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DYNAMIC ANALYSIS OF ELASTO-PLASTIC STRUCTURES

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INTRODUCTION

The basic tool of the theory of linear vibrations was forged by Daniel Bernoulli in 1753, when he enunciated the principle of resolution of vibrations into independent modes. The general theory of vibration of an undamped linear dynamic system with a finite number of degrees of freedom was developed by Lagrange in 1762-1765. The work of these two great mathematicians has served as a basis for most of the developments in the theory of vibrations over the past two centuries. Today, the formal solution of the problem of linear vibrations, both damped and undamped, is complete and has been extended to systems with infintely many degrees of freedom.

The theory of vibrations of nonlinear systems has not fared as well as its linear counterpart. Bernoulli's principle of resolution of vibrations into independent modes, which plays such an important role in linear theory, is inapplicable to nonlinear systems. Exact solutions have been obtained only for a few special cases of single degree of freedom systems. For steady-state forced vibrations of single degree of freedom systems with nonlinear damping, methods have been devised for finding approximately equivalent linear systems.(1) For vibrations due to transient forces it may sometimes be possible to approximate the system by a linear system over a limited range, or by one of the nonlinear systems for which the solution is known. If strong nonlinearities are present, this approach is apt to be unsatisfactory; in such cases, for single degree of freedom systems, the solution can often be obtained by the use of an electronic analog computer. Alternatively, numerical methods can be employed. For a nonlinear system having many degrees of freedom, the complex circuitry required for the analog computer becomes prohibitive, and numerical methods provide the only suitable approach. A high-speed digital computer is essential.

This paper presents two numerical methods for solving the differential equations of motion for nonlinear systems. The differential equations of motion are formulated for a lumped mass multi-story elasto-plastic framework subjected to dynamic lateral forces, taking into account elasto-plastic deformation and viscous damping. Uniqueness of the solution for an elasto-plastic frame is proved.

THE DIFFERENTIAL EQUATIONS OF MOTION FOR DYNAMIC SYSTEMS

To write the differential equations of motion for a damped linear dynamic system, consider a system of n discrete bodies, each of which is connected to every other body and to the base by a system of linear springs and viscous dampers. Such a system, of order two, is shown symbolically in Figure 1. Assume that the springs and dampers are weightless, and that all motion takes place in the x-direction, which is the direction in which the springs and dampers act. Let the motion of the base in the x-direction be any prescribed function of time; and, further, let each body be acted upon by an external force in the x-direction, each force being any prescribed function of time. One then has a linear dynamic system of n degrees of freedom.

Let

m_i = mass of the i^{th} body,

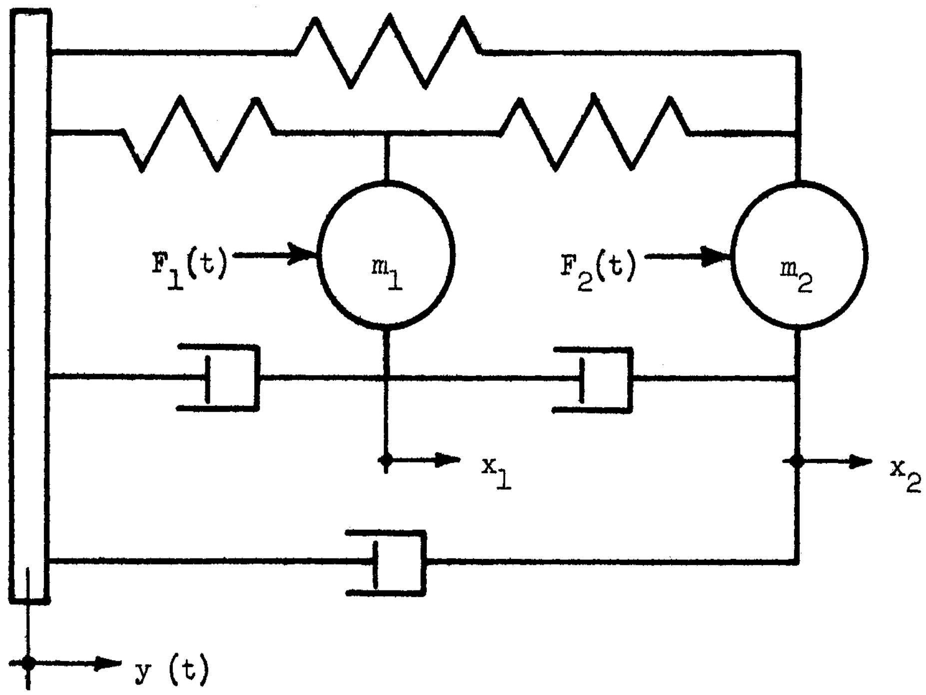
x_i = displacement of the i^{th} body, relative to a coordinate system fixed in the base,

$y(t)$ = displacement of the base, relative to a "fixed" frame of reference,

$F_i(t)$ = external force applied to the i^{th} body,

and let dots denote differentiation with respect to time.

Define the stiffness coefficient k_{ij} as the force exerted on the i^{th} body by the springs when the configuration of the system is $x_j = 1$ unit length; $x_k = 0$, $k \neq j$; and $\dot{x}_k = 0$ for all k. Also, define the damping coefficient c_{ij} as the force exerted on the i^{th} body by the dampers when the



x_1 and x_2 are displacements relative to base

y is displacement of base

Figure 1. Damped Linear Dynamic System.

configuration of the system is $\dot{x}_j = 1$ unit velocity; $\dot{x}_k = 0$, $k \neq j$; and $x_k = 0$ for all k.

Because all the bodies are interconnected by the spring and damper system, a displacement or velocity in any one body will cause spring or damper forces to be exerted on all bodies in the system. For any configuration, the force exerted on the i^{th} body by the spring and damper system is

$$\sum_{j=1}^{n} c_{ij}\dot{x}_j + \sum_{j=1}^{n} k_{ij}x_j$$

The Newtonian equation of motion for the i^{th} body is then

$$m_i\,(\ddot{x}_i + \ddot{y}) + \sum_{j=1}^{n} c_{ij}\dot{x}_j + \sum_{j=1}^{n} k_{ij}x_j = F_i(t) \qquad (1)$$

Thus, for the entire system,

$$m_i\ddot{x}_i + \sum_{j=1}^{n} c_{ij}\dot{x}_j + \sum_{j=1}^{n} k_{ij}x_j = F_i(t) - m_i\ddot{y}(t) \quad (i = 1,2,..n) \qquad (2)$$

Let

$$f_i(t) = F_i(t) - m_i\ddot{y}(t) \qquad (i = 1,2,..n) \qquad (3)$$

Equations (2) then become

$$m_i\ddot{x}_i + \sum_{j=1}^{n} c_{ij}\dot{x}_j + \sum_{j=1}^{n} k_{ij}x_j = f_i(t) \qquad (i = 1,2,..n) \qquad (4)$$

which are the general differential equations of motion for a damped linear dynamic system with n degrees of freedom.

In Equation (4) the term $\sum_{j=1}^{n} c_{ij}\dot{x}_j$ will be called the damping force, and the term $\sum_{j=1}^{n} k_{ij}x_j$ will be called the restoring force. The sum of the damping and restoring forces will be called the resistance force R_i.

In a general nonlinear system the resistance force R_i for each mass may be a nonlinear function of time and of the velocities and displacements of <u>all</u> masses. The equations of motion for a nonlinear system are then

$$m_i\ddot{x}_i + R_i\ (x_1,\ x_2, \ldots\ x_n,\ \dot{x}_1,\ \dot{x}_2, \ldots\ \dot{x}_n,\ t) \quad = \quad f_i(t) \quad (i = 1,2,..n) \quad (5)$$

NUMERICAL SOLUTION OF THE EQUATIONS OF MOTION

There are many methods of approximating the solution to Equations (5) numerically. While these differ in detail, they all employ the same basic idea. Let h be a small finite increment of time. One takes the known discrete values of x_i, $\dot{x}_i$, and $\ddot{x}_i$ $(i = 1,2,..n)$ at times t, t-h, t-2h, etc., as required by the particular numerical process, and the values of f_i, which are explicit functions of time, and projects these to evaluate x_i, $\dot{x}_i$, and $\ddot{x}_i$ at time t+h. In this manner, one advances step by step through the solution.

No one method of numerical integration can be claimed to be the best for all problems. The following two methods have exhibited excellent behavior in machine computation of structural response to dynamic loads.

