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Dynamic Asset Allocation under Inflation

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Abstract

We develop a simple framework for analyzing a finite-horizon investor's asset allocation problem under inflation when only nominal assets are available. The investor's optimal investment strategy and indirect utility are given in simple closed form. Hedge demands depend on the investor's horizon and risk aversion and on the maturities of the bonds included in the portfolio. When short positions are precluded, the optimal strategy consists of investments in cash, equity, and a single nominal bond with optimally chosen maturity. Both the optimal stock-bond mix and the optimal bond maturity depend on the investor's horizon and risk aversion.

An investor who has a long-term but finite horizon and can invest only in nominal bonds or stocks faces a basic problem. Is it better to purchase a zero coupon bond corresponding to the horizon and bear the inflation risk, to follow a policy of rolling over short-term bonds, or to adopt some quite different strategy? Despite the simplicity of this issue, there is still no well-accepted framework for analyzing it because nominal long-term bonds have two important characteristics that cannot be represented adequately within the classical static, single-period, framework introduced by Markowitz (1959). First, the prices of bonds decline as interest rates rise so that, as Merton (1973) originally pointed out, long-term bonds can provide a hedge against adverse shifts in the investor's future investment opportunity set. Secondly, and somewhat weakening the hedging role of long-term bonds, is the sensitivity of their prices to changes in expectations about future inflation. Therefore, a satisfactory counterpart to classical static portfolio theory that would enable us to address the problem faced by the hypothetical long-term investor must satisfy two criteria. It must yield simple closed form expressions for optimal portfolios for investors with different horizons and attitudes towards risk, and it must deal realistically with both the price and return characteristics of long-term bonds, as well as with inflation. In this paper, we develop a simple model that satisfies these criteria. The investor's optimal portfolio is shown to be a sum of two components: first, the mean-variance tangency portfolio and, second, a portfolio that mimics as closely as possible a hypothetical indexed bond with maturity equal to the investment horizon.

The analysis of optimal portfolio strategies for long-lived investors starts with Latane and Tuttle (1967), Mossin (1968), Hakansson (1970), Merton (1969), and Samuelson (1969). These authors were primarily concerned with analyzing the investor's optimal allocation between stock and cash when returns are *i.i.d.* Merton (1971) was the first to consider the effect of a stochastic

investment opportunity set, as well as to demonstrate that this creates a set of “hedge” demands in addition to the standard myopic demand. However, further work on optimal strategies under stochastic investment opportunities languished until revived by empirical work demonstrating apparent asset return predictability in the 1990’s. Brennan *et. al.* (1997) analyzed numerically the portfolio problem of a long-lived investor who can invest in bonds, stock, or cash, when there is stochastic variation in the interest rate, and the equity premium is predictable by the interest rate and the dividend yield. Kim and Omberg (1996) and Wachter (1999) have analyzed the optimal strategy of an investor when the interest rate is constant but the equity premium follows an Ornstein-Uhlenbeck process.¹ Sorensen (2000), Brennan and Xia (1999), and Omberg (1999) also compute optimal dynamic strategies when the interest rate follows a Vasicek (1977) process² and risk premia are constant. Liu (1999), who provides a general treatment of models in which the investment opportunity set has an affine characterization, studies the optimal cash-bond allocation in a model with an affine term structure. Barberis (2000), Kandel and Stambaugh (1996) and Xia (2000) have considered the implications of *uncertain* predictability of asset returns.

None of the above papers allows for stochastic inflation, takes account of borrowing and short sales constraints, or provides a rationale for the bond maturity choice. In this paper, we analyze the portfolio problem of a finite-lived investor³ who can invest in stock or nominal bonds, when the interest rate and the expected rate of inflation follow correlated Ornstein-Uhlenbeck processes and the risk premia are constant.⁴

The composition of the optimal bond portfolio is not determinate within the model when there are no constraints; only the optimal loadings on innovations in the estimated real interest rate, r , and expected rate of inflation, π , can be determined. Calibration of the model to data on U.S.

interest rates, stock returns, and inflation yields mixed results. When the calibration is made to monthly data on bond yields and inflation, strong mean reversion is found for the shadow real interest rate. This makes the optimal portfolio holdings relatively insensitive to the investment horizon beyond five years, and the gains from following a fully dynamic strategy relatively small, except for high levels of risk aversion. When the calibration is made to annual interest rates and inflation over a long period, much less mean reversion is found. With the lower mean reversion parameter, substantial horizon effects appear in the optimal portfolio strategies, and the gains from following the optimal dynamic strategy become large. In both cases, as risk aversion increases, the investor holds less stock and the return on his bond portfolio tends to become less sensitive to innovations in the expected rate of inflation. In the limit, as risk aversion becomes infinite, the stock allocation goes to zero and the investor's dynamic strategy in bonds mimics as closely as possible the returns on an inflation indexed bond with maturity equal to the remaining investment horizon.

