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Dynamic Analysis of a One-Parameter Chaotic System in Complex Field

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ABSTRACT Chaotic dynamics analysis of complex-variable chaotic systems (CVCSs) is an important problem in real secure communication and encryption. In this paper, a simple one-parameter chaotic system in complex field is proposed, whose nonlinear terms are the same as Lorenz system but the linear terms are much simpler. The proposed system has circular equilibria and therefore multi-stability can be measured by phase portraits, bifurcation diagrams and Lyapunov exponent spectrum. Its basin of attraction is filled with initial points leading to chaotic behaviors. The coexistence of infinitely many attractors is found in the proposed system, which is not reported in the existing complex-variable Lorenz system. Finally, two complexity indexes are used to measure dynamic characteristic with respect to parameter.

INDEX TERMS Coexisting attractors, extreme multistability, complex-variable chaotic system, complexity.

I. INTRODUCTION

Up to now, chaos is a very important research topic and received much attention in complicated nonlinear dynamics and chaotic cryptography [1]–[4]. As a model for convection between parallel plates, the Lorenz system with real variables is always a paradigm of chaos. In 1982, the Lorenz system in the complex domain was unexpectedly discovered from laser physics and the geophysical flows [5], [6], and widely attracted attentions of researchers in multiple fields. Since then, complex-variable chaotic systems (CVCSs) have been extensively investigated in various important fields, such as detuned lasers [7], [8], fluids [9], electromagnetic fields [10] and image encryption [11]–[13].

Due to the existence of complex variables, the dynamic behaviors of complex system are more complex and diverse [14], [15]. Real variable is a special case of complex variable, so the conclusions of complex chaotic system are also applicable to real chaotic system. In recent years, the stabilization and synchronization [14]–[19] of CVCSs stimulated a great deal of interest among scientist for the wide scope of applications. Especially in the chaotic secure communication field, the complex variable can increase the content and security of the transmitted information [12], [20].

The coexistence of multi-attractors for the same group of parameters can be observed in a number of nonlinear circuits or systems [1], [21], [22], this phenomenon is called multistability. System displays different attractors depending on the different initial values. The coexistence of attractors exhibits a complex diversity of attractors of a nonlinear dynamical system. At present, the coexistence of attractors becomes one of the hot spots in international nonlinear dynamics research [23], [24]. Some interesting properties of multiple coexisting attractors are found not only in the novel 4D-hyperjerk system [25] and 5-D memristive oscillator, but also in the simple three-dimensional system, such as Sprott B system [26] and Lü system [27]. Recently, it is found that the coexistence of infinitely many attractors may be coined when a periodic function is introduced into the offset-boostable variable [23]. The phenomenon that system has a great many coexisting attractors is defined as extreme multistability. Bao et al. investigated the coexistence of infinitely many attractors through analyzing the stability distribution of line equilibrium point in a memristive circuit [28]. Similarly, extreme multistability is also observed in an extremely simple chaotic system [29]. It is very exciting that a memristive hyperchaotic system without any equilibrium points can still generate coexisting infinitely many attractors [30].

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In the literature mentioned above, the phenomenon of coexisting attractors is widely considered in real-variable systems. However, the coexistence of attractors, particularly that of infinitely many chaotic attractors, is rarely reported in CVCSs so far. CVCSs have more important significance in practical application bacause they possess a diversity of coexisting attractors and exhibit very complicated basins of attraction. The main goal of present work is to investigate coexistence of attractors in one-parameter CVCSs.

The rest of this paper is structured as follows. In Sec. II, we introduce the mathematical model of one-parameter CVCSs and discuss the stability of circular equilibria, dissipation and symmetry. In Sec. III, the dynamical characteristics are investigated and the coexistence of infinitely many chaotic and periodic attractors depending on initial values are therefore discovered. In Sec. IV, spectral entropy (SE) and C_0 complexity algorithms are applied to analyze the complexity of the CVCSs. The last section concludes the paper.