MILNE PREDICTOR-CORRECTOR METHOD

The Milne Predictor-Corrector method[2] is well suited to problems in which (a) damping forces are absent or negligible, and therefore the resistance functions R_i do not involve the velocities $\dot{x}$; and (b) the driving forces f_i can be approximated by smooth curves through sets of discrete points equally spaced in time. Blast problems, for example, often meet these requirements. To employ the Milne Predictor-Corrector method one must know the discrete values of x_i and $\ddot{x}_i$ at the ends of four consecutive time intervals of equal length h. The steps of the method are:

Predictor

$$x_i^*(t+h) = x_i(t) + x_i(t-2h) - x_i(t-3h) + \frac{h^2}{4}\left[5\ddot{x}_i(t) + 2\ddot{x}_i(t-h) + 5\ddot{x}_i(t-2h)\right] \quad (6)$$

Corrector

$$x_i(t+h) = 2x_i(t) - x_i(t-h) + \frac{h^2}{12}\left[\ddot{x}_i^*(t+h) + 10\ddot{x}_i(t) + \ddot{x}_i(t-h)\right] . \quad (7)$$

By recursive use of these two formulas one advances step by step through the solution.

The Milne Predictor-Corrector process does not require calculation of velocities. Also, being noniterative, it is not accompanied by a convergence problem. The dominant error term in the predictor is $+ \frac{17}{240} h^6 x_i^{vi}$, while in the corrector the dominant error term is $- \frac{1}{240} h^6 x_i^{vi}$. The truncation error committed in each step is therefore approximately $\frac{1}{18} |x_i^* - x_i|$. One can easily program the computer to accumulate a set of error functions

$$\epsilon_i = \frac{1}{18} \sum_t |x_i^* - x_i| \quad (8)$$

and halt when any ϵ_i exceeds a present maximum. The error functions are not absolute error bounds, for they consider only the dominant term in the truncation error in each step. Nevertheless, they are worthwhile checks to incorporate in the program.

Because each step of the Milne Predictor-Corrector process uses information from three preceding time steps, some "starter" method must be employed to calculate the initial steps. Also, the length of the time interval h must remain constant. Any time after the sixth cycle the time interval can be doubled in an obvious manner. However, reducing the time interval would require interpolating to find the needed values of displacement at intermediate points of the preceding intervals.

RUNGE-KUTTA METHOD

For problems which involve very irregular driving forces, such as the earthquake response problem, it is advantageous to use a single-step method--that is, one which projects to time t+h from the values of x_i, $\dot{x}_i$, and $\ddot{x}_i$ at time t, and the functions f_i, without using the values of the variables at earlier times. This is advantageous first, because it permits changing the time interval h at any step; and second, because it requires no special starting procedure for the initial steps of the solution. The Runge-Kutta procedure[3] is a single-step method well suited to high-speed computers.

For the equations

$$\dot{y}_i = \dot{y}_i(y_1, y_2, \ldots y_n, t) \qquad (i = 1,2,..n) \qquad (9)$$

the formulas for the Runge-Kutta third order procedure are

$$\begin{aligned} \kappa_{i0} &= h\dot{y}_i(y_1, y_2, \ldots y_n, t) \\ \kappa_{i1} &= h\dot{y}_i(y_1 + p\kappa_{10}, y_2 + p\kappa_{20}, \ldots y_n + p\kappa_{n0}, t + ph) \qquad (10) \\ \kappa_{i2} &= h\dot{y}_i(y_1 + [q-r]\,\kappa_{10} + r\kappa_{11}, \ldots y_n + [q-r]\,\kappa_{n0} + r\kappa_{n1}, t + qh) \end{aligned}$$

and

$$y_i(t+h) = y_i(t) + \ell\kappa_{i0} + m\kappa_{i1} + n\kappa_{i2} \qquad (11)$$

The relations governing the constants ℓ, m, n, p, q, and r are derived by expanding both sides of Equation (11) in power series in h, and equating coefficients of the powers up through h^3. The relations are

$$\begin{aligned} \ell + m + n &= 1 \\ mp + nq &= 1/2 \\ mp^2 + nq^2 &= 1/3 \qquad (12) \\ npr &= 1/6 \quad , \end{aligned}$$

to which there are infinitely many solutions. One of the more useful solutions is

$$
\begin{aligned}
\ell &= 1/4 \\
m &= 0 \\
n &= 3/4 \\
p &= 1/3 \\
q &= 2/3 \\
r &= 2/3
\end{aligned}
\tag{13}
$$

This makes m = (q-r) = 0 and thus reduces the number of arithmetic operations required. A second solution, due to Conte and Reeves,[4] is

$$
\begin{aligned}
\ell &= .62653829327 \\
m &= .85614352807 \\
n &= -.48268182134 \\
p &= .62653829327 \\
q &= .07542588774 \\
r &= -.55111240553
\end{aligned}
\tag{14}
$$

In this solution, $\ell = p = (q-r)$, which leads to a reduced storage requirement for the computer.

To adapt Equations (5) to the Runge-Kutta process, one can write them in the form

$$
\begin{aligned}
\dot{x}_i &= z_i \\
\dot{z}_i &= \frac{1}{m_i}\left[f_i(t) - R_i\ (x_1,\ x_2, \ldots x_n,\ z_1,\ z_2, \ldots z_n,\ t)\right]
\end{aligned}
\tag{15}
$$

The solution is then straightforward.

The numerical methods above are applicable to any system, subject only to the restrictions that the number of degrees of freedom must be finite and the resistance functions R_i must be single-valued and piece-wise continuous.

THE RESISTANCE FORCES FOR AN ELASTO-PLASTIC BENT

The specific problem considered in the remainder of this paper is the response of a multi-story bent to lateral dynamic forces. The bent is a rectangular plane framework of elasto-plastic members, loaded by lateral dynamic forces applied at the joints and in the plane of the bent. Connections may be either pinned or fully restrained. The mass of the structure is assumed to be concentrated at the joints. Damping is assumed to be viscous. Deformations due to axial forces and shear forces in the members are neglected, and the effect of axial forces upon the stiffnesses and plastic hinge moments of the members is neglected. These assumptions yield a dynamic system having degrees of freedom equal to the number of stories in the bent. The principles employed can be extended to more general problems, but such extension is not considered herein.

Consistent with the assumption of viscous damping, the resistance functions R_i are separable into damping forces D_i, which are linear functions of the velocities, and restoring forces Q_i. As in Equations (4), the damping forces are

$$D_i = \sum_{j=1}^{n} c_{ij}\mathring{x}_j \quad . \tag{16}$$

The coefficient c_{ij} is the damping force that would exist at the i^{th} floor if the structure were passing through the equilibrium configuration with unit velocity at the j-th floor and zero velocity at all other floors.

The restoring forces for an elastic frame can be expressed in terms of a matrix of stiffness coefficients k such that

$$Q_i = \sum_{j=1}^{n} k_{ij} x_j \quad , \tag{17}$$

just as in Equation (4). The forces Q_i ($i = 1,2,..n$) are the forces necessary to hold the frame in static equilibrium in the configuration defined by the lateral deflections x_j ($j = 1,2,..n$). The j-th column of the matrix k represents the set of static lateral forces that would be required to hold the frame in equilibrium with unit lateral deflection at the j-th floor and zero lateral deflection at all other floors. A conventional static analysis is sufficient to evaluate the stiffness matrix. The effect of eccentricity of dead load (the "overturning" effect) can be taken into account in evaluating the matrix if desired.

Let the subscripts k and ℓ denote locations in the frame, such that each value of the subscript denotes the end of a specific structural member at a specific joint; and let M_k be the moment at location k, considered positive when it tends to rotate the end of the member clockwise. Then the end moments in the members of an elastic frame can be expressed in terms of a matrix of influence coefficients μ such that

$$M_k = \sum_{j=1}^{n} \mu_{kj} x_j \quad . \tag{18}$$

The j-th column of the matrix μ is the set of end moments that would exist in the members if the frame had unit lateral deflection at the j-th floor and zero lateral deflection at all other floors. The matrix μ can be evaluated by conventional static analysis.

If the lateral deflections become sufficiently large, the elastic limit may be exceeded at one or more locations in the frame. This occurrence will, of course, affect the restoring forces and moments.

For this paper it is assumed that the connections are either pinned or fully restrained, and that all members in the frame, columns and girders alike, have ideal elasto-plastic moment-rotation characteristics. The typical moment-rotation curve follows a linear elastic branch until the moment reaches the plastic hinge moment. Upon further deformation, the moment remains constant and a plastic hinge forms, making a "kink" in the member. Upon reversal of strain after reaching plasticity, the moment-rotation curve follows a path parallel to the original elastic branch. It remains on the second elastic branch until the plastic hinge moment, either positive or negative, is reached, and then follows a plastic branch as before. Extreme reversals cause the moment-rotation curve to follow a hysteresis loop. This idealized behavior is illustrated in Figure 2.

As soon as plasticity is reached at any location, the frame behaves just as though the member were hinged at that location, with a constant moment applied to the hinge. This condition prevails until the next hinge forms or until a decrease in strain occurs at an existing plastic hinge location, causing the hinge to disappear.

