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DUALITY THEOREMS AND AN OPTIMALITY CONDITION FOR NON-DIFFERENTIABLE CONVEX PROGRAMMING

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Abstract

Necessary and sufficient optimality conditions of Kuhn-Tucker type for a convex programming problem with subdifferentiable operator constraints have been obtained. A duality theorem of Wolfe's type has been derived. Assuming that the objective function is strictly convex, a converse duality theorem is obtained. The results are then applied to a programming problem in which the objective function is the sum of a positively homogeneous, lower-semi-continuous, convex function and a continuous convex function.

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0. Introduction

In this paper, we study the following pair of problems:
Problem (P). Minimize $f(x)$ subject to

$$G(x) \leqslant 0 \quad \text{and} \quad x \in A.$$

Problem (D). Maximize $f(x) + \langle z^*, G(x) \rangle$ subject to

$$z^* \geqslant 0, \qquad x \in A \quad \text{and}$$
$$0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A).$$

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Here f is a continuous convex functional defined on a locally convex space X and G is a continuous convex operator, which is regularly subdifferentiable on A, a convex subset of X, defined on X into another locally convex space Z having a closed convex cone defining a partial ordering in Z. $N(x/A)$ denotes the *normal cone to A at x* defined by

$$N(x/A) = \{w^* \in X' \colon \langle w^*, y - x \rangle \leq 0 \text{ for all } y \in A\},$$

where X' is the dual space of X.

$N(x/A)$ is the subdifferential of the *indicator function* of the set A at x, $\delta(x/A)$ defined by

$$\delta(x/A) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

If X and Z are finite dimensional, f and G are differentiable and $A = X$, then this is the problem studied by Wolfe and he has proved a duality theorem in [9]. M. Schechter [7] has derived a duality theorem in Wolfe's problem without assuming the differentiability of the objective function and the constraint functions. If $A = X$, the authors have proved a duality theorem, assuming that f is strictly convex, between the problems (P) and (D) in [5].

In this paper, we shall derive, in Section 2, a set of necessary and sufficient conditions of Kuhn-Tucker type for a point to be optimal for problem (P). We shall use this generalized Kuhn-Tucker theorem to prove a duality and a converse duality theorem between the problems (P) and (D) in Section 3. In Section 4, we apply these theorems in the case of the objective function is the sum of a continuous convex function and a positively homogeneous, lower-semi-continuous, convex function.

1. Preliminaries

In this paper X and X', as well as Z and Z', shall be pairs of real vector spaces in duality, with their respective weak topologies. Thus all the spaces will be locally convex spaces. We denote by $\langle \cdot , \cdot \rangle$ the canonical bilinear form of the dualities between the spaces X and X', as well as Z and Z'. We let $H \subset Z$ be a closed convex cone with non-empty interior defining a partial order in Z—for $x, y \in Z$; $x \leq y$ if $y - x \in H$. For $x, y \in Z$, $x < y$ is equivalent to $y - x$ is an interior point of H. Let H^* stand for the conjugate cone, namely,

$$H^* = \{z^* \in Z' \colon \langle z^*, z \rangle \geq 0 \text{ for every } z \in H\}.$$

Then, H^* defines a partial order in Z'.

Let $G: X \to Z$ be an operator. G is said to be *convex* if

$$G(tx + (1-t)y) \leqslant tG(x) + (1-t)G(y),$$

for all $x, y \in X$ and $0 \leqslant t \leqslant 1$.

A continuous linear operator $T: X \to Z$ is said to be a *subgradient* of G at a point $x_0 \in X$ if

$$T(x - x_0) \leqslant G(x) - G(x_0)$$

for every $x \in X$. The set of all subgradients of G at x_0 is called the *subdifferential* of G at x_0 and is denoted by $\partial G(x_0)$.

The operator $G: X \to Z$ is said to be *regularly subdifferentiable* at x_0 if

$$\partial(z^* \circ G)(x_0) = z^* \circ \partial G(x_0)$$

for every $z^* \in H^*$ [1]. If G is regularly subdifferentiable at every point of a subset A of X, then G is said to be regularly subdifferentiable on A.

We need the following proposition, whose proof can be found in [4].