II. THE PROPOSED COMPLEX-VARIABLE CHAOTIC SYSTEMS

The Sprott B system is algebraically the simplest one [31]. Assigning $x \to -y$, $y \to -x$, $z \to -z$, transform the Sprott B system into

$$\begin{cases} \dot{x} = -x + y, \\ \dot{y} = -xz, \\ \dot{z} = xy - a, \end{cases} \tag{1}$$

where (x, y, z) is the real state vector, and a is a positive constant parameter. The attractors of system (1) are similar to the attractors of Lorenz and Chen systems [32]. And its simplest 4-D extension has coexisting hidden attractors [33]. In this paper, we propose the complex extension of the system:

$$\begin{cases} \dot{z}_1 = -z_1 + z_2, \\ \dot{z}_2 = -z_1 z_3, \\ \dot{z}_3 = \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2) - a, \end{cases} \tag{2}$$

where $z_1 = u_1 + ju_2$ and $z_2 = u_3 + ju_4$ are complex state variables, $z_3 = u_5$ is real state variable, and a is positive constant parameter. It is clear that system (2) has the same nonlinear parts as the complex-variable Lorenz and Chen systems, but has much simpler linear parts. Assume that $a = 4.8$ and the initial values $z_0 = (0.5 + 0.5j, 1 + j, 1)$. Then system (2) has Lyapunov exponents $(0.0526, 0, -0.9533, -47.8200, -116.2680)$, and the corresponding Kaplan-Yorke dimension $D_{KY} = 2.048$. Due to its fractional dimension and positive Lyapunov exponent, system (2) is chaotic and the orbit is a bond chaotic attractor as in Fig. 1. It is apparent that bond-orbital attractor [34] is quite different from butterfly attractor in [32].

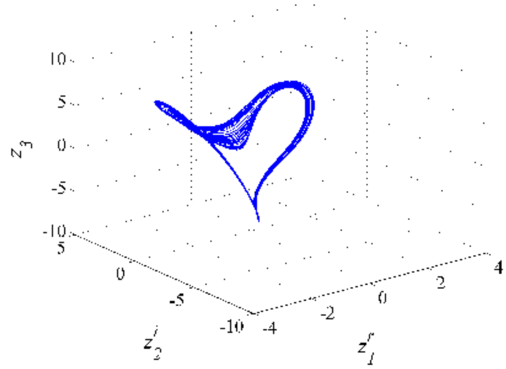


FIGURE 1. 3-D space of the bond chaotic attractor for system (2) when $a = 4.8$.

System (2) can be rewritten as a real-variable system of the form

$$\begin{cases} \dot{u}_1 = -u_1 + u_3, \\ \dot{u}_2 = -u_2 + u_4, \\ \dot{u}_3 = -u_1 u_5, \\ \dot{u}_4 = -u_2 u_5, \\ \dot{u}_5 = u_1 u_3 + u_2 u_4 - a. \end{cases} \tag{3}$$

System (3) has rotational symmetry with respect to the u_5-axis, due to its invariance under the coordinates transform from $(u_1, u_2, u_3, u_4, u_5)$ to $(-u_1, -u_2, -u_3, -u_4, u_5)$. In particular, the symmetry about the u_5-axis for any choice of parameter a is accurate. The corresponding Jacobian matrix of system (3) is

$$J = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -u_5 & 0 & 0 & 0 & -u_1 \\ 0 & -u_5 & 0 & 0 & -u_2 \\ u_3 & u_4 & u_1 & u_2 & 0 \end{bmatrix}.$$

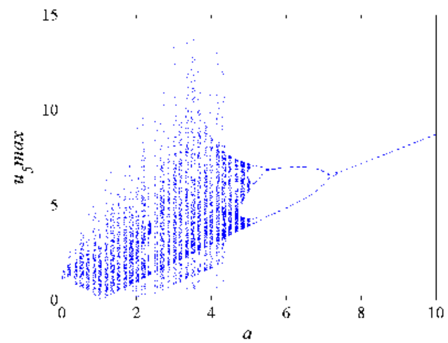
If $u_5 \neq 0$ and $u_1^2 + u_2^2 + u_1\, u_3 + u_2\, u_4 \neq 0$, system (3) has the full rank because the determinant of the Jacobian matrix is $-u_5(u_1^2 + u_2^2 + u_1\, u_3 + u_2\, u_4)$, which means that system (3) is a real five-dimensional system.

