x \geq 0$. Let α be partitioned into $\alpha = (\alpha_0, \alpha_1, \ldots)$ according to levels, where α_i is a vector of m elements. The transition kernel $\tilde{P}(x)$ of the α-dual, or simply the dual, Markov renewal process is defined by $\tilde{P}(x) = \text{diag}^{-1}(\alpha)P'(x)\text{diag}(\alpha)$. We then have the following duality theorem.

Theorem 3
Let $\tilde{R}_{ij}(n,x)$ and $\tilde{G}_{ij}(n,x)$ be the R and G-measures respectively for the dual process. For $i > j \geq 0$,

$$\tilde{G}_{ij}(n,x) = \text{diag}^{-1}(\alpha_i)R'_{ji}(n,x)\,\text{diag}(\alpha_j); \tag{13}$$

and for $0 \leq i < j$,

$$\tilde{R}_{ij}(n,x) = \text{diag}^{-1}(\alpha_i)G'_{ji}(n,x)\,\text{diag}(\alpha_j). \tag{14}$$

Proof
We only prove the first equation and the proof to the second one is similar.

First, by inductive method, it is easy to see that

$$\tilde{P}_{ij}^{(n)}(x) = \text{diag}^{-1}(\alpha_i)\left(P_{ji}^{(n)}(x)\right)'\text{diag}(\alpha_j). \tag{15}$$

holds for $n = 1, 2, \cdots$.

When $l = 1$, the result in (13) is obvious from equation (15). Suppose it is true for $l = 1, 2, \cdots, n-1$, we then need prove that it is still true for $l = n$. In fact, by Lemma 1, equation (15) and the inductive assumption, we have

$$\tilde{G}_{ij}(n, x) = \tilde{P}_{ij}^{(n)}(x) - \sum_{k=0}^{i-1} \sum_{l=1}^{n-1} \tilde{G}_{ik}(l, \cdot) * \tilde{P}_{kj}^{(n-l)}(x)$$

$$= \operatorname{diag}^{-1}(\alpha_i) \left(P_{ji}^{(n)}(x) \right)^{'} \operatorname{diag}(\alpha_j)$$

$$- \sum_{k=0}^{i-1} \sum_{l=1}^{n-1} \left[\operatorname{diag}^{-1}(\alpha_i) \left(R_{ki}(l, \cdot) \right)^{'} \operatorname{diag}(\alpha_k) \right] * \left[\operatorname{diag}^{-1}(\alpha_k) \left(P_{jk}^{(n-l)}(x) \right)^{'} \operatorname{diag}(\alpha_j) \right]$$

$$= \operatorname{diag}^{-1}(\alpha_i) \left(P_{ji}^{(n)}(x) - \sum_{k=0}^{i-1} \sum_{l=1}^{n-1} P_{jk}^{(n-l)}(x) * R_{ki}(l, \cdot) \right)^{'} \operatorname{diag}(\alpha_j)$$

$$= \operatorname{diag}^{-1}(\alpha_i) R_{ji}^{'}(n, x) \operatorname{diag}(\alpha_j).$$

This completes the proof. $\square$

Remark 3

(i)   *For a positive left superregular vector $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ of P, $_0\alpha = (\alpha_1, \alpha_2, \ldots)$ is a positive left superregular vector of $_0P = {}_0P(\infty)$. Therefore, we can also define a dual for $_0P(x)$ when there exists a dual for $P(x)$.*

(ii)   *Because of (i), by using Lemma 2 (instead of Lemma 1), we can prove the same duality results given in (13) and (14) for the values of indices $i \geq 1$ and $j \geq 1$ without dealing with the boundary probabilities in $P(x)$.*

(iii)   *The reverse of (i) is not generally true. Let $_0\alpha = (\alpha_1, \alpha_2, \ldots)$ be a positive left superregular vector of $_0P$. For any positive α_0, the vector $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ may not be a positive left superregular vector of P.*

Remark 4

When the block entries of the transitional probability matrix are of size $m_i \times m_j$, the above results are still valid.