Proposition 1.1. *Let F be a positively homogeneous, lower-semicontinuous, convex function defined on a locally convex space V; and let $u \neq 0$. Then*

$$\partial F(u) = \{u^* \in \partial F(0): F(u) = \langle u, u^* \rangle\}.$$

We shall also need the following definition and a lemma, which can be proved easily.

Definition. Let $f: X \to R$ be a function, and let $a \in X$. f is said to be *strictly convex at a* if

$$f(ta + (1-t)b) < tf(a) + (1-t)f(b)$$

for every $a \neq b \in X$, $0 < t < 1$.

Lemma 1.2. *Let $f: X \to R$ be convex. If f is strictly convex at $a \in X$, then for every $u^* \in \partial f(a)$, we have*

$$f(x) - f(a) > \langle u^*, x - a \rangle$$

for every $x \in X$, $x \neq a$.

2. Necessary and sufficient conditions

Before establishing a necessary and sufficient condition of Kuhn-Tucker type, we shall prove a theorem of Fritz-John type.

THEOREM 2.1. *Let X be a locally convex space and let f be a convex function, continuous at a point of the convex set A and let Z be a locally convex space with a positive cone H with non-empty interior. Let G be a continuous convex operator from X to Z, which is regularly subdifferentiable on A. If x_0 is an optimal solution of the problem* (P), *then there exists $\lambda_0 \geqslant 0$, $z_0^* \in H^*$, not both zero, such that*

$$0 \in \lambda_0 \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$$

and $\langle z_0^*, G(x_0) \rangle = 0$.

PROOF. Consider the set C in $Z \times R$ defined as follows:

$$C = \{(z, a) \in Z \times R: \text{there exists } x \in A \text{ such that } f(x) - f(x_0) < a,\ G(x) \leqslant z\}$$

Since C contains $H \times R^+$ and H has non-empty interior, C has non-empty interior.

The set C is convex, since f and G are convex. Further $(0,0) \notin C$, for if $(0,0) \in C$, then there exists $x \in A$ such that $f(x) - f(x_0) < 0$, and $G(x) \leqslant 0$, which is a contradiction to the assumption that x_0 is an optimal solution of the problem (P). Hence by separation theorem, there exists $(0,0) \neq (z_0^*, \lambda_0) \in Z' \times R$ such that

$$\langle z_0^*, z \rangle + \lambda_0 a \geqslant 0 \quad \text{for every } (z, a) \in C. \tag{1}$$

In particular, for every $a > 0$, $(G(x_0), a) \in C$ and hence we have

$$\langle z_0^*, G(x_0) \rangle + \lambda_0 a \geqslant 0. \tag{2}$$

Letting $a \to 0^+$, we obtain

$$\langle z_0^*, G(x_0) \rangle \geqslant 0. \tag{3}$$

From (2) and (3), we have, by contradiction,

$$\lambda_0 \geqslant 0. \tag{4}$$

Also for every $h \in H$, $(G(x_0) + h, 1) \in C$, so that (1) gives

$$\langle z_0^*, G(x_0) \rangle + \lambda_0 + \langle z_0^*, h \rangle \geqslant 0.$$

That is, $\langle z_0^*, h \rangle \geqslant -[\langle z_0^*, G(x_0) \rangle + \lambda_0]$ for every $h \in H$. Again from (3) and (4), we have by contradiction $z_0^* \in H^*$. But since $G(x_0) \in -H$ and $z_0^* \in H^*$, we have

$$\langle z_0^*, G(x_0) \rangle \leqslant 0. \tag{5}$$

Putting (3) and (5) together, we get

$$\langle z_0^*, G(x_0) \rangle = 0 \tag{6}$$

as desired.

Now $(G(x), f(x) - f(x_0) + \varepsilon) \in C$, for all $\varepsilon > 0$ and for all $x \in A$. Then by (1), we have

$$\langle z_0^*, G(x) \rangle + \lambda_0 (f(x) - f(x_0) + \varepsilon) \geqslant 0 \quad \text{for all } x \in A.$$

Combining with (6), we have

$$\langle z_0^*, G(x) - G(x_0) \rangle + \lambda_0 (f(x) - f(x_0) + \varepsilon) \geqslant 0 \quad \text{for all } x \in A.$$