The foregoing results rely on the assumption that the investor is able to take unlimited short positions. When the investor is constrained to take only long positions, the optimal portfolio can be achieved with positions in only a single bond of the optimally chosen maturity, cash, and stock. The optimal bond maturity depends on both the investor's horizon and risk aversion. As in the unconstrained case, horizon effects are pronounced only for the annual data calibration. For both sets of calibrated parameters, the ratio of bonds to stock increases as the horizon increases, which is at odds with the popular view that long horizon investors should hold more stock. The bond-stock ratio also increases as risk aversion increases, which is consistent with the portfolio recommendations of popular financial advisors that have puzzled Canner, Mankiw, and Weil

(1997). The maturity of the optimal bond decreases as risk aversion increases.

There are at least two possible explanations for the differences between the monthly and the annual calibrations. The first is that the expected rate of inflation series estimated using monthly data on bond yields and price index changes is too smooth because non-price signals about the expected rate of inflation are ignored; this would cause the high frequency changes in the true market assessment of the expected rate of inflation, which are reflected in bond yields, to be impounded in our estimates of the real interest rate. The second possibility is that, whereas we use a one-factor model to describe the dynamics of the real interest rate, these may be better represented by a two-factor model in which one factor has high frequency⁵ and the other low frequency; in this case, the monthly calibration may be picking up the high frequency factor while the annual calibration may place more emphasis on the low frequency component.

In the paper that is closest to this one, Campbell and Viceira (1999) (CV) develop an approximately optimal portfolio strategy for an infinitely lived investor with recursive utility, in a discrete time setting in which the real interest rate and the expected rate of inflation follow similar stochastic processes. The major differences between their paper and this one are that: first, unlike CV, we are interested in the problem faced by a *finite horizon* investor and are therefore able to give explicit consideration to the effect of the horizon on both the optimal bond-stock mix and the maturity of the optimal bond portfolio. Secondly, we develop closed form expressions for the investor's optimal policy and indirect utility function, whereas the CV formulation relies on linear approximation and numerical analysis.⁶ Our closed form solutions enable us to analyze the welfare loss of the myopic policy and to characterize the dependence of the portfolio hedge demand on the characteristics of the investment opportunity set, such as the horizon and intensity

of mean reversion of the real interest rate. Finally, in considering the problem in which the investor is subject to short sales constraints, CV take the maturity of the investible bond as given, whereas we allow for the optimal choice of bond maturity and are therefore able to consider the effect of risk aversion and horizon on the optimal maturity. In general, the introduction of constraints on the size of positions makes the choice of the maturity of the bonds to be included in the portfolio a critical element of the portfolio decision.

We present the basic model of stochastic real interest rates, inflation, and expected stock returns in Section I. The optimal portfolio problem is derived in Section II. We calibrate the model to the U.S. postwar nominal interest rate, inflation, and stock return data in Section III. Some representative calculations and discussions are offered in Section IV. We summarize the results and conclude the paper in Section V.

I. Investment Opportunities

We assume that the investor can invest in a nominal instantaneous risk free asset, a stock, and in nominal bonds with different maturities. The real returns on the nominal bonds are risky, because both the rate of inflation and the (shadow) real interest rate are stochastic.⁷

The (commodity) price level, Π , follows a diffusion process:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_{\Pi} dz_{\Pi}, \tag{1}$$

so that the (realized) rate of inflation is locally stochastic. The instantaneous expected rate of

inflation, or proportional drift of the price level, π , follows an Ornstein-Uhlenbeck process:

$$d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi. \quad (2)$$

The investment opportunities depend on the (real) pricing kernel of the economy, M_t , which determines the expected returns on all securities:

$$\frac{dM}{M} = -r dt + \phi_S dz_S + \phi_r dz_r + \phi_\pi dz_\pi + \phi_u dz_u = -r dt + \phi' dz + \phi_u dz_u, \quad (3)$$

