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Duality Theorems for Continuous Linear Programming Problems

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§ 1. Introduction

Continuous linear programmings were first considered by W.F. Tyndall [7] as a generalization of "bottle-neck problems" in dynamic programming. N. Levinson [6], M.A. Hanson [3] and M.A. Hanson and B. Mond [4] generalized the results in [7].

In this paper we shall apply the theory of infinite linear programming studied by K.S. Kretschmer [5] and M. Yamasaki [8] to the investigation of the continuous linear programmings. Our main purpose is to improve the duality theorems in [6] and [7] obtained by approximation from the classical finite duality theorem.

In order to state the continuous linear programmings, we shall introduce some notation. If $D(t)$ is a matrix on the interval $[0, T]$ $(0<T<\infty)$ in the real line with entries $d_{ij}(t)$ and $g(t)$ is a scalar on $[0, T]$ such that every entry satisfies

$$d_{ij}(t) \leqq g(t),$$

then the notation

$$D(t) \leqq g(t)$$

will be used. If $\tilde{D}(t)$ is a matrix on $[0, T]$ with the same number of rows and columns as $D(t)$, then $D(t) \leqq \tilde{D}(t)$ means that $d_{ij}(t) \leqq \tilde{d}_{ij}(t)$ for all entries. For a matrix $D=(d_{ij})$ and a vector $d=(d_i)$, we set

$$|D| = \sum_{i,j} |d_{ij}| \text{ and } |d| = \sum_{i} |d_i|.$$

For an n vector d, an m vector e and an $n \times m$ matrix D, let dD and De denote the vector-matrix products. Note that we do not use the familiar notation Dd^T. For two n vectors $x(t)=(x_i(t))$ and $y(t)=(y_i(t))$, we set

$$x(t) \cdot y(t) = \sum_{i=1}^{n} x_i(t) y_i(t).$$

In this paper we always assume that

$$B(t)=(b_{ij}(t)) \text{ is an } n \times m \text{ matrix on } [0, T],$$

$c(t)=(c_i(t))$ is an n vector on $[0, T]$,

$a(t)=(a_j(t))$ is an m vector on $[0, T]$,

$K(t, s)=(k_{ij}(t, s))$ is an $n\times m$ matrix on $[0, T]\times[0, T]$,

where $b_{ij}(t)$, $c_i(t)$, $a_j(t)$ and $k_{ij}(t, s)$ are bounded real-valued functions which are measurable with respect to the Lebesgue measures on the real line and the plane respectively.

A bounded measurable n vector $x(t)$ on $[0, T]$ is said to be *feasible* for the *primal program* of the (original) continuous linear programmings if $x(t)\geqq 0$ and

$$x(t)B(t)\geqq a(t)+\int_t^T x(s)K(s, t)\,ds.$$

The set of feasible vectors for the primal program is denoted by $S(N)$. The *value* of the primal program is defined by

$$N=\inf\left\{\int_0^T x(t)\cdot c(t)dt;\ x\in S(N)\right\} \quad \text{if } S(N)\neq\phi,$$

and

$$N=\infty \qquad\qquad \text{if } S(N)=\phi,$$

where ϕ denotes the empty set. A bounded measurable m vector $w(t)$ on $[0, T]$ is said to be feasible for the *dual program* of the continuous linear programmings if $w(t)\geqq 0$ and

$$B(t)w(t)\leqq c(t)+\int_0^t K(t, s)w(s)\,ds.$$

The set of feasible vectors for the dual program is denoted by $S(N')$. The value of the dual program is defined by

$$N'=\sup\left\{\int_0^T w(t)\cdot a(t)\,dt;\ w\in S(N')\right\} \quad \text{if } S(N')\neq\phi,$$

and

$$N'=-\infty \qquad\qquad \text{if } S(N')=\phi.$$

We shall always assume the following conditions as in [6]:

(N. 1) *$c(t)\geqq 0$ and $K(t, s)\geqq 0$.*

(N. 2) *There exists $\beta>0$ such that for each i, j and t either $b_{ij}(t)=0$ or else $b_{ij}(t)\geqq\beta$.*
Also for each t and j, there exists $i_j=i_j(t)$ such that

$$b_{i_j j}(t)\geqq\beta.$$

§ 2. Generalized continuous linear programmings

We shall first recall the theory of infinite linear programmings studied in [5] and [8].