As $\varepsilon \to 0$, we have

$$\langle z_0^*, G(x) - G(x_0) \rangle + \lambda_0 (f(x) - f(x_0)) \geqslant 0 \quad \text{for all } x \in A.$$

That is

(7) $$\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle \leqslant \lambda_0 f(x) + \langle z_0^*, G(x) \rangle \quad \text{for all } x \in A.$$

Hence x_0 minimizes the function $\lambda_0 f(x) + \langle z_0^*, G(x) \rangle$ on A. That is x_0 is a solution of the problem:

$$\underset{x \in X}{\text{minimize}}\, \lambda_0 f(x) + \langle z_0^*, G(x) \rangle + \delta(x/A).$$

Therefore, by Proposition 1, page 81 in [3], we have

$$0 \in \partial\big(\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle + \delta(x_0/A)\big).$$

Since, f and G are continuous and G is regularly subdifferentiable on A, by the Moreau-Rockafeller theorem [6],

$$0 \in \lambda_0 \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A).$$

Hence the theorem.

We shall now prove a theorem of Kuhn-Tucker type.

THEOREM 2.2. *In addition to the assumptions of Theorem* 2.1, *if we further assume that Stater's constraint qualification is satisfied* (*that is, there exists* $x' \in A$ *such that* $G(x') < 0$), *then* $\lambda_0 \neq 0$ *and one can set* $\lambda_0 = 1$. *In this case, the necessary and sufficient condition for* x_0 *to be an optimal solution of the problem* (P) *is that there exists an* $z_0^* \in H^*$ *such that*

(8) $$0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A) \quad \textit{and} \quad \langle z_0^*, G(x_0) \rangle = 0.$$

PROOF. Suppose Slater's constraint qualification is satisfied. Then there exists $x' \in A$ such that $G(x') < 0$.

Since all the conditions of Theorem 2.1 are satisfied, we have by (7) in the proof of Theorem 2.1, there exists $\lambda_0 \geqslant 0$, $z_0^* \in H^*$, not both zero such that

$$\lambda_0 f(x_0) + \langle z_0^*, G(x_0) \rangle \leqslant \lambda_0 f(x) + \langle z_0^*, G(x) \rangle$$

for all $x \in A$ and $\langle z_0^*, G(x_0) \rangle = 0$.

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If $\lambda_0 = 0$, then $z_0^* \neq 0$, $z_0^* \in H^*$ and we have

$$\lambda_0 f(x') + \langle z_0^*, G(x')\rangle = \langle z_0^*, G(x')\rangle < 0 = \lambda_0 f(x_0) + \langle z_0^*, G(x_0)\rangle$$

and this contradicts (7). Therefore $\lambda_0 \neq 0$. Hence we can set $\lambda_0 = 1$ and the relations (8) are satisfied.

Conversely, suppose $x_0 \in A$ such that $G(x_0) \leqslant 0$, $z_0^* \in H^*$ satisfy relations (8). Now (8) implies by the Moreau-Rockafellar theorem [6]

$$0 \in \partial(f + z_0^* \circ G + \delta(\cdot/A))(x_0).$$

Then by Proposition 1, page 81 in [3], we have x_0 is an optimal solution of the problem

$$\underset{x \in X}{\text{minimize}}\, f(x) + z_0^* \circ G(x) + \delta(x/A).$$

This implies

$$f(x_0) + z_0^* \circ G(x_0) \leqslant f(x) + z_0^* \circ G(x) + \delta(x/A)$$

for every $x \in X$, as $x_0 \in A$. Hence,

$$f(x_0) + z_0^* \circ G(x_0) \leqslant f(x) + z_0^* \circ G(x) \tag{9}$$

for every $x \in A$. Then for any $x \in A$ satisfying $G(x) \leqslant 0$, we have

$$\begin{aligned} f(x_0) = f(x_0) + \langle z_0^*, G(x_0)\rangle &\leqslant f(x) + \langle z_0^*, G(x)\rangle, \quad \text{by (9)} \\ &\leqslant f(x). \end{aligned}$$

This means that x_0 is an optimal solution of problem (P).

REMARK. If $Z = R^m$, then Theorems 2.1 and 2.2 reduce to Theorems 1.1 and 1.2 in [8] proved by M. Schechter using the theory of Dubovitski-Milyutin [2]. If $A = X$, then Theorem 2.2 becomes Theorem 2 in [4].