3 Duality of Markov renewal kernels with repeating blocks

In this section, we discuss the dual of a kernel with repeating blocks. We assume that the transition kernel in (4) is given by

$$P(x) = \begin{bmatrix} P_{00}(x) & P_{01}(x) & P_{02}(x) & P_{0,3}(x) & \cdots & \cdots \\ P_{10}(x) & A_0(x) & A_1(x) & A_2(x) & \cdots & \cdots \\ P_{20}(x) & A_{-1}(x) & A_0(x) & A_1(x) & \cdots & \cdots \\ P_{30}(x) & A_{-2}(x) & A_{-1}(x) & A_0(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{16}$$

If $P_{0j}(x) = 0$ for $j \geq 2$ and $A_i(x) = 0$ for $i \geq 2$, $P(x)$ is of $GI/M/1$ type; If $P_{i0}(x) = 0$ for $i \geq 2$ and $A_i(x) = 0$ for $i \leq -2$, $P(x)$ is of $M/G/1$ type. Duality on $GI/M/1$ and $M/G/1$ paradigms has been studied by Ramaswami [6] and Asmussen and Ramaswami [2]. We will study duality properties of a renewal kernel with repeating rows defined in (16) in a different way from that in [6] and [2]. Firstly, the definition adopted in this paper coincides the classical duality; secondly, no transformations are involved in any of our arguments and results; and thirdly, the probabilistic proofs are simple and direct.

We begin the study with showing that the repeating property of $P(x)$ is inherited in some way by the R and G-measures of $P(x)$.

Lemma 4

For $i > j > 0$ and for fixed values of n and x, the G-measure $G_{ij}(n,x)$ only depends on the value of $i - j$; or $G_{ij}(n,x) = G_{i+k,\,j+k}(n,s)$ for any $k \geq 0$. For $j > i > 0$ and for fixed values of n and x, the R-measure $R_{ij}(n,x)$ only depends on the value of $j - i$; or $R_{ij}(n,x) = R_{i+k,\,j+k}(n,x)$ for any $k \geq 0$.

Proof
Since the repeating structure of $P(x)$, we can define

$$P_{l;u,v}(x) = P\{(X_{n+1}, Y_{n+1}) = (j,v), t_{n+1} \leq x | (X_n, Y_n) = (i,u)\}$$

for any $i > 0$ and $j > 0$ such that $j - i = l$, and denote the $m \times m$ matrix of entries $P_{l;u,v}(x)$ by $P_l(x)$. Then for $u, v = 1, 2, \cdots, m$, by the repeating structure of $P(x)$ again, we have

$$
\begin{aligned}
& G_{(i,u)(j,v)}(n,x) \\
= & \sum_{k_{n-1}=i}^{\infty} \sum_{s_{n-1}=1}^{m} \cdots \sum_{k_1=i}^{\infty} \sum_{s_1=1}^{m} P_{(i,u)(k_1,s_1)}(\cdot) * P_{(k_1,s_1)(k_2,s_2)}(\cdot) * \cdots * P_{(k_{n-1},s_{n-1})(j,u)}(x) \\
= & \sum_{l_{n-1}=0}^{\infty} \sum_{s_{n-1}=1}^{m} \cdots \sum_{l_1=0}^{\infty} \sum_{s_1=1}^{m} P_{l_1,(u,s_1)}(\cdot) * P_{l_2-l_1,(s_1,s_2)}(\cdot) * \cdots * P_{j-i-l_{n-1},(s_{n-1},v)}(x),
\end{aligned}
$$

or

$$G_{ij}(n,x) = \sum_{l_1=0}^{\infty} \cdots \sum_{l_{n-1}=0}^{\infty} P_{l_1}(\cdot) * P_{l_2-l_1}(\cdot) * \cdots * P_{l_{n-1}-l_{n-2}}(\cdot) * P_{(j-i)-l_{n-1}}(x).$$

Similarly, we can prove that

$$R_{ij}(n,x) = \sum_{l_{n-1}=0}^{\infty} \cdots \sum_{l_1=0}^{\infty} P_{l_1+(j-i)}(\cdot) * P_{l_2-l_1}(\cdot) * \cdots * P_{l_{n-1}-l_{n-2}}(\cdot) * P_{-l_{n-1}}(x).$$

$\square$

Remark 5
Lemma 4 says that the non-boundary components of the R and G measures reveal also a repeating property.