Let X and Y be (real) linear spaces paired under the bilinear functional $((\ ,\ ))_1$ and Z and W be linear spaces paired under the bilinear functional $((\ ,\ ))_2$. The weak topology on X is denoted by $w(X, Y)$ and the Mackey topology on X is denoted by $s(X, Y)$.

A linear program for these paired spaces is a quintuple (A, P, Q, y_0, z_0). In this quintuple, A is a linear transformation from X into Z which is $w(X, Y)-w(Z, W)$ continuous, P is a convex cone in X which is $w(X, Y)$-closed, Q is a convex cone in Z which is $w(Z, W)$-closed, y_0 is an element of Y, and z_0 is an element of Z. We say that x is *feasible* for the program (A, P, Q, y_0, z_0) if $x \in P$ and $Ax-z_0 \in Q$. The set of feasible elements for the program is denoted by $S(M)$. The *value* of the program is defined by

$$M=\inf\{((x,\ y_0))_1;\ x\in S(M)\} \quad \text{if } S(M)\neq\phi,$$

and

$$M=\infty \quad \text{if } S(M)=\phi.$$

The *dual program* is the program $(A^*, Q^+, -P^+, -z_0, y_0)$ for W and Z paired under ${}_2((\ ,\ ))$ and for Y and X paired under ${}_1((\ ,\ ))$, where A^* is the dual transformation of A, i.e., $((x, A^*w))_1=((Ax, w))_2$ for all $x \in X$ and $w \in W$, and P^+ and Q^+ are defined by

$$P^+=\{y\in Y;\ ((x,\ y))_1\geqq 0 \quad \text{for all } x\in P\},$$

$$Q^+=\{w\in W;\ ((z,\ w))_2\geqq 0 \quad \text{for all } z\in Q\}.$$

The bilinear functionals ${}_2((\ ,\ ))$ and ${}_1((\ ,\ ))$ are defined by ${}_2((w, z))=((z, w))_2$ for all $w \in W$ and $z \in Z$ and ${}_1((y, x))=((x, y))_1$ for all $y \in Y$ and $x \in X$. We say that w is feasible for the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ if $w \in Q^+$ and $y_0-A^*w \in P^+$. The set of feasible elements for the dual program is denoted by $S(M')$. The value of the dual program is defined by

$$M'=\sup\{((z_0,\ w))_2;\ w\in S(M')\} \quad \text{if } S(M')\neq\phi,$$

and

$$M'=-\infty \quad \text{if } S(M')=\phi.$$

The set of real numbers are denoted by R and the set of non-negative real numbers by R_0. Let $X\times R$ and $Y\times R$ be paired under the bilinear functional $((\ ,\ ))$ defined by

$$(((x, r), (y, s))) = ((x, y))_1 + rs$$

for all $(x, r) \in X \times R$ and $(y, s) \in Y \times R$. Let G be the set in $Y \times R$ defined by

$$G = \{(A^*w + y, r - ((z_0, w))_2);\ y \in P^+, w \in Q^+ \text{ and } r \in R_0\}.$$

Kretschmer proved

THEOREM 1.[1] *If M is finite and the set G is $w(Y \times R, X \times R)$-closed, then $M = M'$ holds and there exists $\bar{w} \in Q^+$ such that*

$$y_0 - A^*\bar{w} \in P^+ \quad \text{and} \quad ((z_0, \bar{w}))_2 = M'.$$

Let us denote by $L_m^2[0, T]$ the m product of $L^2[0, T]$, the space of all real-valued functions on $[0, T]$ which are square integrable. For $f \in L^2[0, T]$, we set

$$\|f\| = \left(\int_0^T f(t)^2 dt\right)^{1/2}.$$

Hereafter we choose

$$X = Y = L_n^2[0, T],\ Z = W = L_m^2[0, T],$$

$$((x, y))_1 = \int_0^T x(t) \cdot y(t) dt \quad \text{for } x \in X \text{ and } y \in Y,$$