3. Duality and converse duality theorems

Using the necessary conditions of the previous section, we prove a duality theorem and a converse duality theorem between the problems (P) and (D). We assume that the Slater's constraint qualification is satisfied.

THEOREM 3.1 (Duality). *If x_0 is an optimal solution of* (P), *then there exists an z_0^* such that (x_0, z_0^*) is optimal for* (D). *Further, the two problems have the same extremal values.*

PROOF. Since x_0 is an optimal solution of (P), Theorem 2.2 guarantees the existence of feasible solutions to problem (D).

Let (x, z^*) be a feasible solution for problem (D). Then $z^* \geqslant 0$ and $0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A)$. This implies that there exist $x^* \in \partial f(x)$, $T \in \partial G(x)$ and $y^* \in N(x/A)$ such that $0 = x^* + z^* \circ T + y^*$. Now,

$$\begin{aligned}
f(x_0) - [f(x) + \langle z^*, G(x)\rangle]
&= [f(x_0) - f(x)] - \langle z^*, G(x)\rangle \\
&\geqslant \langle x^*, x_0 - x\rangle - \langle z^*, G(x)\rangle, \quad \text{since } x^* \in \partial f(x) \\
&= -\langle z^* \circ T + y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\
&= -\langle z^*, T(x_0 - x)\rangle - \langle y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\
&\geqslant -\langle z^*, G(x_0) - G(x)\rangle - \langle y^*, x_0 - x\rangle - \langle z^*, G(x)\rangle \\
&= -\langle z^*, G(x_0)\rangle - \langle y^*, x_0 - x\rangle \\
&\geqslant 0 \quad (\text{since } z^* \geqslant 0,\ G(x_0) \leqslant 0 \text{ and } y^* \in N(x/A)).
\end{aligned}$$

Thus,

$$f(x_0) \geqslant f(x) + \langle z^*, G(x)\rangle \tag{1}$$

for any feasible solution (x, z^*) for problem (D). Since x_0 is an optimal solution of (P), we have from Theorem 2, that there exists $z_0^* \in H^*$ such that $\langle z_0^*, G(x_0)\rangle = 0$ and $0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$. In other words, (x_0, z_0^*) is a feasible solution for (D). Hence

$$f(x_0) = f(x_0) + \langle z_0^*, G(x_0)\rangle. \tag{2}$$

Thus, from (1) and (2), (x_0, z_0^*) is an optimal solution of problem (D), and that the two problems have the same extremal value.

THEOREM 3.2 (Converse Duality). *Let us assume that the primal problem* (P) *has a solution* $\bar{x}$. *If* (x_0, z_0^*) *is an optimal solution of the dual problem* (D), *and if* f *is strictly convex at* x_0, *then* $x_0 = \bar{x}$. *Hence* x_0 *solves the problem* (P). *Furthermore, the extremal values of the two problems are same.*

PROOF. Suppose $x_0 \neq \bar{x}$. Since $\bar{x}$ is a solution of (P), it follows from the duality Theorem 3.1, there exists $\bar{z}^* \in H^*$ such that $(\bar{x}, \bar{z}^*)$ is optimal for (D).

Let $L(x, z^*) = f(x) + \langle z^*, G(x)\rangle$ be the Lagrangian of (P). Then,

$$L(\bar{x}, \bar{z}^*) = L(x_0, z_0^*) = \max_{(x, z^*) \in K} L(x, z^*)$$

where $K = \{(x, z^*)\colon x \in A,\ z^* \in H^* \text{ and } 0 \in \partial f(x) + z^* \circ \partial G(x) + N(x/A)\}$. Note that $(\bar{x}, \bar{z}^*) \in K$.