Because of Lemma 4, we can define

$$G_{i-j}(n,x) = G_{ij}(n,x), \quad \text{for } i > j > 0,$$

and

$$R_{j-i}(n,x) = R_{ij}(n,x), \quad \text{for } j > i > 0.$$

The (u,v)th entry of $G_k(n,x)$ and $R_k(n,x)$ are denoted by $G_{k;u,v}(n,x)$ and $R_{k;u,v}(n,x)$. To study the non-boundary components of the R and G measures of $P(x)$, we will consider the non-boundary transition probabilities only, or $_0P(x)$. Since the transition kernel has repeating rows, we confine ourselves to such positive left superregular vectors $_0\alpha = (\alpha_1, \alpha_2, \ldots)$ of $_0P$ that $\alpha_1 = \alpha_2 = \cdots = \pi$. The following lemma provides the existence and uniqueness of such a superregular vector.

Lemma 5
$_0\alpha = (\pi, \pi, \ldots)$ *is a left superregular vector of $_0P$ if and only if π is a left superregular vector of $A = \sum_{k=-\infty}^{\infty} A_k$, where $A_k = A_k(\infty)$; or $\pi \geq \pi A$. If A is stochastic and irreducible, then π is the unique positive left superregular vector of A.*

Proof
The first half of the conclusions follows directly from the definition of superregular vectors and the second half from the uniqueness theorem of superregular vectors of a recurrent Markov chain. $\square$

Remark 6
Since $_0P(x)$ is defined on non-boundary levels $1, 2, \ldots$, the dual $_0\tilde{P}(x)$ is also defined on the non-boundary levels. We can extend this dual to be a transition kernel defined on all levels including level 0. We use $\tilde{P}(x)$ for any of such extensions.

The next theorem gives the main duality result about R and G-measures of a transition kernel with repeating blocks.

Theorem 6
Consider a Markov renewal process with transition kernel (16). Let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ be a positive left superregular vector of A and let $R_k(n, x)$ and $G_k(n, x)$, $k \geq 1$, be the non-boundary components of the R and G-measures of $P(x)$. Let $\tilde{P}(x)$ be an extension of the ${}_0\alpha = (\pi, \pi, \ldots)$-dual of ${}_0P(x)$:

$$\tilde{P}(x) = \begin{bmatrix} \tilde{P}_{00}(x) & \tilde{P}_{01}(x) & \tilde{P}_{02}(x) & \tilde{P}_{0,3}(x) & \cdots & \cdots \\ \tilde{P}_{10}(x) & B_0(x) & B_1(x) & B_2(x) & \cdots & \cdots \\ \tilde{P}_{20}(x) & B_{-1}(x) & B_0(x) & B_1(x) & \cdots & \cdots \\ \tilde{P}_{30}(x) & B_{-2}(x) & B_{-1}(x) & B_0(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{17}$$

where $B_k(x) = \Delta^{-1} A'_{-k}(x)\Delta$ and $\Delta = \, diag\,(\pi_i)$. Let the non-boundary components of the R and G-measures for the dual extension $\tilde{P}(x)$ be $\tilde{R}_k(n, x)$ and $\tilde{G}_k(n, x)$, $k \geq 1$. Then, the duality relationship of the R and G-measures between the Markov renewal process $P(x)$ and its dual extension is given by

$$\tilde{G}_k(n, x) = \Delta^{-1} R'_k(n, x)\Delta \tag{18}$$

and

$$\tilde{R}_k(n, x) = \Delta^{-1} G'_k(n, x)\Delta \tag{19}$$

for all $k \geq 1$.

Proof
It follows from Theorem 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We now give the relationship of the recurrence and transience between a Markov chain $P = P(\infty)$ and its dual extension $\tilde{P} = \tilde{P}(\infty)$.