where $\phi = [\phi_S, \phi_r, \phi_\pi]'$ and $dz = [dz_S, dz_r, dz_\pi]'$. The expression, ϕ_i ($i = S, r, \pi, u$), representing the constant loadings on the stochastic innovations in the economy, determines the associated prices of risk, $\lambda_S, \lambda_r, \lambda_\pi$ and λ_u , which are constant because the ϕ 's are constant. The variable, dz_S , represents the increment to the Brownian motion that drives the stock return in equation (5) below, and dz_u , which is defined in equation (7) below, is proportional to the component of the inflation rate, $\frac{d\Pi}{\Pi}$, that is orthogonal to dz and therefore to the nominal returns on all assets: it drives the component of inflation that is not spanned by the asset returns and therefore cannot be hedged. We denote the drift of the pricing kernel by $-r$ because it is well known that, in an economy with a fully indexed riskless asset, the instantaneous (real) riskless interest rate is equal to (the negative of) the drift of the pricing kernel.

Following Vasicek (1977), we assume that r follows the Ornstein-Uhlenbeck process:

$$dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r. \quad (4)$$

If an instantaneously riskless *real* asset existed, then its instantaneous real rate of return would be r . However, we assume that, in this economy, no instantaneously riskless real asset exists⁸ and that the investor is able to invest only in a single stock index and in nominal bonds with different maturities. The nominal stock price is assumed to follow a Geometric Brownian motion,

$$\frac{dS}{S} = (R_f + \sigma_S \lambda_S)dt + \sigma_S dz_S, \quad (5)$$

where λ_S is the constant unit risk premium associated with the innovation, dz_S , and R_f is the nominal interest rate.

Since the Brownian increments in equation (3), dz and dz_u , are orthogonal, the pricing kernel relative can be rewritten as the product of two independent stochastic integrals:

$$\begin{aligned} M_s/M_t &= \exp \left\{ \int_t^s \left(-r(u) - \frac{1}{2} \phi' \rho \phi \right) du + \int_t^s \phi' dz \right\} \exp \left\{ \int_t^s \left(-\frac{1}{2} \phi_u^2 \right) du + \int_t^s \phi_u dz_u \right\} \\ &\equiv \zeta_1(t, s) \zeta_2(t, s), \end{aligned} \quad (6)$$

where ρ is the correlation matrix of dz_S , dz_r and dz_π with rows $[1, \rho_{Sr}, \rho_{S\pi}]$, $[\rho_{Sr}, 1, \rho_{r\pi}]$ and $[\rho_{S\pi}, \rho_{r\pi}, 1]$, and $\zeta_1(t, s)$ and $\zeta_2(t, s)$ are orthogonal.

In general, the realized rate of inflation given by equation (1) will not be perfectly correlated with the change in the expected rate of inflation given by equation (2). However, it can be shown that if the expected rate of inflation is not observable but must be inferred from observation of the price level itself (and not from any other information), then the change in (the investor's assessment of) the expected rate of inflation will be perfectly correlated with the realized rate of inflation: for reasons that will be apparent below, we call this the "complete markets case." In

the general case, the innovation in the rate of inflation can be written as a linear function of the innovations dz_S , dz_r , and dz_π and the projection residual, $\xi_u dz_u$:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi dz_\Pi = \pi dt + \xi_S dz_S + \xi_r dz_r + \xi_\pi dz_\pi + \xi_u dz_u \equiv \pi dt + \xi' dz + \xi_u dz_u. \quad (7)$$

If $\xi_S = \xi_r = \xi_u = 0$, $d\Pi/\Pi$ is perfectly correlated with $d\pi/\pi$ and the market is complete.

Equation (7) implies that the price level relative, Π_s/Π_t ($s > t$), can also be written as the product of two independent stochastic integrals:

$$\begin{aligned} \Pi_s/\Pi_t &= \exp \left\{ \int_t^s \left(\pi(u) - \frac{1}{2} \xi' \rho \xi \right) du + \int_t^s \xi' dz \right\} \exp \left\{ \int_t^s \left(-\frac{1}{2} \xi_u^2 \right) du + \int_t^s \xi_u dz_u \right\} \\ &\equiv \eta_1(t, s) \eta_2(t, s), \end{aligned} \quad (8)$$

where $\eta_1(t, s)$ and $\eta_2(t, s)$ are orthogonal so that $\sigma_\Pi^2 = \xi' \rho \xi + \xi_u^2$. $\eta_1(t, s)$ is the component of the price level change that can be hedged by investing in the available securities, while $\eta_2(t, s)$ is the unhedgeable component.