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Since $(x_0, z_0^*) \in K$, we have $0 \in \partial f(x_0) + z_0^* \circ \partial G(x_0) + N(x_0/A)$. Hence there exist $x^* \in \partial f(x_0)$, $T \in \partial G(x_0)$ and $y^* \in N(x_0/A)$ such that $0 = x^* + z_0^* \circ T + y^*$. Now,

$$\begin{aligned} L(\bar{x}, z_0^*) - L(x_0, z_0^*) &= f(\bar{x}) + \langle z_0^*, G(\bar{x})\rangle - f(x_0) - \langle z_0^*, G(x_0)\rangle \\ &= f(\bar{x}) - f(x_0) + \langle z_0^*, G(\bar{x}) - G(x_0)\rangle \\ &> \langle x^*, \bar{x} - x_0\rangle + \langle z_0^*, G(\bar{x}) - G(x_0)\rangle, \quad \text{by Lemma 1.2,} \\ &\geqslant \langle x^*, \bar{x} - x_0\rangle + \langle z_0^*, T(\bar{x}) - T(x_0)\rangle, \quad \text{since } T \in \partial G(x_0) \\ &= \langle x^*, \bar{x} - x_0\rangle + \langle z_0^* \circ T, \bar{x} - x_0\rangle \\ &= +\langle x^* + z_0^* \circ T, \bar{x} - x_0\rangle \\ &= -\langle y^*, \bar{x} - x_0\rangle \quad \text{by (1)} \\ &\geqslant 0, \quad \text{since } y^* \in N(x_0/A). \end{aligned}$$

It follows that, $L(\bar{x}, z_0^*) > L(x_0, z_0^*) = L(\bar{x}, \bar{z}^*)$. That is,

$$f(\bar{x}) + \langle z_0^*, G(\bar{x})\rangle > f(\bar{x}) + \langle \bar{z}^*, G(\bar{x})\rangle. \tag{3}$$

By hypothesis, since $\bar{x}$ is a solution of (P), it follows from Theorem 2, $\langle \bar{z}^*, G(\bar{x})\rangle = 0$. Hence, by (3), $\langle z_0^*, G(\bar{x})\rangle > 0$, which is a contradiction to the fact that $z_0^* \in H^*$, $G(\bar{x}) \leqslant 0$. Hence, $\bar{x} = x_0$ and x_0 solves the problem (P).

Further, we have, $f(x_0) = f(\bar{x}) = f(\bar{x}) + \langle \bar{z}^*, G(\bar{x})\rangle = L(\bar{x}, \bar{z}^*) = L(x_0, z_0^*) = f(x_0) + \langle z_0^*, G(x_0)\rangle$. Hence, the extremal values of the two problems are equal.

4. Applications

We shall now specialize the theorems derived in Section 3 to the case where the objective function is the sum of a positively homogeneous, lower-semi-continuous convex function and a continuous convex function.

Let the objective function $f: X \to R$ be of the form $f = f_1 + f_2$, where f_1 is a continuous convex function and f_2 is a positively homogeneous lower-semi-continuous convex function. Then the problem (P) becomes

$$(\mathrm{P}_1): \text{Minimize} \quad f_1(x) + f_2(x) \quad \text{subject to}$$
$$G(x) \leqslant 0, \quad \text{and} \quad x \in A.$$

Let us now construct the dual problem (D_1) using the above argument.

$$(\mathrm{D}_1): \text{Maximize} \quad f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle \quad \text{subject to}$$
$$s^* \geqslant 0, u^* \in \partial f_2(0), \langle u^*, x\rangle = f_2(x), x \in A \quad \text{and}$$
$$0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A).$$

We will now show that the duality theorem still holds even if one of the constraints is removed from the dual problem (D_1).

$$(D_2)\text{: Maximize } f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle \quad \text{subject to}$$
$$z^* \geqslant u^* \in \partial f_2(0),\, x \in A \quad \text{and}$$
$$0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A).$$

THEOREM 4.1. *If x_0 is an optimal solution of (P_1), then there exist z_0^*, u_0^* and w_0^* such that $(x_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_2). Further, the two problems have the same extremal values.*

PROOF. Since x_0 is optimal for (P_1), by Theorem 2.2 there exists an $z^* \in H^*$ such that $\langle z^*, G(x_0)\rangle = 0$ and $0 \in \partial(f_1 + f_2)(x_0) + z^* \circ \partial G(x_0) + N(x_0/A)$. But $\partial(f_1 + f_2)(x_0) = \partial f_1(x_0) + \partial f_2(x)$ by the Moreau-Rockafellar theorem [6]. Also, $\partial f_2(x_0) = \{u^* \in \partial f_2(0)\colon f_2(x_0) = \langle u^*, x_0\rangle\}$, by Proposition 1.1. Therefore,

$$0 \in \partial f_1(x_0) + \{u^* \in \partial f_2(0)\colon f_2(x_0) = \langle u^*, x_0\rangle\} + z^* \circ \partial G(x_0) + N(x_0/A).$$

Hence, there is $u^* \in \partial f_2(0)$ satisfying $f_2(x_0) = \langle u^*, x_0\rangle$ such that $0 \in \partial f_1(x_0) + u^* + z^* \circ \partial G(x_0) + N(x_0/A)$. Thus feasible solutions to problem (D_2) exist.