Theorem 7
Let $P = P(\infty)$ as given in (16) be stochastic and irreducible, and let $A = \sum_{k=-\infty}^{\infty} A_k = \sum_{k=-\infty}^{\infty} A_k(\infty)$ be also stochastic and irreducible. Let $\tilde{P}(x)$ be a dual extension of the unique π-dual ${}_0\tilde{P}(x)$, where π is the unique positive superregular vector of A, such that $\sum_{k=0}^{\infty} k\tilde{P}_{0k}(\infty)e < \infty$ and that $\tilde{P} = \tilde{P}(\infty)$ is stochastic and irreducible. If $\sum_{k=0}^{\infty} kP_{0k}(\infty)e < \infty$, then

1. *P is positive recurrent if and only if $\tilde{P}$ is transient;*

2. *P is null recurrent if and only if $\tilde{P}$ is null recurrent; and*

3. *P is transient if and only if $\tilde{P}$ is positive recurrent.*

Proof
Since A is stochastic and irreducible, we know $B = \sum_{k=-\infty}^{\infty} B_k = \sum_{k=-\infty}^{\infty} B_k(\infty)$ is also stochastic and irreducible. Denote by π_A the unique stationary distribution of A.

It is easy to know that π_A is also the unique stationary distribution, π_B, of B. Let $\Delta = diag(\pi_s)$, $\mu_A = \sum_{k=-\infty}^{+\infty} kA_k e$ and $\mu_B = \sum_{k=-\infty}^{+\infty} kB_k e$ where e is a column vector of ones, we then have

$$
\begin{aligned}
\pi_B \mu_B &= (\mu_B)'(\pi_B)' = \sum_{k=-\infty}^{+\infty} ke'(B_k)'(\pi_B)' \\
&= \sum_{k=-\infty}^{+\infty} ke'\Delta A_{-k}\Delta^{-1}(\pi_B)' \\
&= \sum_{k=-\infty}^{+\infty} k\pi_A A_{-k} e \\
&= -\pi_A \sum_{k=-\infty}^{+\infty} kA_k e \\
&= -\pi_A \mu_A.
\end{aligned}
$$

Therefore, by the result given on page 238 of [1], the proof now has been completed. $\square$

By using the above duality theorem, we can prove a useful result, which characterizes the recurrence and transience of a stochastic matrix P with repeating block rows in terms of the R and G-measures of P. We first give the following lemma.

Lemma 8
For any stochastic or substochastic matrix $P = P(\infty)$ with repeating rows given in (16), if π is any positive left superregular vector of $A = \sum_{k=-\infty}^{\infty} A_k$, then $\pi R \leq \pi$ with $R = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} R_k(n, \infty)$.

Proof
Let $\tilde{P}$ be a stochastic irreducible dual extension of $_0P$ and let $\tilde{G}$ be the G-measure of $\tilde{P}$. The lemma follows directly from the fact $\tilde{G}e \leq e$ (see Remark 2.15 of [7] or notice the probabilistic interpretation of $\tilde{G}$) and $R_k(n, x) = \Delta^{-1}\tilde{G}'_k(n, x)\Delta$ (see Theorem 6). $\square$

Theorem 9
For any stochastic and irreducible transition matrix $P = P(\infty)$ in the form of in (16), let $R = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} R_k(n, \infty)$ and $G = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} G_k(n, \infty)$ be the sums of the R and G-measures of P respectively. If $\sum_{k=0}^{\infty} kP_{0k}(\infty)e < \infty$ and if $A = \sum_{k=-\infty}^{\infty} A_k$ is stochastic and irreducible with the stationary distribution π, then

1. *P is null recurrent if and only if $\pi R = \pi$ and $Ge = e$;*

2. *P is positive recurrent if and only if $\pi R < \pi$; and*

3. *P is transient if and only if $Ge < e$.*

Here, $x < y$ means that each of the elements in vector x is less than or equal to the corresponding element of vector y and that there exists at least one element of x which is strictly less than the corresponding element of y.

Proof

Suppose that P is null recurrent. Then G is stochastic or $Ge = e$ from Theorem 3.4 of [7] and $\tilde{G}$ of any stochastic irreducible dual extension $\tilde{P}$ is also stochastic. $\pi R = \pi$ now follows from that $\tilde{G}$ is stochastic and $R = \Delta^{-1}\tilde{G}'\Delta$. Suppose now that $\pi R = \pi$ and $Ge = e$. Then P is recurrent from Theorem 3.4 of [7]. $\pi R = \pi$, together with $G = \Delta^{-1}\tilde{R}'\Delta$, leads to the fact that $\tilde{G}e = e$. Therefore, any stochastic irreducible dual extension $\tilde{P}$ is also recurrent from Theorem 3.4 of [7]. It follows from Theorem 6 that P can only be null recurrent. The other two statements can be similarly proved using Theorem 6 and Theorem 3.4 of [7]. $\qquad\square$

Corollary 10
At least one of $Ge = e$ and $\pi R = \pi$ holds.