$$((z, w))_2 = \int_0^T z(t) \cdot w(t) dt \quad \text{for } z \in Z \text{ and } w \in W,$$

$$P = \{x \in X;\ x(t) \geqq 0 \quad \text{a.e.}^{2)}\},$$

$$Q = \{z \in Z;\ z(t) \geqq 0 \quad \text{a.e.}\},$$

$$y_0 = c,\ z_0 = a,$$

$$Ax(t) = x(t) B(t) - \int_t^T x(s) K(s, t)\, ds.$$

Then the quintuple (A, P, Q, c, a) is a linear program and called the primal program of the *generalized* continuous linear programmings. We can easily verify that

$$A^*w(t) = B(t) w(t) - \int_0^t K(t, s) w(s)\, ds.$$

Let M and M' be the values of the primal and the dual of the generalized continuous linear programmings respectively. Then it is always valid that

1) [5], Theorem 3.

2) =almost everywhere with respect to the Lebesgue measure on the real line.

$$N' \leqq M' \leqq M \leqq N.$$[3]

Let μ and α be positive numbers such that

$$|K(t, s)| \leqq \mu \text{ on } [0, T] \times [0, T],$$
$$|a(t)| \leqq \alpha \quad \text{ on } [0, T],$$

and let

$$h(t) = (\alpha/\beta) \exp [\mu(T-t)/\beta].$$

Denote by $x_h(t)$ the n vector with all components equal to $h(t)$. Making use of conditions (N. 1) and (N. 2), Levinson showed that $0 \in S(N')$ and $x_h \in S(N)$.[4] Consequently M and M' are finite.

We shall prepare

LEMMA 1.[5] *Let the integrable function $g(t) \geqq 0$ satisfy*

$$g(t) \leqq \rho_1 + \rho_2 \int_0^t g(s)\, ds \quad \text{a.e. on } [0, T],$$

where $\rho_1 \geqq 0$ and $\rho_2 > 0$. Then we have

$$g(t) \leqq \rho_1 \exp [\rho_2 t] \quad \text{a.e. on } [0, T].$$

LEMMA 2. *Let two functions $f(t)$ and $q(t)$ of $L^2[0, T]$ satisfy*

$$0 \leqq f(t) \leqq q(t) + \rho \int_0^t f(s)\, ds \quad \text{a.e. on } [0, T],$$

where $\rho > 0$. Then we have

$$||f|| \leqq 2^{1/2} ||q|| \exp [\rho^2 T^2].$$

PROOF. From the given relation, it follows that

$$f(t)^2 \leqq \left[q(t) + \rho \int_0^t f(s)\, ds \right]^2$$
$$\leqq 2q(t)^2 + 2\rho^2 \left[\int_0^t f(s)\, ds \right]^2$$
$$\leqq 2q(t)^2 + 2\rho^2 T \int_0^t f(s)^2\, ds$$

almost everywhere on $[0, T]$. Writing $g(t) = \int_0^t f(s)^2 ds$ and integrating both sides of the above inequality, we have

3) cf. [8], p. 336, Theorem 6.
4) [6], p. 74 and p. 78
5) [6], p. 75, Gronwall's lemma.

$$0 \leqq g(t) \leqq 2\,\|q\|^2 + 2\rho^2 T \int_0^t g(s)\,ds.$$

By means of Lemma 1, we have

$$g(t) \leqq 2\|q\|^2 \exp[2\rho^2 Tt] \leqq 2\|q\|^2 \exp[2\rho^2 T^2],$$

and hence

$$\|f\|^2 \leqq 2\,\|q\|^2 \exp[2\rho^2 T^2].$$

Now we shall prove

THEOREM 2. *It is valid that $M = M'$ and there exists $\bar{w} \in S(M')$ such that $M' = ((a, \bar{w}))_2$, i.e., $\bar{w} \in L^2_m[0, T]$ satisfies that*