Let (x, z^*, u^*, w^*) be any feasible solution for (D_2). Then $z^* \in H^*$, $u^* \in \partial f_2(0)$ and there exist $x^* \in \partial f_1(x)$, $T \in \partial G(x)$ and $w^* \in N(x/A)$ such that

$$(1) \qquad 0 = x^* + u^* + z^* \circ T + w^*.$$

Now, using the idea of subdifferential calculus, the definition of normal cone and the relation (1), we can easily prove

$$f_1(x_0) + f_2(x_0) \geqslant f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle$$

for every feasible solution (x, z^*, u^*, w^*) of (D_2). Now, since x_0 is optimal for (P_1), then there are $z_0^* \in H^*$, $u_0^* \in \partial f_2(0)$ satisfying $f_2(x_0) = \langle u_0^*, x_0\rangle$ such that $0 \in \partial f_1(x_0) + u_0^* + z_0^* \circ \partial G(x_0) + N(x_0/A)$ and such that $\langle z_0^*, G(x_0)\rangle = 0$. Hence $f_1(x_0) + f_2(x_0) + \langle z_0^*, G(x_0)\rangle \geqslant f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle$ for every feasible solution (x, z^*, u^*, w^*) of (D_2). That is, $(u_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_2). Further, it is clear that the extremal values of the two problems are the same.

REMARK. The $(u_0, z_0^*, u_0^*, w_0^*)$ which optimizes D_2, in fact, also optimizes D_1.

THEOREM 4.2. *Let $\bar{x}$ be an optimal solution of (P_1). If $(x_0, z_0^*, u_0^*, w_0^*)$ is optimal for (D_1) and if f_1 is strictly convex at x_0, then $x_0 = \bar{x}$. Hence x_0 solves (P_1). Further, the extremal values of the two problems are equal.*

PROOF. Suppose $x_0 \neq \bar{x}$. Since $\bar{x}$ is a solution of (P_1), it follows from the duality Theorem 4.1, there exist $\bar{z}^* \in H^*$, $\bar{u}^* \in \partial f_2(0)$ satisfying $f_2(\bar{x}) = \langle \bar{u}^*, \bar{x}\rangle$ and $\bar{w}^* \in N(\bar{x}/A)$ such that $0 \in \partial f_1(\bar{x}) + \bar{u}^* + \bar{z}^* \circ \partial G(\bar{x}) + \bar{w}^*$. That is, $(\bar{x}, \bar{z}^*, \bar{u}^*, \bar{w}^*)$ is optimal for (D_1).

Let $\phi(x, z^*, u^*) = f_1(x) + \langle u^*, x\rangle + \langle z^*, G(x)\rangle$. Hence,

$$\Phi(\bar{x}, \bar{z}^*, \bar{u}^*) = \phi(x_0, z_0^*, u_0^*) = \max_{(x, z^*, u^*)\in N} \phi(x, z^*, u^*)$$

where $N = \{(x, z^*, u^*)\colon x \in A,\ z^* \in H^*,\ u^* \in \partial f_2(0)$ satisfying $f_2(x) = \langle u^*, x\rangle$ such that $0 \in \partial f_1(x) + u^* + z^* \circ \partial G(x) + N(x/A)\}$. Note that $(x_0, z_0^*, u_0^*) \in N$.