The definition of the pricing kernel implies that $P(t, T)$, the nominal price at time t of a bond which matures at time T with a nominal payoff of \$1, is given by:

$$P(t, T) = E_t \left[\frac{M_T/M_t}{\Pi_T/\Pi_t} \right] = E_t \left[\frac{\zeta_1(t, T)}{\eta_1(t, T)} \right] E_t \left[\frac{\zeta_2(t, T)}{\eta_2(t, T)} \right]. \quad (9)$$

It is shown in Appendix A that $P(t, T)$ is an exponential affine function of r_t , the (shadow) real

interest rate, and π_t , the investor's assessment of the current expected rate of inflation:

$$P(t, T) = \exp \{A(t, T) - B(t, T)r_t - C(t, T)\pi_t\}, \quad (10)$$

where $A(t, T)$, $B(t, T)$ and $C(t, T)$ are time dependent constants, expressions for which are given in Appendix A.

Using Ito's Lemma and the expressions for $A(t, T)$, $B(t, T)$, and $C(t, T)$, the stochastic process for the bond price can be written as:

$$\begin{aligned} \frac{dP}{P} &= [r + \pi - B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi - \xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi - \xi_u\lambda_u] dt \\ &- B\sigma_r dz_r - C\sigma_\pi dz_\pi, \end{aligned} \quad (11)$$

where the λ 's are market prices of risk associated with stock return, the innovations of real interest rate, and the innovations of expected and unexpected inflation. The expressions for λ 's are given in equations (A13 - A16).

There is a simple linear relation between the market price of risk vector, λ , and the factor loadings of the real pricing kernel in equation (3). Let Λ be the vector of nominal risk premiums for the stock and two nominal bonds with maturities T_1 and T_2 , then $\Lambda \equiv [\sigma_S\lambda_S, -B(t, T_1)\lambda_r - C(t, T_1)\lambda_\pi, -B(t, T_2)\lambda_r - C(t, T_2)\lambda_\pi]'$. Define the factor loadings matrix of the three securities as σ where the first row is $(\sigma_S, 0, 0)$, and the second and third rows are $(0, -B(t, T_j)\sigma_r, -C(t, T_j)\sigma_\pi)$ ($j = 1, 2$). The real risk premium on the three securities is equal to the covariance between the real return of the (nominal) security and the real pricing kernel, so the unit market price of the risk vector is related to the risk premium vector by $\Lambda = \sigma\lambda$.

Equivalently, the factor loadings of the real pricing kernel are related to the risk premium vector by $\phi = \xi - \rho^{-1}\lambda$.

The instantaneous nominal riskfree interest rate, R_f , is obtained by taking the limit of the return on the nominal bond in equation (11) by letting $(t - T) \rightarrow 0$:

$$R_f = r + \pi - \xi_S \lambda_S - \xi_r \lambda_r - \xi_\pi \lambda_\pi - \xi_u \lambda_u, \quad (12)$$

and we call $-\xi_S \lambda_S - \xi_r \lambda_r - \xi_\pi \lambda_\pi - \xi_u \lambda_u$ the risk premium for the nominal instantaneous risk-free asset. The Fisher equation does not hold in this economy unless all the market prices of risk, $\lambda_S, \lambda_r, \lambda_\pi$, and λ_u , are zero, so that the nominal risk free asset has a zero risk premium. Using the definition of the nominal risk free interest rate in equation (12), the nominal return on the nominal bond in equation (11) can also be simplified as:

$$\frac{dP}{P} = [R_f - B\sigma_r \lambda_r - C\sigma_\pi \lambda_\pi] dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi, \quad (13)$$

which shows that the nominal risk premium on a bond depends only on its exposure to innovations in the real interest rate and expected rate of inflation.

II. Optimal Portfolio Choice

Turning now to the issue of optimal portfolio strategies for long-lived investors, we shall consider two classical cases. In the first, the investor is assumed to be concerned with maximizing the expected utility of wealth on some fixed horizon date, T . This problem has the merits both of

simplicity, for it admits a closed form solution, and of clarifying the role of the horizon, for with this simple objective, there is no ambiguity about the duration of the consumption stream that is being financed. The problem can be thought of as corresponding to that faced by an individual who has set aside predetermined savings for retirement and wishes to maximize the expected utility of wealth on his retirement date;⁹ we are simplifying the full problem by ignoring the optimal investment and consumption plan during retirement. The second case that we consider is that of an investor who is concerned with maximizing the expected value of a time-additive utility function defined over lifetime consumption. This problem, which is only slightly more complicated, corresponds to the consumption-portfolio choice problem of an individual who is retired and faces a known date of death with no bequest motives.