The rate of hypervolume contraction is given by the Lie derivative

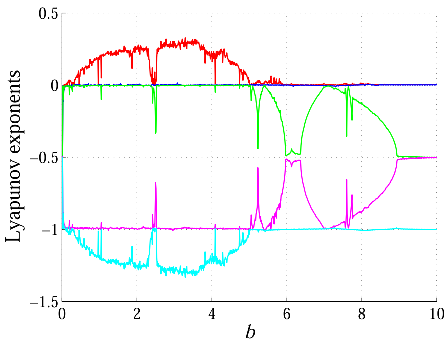
$$\nabla V = \sum_{l=1}^{5} \frac{\partial \dot{u}_l}{\partial u_l} = -2.$$

Hence, system (3) is dissipative. This means each volume element V_0 containing the system trajectories shrinks to zero when $t \to \infty$ at an exponential rate -2. From the algebraic equations: $-u_1 + u_3 = 0$, $-u_2 + u_4 = 0$, $-u_1 u_5 = 0$, $-u_2 u_5 = 0$, $u_1 u_3 + u_2 u_4 - a = 0$, one can derive that system (3) has infinite equilibrium points. Let $u_1 = u_3 = \sqrt{a}\cos\theta$ and $u_2 = u_4 = \sqrt{a}\sin\theta$, where $\theta \in [0, 2\pi]$, the non-isolated (nontrivial) fixed points can be described as

$$E = (\sqrt{a}\cos\theta, \sqrt{a}\sin\theta, \sqrt{a}\cos\theta, \sqrt{a}\sin\theta, 0).$$



(a) Bifurcation diagram



(b) Lyapunov exponent spectrum

FIGURE 2. Bifurcation diagram and Lyapunov exponent spectrum versus parameter *a* of system (3).

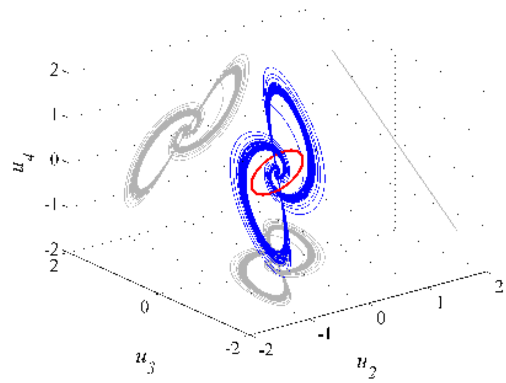


FIGURE 3. 3D views of chaotic attractors of system (3) when $a = 0.12$.

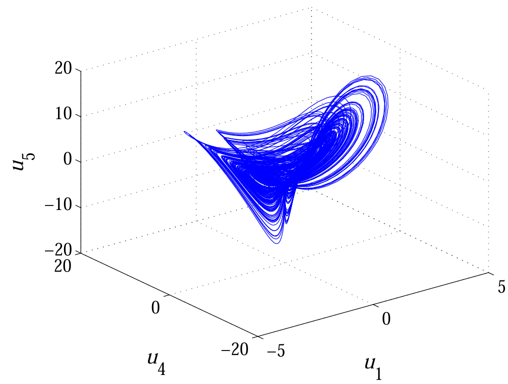


FIGURE 4. 3D views of the two-butterfly chaotic attractor of system (3) when $a = 3.61$.

The characteristic equation of the corresponding Jacobian matrix is obtained as $\mu(\mu+1)(\mu^3+\mu^2+a\mu+2\,a)=0$. According to Routh-Hurwitz criterion, system (3) has one eigenvalue 0, two negative real eigenvalues and a pair of conjugate complex eigenvalues with positive real part provided $a > 0$. For example, when $a = 1$, eigenvalues are $\mu_1 = -1.3532, \mu_2 = -1, \mu_3 = 0, \mu_4 = 0.1766 + 1.2028j, \mu_5 = 0.1766 - 1.2028j$, which means that circular equilibria are non-hyperbolic and saddle-foci with index 2 (unstable).

III. CHAOTIC ATTRACTORS

To verify existence of the chaotic attractors, we use two effective indexes, Lyapunov exponents and bifurcation diagram, to characterize statistical properties. Lyapunov exponents are the average exponential divergence rate of two closed orbits, which are used to quantitatively express the stability of the orbits and its sensitivity to the initial conditions. For chaotic attractors, it possess the positive Lyapunov exponents. According to the definition, Lyapunov exponents are adopted as

$$\lambda_l = \lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{\delta u_l(u_0,t)}{\delta u_l(u_0,0)}\right|. \tag{4}$$