Since $(x_0, z_0^*, u_0^*) \in N$, we have $0 \in \partial f_1(x_0) + u_0^* + z_0^* \circ \partial G(u_0) + N(x_0/A)$. Hence, there exist $x^* \in \partial f_1(x_0)$, $T \in \partial G(u_0)$ and $w^* \in N(x_0/A)$ such that

$$0 = x^* + u_0^* + z_0^* \circ T + w^*. \tag{3}$$

Using the idea of subdifferential calculus, definition of normal cone and using the Lemma 1.2 and relation (3), we can prove,

$$\begin{aligned}\phi(\bar{x}, z_0^*, \bar{u}^*) - \phi(x_0, z_0^*, u_0^*) &> -\langle u_0^*, \bar{x}\rangle + \langle \bar{u}^*, \bar{x}\rangle - \langle w^*, \bar{x} - x_0\rangle \\ &\geqslant -f_2(\bar{x}) + f_2(\bar{x}) - \langle w^*, \bar{x} - x_0\rangle,\end{aligned}$$

since $\bar{u}^* \in \partial f_2(0)$ satisfying $f_2(\bar{x}) = \langle \bar{u}^*, \bar{x}\rangle$ and $u_0^* \in \partial f_2(0)$ which implies that $f_2(\bar{x}) \geqslant \langle u_0^*, \bar{x}\rangle = -\langle w^*, \bar{x} - x_0\rangle \geqslant 0$, since $w^* \in N(x_0/A)$. Therefore, $\phi(\bar{x}, z_0^*, \bar{u}^*) > \phi(x_0, z_0^*, u_0^*) = \phi(\bar{x}, \bar{z}^*, \bar{u}^*)$. That is,

$$f_1(\bar{x}) + \langle \bar{u}^*, \bar{x}\rangle + \langle z_0^*, G(\bar{x})\rangle > f_1(\bar{x}) + \langle \bar{u}^*, \bar{x}\rangle + \langle \bar{z}^*, G(\bar{x})\rangle. \tag{4}$$

Since $\bar{x}$ is an optimal solution of (P_1), it follows from Theorem 2.2, $\langle \bar{z}^*, G(\bar{x})\rangle = 0$. Hence $\langle z_0^*, G(\bar{x})\rangle > 0$, from (4) which is not possible because $z_0^* \in H^*$, $G(\bar{x}) \leqslant 0$. Hence $\bar{x} = x_0$, and x_0 solves the problem (P_1).

Further, we have,

$$\begin{aligned}f_1(x_0) + f_2(x_0) &= f_1(\bar{x}) + f_2(\bar{x}) \\ &= f_1(\bar{x}) + \langle \bar{u}^*, \bar{x}\rangle + \langle \bar{z}^*, G(\bar{x})\rangle \\ &= \phi(\bar{x}, \bar{z}^*, \bar{u}^*) = \phi(x_0, z_0^*, u_0^*) \\ &= f_1(x_0) + \langle u_0^*, x_0\rangle + \langle z_0^*, G(x_0)\rangle.\end{aligned}$$

Hence, the extremal values of the two problems are equal.

REMARK. We are not able to prove a converse duality between (P_1) and (D_2).

Special cases of problems of type (P_1) with finite dimensional applications have been discussed in [7, 8, 5].

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References

[1] V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces* (Sijthoff and Noordhoff, The Netherlands, 1978).
[2] I. V. Girsenov, *Lectures on mathematical theory of extremum problems*, (Springer-Verlag, New York, 1972).
[3] A. D. Ioffe and V. M. Tihomirov, *Theory of extremal problems* (Studies in Mathematics and its Applications, 6, North-Holland, Amsterdam, New York, Oxford, 1979).
[4] P. Kanniappan and Sundaram M. A. Sastry, 'A duality theorem for non-differentiable convex programming with operatorial constraints', *Bull. Austral. Math. Soc.* **22** (1980), 145–152.
[5] P. Kanniappan and Sundaram M. A. Sastry, 'A subgradient converse duality theorem for a convex programming', communicated to *J. Math. Anal. Appl.*
[6] R. T. Rockafellar, 'Extension of Fenchel's duality theorem for convex functions,' *Duke Math. J.* **33** (1966), 81–89.
[7] M. Schechter, 'A subgradient duality theorem', *J. Math. Anal. Appl.* **61** (1977), 850–855.
[8] M. Schechter, 'More on subgradient duality', *J. Math. Anal. Appl.* **71** (1979), 251–262.
[9] P. Wolfe, 'A duality theorem for non-linear programming', *Quart. Appl. Math.* **19** (1961), 239–244.

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