Cohen, Levy and Smollen developed a procedure for adapting the method of normal modes to an elasto-plastic frame with infinitely rigid girders,[5] and Schenker indicated a method of extending this to include

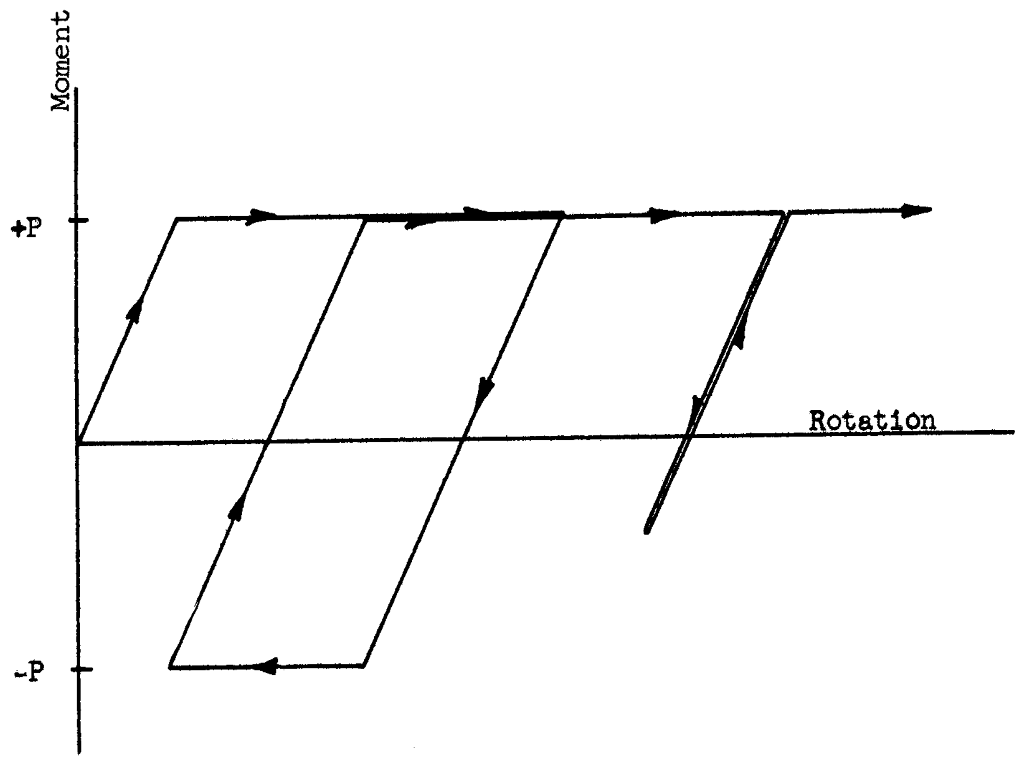
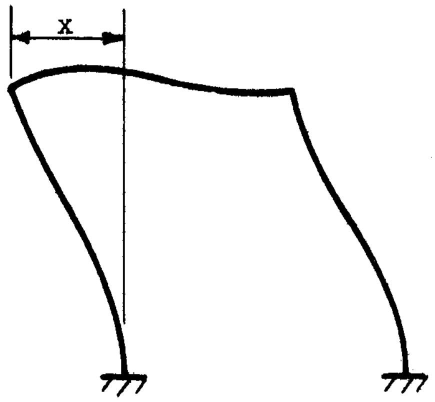


Figure 2. Idealized Moment - Rotation Diagram

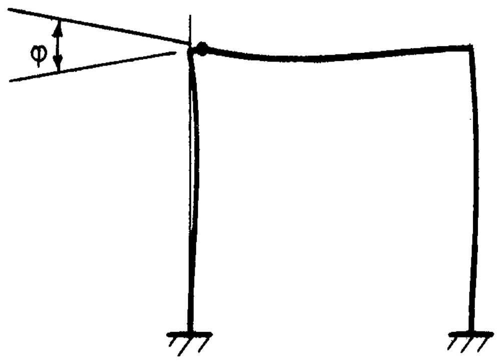
the frame with flexible girders.[6] In both procedures, an elastic solution is carried out by the method of normal modes to the point where plasticity is reached at some location. The frame is then modified by inserting a real hinge at the plastic hinge location, the new normal modes and frequencies are computed, and the solution is continued as a superposition of the conditions at the instant of transition and the changes found by solving the modified frame. Each time a new hinge forms or an old one disappears, the frame is modified accordingly. In this way a continuous solution can be found.

For a step by step numerical solution, an analogous and somewhat simpler procedure can be devised. The conditions at the end of a time step can be found as a superposition of the conditions at the beginning of the step and the changes during the step. To find the changes during the time step, one first computes the "elastic increments"; that is, the changes which would be caused by the motion during the time step if plastic hinge rotations did not occur. The resulting moments may exceed the plastic hinge moments at one or more locations in the frame. To remedy this, "corrector" solutions are superimposed in which the frame has hinges at these locations, and these hinges are rotated in such a way that the total moments, obtained from superposition of the elastic and corrector solutions, nowhere exceed the plastic hinge moments. The superposition of elastic and corrector solutions is illustrated in Figure 3.

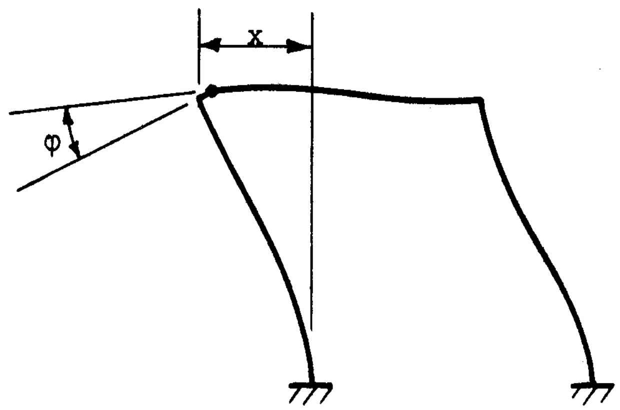
The effects of plastic hinge rotations upon the forces at the floors and the bending moments in the frame members can be expressed in terms of two matrices of influence coefficients b and ν, evaluated as described below.



Elastic



Corrector



Elasto-Plastic

Figure 3. Elasto-Plastic Solution by Superposition.

Let φ_ℓ be the plastic hinge rotation at location ℓ, taken as positive when it corresponds to clockwise rotation of the end of the member. (The hinge rotation in Figure 3 is positive.) Take the bent in its initial (unstrained) position, insert a hinge at location ℓ, and rotate the hinge through an angle $\varphi_\ell = 1$, with zero lateral deflection at all joints. Evaluate the lateral forces at all floors and the end moments for the members in the frame. The lateral force at the i-th floor is the influence coefficient $b_{i\ell}$, and the moment at location k is the influence coefficient $\nu_{k\ell}$. Both matrices b and ν can be evaluated by conventional static analysis. It will be seen that only those locations at which plastic hinges form are significant to the dynamic response.

If the lateral deflections x and the plastic hinge rotations φ were known for some instant of time, one could evaluate the restoring forces and moments as

$$Q_i = \sum_j k_{ij} x_j + \sum_\ell b_{i\ell} \varphi_\ell \tag{19}$$

and

$$M_k = \sum_j \mu_{kj} x_j + \sum_\ell \nu_{k\ell} \varphi_\ell \tag{20}$$

The deflections x can be found by integrating the equations of motion, but one must also find the plastic hinge rotations which satisfy all the constraints of the elasto-plastic system. An iterative method of solution is developed below.

Consider the time rates of change of the forces Q_i, the moments M_k, and the plastic hinge rotations φ_k. Let the plastic hinge moment at location k be P_k. The elasto-plastic constraints are:

1. The moment M cannot exceed the plastic hinge moment P in magnitude.
2. If a positive plastic hinge exists, the moment must be either constant or decreasing. If the moment is constant, the hinge may rotate in a direction consistent with the moment. If the moment is decreasing, the hinge cannot rotate.
3. Equivalent conditions exist for a negative plastic hinge.
4. If the moment is less than the plastic hinge moment in magnitude, the hinge cannot rotate.

These conditions may be expressed mathematically as the following set of constraints:

$$\begin{aligned} &1.\quad -P_k \le M_k \le P_k \\ &2.\quad \text{If } M_k = P_k,\ \text{then } \dot{M}_k \le 0,\ \mathring{\varphi}_k \le 0,\ \text{and } \mathring{M}_k \dot{\varphi}_k = 0 \\ &3.\quad \text{If } M_k = -P_k,\ \text{then } \dot{M}_k \ge 0,\ \mathring{\varphi}_k \ge 0,\ \text{and } \dot{M}_k \dot{\varphi}_k = 0 \\ &4.\quad \text{If } -P_k < M_k < P_k,\ \text{then } \mathring{\varphi}_k = 0 \end{aligned} \tag{21}$$

Differentiating Equation (20), one gets

$$\mathring{M}_k = \sum_j \mu_{kj} \mathring{x}_j + \sum_\ell \nu_{k\ell} \mathring{\varphi}_\ell \tag{22}$$

The following uniqueness theorem can be established: Given any rigid-jointed rectangular plane framework of ideal elasto-plastic flexural members, and given any $\mathring{x}$'s whatever, and any M's not exceeding the plastic hinge moments,* there exists one and only one set of $\mathring{M}$'s and $\mathring{\varphi}$'s which satisfies Equation (22) and Constraints (21). The proof follows.