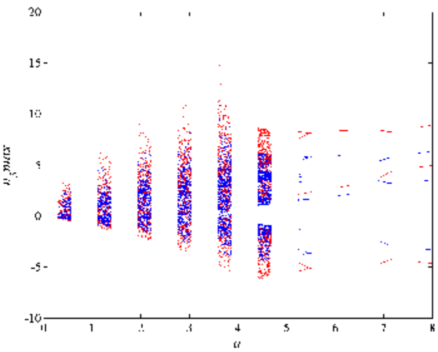
For the initial condition $U_0 = (0.5, 0.5, 1, 1, 1)$, the bifurcation diagram and Lyapunov exponent spectrum versus parameter a are plotted in Fig. 3. It can be seen from Fig. 2(a) that the bifurcation of system (3) with respect to a is reverse period-doubling bifurcation. Fig. 2(b) shows that system (3) has periodic behaviors when $a \in (0.24, 0.32), (2.4, 2.53)$ and $(5.02, 10)$. The chaotic attractors are clear for $a \in (0, 0.23)$, $(0.33, 2.39)$ and $(2.54, 5.01)$, respectively. Note that there exist always two zero Lyapunov exponents in Fig. 2(b).

The Lyapunov exponents of system (3) are $(0.0197, 0, -0.0113, -0.987, -1.0127)$ when $a = 0.12$. The chaotic attractor and the infinite equilibria when $a = 0.12$ are shown in Fig 3, where red circle represents the equilibrium points, gray plots are projections of the attractor onto the different planes. Interestingly, a two-butterfly chaotic attractor with Lyapunov exponents $(0.2866, 0, 0, -0.9983, -1.2876)$ is found when $a = 3.61$, as shown in Fig. 4.

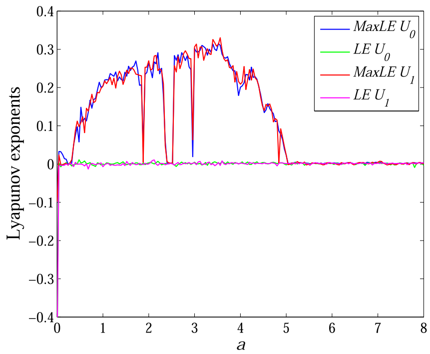
IV. COEXISTING ATTRACTORS

A. COEXISTENCE OF TWO ATTRACTORS

The coexisting bifurcation diagram and coexisting Lyapunov exponent spectrum are used to examine the coexisting attractors of the system. The coexisting bifurcation diagram about state variable u_3 and coexisting Lyapunov exponent spectrum versus $a \in (0, 8)$ are plotted in Fig. 5, respectively. As shown in Fig. 5(a), the initial values for the blue orbit is $U_0 = (0.5, 0.5, 1, 1, 1)$, while those for the blue one is $U_1 = (0.5, 0, 0.5, 0.1, 0.1)$. It is obvious that $|\max(u_3)|$ starting from U_1 is larger than that from U_0 at $a \in (0, 0.23)$,$(0.33, 2.39)$ and $(2.54, 5.01)$, which means that there are two different amplitudes attractors. In Fig. 5(b),

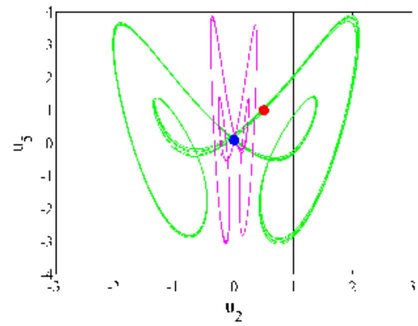


(a) Coexisting bifurcation diagram

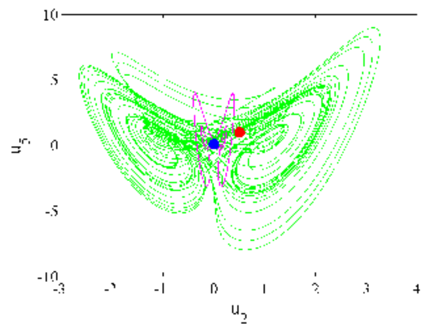


(b) Coexisting Lyapunov exponent spectrum

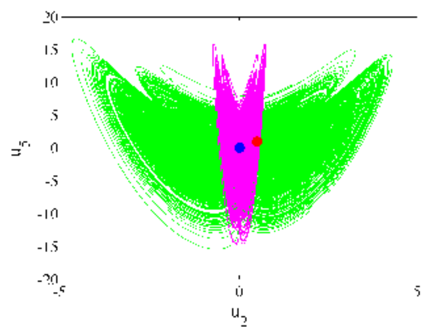
FIGURE 5. **Coexisting bifurcation diagram and coexisting Lyapunov exponent spectrum of system (3) for the initial conditions $U_0 = (0.5, 0.5, 1, 1, 1)$ and $U_1 = (0.5, 0, 0.5, 0.1, 0.1)$.**