$$\bar{w}(t) \geqq 0 \quad a.e.\ on\ [0, T],$$

$$B(t)\bar{w}(t) \leqq c(t) + \int_0^t K(t, s)\bar{w}(s)\,ds \quad a.e.\ on\ [0, T],$$

$$M' = \int_0^T a(t) \cdot \bar{w}(t)\,dt.$$

PROOF. In order to apply Theorem 1, it suffices to show that the set G is $w(Y \times R, X \times R)$-closed. Since G is convex, it is enough to verify that G is $s(Y \times R, X \times R)$-closed ([1], p. 67, Proposition 4). Since $Y \times R$ is a Banach space with respect to the norm defined by $\sum_{i=1}^n \|y_i\| + |r|$ for $y = (y_i) \in Y$ and $r \in R$ and $X \times R$ is the strong dual of $Y \times R$, we see that $s(Y \times R, X \times R)$ coincides with the topology of $Y \times R$ induced by the norm ([1], p. 71, Proposition 6). Let $\{(y^{(k)}, r^{(k)})\}$ be a sequence in G which $s(Y \times R, X \times R)$-converges to $(y, r) \in Y \times R$. Then there exists $w^{(k)} \in Q^+$ such that

$$y^{(k)} - A^* w^{(k)} \in P^+ \text{ and } ((a, w^{(k)}))_2 \geqq -r^{(k)}.$$

Namely we have

$$(1) \qquad B(t) w^{(k)}(t) \leqq y^{(k)}(t) + \int_0^t K(t, s) w^{(k)}(s)\,ds \quad \text{a.e.}$$

Multiplying the both sides of (1) by the n vector $e(t)$ with all components equal to 1, we have by condition (N. 2) that

$$\beta\,|w^{(k)}(t)| \leqq |\,y^{(k)}(t)| + n\mu \int_0^t |w^{(k)}(s)|\,ds \quad \text{a.e. on } [0, T].$$

It follows from Lemma 2 that

$$\|w_j^{(k)}\| \leqq \|\,|w^{(k)}|\,\| \leqq 2^{1/2} \beta^{-1} \|\,|\,y^{(k)}|\,\| \exp[(n\beta^{-1}\mu T)^2]$$

$$\leqq 2^{1/2}\beta^{-1}\exp[(n\beta^{-1}\mu T)^2]\sum_{i=1}^{n}\|y_i^{(k)}\|.$$

Since $\|y_i^{(k)}-y_i\|\to 0$ as $k\to\infty (i=1, 2,\dots, n)$, we see that $\{\|w_j^{(k)}\|; j=1,\dots, m, k=1, 2,\dots\}$ is bounded. From the fact that every closed ball $\{x\in L^2[0, T]; \|x\|\leqq d\}(d>0)$ is weakly sequentially compact ([2], p. 68, Theorem 28), we can find a $w(W, Z)$-convergent subsequence of $\{w^{(k)}\}$. Denote it again by $\{w^{(k)}\}$ and let w be the limit. Then we have $w\in Q^+$,

$$((a, w))_2=\lim_{k\to\infty}((a, w^{(k)}))_2\geqq\lim_{k\to\infty}(-r^{(k)})=-r,$$

$$((x, y-A^*w))_1=\lim_{k\to\infty}((x, y^{(k)}))_1-\lim_{k\to\infty}((Ax, w^{(k)}))_2$$

$$=\lim_{k\to\infty}((x, y^{(k)}-A^*w^{(k)}))_1\geqq 0$$

for all $x\in P$, and hence $y-A^*w\in P^+$. Therefore $(y, r)\in G$ and G is $w(Y\times R, X\times R)$-closed.

§ 3. Duality theorems for the continuous linear programmings

In this section we shall apply Theorem 2 to the study of the duality theorem for the continuous linear programmings.

We have

THEOREM 3. *It is valid that* $M'=N'$ *and there exists* $v\in S(N')$ *such that* $N'=((a, v))_2$.