* The case in which all members at a joint become hinges simultaneously is excluded.

First, observe that if, for one or more values of k, $-P_k < M_k < P_k$, then $\mathring{\varphi}_k = 0$ and there is no constraint on $\mathring{M}_k$. Thus the solution for $\mathring{\varphi}_k$ is unique, it does not affect other locations, and the effect of conditions at other locations upon $\mathring{M}_k$ is immaterial. Hence it is sufficient to consider only those locations for which $|M_k| = P_k$.

Define

$$\begin{aligned} y_k &= -(\operatorname{Sgn} M_k)\mathring{M}_k \\ a_k &= (\operatorname{Sgn} M_k)\mu_{kj}\mathring{x}_j \\ \xi_{k\ell} &= (\operatorname{Sgn} M_k)(\operatorname{Sgn} M_\ell)\ \nu_{k\ell} \\ z_\ell &= -(\operatorname{Sgn} M_\ell)\mathring{\varphi}_\ell \quad . \end{aligned} \tag{23}$$

Equation (22) then can be written

$$\sum_\ell \xi_{k\ell} z_\ell - \sum_\ell \delta_{k\ell} y_\ell = a_k \quad , \tag{24}$$

where $\delta_{k\ell}$ is the Kronecker delta,

$$\delta_{k\ell} = 1,\ k = \ell,$$

$$\delta_{k\ell} = 0,\ k \neq \ell\ .$$

Constraints (21) become

$$\begin{aligned} y_k &\geq 0, \\ z_k &\geq 0, \\ y_k z_k &= 0. \end{aligned} \tag{25}$$

The vector a is unrestricted. To establish uniqueness, it must be shown that the matrix ξ has properties which guarantee that, corresponding to any vector a, there exists one and only one pair of vectors y and z that satisfies Equation (24) and Constraints (25). This amounts to a partition problem in n-dimensional Euclidean space.

Consider the columns of the matrices $-\delta$ and ξ as vectors $\eta_1, \eta_2, \ldots \eta_n; \zeta_1, \zeta_2, \ldots \zeta_n$. That is, let $\{\eta_i\} = \{-\delta_{ji}\}$ and $\{\zeta_i\} = \{\xi_{ji}\}$. Then let $\alpha_1, \alpha_2, \ldots \alpha_n$ be a set of vectors such that every α_i is either η_i or ζ_i. There are 2^n such sets. If the η's and ζ's partition the space, every vector a in the space can be formed by linear combination of one and only one set of α_i's with nonnegative coefficients.* In other words, for any given vector a there is one and only one set of α_i's for which

$$\{a\} = \sum_i q_i \{\alpha_i\} \quad , \tag{26}$$

where every q_i is nonnegative. If this is so, then for those i for which $\alpha_i = \eta_i$, one has $y_i = q_i$ and $z_i = 0$; and for those i for which $\alpha_i = \zeta_i$, one has $y_i = 0$ and $z_i = q_i$. Thus the vectors y and z corresponding to any given vector a are unique and satisfy the constraints.

To illustrate this concept, consider the two-dimensional case. Suppose positive plastic hinges exist at locations 1 and 2, and all other locations are elastic. Further, suppose that the matrix ν is (see example, page 32),

$$\nu_{k\ell} = \begin{bmatrix} 69,700 & -16,580 \\ -16,580 & 105,560 \end{bmatrix} \text{ in-kips/radian}$$

In this case, $\nu_{k\ell} = \xi_{k\ell}$, and Equation (24) becomes

$$\begin{bmatrix} 69,700 & -16,580 \\ -16,580 & 105,560 \end{bmatrix} \{z\} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \{y\} = \{a\}$$

The columns of the matrices in this equation are plotted as the vectors ζ_1, ζ_2, η_1, and η_2 in Figure 4. The four vectors partition the plane so that

---

* If any coefficient is zero, say $q_m = 0$, obviously one can have either $\alpha_m = \eta_m$ or $\alpha_m = \zeta_m$. However, in this case $y_m = z_m = 0$ and uniqueness is preserved.

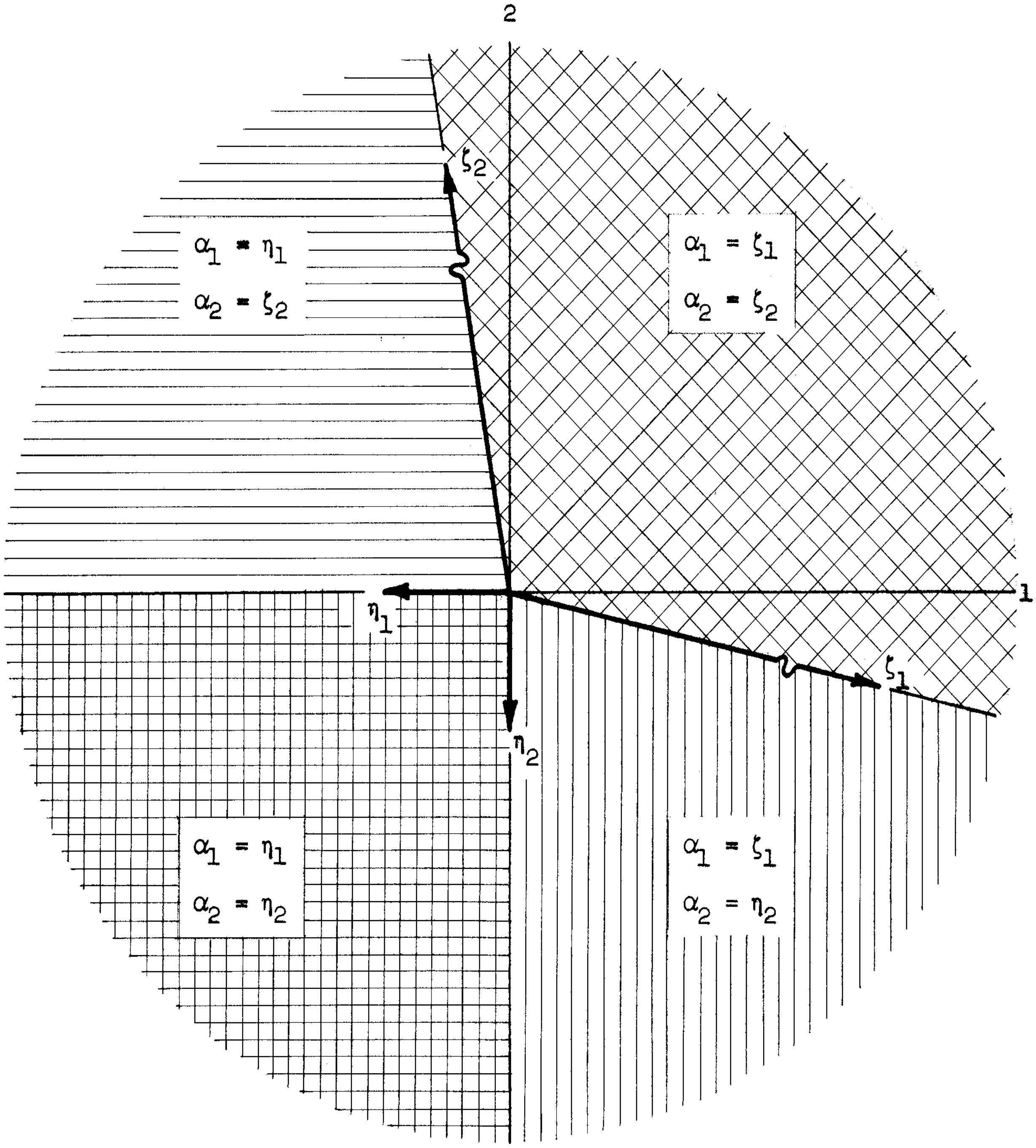


Figure 4. Partitioning Vectors.

any vector $\{a\}$ in the entire plane can be formed by linear combination of one and only one set of α_i's with nonnegative coefficients.

By virtue of a recent theorem due to Samelson, Thrall and Wesler,(7) necessary and sufficient conditions for the columns of ξ and $-\delta$ to partition the space are that every principal minor of the matrix ξ be positive.

Consider now the frame with real hinges at one or more locations, in the equilibrium configuration. This is the frame used for the "corrector" solutions, above. Let the hinged joints in this frame undergo hinge rotations φ, with lateral deflection of the floors prevented. The resulting strain energy in the frame is

$$E = \frac{1}{2} \sum_k \varphi_k M_k \quad . \tag{27}$$

The moments are

$$M_k = \sum_\ell \nu_{k\ell} \varphi_\ell \quad . \tag{28}$$

Combining Equations (27) and (28), one gets

$$E = \frac{1}{2} \sum_k \sum_\ell \nu_{k\ell} \varphi_k \varphi_\ell \tag{29}$$

Because Equation (29) is a strain energy equation, it is a positive definite quadratic form. The principal minors of ν are therefore positive. Finally, since ξ was obtained from ν simply by changing the signs of the off-diagonal elements of certain rows <u>and the corresponding columns</u> (see Equations (23)), the principal minors of ξ are positive. It follows from the Samelson-Thrall-Wesler theorem that a unique solution exists.