There are four potential sources of uncertainty in the model economy that we have described: real interest rate risk represented by the innovations in r as shown in Equation (4); inflation risk due to unanticipated changes in the price level as shown in Equation (7); unanticipated changes in the expected rate of inflation as shown in Equation (2); and finally, the unanticipated stock return shown in Equation (5).

If $\xi_u = 0$, the change in the price level is an exact linear function of dz_S , dz_r , and dz_π , and one dimension of risk faced by the investor is eliminated.¹⁰ Then the market is complete if there are at least four securities whose instantaneous variance-covariance matrix has rank three.¹¹ Since the variance-covariance matrix of real returns on cash, stock, and any two finite maturity bonds with different maturities has rank three, the market is complete whenever $\xi_u = 0$. In the complete market setting, the investor's optimal portfolio problem may be solved using the martingale pricing approach of Cox and Huang (1989).

If the market is incomplete, ($\xi_u \neq 0$), ϕ_u cannot be derived from the observable security prices, but any specific value of ϕ_u determines a unique pricing kernel. If a wealth (consumption) plan is feasible — i.e., can be financed by a trading strategy in the available (nominal) securities — then the static budget constraint can be represented by the expectation of the product of payoffs and the pricing kernel defined by any given value of ϕ_u . Therefore, the original dynamic portfolio choice problem can be mapped into the static variational problem of Cox and Huang (1989) using an arbitrary value of ϕ_u .¹² We first solve the optimal terminal wealth and consumption allocation under the static budget constraint for a given ϕ_u and then verify that the optimal terminal wealth and consumption are financed by feasible trading strategies.

Although it is no longer possible to construct a portfolio whose return is riskless in real terms when the market is incomplete, the unhedgeable component of inflation is orthogonal to all available asset returns. Moreover, the homogeneity of both the utility function and the inflation process implies that the optimal proportional allocation is independent of real wealth and the price level. Therefore, the market incompleteness caused by non-hedgeable inflation has no effect on the optimal proportional portfolio allocation.

Lemma 1: *For an investor with isoelastic utility, the optimal proportional portfolio allocation is independent of real wealth, $w_t \equiv \frac{W_t}{\Pi_t}$, and the price level, Π_t .*

The Lemma allows us to characterize the investor's problem in the incomplete market setting via the extended Cox-Huang method developed by Pagés (1987), He and Pearson (1991), and Karatzas *et. al.* (1991). In order to represent the market incompleteness caused by non-hedgeable inflation, fix a probability space $(\mathcal{O}, \mathcal{F}, \mathcal{P})$, where $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$. The vector of standard Brownian motions $dz \equiv [dz_S, dz_r, dz_\pi]'$ is defined on \mathcal{O}_1 , while $\xi_u dz_u$, the component of realized inflation

that is orthogonal to all security returns and therefore non-hedgeable, is defined on \mathcal{O}_2 . The standard filtration \mathbf{F}_1 is generated by $\{dz_s : 0 \leq s \leq t\}$ and \mathbf{F}_2 by dz_u , and $\mathbf{F} \equiv \mathbf{F}_1 \times \mathbf{F}_2$. It follows from the Lemma that an optimal trading strategy is a process x , which specifies the proportion of wealth invested in each security and is adapted to \mathbf{F}_1 .

A. State Contingent Wealth and Consumption

Consider first the problem of an investor with an iso-elastic utility function who is concerned with maximizing the expected utility of real wealth at time T . The investor's optimal portfolio choice problem can be mapped into the following static variational problem:

$$\max_{W(\tau): t \leq \tau \leq T} E_t \left\{ \frac{(W_T/\Pi_T)^{1-\gamma}}{1-\gamma} \right\}, \quad (14)$$

$$\text{s.t. (1) } E_t \left[\frac{M_T}{M_t} (W_T/\Pi_T) \right] = W_t/\Pi_t \equiv w_t, \quad (15)$$

$$(2) W_T \text{ is financed by a feasible trading strategy with initial investment } W_0,$$

where equation (15) is the static budget constraint. The following theorem presents the solution of this static variational problem.