PROOF. On account of Theorem 2, there exists $\bar{w}\in S(M')$ such that $M'=((a, \bar{w}))_2$. Define $v(t)$ by

$$v(t)=\begin{cases}0 & \text{on } E,\\ \bar{w}(t) & \text{on } [0, T]-E,\end{cases}$$

where

$$E=\{t\in[0, T]; \bar{w}(t)<0 \text{ or } B(t)\bar{w}(t)-\int_0^t K(t, s)\bar{w}(s)ds>c(t)\}.$$

We shall show that $v\in S(N')$. Clearly $v(t)$ is non-negative and measurable and satisfies

$$B(t)v(t)\leqq c(t)+\int_0^t K(t, s)v(s)\,ds \quad \text{on } [0, T], \tag{2}$$

since $c(t)\geqq 0$ by condition (N. 1). Let ν be a positive number such that $|c(t)|\leqq\nu$ on $[0, T]$ and $e(t)$ the n vector with all components equal to 1. Multiplying both sides of (2) by $e(t)$, we have

$$\beta|v(t)| \leqq |c(t)| + n\mu\int_0^t |v(s)|\,ds$$

$$\leqq \nu + n\mu T\||v|\|^2,$$

which shows that $v(t)$ is bounded and hence $v \in S(N')$. Since E is a set of zero measure, we have

$$M' = ((a, \bar{w}))_2 = ((a, v))_2 \leqq N',$$

and hence $M' = N' = ((a, v))_2$.

THEOREM 4. *It is valid that $M=N$ and there exists $u \in S(N)$ such that $N = ((u, c))_1$.*

PROOF. Let $\{x^{(k)}\}$ be a sequence in $S(M)$ such that $((x^{(k)}, c))_1$ tends to M as $k \to \infty$. Define $\bar{x}^{(k)}(t)$ by

$$\bar{x}_i^{(k)}(t) = \min(x_i^{(k)}(t), h(t)) \quad (i=1,\dots, n).$$

By the same argument as in the proof of Lemma 3. 1 in [6], we see that $\bar{x}^{(k)} \in S(M)$ and $((\bar{x}^{(k)}, c))_1$ tends to M as $k \to \infty$. Since $\|\bar{x}_i^{(k)}\| \leqq \|h\| < \infty (i=1, \dots, n, k=1, 2, \dots)$, we can find a $w(X, Y)$-convergent subsequence of $\{\bar{x}^{(k)}\}$. Denote it again by $\{\bar{x}^{(k)}\}$ and let $\bar{x}$ be the limit. By the same reasoning as in the proof of Theorem 2 in [6], we can prove that $\bar{x} \in S(M)$, $x_h - \bar{x} \in P$ and $M = ((\bar{x}, c))_1$. Define $u(t)$ by

$$u(t) = \begin{cases} x_h(t) & \text{on } F, \\ \bar{x}(t) & \text{on } [0, T] - F, \end{cases}$$

where

$$F = \{t \in [0, T];\ \bar{x}(t) < 0 \text{ or } \bar{x}(t) > x_h(t) \text{ or}$$

$$\bar{x}(t) B(t) - \int_t^T \bar{x}(s) K(s, t)\,ds < a(t)\}.$$

Then we see that $u \in S(N)$. Since the measure of F is equal to zero, we have

$$M = ((\bar{x}, c))_1 = ((u, c))_1 \geqq N,$$

and hence $M = N = ((u, c))_1$.

According to Theorems 2, 3 and 4, we have

THEOREM 5. *It is valid that $N = N'$ and there exist $u \in S(N)$ and $v \in S(N')$ such that*

$$\int_0^T u(t)\cdot c(t)\,dt=\int_0^T v(t)\cdot a(t)\,dt.$$

Levinson proved this theorem under additional conditions that $B(t)$, $c(t)$, $a(t)$ and $K(t, s)$ are continuous (Theorem 3 in [6]). Tyndall proved this theorem in the case where $B(t)$ and $K(t, s)$ are constant matrices. We remark that the above result is an answer to Tyndall's conjecture in Mathematical Review 37 (1969) ♯2527 (see also [4]).

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Note added in proof.

After our paper was sent for printing the following related papers drew our attention.

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