The solution can be found by the following iterative scheme. Let z^p be the p-th approximation to the vector z, with all elements of z^p nonnegative. (Zero is a convenient and satisfactory first approximation.)

For convenience in notation, define

$$\xi_{k0} = \xi_{kn+1} = z_0 = z_{n+1} = 0$$

Starting with k = 1, let

$$u_k^{p+1} = a_k - \sum_{\ell=0}^{k-1} \xi_{k\ell} z_\ell^{p+1} - \sum_{\ell=k+1}^{n+1} \xi_{k\ell} z_\ell^p \tag{30}$$

Then,

$$\text{if} \quad u_k^{p+1} > 0, \text{ let } \quad z_k^{p+1} = \frac{u_k^{p+1}}{\xi_{kk}} \quad , \tag{31}$$

$$\text{and if} \quad u_k^{p+1} \leq 0, \text{ let } \quad z_k^{p+1} = 0 \quad .$$

Repeat this for k = 2, k = 3,...k = n. Then let

$$y_k^{p+1} = \sum_{\ell=1}^{n} \xi_{k\ell} z_\ell^{p+1} - a_k \qquad (k = 1,2,..n). \tag{32}$$

The (p+1)st approximation to the vectors y and z is then complete. It satisfies Equation (24) and the constraint $z_k \geq 0$, but may fail to satisfy the other constraints in (25). The process is repeated until the amount by which the approximation fails to satisfy all the constraints is insignificant.

Once the vector z is determined (and therefore the $\mathring{\phi}$'s) the rates of change of the restoring forces are found by the equation

$$\mathring{Q}_i = \sum_j k_{ij} \mathring{x}_j + \sum_\ell b_{i\ell} \mathring{\phi}_\ell \quad . \tag{33}$$

The foregoing procedure is, in essence, an adaptation of the Gauss-Seidel iterative method of solving linear equations. Equation (24) and Constraints (25) reduce finally to a system of linear equations, since at least one of z_k, y_k must be zero for each value of k. The nonzero elements form a system of n (or fewer) linear equations in the same number of unknowns. In this application, the Gauss-Seidel procedure is used to set up the equations as well as to solve them. The procedure lends itself well to machine computation, and it turns out that convergence is quite rapid.

Using this procedure, at each step one must perform three matrix multiplications and solve a system of equations equal in number to the number of plastic hinges. An obvious alternate is moment distribution, whereby at each step one must solve a set of equations equal in number to twice the number of members in the structure.

In developing the above procedure, rates of change of deflection, hinge rotation, and moment have been used. As soon as one introduces finite time intervals in place of differentials, one loses the claim to uniqueness, since it cannot be claimed that reversals in strain do not occur within the time interval. However, by taking the intervals sufficiently small and considering the strain rates as everywhere monotonic within any one time interval, the exact, unique solution of the differential system can be approximated to any desired degree of accuracy by the finite difference system.

To set up the finite difference system, let the subscripts s and s+1 denote a condition at the end of the s^{th} and $s+1^{st}$ time step; e.g., $x_{i,s+1}$ is the value of the variable x_i at the end of the $s+1^{st}$ time step. Also, let

$$\begin{aligned} \Delta x_i &= x_{i,s+1} - x_{i,s} , \\ \Delta Q_i &= Q_{i,s+1} - Q_{i,s} , \\ \Delta M_i &= M_{k,s+1} - M_{k,s} , \text{ and} \\ \Delta \varphi_k &= \varphi_{k,s+1} - \varphi_{k,s} . \end{aligned} \tag{34}$$

Then the finite difference equations are

$$\Delta Q_i = \sum_j k_{ij} \Delta x_j + \sum_\ell b_{i\ell} \Delta \varphi_\ell \tag{35}$$

and

$$\Delta M_k = \sum_j \mu_{kj} \Delta x_j + \sum_\ell \nu_{k\ell} \Delta \varphi_\ell \quad , \tag{36}$$

which correspond to Equations (33) and (22).

The elasto-plastic constraints are:

1. If $0 \leq M_{k,s} \leq P_k$,

   then either $\Delta M_k = P_k - M_{k,s}$ and $\Delta\varphi_k \leq 0$ (37)

   or $-P_k - M_{k,s} < \Delta M_k < P_k - M_{k,s}$ and $\Delta\varphi_k = 0$

and

2. If $-P \leq M_{k,s} \leq 0$,

   then either $\Delta M_k = -P_k - M_{k,s}$ and $\Delta\varphi_k \geq 0$ (38)

   or $-P_k - M_{k,s} < \Delta M_k < P_k - M_{k,s}$ and $\Delta\varphi_k = 0$.

In the finite difference system, by taking Δx_j sufficiently large, one can construct mathematically a situation for which no solution exists. For example, if, for some value of k, $M_{k,s}$ were negative, one could choose the Δx_j so that $\sum_j \mu_{kj} \Delta_j$ would be a very large positive quantity, so large that a solution of Equation (36) would require either that ΔM_k exceed $P_k - M_{k,s}$ or that $\Delta\varphi_k$ be negative, both of which are in violation of Constraints (38) for negative $M_{k,s}$. The occurrence of such a situation in the course of an actual numerical integration would mean that very large changes in moment were taking place within a single time step--a condition that would rapidly destroy the significance of the numerical results.

Should such a situation arise, it would simply mean that the time interval was taken too large in the first place.

ADAPTATION FOR THE COMPUTER

In using the computer, both storage requirements and computation time can be reduced by expressing the moments in nondimensional form. To do this, one can define

$$\begin{aligned} \rho_k &= \frac{M_k}{P_k} \\ \psi_k &= \frac{\nu_{kk}\varphi_k}{P_k} \\ \beta_{i\ell} &= \frac{b_{i\ell}P_\ell}{\nu_{\ell\ell}} \\ \lambda_{kj} &= \frac{\mu_{kj}}{P_k} \\ \sigma_{k\ell} &= \frac{\nu_{k\ell}}{\nu_{\ell\ell}}\frac{P_\ell}{P_k} \end{aligned} \tag{39}$$

These transformations do not affect the uniqueness of the solution. In this notation, Equations (35) and (36) become

$$\Delta Q_i = \sum_j k_{ij}\Delta x_j + \sum_\ell \beta_{i\ell}\Delta\psi_\ell \tag{40}$$

$$\Delta\rho_k = \sum_j \lambda_{kj}\Delta x_j + \sum_\ell \sigma_{k\ell}\Delta\psi_\ell \tag{41}$$

and the elasto-plastic constraints (37) and (38) become

1. If $0 \leq \rho_{k,s} \leq 1$, then
either $\Delta\rho_k = (1 - \rho_{k,s})$ and $\Delta\psi_k \leq 0$ (42)
or $(-1 - \rho_{k,s}) < \Delta\rho_k < (1 - \rho_{k,s})$ and $\Delta\psi_k = 0$

2. If $-1 \leq \rho_{k,s} \leq 0$, then
either $\Delta\rho_k = (-1 - \rho_{k,s})$ and $\Delta\psi_k \geq 0$ (43)
or $(-1 - \rho_{k,s}) < \Delta\rho_k < (1 - \rho_{k,s})$ and $\Delta\psi_k = 0$.

The iterative procedure of Equations (30), (31) and (32) is readily adapted to this formulation of the problem. Let

r = number of moment locations (the order of the matrix σ)

and

$\Delta\psi_k^p$ = p-th approximation to $\Delta\psi_k$.

Define

$$\Delta\psi_k^0 = 0 \text{ for all } k ,$$
$$\sigma_{k0} = \sigma_{kr+1} = 0 \text{ for all } k ,$$
$$\Delta\psi_0^p = \Delta\psi_{r+1}^p = 0 \text{ for all } p .$$

Then, starting with $k = 1$ and $p = 1$, compute

$$u_k^p = \sum_j \lambda_{kj}\Delta x_j + \sum_{\ell=0}^{k-1} \sigma_{k\ell}\Delta\psi_\ell^p + \sum_{\ell=k+1}^{r+1} \sigma_{k\ell}\Delta\psi_\ell^{p-1} \tag{44}$$

If $(\rho_{k,s} + u_k{}^p) > 1$, let $\Delta\psi_k^p = (1 - \rho_{k,s} - u_k{}^p)$;

if $-1 \leq (\rho_{k,s} + u_k{}^p) \leq 1$, let $\Delta\psi_k^p = 0$; and (45)

if $(\rho_{k,s} + u_k{}^p) < -1$, let $\Delta\psi_k^p = (-1 - \rho_{k,s} - u_k{}^p)$.