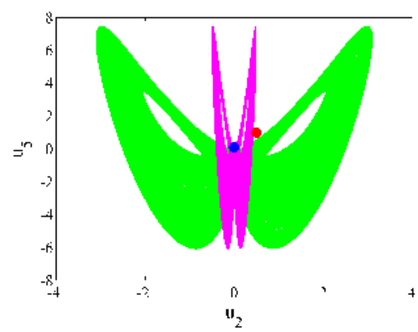
(a) Coexistence of periodic attractors ($a = 2.4$)



(b) Coexistence of chaotic attractor and periodic attractor ($a = 2.5$)



(c) Coexistence of two-butterfly attractors ($a = 3.61$)



(d) Coexistence of chaotic attractors($a = 4.56$)

FIGURE 6. **Coexistence of attractors in the (u_2, u_5) plane for the initial conditions $U_0 = (0.5, 0.5, 1, 1, 1)$ and $U_1 = (0.5, 0, 0.5, 0.1, 0.1)$.**

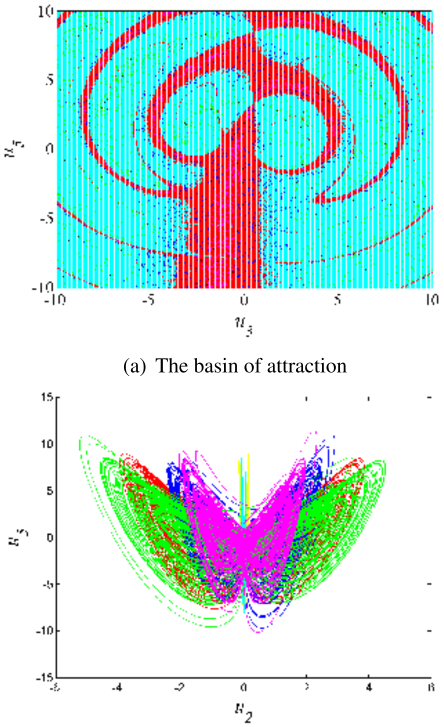
the blue and green trajectories express the first two Lyapunov exponents of the system with initial value U_0, while the red and pink trajectories represent the case with U_1.

It is very interesting that coexisting butterfly attractors with asymmetric initial conditions are shown in Fig. 6, where green attractors start form red initial conditions U_0 and pink attractors start form blue U_1. Fig. 6 illustrates the coexistence of periodic attractors when $a = 2.4$, coexistence of periodic attractor and chaotic attractor when $a = 2.5$, coexistence of two-butterfly chaotic attractors when $a = 3.61$ and coexistence of chaotic attractors when $a = 4.56$.

B. COEXISTENCE OF SIX CHAOTIC ATTRACTORS

The basin of attraction is a useful method to examine the coexisting attractors of the system, which changes with the initial values [28], [33]. When parameter of the system is set as $a = 4$ and the initial value is given as $(0.5, 1, u_3, 1, u_5)$, the basin of attraction with respect to initial values is shown in Fig. 7(a). The chaotic sea is filled with points leading to chaotic attractors when $u_3 \in [-10, 10]$ and $u_5 \in [-10, 10]$, where different colors represent different chaotic attractors in Fig. 7(a), respectively.

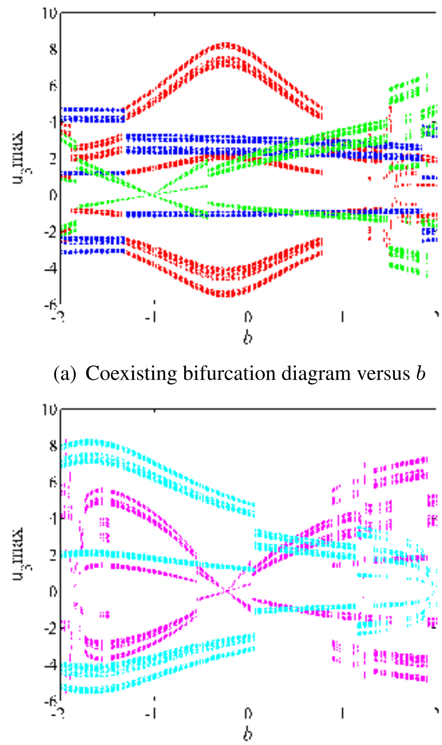
Fig. 7(b) depicts six typical chaotic attractors which are observed in the basin of attraction. And the initial values of the attractors are selected from different color points in the basin of attraction. In Fig. 7(b), the red attractor corresponds to the initial value $(0.5, 1, 1, 1, -6)$ with LEs $(0.2066, 0, 0, -1.0036, -1.2121)$, the blue attractor corresponds to the initial value $(0.5, 1, 4, 1, -6)$ with