Remark 7
If A is no stochastic, then P is ergodic in general.

Remark 8
Under the assumptions made in Theorem 7, the following are true:

1. *Let $R = \sum_{k=1}^{m} R_k$, where $R_k = \sum_{n=1}^{\infty} R_k(n, \infty)$. The sum of every column of R is bounded up by $\pi_{\max}/\pi_{\min}$, where $\pi_{\max} = \max\{\pi_1, \pi_2, \ldots, \pi_m\}$ and $\pi_{\min} = \min\{\pi_1, \pi_2, \ldots, \pi_m\}$; that is,*

$$e'R \leq \frac{\pi_{\max}}{\pi_{\min}}e',$$

   where e is a row vector of ones.

2. *If P is recurrent, then every column sum of R is also bounded below by $\pi_{\min}/\pi_{\max}$; that is,*

$$\frac{\pi_{\min}}{\pi_{\max}}e' \leq e'R.$$

4   A batch arrival queue

In this section, we provide an example of applications by considering the queueing model $GI^X/GE_m/1$, where the service time is a generalized Erlang random variable, or it is the sum of m exponential random variables with parameters λ_i, $i = 1, 2, \ldots, m$. If we let the level be the number of customers in the system and the phase the number of service phases completed of the customer in service, we then have the transition probability matrix P of the imbedded Markov chain, with state space $\{0, (i, u) | i = 1, 2, \ldots$ and $j = 1, 2, \ldots, m\}$, at the epochs immediately before the arrivals of the $GI^X/GE_m/1$ queue. Assume that the batch arrival size is a random variable with probability mass function a_k, $k = 1, 2, \ldots,$

with mean batch size $\bar{a} < \infty$. Then P is given by

$$
P = \begin{bmatrix}
P_{00} & P_{01} & P_{02} & P_{0,3} & \cdots & \cdots \\
P_{10} & A_0 & A_1 & A_2 & \cdots & \cdots \\
P_{20} & A_{-1} & A_0 & A_1 & \cdots & \cdots \\
P_{30} & A_{-2} & A_{-1} & A_0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
$$

where P_{00} is a scalar and all A_k are $m \times m$ matrices. Since details of the expressions for the boundary probabilities P_{0j} and P_{i0} will not be needed for our purpose here, we only provide expressions for A_k here. Let $\beta_k^{(r)}$ be the conditional probability that the total number of phases served during an interarrival time is k given that at the beginning of the interarrival time the service is in phase r. The (u,v)th entry $A_{l;uv}$ of A_l is given by

$$
A_{-l;u,v} = \sum_{k=1}^{\infty} a_k \beta_{(k+l)m+v-u}^{(r)}, \quad l \geq 0;
$$

$$
A_{l;u,v} = \begin{cases}
\sum_{k=l}^{\infty} a_k \beta_{(k-l)m+v-u}^{(r)}, & l > 0, v \geq u, \\
\sum_{k=l+1}^{\infty} a_k \beta_{(k-l)m+v-u}^{(r)}, & l > 0, v < u.
\end{cases}
$$

Theorem 11
$A = \sum_{k=-\infty}^{\infty} A_k$ *is stochastic and irreducible and the unique positive left superregular (and also regular) vector of A is given by*

$$
\pi_i = \left(\frac{1}{\lambda_i} \right) \bigg/ \left(\sum_{j=1}^{m} \frac{1}{\lambda_j} \right), \quad i = 1, 2, \ldots, m.
$$

Proof
By a directly calculation, we know that the (u,v)th entry $a_{u,v}$ of A is given by

$$
a_{u,v} = \begin{cases}
\sum_{j=0}^{\infty} \beta_{jr+v-u}^{(u)}, & if \quad v \geq u, \\
\sum_{j=1}^{\infty} \beta_{jr+v-u}^{(u)}, & if \quad v < u.
\end{cases}
$$