Repeat this for $k = 2$, $k = 3, \ldots k = r$. This completes the p-th approximation. Then let

$$e^p = \sum_{k=1}^{r} |\Delta\psi_k^p - \Delta\psi_k^{p-1}| \quad , \tag{46}$$

which is an indicator of the amount by which the solution fails to satisfy the constraints (42) and (43). If e^p is larger than some predetermined error test δ, one returns to k=1 and computes the p+1 st approximation. When the convergence test is satisfied, one accepts the last approximation as the correct solution for the $\Delta\psi_k$ and computes ΔQ_i and $\Delta\rho_k$ from Equations (40) and (41

NUMERICAL EXAMPLE

The following numerical example illustrates the foregoing processes. Consider the undamped bent shown in Figure 5. Taking f_y = 36,000 psi, one gets for plastic hinge moments

$$P_1 = P_2 = P_3 = 1398 \text{ in-kips}$$

$$P_4 = 2072 \text{ in-kips}$$

$$P_5 = 1583 \text{ in-kips} \quad .$$

Because the bent is symmetric and the effects of axial forces on the member stiffnesses are ignored, the stress distribution will also be symmetric. Hence only half the moments need be recorded. Moreover, the moment at location 5 cannot reach plasticity and therefore does not need to be recorded.

The matrices of influence coefficients k_{ij}, $b_{i\ell}$, μ_{kj}, and $\nu_{k\ell}$ which describe the elasto-plastic properties of the system are the forces and moments shown in Figures 6 and 7. The effect of eccentricity of the dead load on the moments may be taken into account if desired. It is neglected in this example. The complete matrices are

$$k_{ij} = \begin{bmatrix} 35.73 & -24.11 \\ -24.11 & 20.20 \end{bmatrix} \text{kip in}^{-1}$$

$$b_{i\ell} = \begin{bmatrix} 1425 & 2019 & -1674 & -345 \\ -1325 & -1584 & 563 & 1021 \end{bmatrix} \text{kips/radian}$$

$$\mu_{kj} = \begin{bmatrix} 726.1 & -662.6 \\ 1009.6 & -791.8 \\ -836.9 & 281.3 \\ -172.7 & 510.5 \end{bmatrix} \quad \text{in-kips/inch}$$

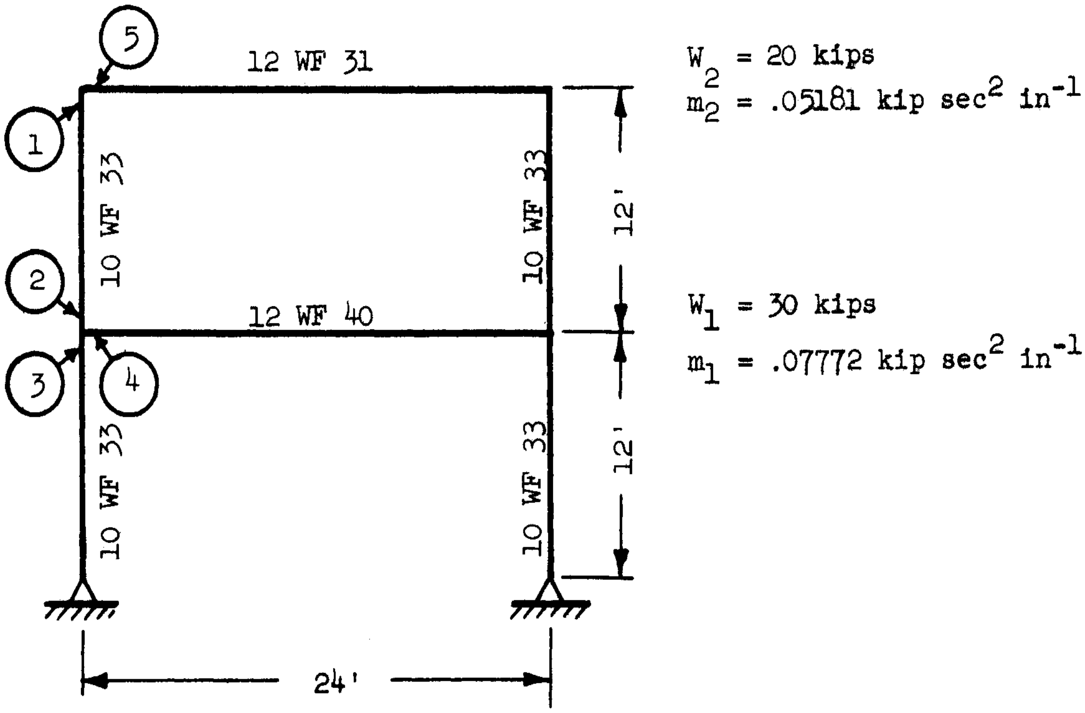


Figure 5. Bent for Numerical Example.