Theorem 1: optimal terminal wealth allocation

For an investor concerned with maximizing the expected value of an isoelastic utility function defined over wealth at time T in the incomplete markets setting in which $\xi_u \neq 0$:

(i) The optimal terminal real wealth allocation, $w_T^ \equiv W_T^*(M_T, \Pi_T)/\Pi_T$, is*

$$w_T^* = w_t \eta_2^{-1}(t, T) (\zeta_1(t, T))^{-\frac{1}{\gamma}} F_1(t, T)^{-1} F_2(t, T)^{-1}, \quad (16)$$

where

$$F_1(t, T) = E_t \left[\zeta_1(t, T)^{1-\frac{1}{\gamma}} \right] = \exp^{\frac{1-\gamma}{\gamma} [B(t, T)r_t + a_0(t, T)]} \quad (17)$$

$$F_2(t, T) = E_t [\zeta_2(t, T)/\eta_2(t, T)] = \exp^{(\xi_u^2 - \xi_u \phi_u)(T-t)} \quad (18)$$

$$\begin{aligned} a_0(t, T) &= \frac{\phi' \rho \phi(T-t)}{2\gamma} + \left[\bar{r} - \frac{(1-\gamma)\sigma_r \phi' \rho e_2}{\gamma \kappa} \right] [(T-t) - B(t, T)] \\ &+ \frac{(1-\gamma)\sigma_r^2}{4\gamma \kappa^3} [2\kappa(T-t) - 3 + 4e^{\kappa(t-T)} - e^{2\kappa(t-T)}]. \end{aligned} \quad (19)$$

(ii) The indirect utility function, $J(W_t, r_t, \Pi_t, t)$, is separable in real wealth, w_t , and can be written as:

$$J(W, r, \Pi, t) \equiv E_t \left\{ \frac{(w_T^*)^{1-\gamma}}{1-\gamma} \right\} = \left(\frac{(w_t)^{1-\gamma}}{1-\gamma} \right) \psi_1(r, t, T), \quad (20)$$

where w_T^* is the real wealth at period T under the optimal policy, and

(iii) $\psi_1(r, t, T)$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon:

$$\psi_1(r, t, T) = \exp^{(1-\gamma)[B(t, T)r_t + a_1(t, T)]}, \quad (21)$$

where $B(t, T) \equiv \kappa^{-1}(1 - e^{-\kappa(T-t)})$ and

$$a_1(t, T) = a_0(t, T) + \left[\phi_u \xi_u - \frac{\gamma}{2} \xi_u^2 \right] (T-t). \quad (22)$$

The complete markets problem ($\xi_u = 0$) is a special case of Theorem 1 in which $\eta_2(t, T) =$

$F_2(t, T) = 1$, so that the optimal real wealth allocation and indirect utility simplify to:

$$w^*_T(M_T, \Pi_T) = w_t (\zeta_1(t, T))^{-\frac{1}{\gamma}} F_1(t, T)^{-1}, \quad (23)$$

$$J(r, W, \Pi, t) = \left(\frac{w_t^{1-\gamma}}{1-\gamma} \right) \exp^{(1-\gamma)[B(t, T)r_t + a_0(t, T)]}. \quad (24)$$

The problem for the interim consumption case can be written as:

$$\max_{C(s): t \leq s \leq T} E_t \left\{ \int_t^T \frac{(C(s)/\Pi_s)^{1-\gamma}}{1-\gamma} ds \right\}, \quad (25)$$

$$\text{s.t. (1) } E_t \left[\int_t^T \frac{M_s C(s)}{M_t \Pi_s} ds \right] = W_t / \Pi_t \equiv w_t, \quad (26)$$

(2) C is financed by a feasible trading strategy with initial investment W_0 ,

where $C(\tau)$ is the investor's (nominal) consumption flow at time $\tau : \tau \in [t, T]$. The optimal consumption plan and indirect utility function are characterized in the following theorem:

Theorem 2: optimal allocation of lifetime consumption

For an investor concerned with maximizing the expected value of an isoelastic utility function defined over lifetime consumption in the incomplete markets setting in which $\xi_u \neq 0$:

(i) *The optimal real consumption program, $c^*(s) \equiv C^*(s)/\Pi_s$, is:*

$$c^*(s) = \eta_2^{-1}(t, s) (\zeta_1(t, s))^{-\frac{1}{\gamma}} Q_1^{-1}(t, T) w_t, \quad (27)$$

where w_t is real wealth at time t , and $Q_1(t, T)$ is a constant chosen to satisfy the budget

constraint:

$$Q_1(t, T) = \int_t^T F_1(t, s) F_2(t, s) ds = \int_t^T q(t, s) \exp \left\{ \frac{1-\gamma}{\gamma} [B(t, s)r_t + a_1(t, s)] \right\} ds, \quad (28)$$

and $q(t, s) = \exp \left\{ \left[\frac{(3-\gamma)\xi_u^u}{2} - \frac{\phi_u \xi_u}{\gamma} \right] (s-t) \right\}$.