(a) The basin of attraction

(b) Coexisting chaotic attractors

FIGURE 7. The basin of attraction and the coexisting chaotic attractors with initial condition $(0.5, 1, u_3, 1, u_5)$ when $a = 4$.



(a) Coexisting bifurcation diagram versus b

(b) Coexisting bifurcation diagram versus b

FIGURE 8. Coexisting bifurcation diagram versus initial conditions when $a = 5$.

LEs $(0.2371, 0, 0, -0.9983, -1.2396)$, the green attractor corresponds to the initial value $(0.5, 1, -2.8, 1, 6)$ with LEs $(0.2269, 0, 0, -0.9977, -1.2330)$, the yellow attractor corresponds to the initial value $(0.5, 1, 6, 1, 6)$ with LEs $(0.2053, 0, 0, -1.0034, -1.2048)$, the cyan attractor corresponds to the initial value $(0.5, 1, 8, 1, -3)$ with LEs $(0.2497, 0, 0, -0.9978, -1.2482)$ and the pink attractor corresponds to the initial value $(0.5, 1, 6.8, 1, -10)$ with LEs $(0.2061, 0, 0, -1.0012, -1.2001)$, respectively.

C. THE COEXISTENCE OF INFINITELY MANY ATTRACTORS

Extreme multi-stability means that the coexistence of an infinite number of attractors exists in a nonlinear system. And it can be obtained in a nonlinear dynamical circuit or system [28], [33], [35]. To further confirm the infinite number of coexisting attractors, the coexisting bifurcation diagrams with the change of initial conditions and phase diagram of attractors with different amplitudes are drawn.

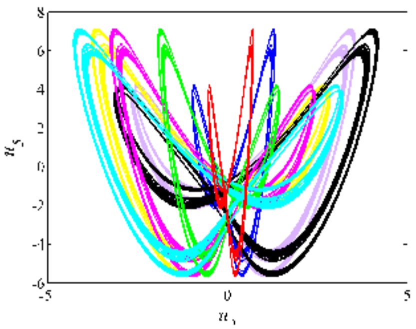
Here, the parameter $a = 5$ is set and the initial conditions versus b are considered. When b is gradually increased from -2 to 2, coexisting bifurcation diagrams are shown in Fig. 8, where the green, cyan, pink, red and blue branches are yielded from initial conditions $(b, 0.5, 0.5, 1, 1)$, $(0.5, b, 0.5, 1, 1)$, $(0.5, 0.5, b, 1, 1)$, $(0.5, 0.5, 0.5, b, 1)$ and $(0.5, 0.5, 0.5, 1, b)$, respectively. The coexistence of attractors with different amplitudes is found, which means that coexistence phenomenon of infinitely many attractors in the proposed system. We only pick initial values selected randomly in Fig. 8, and the coexistence of infinitely many attractors as plotted in Fig. 9.

In Fig. 9, the coexistence of infinitely many chaotic attractors at $a = 5$ and the the coexistence of infinitely many periodic attractors at $a = 7$ with different initial conditions are discovered, where the black attractors match the initial value $(0.5, 0.5, 0.5, 2, 1)$, the yellow attractors match $(0.5, 0.5, 0.5, 1, 1.5)$, the blue attractors match $(0.5, 0.5, 0.5, 0, 1)$, the purple attractors match $(0.5, 0.5, 0.5, 1, -1.6)$, the pink attractors match $(0.5, 0.5, -1.4, 1, 1)$, the green attractors match $(0.5, 0.5, 0.5, 1, 1)$, the cyan attractors match $(0.5, 2, 0.5, 1, 1)$ and the red attractors match $(0.5, -1.5, 0.5, 1, 1)$. In fact, for more different initial values, more attractors could be found. Fig. 9 clearly disclose the coexistence of infinite many chaotic and periodic attractors, which indicates that there indeed exists extreme multistability in system.