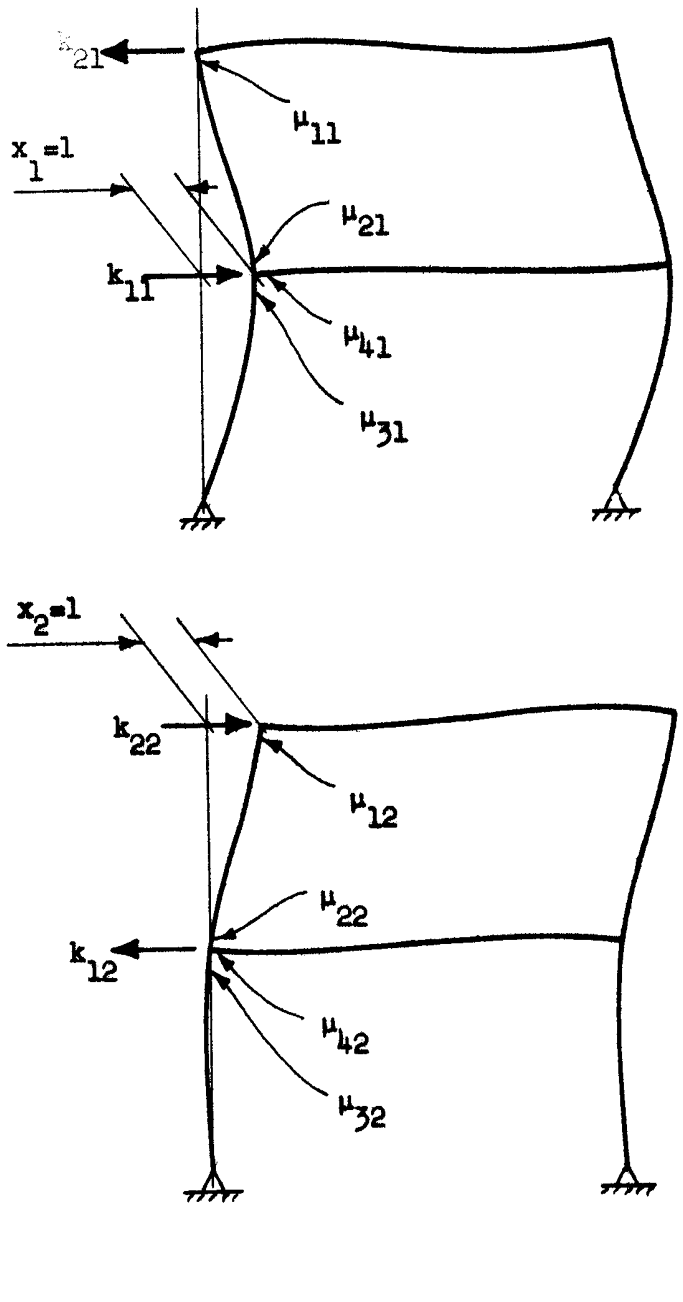


Figure 6. Matrix Elements k_{ij} and μ_{kj}

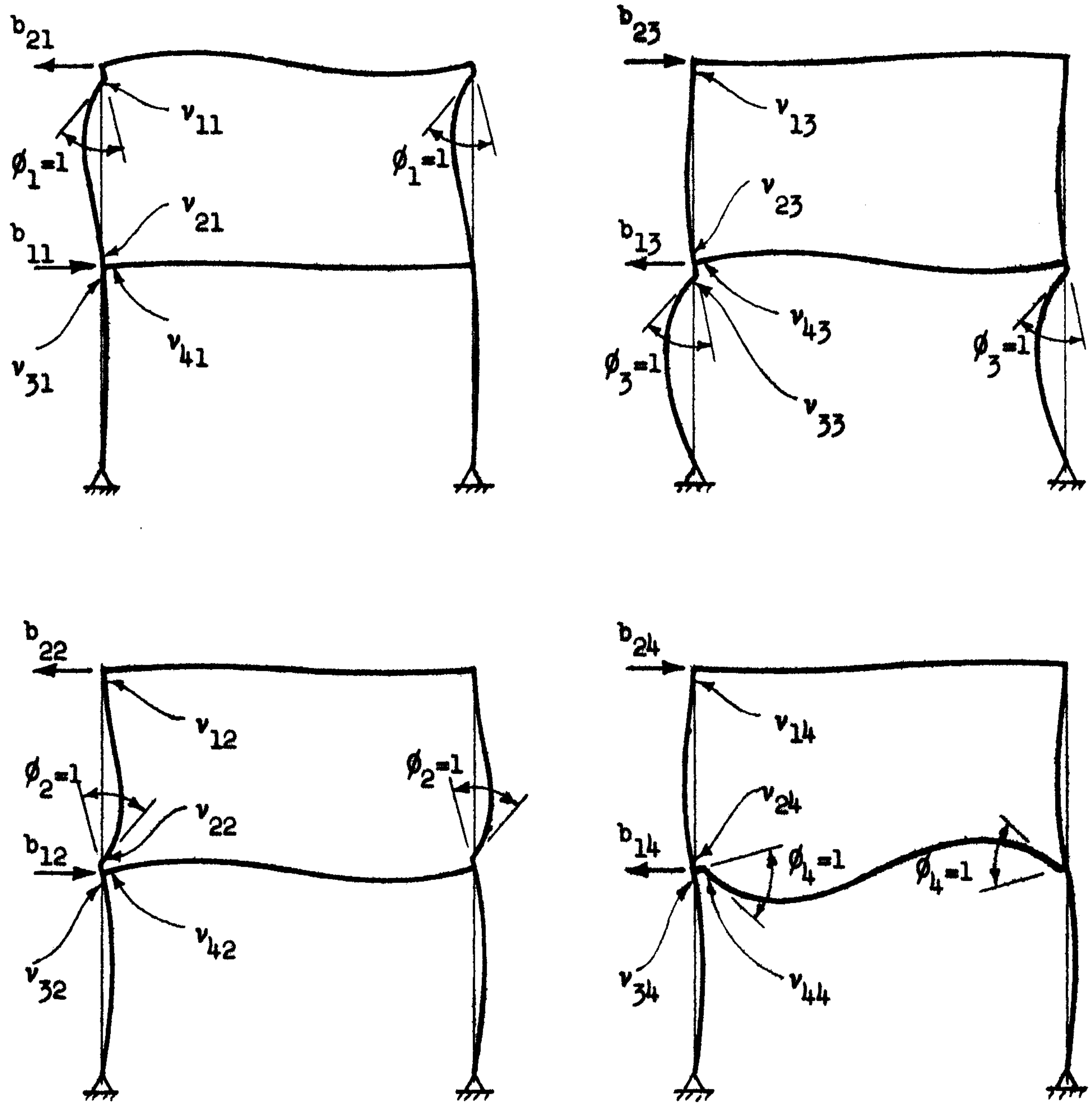


Figure 7. Matrix Elements $b_{i\ell}$ and $v_{k\ell}$

$$\nu_{k\ell} = \begin{bmatrix} 69,700 & 25,720 & -9,140 & -16,580 \\ 25,720 & 88,300 & -31,370 & -56,930 \\ -9,140 & -31,370 & 80,010 & -48,640 \\ -16,580 & -56,930 & -48,640 & 105,560 \end{bmatrix} \text{ in-kips/radian}$$

Converting the moments to nondimensional form according to Equations (39), one obtains

$$\beta_{i\ell} = \begin{bmatrix} 29.12 & 31.97 & -29.25 & -6.78 \\ -26.58 & -25.08 & 9.83 & 20.04 \end{bmatrix} \text{ kip}$$

$$\lambda_{kj} = \begin{bmatrix} .5194 & -.4740 \\ .7222 & -.5664 \\ -.5987 & .2012 \\ -.0833 & .2464 \end{bmatrix} \text{ in}^{-1}$$

$$\sigma_{k\ell} = \begin{bmatrix} 1.0000 & .2913 & -.1142 & -.2328 \\ .3690 & 1.0000 & -.3921 & -.7993 \\ -.1311 & -.3553 & 1.0000 & -.6829 \\ -.1605 & -.4350 & -.4102 & 1.0000 \end{bmatrix}$$

Suppose that at time t_0 the system is in the configuration

$$x_1 = 2.600 \text{ in} \qquad \dot{x}_1 = 10.00 \text{ in/sec.}$$

$$x_2 = 4.900 \text{ in} \qquad \dot{x}_2 = 25.00 \text{ in/sec.}$$

and that plasticity has not yet been encountered at any location. Suppose also that the driving forces at time t_0 are

$$f_1 = f_2 = 10.00 \text{ kips,}$$

and that these forces decay linearly to zero at time $t_0 + 0.1$ second. The complete process for determing inelastic behavior and executing one time step of Runge-Kutta integration is given below.

The Runge-Kutta constants for this example are those given in Equation (13), namely,

$$\ell = 1/4 \qquad p = 1/3$$
$$m = 0 \qquad q = 2/3$$
$$n = 3/4 \qquad r = 2/3 \quad ,$$

and a time interval of h = 0.02 sec. is used. These initial conditions and time interval have been chosen for illustrative purposes, and are not necessarily typical or optimum.

Because plasticity has not yet been reached, the restoring forces at time t_0 are given by Equation (17),

$$Q_i = \sum_j k_{ij} x_j = \begin{bmatrix} 35.73 & -24.11 \\ -24.11 & 20.20 \end{bmatrix} \begin{bmatrix} 2.600 \\ 4.900 \end{bmatrix}$$

or

$$Q_1 = -25.24 \text{ kips}$$
$$Q_2 = 36.29 \text{ kips}$$

and the nondimensional moments at time t_0 are

$$\rho_k = \sum_j \lambda_{kj} x_j = \begin{bmatrix} .5194 & -.4740 \\ .7222 & -.5664 \\ -.5987 & .2012 \\ -.0833 & .2464 \end{bmatrix} \begin{bmatrix} 2.600 \\ 4.900 \end{bmatrix}$$

or
$$\rho_1 = -.9722$$
$$\rho_2 = -.8976$$
$$\rho_3 = -.5707$$
$$\rho_4 = .9908$$

From Equations (15) and the initial conditions,

$$\dot{x}_1\ (t_0) = 10.00 \text{ in/sec}$$
$$\dot{x}_2\ (t_0) = 25.00 \text{ in/sec}$$
$$\dot{z}_1\ (t_0) = \frac{1}{.07772}(10 + 25.24) = +453.4 \text{ in/sec}^2$$
$$\dot{z}_2\ (t_0) = \frac{1}{.05181}(10 - 36.29) = -507.4 \text{ in/sec}^2$$

The first of Equations (10) yields

$$\kappa^x_{10} = h\ \dot{x}_1\ (t_0) = (.02)(10.00) = 0.200 \text{ in}$$
$$\kappa^x_{20} = h\ \dot{x}_2\ (t_0) = (.02)(25.00) = 0.500 \text{ in}$$
$$\kappa^z_{10} = h\ \dot{z}_1\ (t_0) = (.02)(453.4) = 9.068 \text{ in/sec}$$
$$\kappa^z_{20} = h\ \dot{z}_2\ (t_0) = (.02)(-507.4) = -10.148 \text{ in/sec.}$$

The superscripts x and z identify the variable associated with κ. Before proceeding to the second of Equations (10), one must evaluate the functions $Q(x_i + \frac{1}{3}\kappa^x_{i0})$, which in turn require the evaluation of the $\Delta\psi$'s. The first terms in Equations (44) are

$$\sum_j \lambda_{kj}\Delta x_j = \sum_j \lambda_{kj}(\tfrac{1}{3}\kappa_{j0})$$

$$= \frac{1}{3}\begin{bmatrix} .5194 & -.4740 \\ .7222 & -.5664 \\ -.5987 & .2012 \\ -.0833 & .2464 \end{bmatrix}\begin{bmatrix} .200 \\ .500 \end{bmatrix} = \begin{bmatrix} -.0444 \\ -.0462 \\ -.0064 \\ .0355 \end{bmatrix}$$

The iteration of Equations (44) and (45) yields, for p = 1,

$$u_1^1 = -.0444$$

$$\rho_1 + u_1^1 = -.9722 - .0444 = -1.0166$$

$$\Delta\psi_1^1 = +.0166$$

$$u_2^1 = -.0462 + (.3690)(.0166) = -.0401$$

$$\rho_2 + u_2^1 = -.8976 - .0401 = -.9377$$

$$\Delta\psi_2^1 = 0$$

$$u_3^1 = -.0064 + (-.1311)(.0166) = -.0086$$

$$\rho_3 + u_3^1 = -.5707 - .0086 = -.5793$$

$$\Delta\psi_3^1 = 0$$

$$u_4^1 = +.0355 + (-.1605)(.0166) = +.0328$$

$$\rho_4 + u_4^1 = .9908 + .0328 = 1.0236$$

$$\Delta\psi_4^1 = -.0236$$

From Equation (46),

$$e^1 = \sum_i |\Delta\psi_i^1 - \Delta\psi_i^0| = .0402 \quad ,$$

and the process must be repeated. The second cycle gives

$$\Delta\psi_1^2 = .0111$$

$$\Delta\psi_2^2 = 0$$

$$\Delta\psi_3^2 = 0$$

$$\Delta\psi_4^2 = -.0245$$

and $$e^2 = .0064 \quad .$$

As the process is repeated, e^p approaches zero and the iteration converges to

$$\Delta\psi_1 = .0109 \qquad \rho_1 + \Delta\rho_1 = -1.0000$$
$$\Delta\psi_2 = 0 \qquad \rho_2 + \Delta\rho_2 = -.9201$$
$$\Delta\psi_3 = 0 \qquad \rho_3 + \Delta\rho_3 = -.5617$$
$$\Delta\psi_4 = -.0246 \qquad \rho_4 + \Delta\rho_4 = 1.0000$$