(ii) The indirect utility function $J(W_t, r_t, \Pi_t, t)$ is separable in w_t and can be written as:

$$J(r, W, \Pi, t) \equiv E_t \left\{ \int_t^T \frac{c^*(s)^{1-\gamma}}{1-\gamma} ds \right\} = \left(\frac{w_t^{1-\gamma}}{1-\gamma} \right) \psi_2(r, t, T), \quad (29)$$

where

(iii) $\psi_2(r, t, T)$ represents the contribution to expected utility of the remaining investment opportunities up to the horizon:

$$\psi_2 = \left[\int_t^T q(t, s) \exp \frac{1-\gamma}{\gamma} [B(t, s)r_t + a_1(t, s)] ds \right]^{\gamma-1} \left[\int_t^T q^{1-\gamma}(t, s) \exp \frac{1-\gamma}{\gamma} [B(t, s)r_t + a_1(t, s)] ds \right], \quad (30)$$

where $B(t, s) = \kappa^{-1} (1 - e^{\kappa(s-t)})$ and $a_1(t, s)$ is given in (22) by replacing T with s .

The complete markets problem ($\xi_u = 0$) is again a special case of Theorem 2 in which $\eta_2(t, s) = F_2(t, s) = 1$, so that the optimal consumption and indirect utility simplify to:

$$c^*(s) = Q(t, T)^{-1} (\zeta_1(t, s))^{-\frac{1}{\gamma}} w_t \quad (31)$$

$$J(r, W, \Pi, t) = \left(\frac{w_t^{1-\gamma}}{1-\gamma} \right) Q(t, T)^\gamma \quad (32)$$

$$Q(t, T) = \int_t^T \exp \frac{1-\gamma}{\gamma} [B(t, s)r_t + a_0(t, s)] ds, \quad (33)$$

where $Q(t, T)$ is derived from $Q_1(t, T)$ by setting $\xi_u = 0$ so that $q(t, s) = 1$ and $a_1(t, T) = a_0(t, T)$.

In the complete market setting, the optimal consumption allocation $c^*(s)$ given by equation (31) is of similar form to the optimal terminal wealth allocation $w^*(T)$ in equation (23), and the problem with interim consumption can be interpreted as a summation of terminal wealth problems with the horizons s varying from t to T and initial wealth allocated to horizon s equal to $\frac{F_1(t, s)}{\int_t^T F_1(t, s) ds} w_t$.¹³

The interim consumption problem in the incomplete market setting can also be interpreted as a two-stage problem in which the investor first allocates his initial wealth across horizons s ($s \in [t, T]$), $\frac{F_1(t, s)F_2(t, s)}{\int_t^T F_1(t, s)F_2(t, s) ds} w_t$, and then carries out the terminal-wealth optimization problem for each s . The expected utility for the interim consumption problem is still the sum of the expected utilities of the individual terminal-wealth problems, although the result no longer simplifies as in the complete market case because of r_p .

B. Unconstrained Optimal Portfolio Strategies

The optimal portfolio strategy replicates the optimal terminal wealth and consumption allocations by dynamically trading the available nominal securities. Although market incompleteness affects investor utility, the optimal strategies are of the same form in the complete and incomplete markets settings, as seen in the following theorems.

Theorem 3: optimal portfolio strategy for terminal wealth problem

The vector of optimal proportional wealth allocation to the stock and two nominal bonds for

problem (14)-(15), $\mathbf{x}^* \equiv (x_s^*, x_1^*, x_2^*)'$, is given by:

$$\mathbf{x}^* = \frac{1}{\gamma} \Omega^{-1} \Lambda + \frac{(1-\gamma)B(t, T)}{\gamma} (\Omega^{-1} \sigma \rho e_2 \sigma_r) - \frac{1-\gamma}{\gamma} (\Omega^{-1} \sigma \rho \xi) \quad (34)$$

$$= \frac{1}{\gamma} \Omega^{-1} \Lambda + \left(1 - \frac{1}{\gamma}\right) \Omega^{-1} \sigma \rho (\xi_S, \xi_r - B(t, T) \sigma_r, \xi_\pi)', \quad (35)$$

where $e_2 = [0, 1, 0]'$, Λ is the (3×1) vector of risk premia of the stock and two nominal bonds, and $\Omega \equiv \sigma \rho \sigma'$ is the (3×3) variance covariance matrix of the nominal security returns. The variable $\sigma \rho e_2 \sigma_r$ denotes the vector of covariances between the security returns and the real interest rate, and the variable $\sigma \rho \xi$ represents the vector of covariances between the security returns and realized inflation.

The balance of the portfolio, $\mathbf{1} - \mathbf{x}'\mathbf{i}$, is invested in the nominal riskless asset at the rate R_f .