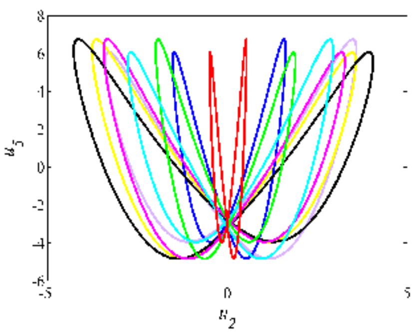
It is worth mentioning that the proposed system (2) has the coexistence of infinitely many attractors because of its infinite equilibrium points. This property of extreme multi-stability is not reported in the Lorenz CVCSs in [15], [16], [19] and the real-variable systems in [28], [31]–[33], [35].

V. SPECTRAL ENTROPY (SE) AND C_0 COMPLEXITY

Complexity is an important reference to measure dynamic characteristics of a chaotic system. Up to now, there are several main algorithms for measuring the complexity of

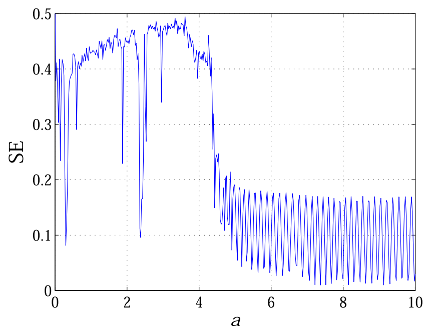


(a) The coexistence of infinitely many chaotic attractors

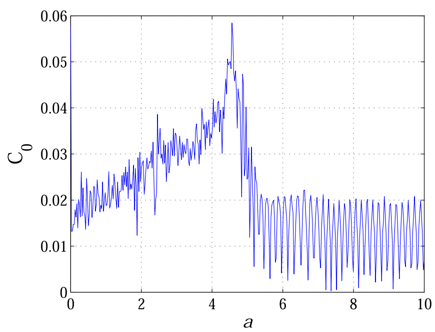


(b) The coexistence of infinitely many periodic attractors

FIGURE 9. Coexisting infinitely many attractors when $a = 5$ and $a = 7$, respectively.



(a) SE complexity



(b) C_0 complexity

FIGURE 10. SE and C_0 complexity versus parameter a for the initial condition $U_0 = (0.5, 0.5, 1, 1, 1)$.

chaotic sequences, such as intensive statistical method [36], multiscale entropy (MSE) [37], spectral entropy (SE) [38], and C_0 complexity [39]. In particular, C_0 and SE complexity algorithms often have been used to analyze information security of the system in chaotic secure communication, because of their less parameters, faster calculation speed, and higher accuracy [20]. The SE value is obtained through the energy distribution in the Fourier transform domain and Shannon entropy. The main idea of C_0 complexity is to decompose the sequence into regular and irregular components. And its measure value is the proportion of irregular components in the sequence. Here, the SE and C_0 complexity of the complex system versus parameter a at the initial condition $U_0 = (0.5, 0.5, 1, 1, 1)$ are shown in Fig. 10, they have the similar changing trend, although the values are different. Compared with the Lyapunov exponent spectrum and bifurcation diagram in Fig. 10, SE and C_0 complexity are consistent, which means that complexity can also reflect the dynamic characteristics of a chaotic complex system.

The value of SE complexity for proposed CVCSs has obvious change for the period, whereas it has relatively small change for the chaos like the real-variable system [21]. The value of C_0 complexity has obvious change for both period and chaos. It is noted that the SE complexity of two-butterfly attractor at $a = 3.61$ is biggest in Fig. 10(a), and the C_0 complexity of chaotic attractor with $a = 4.56$ is the biggest in Fig. 10(b).

VI. CONCLUSION

This paper studied dynamical properties of complex-variable chaotic systems with circular equilibria. Numerical simulations displayed that the proposed system had a number of rich and interesting dynamic phenomenon. Furthermore, chaotic sea about chaotic attractors was revealed in the basin of attraction. In particular, the coexistence of infinitely many chaotic and periodic attractors with different values of control parameter were discovered in coexisting bifurcation diagram versus initial conditions. The results of spectral entropy (SE) and C_0 complexity were matched well with Lyapunov exponent spectrum and bifurcation model. Only some preliminary and empirical experimental results were reported, much more underlying dynamical properties and their applications deserve further exploration.

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