It is easy to see A is stochastic since $\sum_{v=1}^{m} a_{u,v} = \sum_{k=0}^{\infty} \beta_k^{(u)} = 1$ for $u = 1, 2, \cdots, m$. To prove $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ is the stationary distribution of A, it is only needed to prove $\pi = \pi A$, or

$$
\pi_1 = \sum_{j=0}^{\infty} \left[\pi_1 \beta_{jm}^{(1)} + \pi_2 \beta_{jm+m-1}^{(2)} + \cdots + \pi_m \beta_{jm+1}^{(m)} \right],
$$

$$
\pi_2 = \sum_{j=0}^{\infty} \left[\pi_1 \beta_{jm+1}^{(1)} + \pi_2 \beta_{jm}^{(2)} + \cdots + \pi_m \beta_{jm+2}^{(m)} \right],
$$

$$
\vdots
$$

$$\pi_m = \sum_{j=0}^{\infty} \left[\pi_1 \beta_{jm+m-1}^{(1)} + \pi_2 \beta_{jm+m-2}^{(2)} + \cdots + \pi_m \beta_{jm}^{(m)} \right].$$

Let B and E be the phase of the customer in service at the beginning and at the end, respectively, of an interarrival time, and let N_{BE} be the number of customers left the system during an interarrival time. It is well known that $P\{B = u\} = P\{E = u\} = \pi_k$. Since

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \left[\pi_1 \beta_{jm}^{(1)} + \pi_2 \beta_{jm+m-1}^{(2)} + \cdots + \pi_m \beta_{jm+1}^{(m)} \right] \\
= \; & \sum_{j=0}^{\infty} \left[\sum_{k=1}^{m} P\{B = k, E = 1, N_{BE} = j\} \right] \\
= \; & \sum_{j=0}^{\infty} P\{E = 1, N_{BE} = j\} \\
= \; & P\{E = 1\} \\
= \; & \pi_1
\end{aligned}
$$

and for $l = 2, 3, \cdots, m$,

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \left[\pi_1 \beta_{jm+l-1}^{(1)} + \pi_2 \beta_{jm+l-2}^{(2)} + \cdots + \pi_m \beta_{jm+l}^{(m)} \right] \\
= \; & \sum_{j=0}^{\infty} \left[\sum_{k=1}^{m} P\{B = k, E = l, N_{BE} = j\} \right] \\
= \; & \sum_{j=0}^{\infty} P\{E = l, N_{BE} = j\} \\
= \; & P\{E = l\} \\
= \; & \pi_l,
\end{aligned}
$$

we then proved that π is indeed the stationary distribution of A. $\qquad \square$

We now consider another queueing model which will be proved to be a dual extension as discussed in previous sections. The queueing model is a modified $GE_m/G^X/1$ queue with interarrival time parameters $\mu_i = \lambda_{m+1-i}$, $i = 1, 2, \ldots, m$, such that there is a storage of infinite capacity filled up by customers waiting for service. For this model, we consider the imbedded epochs immediately after service completions. The level is the number of customers in the system again while the phase now is the number of phases completed of the current arrival. At the beginning of each service, with probability a_i, $i = 1, 2, \ldots$, i customers will be put in service as a batch. If the number of customers k in the system is smaller than i, the extra service capacity $i - k$ will go customers in the storage to make the batch size equal to i. Up to $i - k$, if there are, arrivals during the service time will be returned to the storage first and then all the other arrivals join the queue. Then, the imbedded Markov chain has the some states as in the previous model

and is given by

$$
\tilde{P} = \begin{bmatrix}
\tilde{P}_{00} & \tilde{P}_{01} & \tilde{P}_{02} & \tilde{P}_{0,3} & \cdots & \cdots \\
\tilde{P}_{10} & B_0 & B_1 & B_2 & \cdots & \cdots \\
\tilde{P}_{20} & B_{-1} & B_0 & B_1 & \cdots & \cdots \\
\tilde{P}_{30} & B_{-2} & B_{-1} & B_0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
$$

where $\tilde{P}_{00}$ is a scalar and all B_k are $m \times m$ matrices. Notice that we used the same notation as for a dual of P here. That is what we want to show in the following theorem.