Equation (40) then yields

$$\Delta Q_i = \sum_j k_{ij}(\frac{1}{3}\kappa^x_{j0}) + \sum_\ell \beta_{i\ell}\Delta\psi_\ell$$

or

$$\Delta Q_1 = -1.636 + 0.484 = -1.15 \text{ kips}$$
$$\Delta Q_2 = 1.759 - 0.783 = 0.98 \text{ kips}$$

and

$$Q_1 + \Delta Q_1 = -26.39 \text{ kips}$$
$$Q_2 + \Delta Q_2 = 37.27 \text{ kips} .$$

One can now return to the second of Equations (10) and compute

$$\kappa^x_{i1} = h\left[z_i + \frac{1}{3}\kappa^z_{i0}\right] \quad \text{and}$$

$$\kappa^z_{i1} = \frac{h}{m_i}\left[f_i\,(t_0 + \frac{h}{3}) - Q_i\,(x_j + \frac{1}{3}\kappa^x_{j0})\right] .$$

The results are

$$\kappa^x_{11} = (.02)(10.00 + \frac{9.068}{3}) = 0.260 \text{ in}$$

$$\kappa^x_{21} = (.02)(25.00 - \frac{10.148}{3}) = 0.432 \text{ in}$$

$$\kappa^z_{11} = (\frac{.02}{.07772})\left[(10.00 - \frac{1}{3} \times 2.00) + 26.39\right] = 9.193 \text{ in/sec}$$

$$\kappa^z_{21} = (\frac{.02}{.05181})\left[(10.00 - \frac{1}{3} \times 2.00) - 37.27\right] = 10.784 \text{ in/sec.}$$

Before going to the last of Equations (10) one must evaluate $Q(x_i + \frac{2}{3}\kappa^x_{i1})$. Using Equations (40), (41), (44), (45), and (46) again, just as illustrated earlier, one gets

$$\Delta\psi_1 = .0080 \qquad \rho_1 + \Delta\rho_1 = -1.0000$$

$$\Delta\psi_2 = 0 \qquad \rho_2 + \Delta\rho_2 = -.8958$$

$$\Delta\psi_3 = 0 \qquad \rho_3 + \Delta\rho_3 = -.5861$$

$$\Delta\psi_4 = -.0460 \qquad \rho_4 + \Delta\rho_4 = 1.0000$$

$$\Delta Q_1 = -0.21 \text{ kips} \qquad Q_1 + \Delta Q_1 = -25.45 \text{ kips}$$

$$\Delta Q_2 = 0.50 \text{ kips} \qquad Q_2 + \Delta Q_2 = 36.79 \text{ kips}$$

The last of Equations (10) is now executed.

$$\kappa^x_{12} = (.02)(10.00 + \frac{2}{3} \times 9.193) = 0.323 \text{ in}$$

$$\kappa^x_{22} = (.02)(25.00 - \frac{2}{3} \times 10.784) = 0.356 \text{ in}$$

$$\kappa^z_{12} = (\frac{.02}{.07772}) \left[(10.00 - \frac{2}{3} \times 2.00) + 25.45\right] = 8.779 \text{ in/sec}$$

$$\kappa^z_{22} = (\frac{.02}{.05181}) \left[(10.00 - \frac{2}{3} \times 2.00) - 36.79\right] = -10.856 \text{ in/sec.}$$

Equation (11) now yields the displacements and velocities for the end of the time step.

$$x_1 (t_0+h) = x_1 (t_0) + \frac{1}{4}\kappa^x_{10} + \frac{3}{4}\kappa^x_{12} = 2.892 \text{ in}$$

$$x_2 (t_0+h) = 5.292 \text{ in}$$

$$\dot{x}_1 (t_0+h) = z_1(t_0) + \frac{1}{4}\kappa^z_{10} + \frac{3}{4}\kappa^z_{12} = 18.85 \text{ in/sec}$$

$$\dot{x}_2 (t_0+h) = 14.32 \text{ in/sec} .$$

The terminal values of Q and ρ for this time step must now be computed from Equations (40), (41), (44), (45), and (46), as before. The results are

$$\Delta\psi_1 = 0 \qquad \rho_1\ (t_0+h) = -.9916$$

$$\Delta\psi_2 = 0 \qquad \rho_2\ (t_0+h) = -.8584$$

$$\Delta\psi_3 = 0 \qquad \rho_3\ (t_0+h) = -.6236$$

$$\Delta\psi_4 = -.0630 \qquad \rho_4\ (t_0+h) = 1.0000$$

$$Q_1\ (t_0+h) = -23.83 \text{ kips}$$

$$Q_2\ (t_0+h) = 35.91 \text{ kips} .$$

One now advances to the next time step, taking these terminal values as the initial values for the next step.

APPLICATION

The multi-degree elasto-plastic response problem has been programmed for solution on an IBM Type 650 computer at the University of Michigan. Both the Milne Predictor-Corrector Method and the Runge-Kutta Method have been used. The programs were written for the basic IBM 650, without index registers, automatic address modification, or floating point hardware. If the programs were to be rewritten today, they could be simplified a great deal by taking advantage of these improvements.

The 2,000-word memory was able to accommodate a system with as many as 16 degrees of freedom and 16 plastic hinges using the Milne method; and with the Runge-Kutta method the system could go to 12 degrees of freedom and 16 plastic hinges. The more severe limitation for Runge-Kutta occurred partly because viscous damping and a form of inelastic deformation not considered in this paper were taken into account in that program.

The four matrices, k_{ij}, $\beta_{i\ell}$, λ_{kj}, and $\sigma_{k\ell}$, were stored in the memory initially as part of the data for the main program. A separate program was written to evaluate all of these matrices. Input for the matrix evaluation program comprised the number of stories, number of bays, story weights, member stiffnesses, plastic hinge locations, and plastic hinge moments. Acting on this information, the machine wrote the slope deflection equations, solved them by iteration, evaluated the matrix elements, and printed the matrices on load cards ready for use with the main program.

For the matrices $\beta_{i\ell}$, λ_{kj} and $\sigma_{k\ell}$, one need consider only those locations at which plastic hinges form during the response. There is no need to calculate or record the moments at locations that remain elastic.

Thus the limitation of 16 plastic hinges is not as severe as it might seem. Intuition and experience with smaller systems may enable one to predict which locations are most likely to become plastic during the response. The unlikely locations can be omitted from the matrices. One can process the output data after the run is complete to check whether or not all of the unrecorded locations did, in fact, remain elastic.

The programs were applied to the problems of structural response to blast and earthquake. In each case the driving forces were put in punched cards which were read by the computer as the solution progressed. For the blast problem, the equations governing the dynamic pressures were given to the machine in a separate program and the machine calculated the driving forces and converted them to punched cards ready for unput for the main program. For the earthquake problem, recorded accelerograms were approximated by piece-wise linear functions, and the time-acceleration coordinates of the intersection points of successive line segments were put in punched card form. The machine accepted this input data and interpolated where necessary to find the desired driving forces.

CONCLUSIONS

Experience with both the blast and earthquake problems shows that the methods of analysis presented herein are practical and can be carried out on equipment that exists today. Tomorrow's equipment, or even a late version of todays equipment, greatly facilitates programming and permits many of the restrictions to be relaxed.

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NOTATION

Symbols are defined where they first appear in the text. Those used frequently are summarized here for reference.

i, j	subscripts associated with mass or force
k, ℓ	subscripts associated with bending moment or plastic hinge
h	time interval in numerical integration
ℓ, m, n, p, q, r	constants in Runge-Kutta method of integration
f_i	driving force
m_i	mass
x_i	displacement
D_i	damping force
M_k	bending moment
P_k	plastic hinge moment
Q_i	restoring force
R_i	resistance force
φ_k	plastic hinge rotation
ρ_k	dimensionless bending moment
ψ_k	plastic hinge rotation (with dimensionless bending moment)
$b_{i\ell}$	rotation-force coefficient
c_{ij}	damping coefficient
k_{ij}	stiffness coefficient
$\beta_{i\ell}$	rotation-force coefficient (with dimensionless moments)
$\delta_{k\ell}$	Kronecker delta
λ_{kj}	displacement - moment coefficient (dimensionless moments)

μ_{kj}	displacement - moment coefficient
$\nu_{k\ell}$	rotation - moment coefficient
$\sigma_{k\ell}$	rotation - moment coefficient (dimensionless moments)

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