Theorem 12
The imbedded Markov chain $\tilde{P}$ is indeed a dual extension of of the $(\pi, \pi, \ldots)$-dual ${}_0\tilde{P}$.

Proof
Let $\tilde{\beta}_k^{(r)}$ be the conditional probability that the number of phases arrived during a service time is k given that at the beginning of the service time the arriving customer is in phase r. Then the (u, v)th entry $B_{l;u,v}$ of B_l is given by

$$
B_{l;u,v} = \sum_{k=1}^{\infty} a_k \tilde{\beta}_{(k+l)m+u-v}^{(u)}, \quad l \geq 0;
$$

$$
B_{-l;u,v} = \begin{cases}
\sum_{k=l}^{\infty} a_k \tilde{\beta}_{(k-l)m+u-v}^{(u)}, & l > 0, u \geq v, \\
\sum_{k=l+1}^{\infty} a_k \tilde{\beta}_{(k-l)m+u-v}^{(u)}, & l > 0, u < v.
\end{cases}
$$

By a similar argument as that in Theorem 11, we know that $B = \sum_{k=-\infty}^{\infty} B_k$ is stochastic and irreducible, and has the same stationary distribution as that of A. For Markov chain $\tilde{P}$, let Y and Y' be the phase in which the arrival is at the beginning and at the end, respectively, of a service time. For Markov chain P, let X and X' be the phase in which the service is at the beginning and at the end, respectively, of an interarrival time. Then, for any positive integer k, and for any $u, v = 1, 2, \cdots, m$,

$$
\begin{aligned}
& \pi_u \tilde{\beta}_k(u) \\
= & P\{Y = u, Y' = v, \text{ with } v = k - u \bmod (m)\} \\
= & P\{X = v, X' = u, \text{ with } u = k - v \bmod (m)\} \\
= & \pi_v \beta_k(v).
\end{aligned}
$$

By the expressions of A_l and B_l, we know $\mathrm{diag}\,(\pi_s) B_k = A'_{-k} \mathrm{diag}\,(\pi_s)$ for $k = 1, 2, \cdots$. Therefore, we complete the proof. $\square$

Corollary 13
Let R and G, and $\tilde{R}$ and $\tilde{G}$ be the R and G-measures of the $GI^X/GE_m/1$ model and the modified $GE_m/G^X/1$ model defined above respectively. Then,

$$
R = \mathrm{diag}^{-1}(\pi_s) \tilde{G}' \, \mathrm{diag}\,(\pi_s)
$$

and

$$G = \ diag^{-1}(\pi_s)\tilde{R}' \ diag\,(\pi_s),$$

where π_s is given in Theorem 11.

5 Conclusion

In this paper, we extended the classical definition of duality to allow to study the dual ignoring boundaries. Under this treatment, both the classical dual and the dual introduced by Ramaswami are unified. We provided duality results between the R and G-measures. For a transition kernel with repeating blocks, we proved a duality theorem to characterize the recurrence and transience of the original and the dual ignoring boundaries processes. As examples of applications, we proved a necessary and sufficient condition under which a Markov chain with repeating blocks is either positive recurrent, null recurrent or transient, and we studied a batch arrival queueing model and its dual.

Acknowledgements

References

[1] S. Asmussen, *Applied Probability and queues* , John Wiley & Sons, 1987.

[2] S. Asmussen and V. Ramaswami, Probabilistic interpretations of some duality results for the matrix paradigms in queueing theory, *Stochastic Models*, **6(4)** (1990) 715–733.

[3] J.G. Kemeny, J.L. Snell and A.W. Knapp, *Denumerable Markov Chains,* 2nd ed., Springer-Verlag, New York, 1976.

[4] M.F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore, 1981.

[5] M.F. Neuts, *Structured Stochastic Matrices of M/G/1 Type and Their Applications*, Marcel Decker Inc., New York, 1989.

[6] V. Ramaswami, A duality theorem for the matrix paradigms in queueing theory, *Stochastic Models*, **6(1)** (1990) 151–161.

[7] Y.Q. Zhao, W. Li and W.J. Braun, Infinite block-structured transition matrices and their properties, *Adv. Appl. Prob.*, (1998) to appear.