Theorem 3 establishes that there is a *feasible* trading strategy that finances the optimal wealth and consumption allocations derived in Theorems 1 and 2. Therefore, the optimal strategies given in Theorem 3 are indeed the solution to the original portfolio choice problem.

Equation (34) expresses the optimal portfolio as the sum of three portfolios, in a form that is familiar from Merton (1973). The first portfolio is proportional to the nominal mean-variance tangency portfolio,¹⁴ and the amount invested in it is inversely related to the investor's relative risk aversion. The second portfolio, $\Omega^{-1} \sigma \rho e_2 \sigma_r$, is the one with the largest correlation with the state variable r , and the third portfolio, $\Omega^{-1} \sigma \rho \xi$, has the highest correlation with the inflation realization.

Equation (35) expresses the optimal portfolio as the weighted sum of the nominal mean-variance tangency portfolio and the "minimum risk" portfolio. This is the portfolio that would

be held by an infinitely risk averse investor ($\gamma \rightarrow \infty$) with horizon T , and it has the highest correlation with the return on an indexed bond, with a maturity equal to the remaining horizon: the return on this indexed bond has a real interest rate sensitivity of $B(t, T)$ and no sensitivity to expected or unexpected inflation.

While the optimal portfolio allocation depends on the maturities of the two bonds that are chosen for the portfolio, the return characteristics of the portfolio are completely described by its loadings on the stock return and the innovations in r and π , and these are determined completely by the investor's horizon and risk aversion. They are characterized in the following Proposition.

Proposition 1: optimal factor loadings for nominal portfolio returns

For an investor with risk aversion γ and horizon T , the optimal stock allocation and loadings on the innovations in the real interest rate and expected inflation are

$$x_S(\gamma) = \frac{1}{\gamma} \frac{\xi_S - \phi_S}{\sigma_S} + \left(1 - \frac{1}{\gamma}\right) x_S(\infty), \quad (36)$$

$$B_p(\gamma, T) = \frac{1}{\gamma} \frac{\xi_r - \phi_r}{\sigma_r} + \left(1 - \frac{1}{\gamma}\right) B_p(\infty, T), \quad (37)$$

$$C_p(\gamma, T) = \frac{1}{\gamma} \frac{\xi_\pi - \phi_\pi}{\sigma_\pi} + \left(1 - \frac{1}{\gamma}\right) C_p(\infty, T), \quad (38)$$

where

$$x_S(\infty) = \xi_S/\sigma_S, \quad B_p(\infty, T) = (\xi_r - B(t, T)\sigma_r)/\sigma_r, \quad C_p(\infty, T) = \xi_\pi/\sigma_\pi \quad (39)$$

are the stock allocation and loadings for an investor with very large risk aversion ($\gamma \rightarrow \infty$) and horizon T .

To understand the proposition, consider the price process of a hypothetical indexed bond with maturity equal to the investment horizon T :

$$\frac{dp^*}{p^*} = \left[r - B(t, T)\sigma_r\hat{\lambda}_r \right] dt - B(t, T)\sigma_r dz_r, \quad (40)$$

where $\hat{\lambda}_r \equiv \text{Cov}\left(-\frac{dM}{M}, dz_r\right) = -\phi'\rho e_2$ is the real risk premium per unit risk of dz_r . Its nominal return is calculated by applying Ito's Lemma to its nominal value, $P^* \equiv \Pi p^*$:

$$\begin{aligned} \frac{dP^*}{P^*} &= [r + \pi - B(t, T)\sigma_r\lambda_r] dt + \left(\frac{\xi_S}{\sigma_S}\right) \sigma_S dz_S \\ &+ \left(\frac{\xi_r}{\sigma_r} - B(t, T)\right) \sigma_r dz_r + \left(\frac{\xi_\pi}{\sigma_\pi}\right) \sigma_\pi dz_\pi + \xi_u dz_u. \end{aligned} \quad (41)$$

The loadings in equation (41) are the same as those of the portfolio chosen by the highly risk averse investor in equations (39), so that a highly risk averse investor chooses loadings on the innovations that match those of the hypothetical indexed bond with maturity T , leaving himself exposed only to the unhedgeable component of inflation $\xi_u dz_u$. We call the portfolio that replicates an index bond up to the unhedgeable inflation risk, a pseudo index bond. The Proposition shows that the optimal factor loadings for an investor with finite risk aversion can be written as a weighted average of the loadings of the nominal return of the mean variance tangency portfolio and the loadings of the nominal return of the pseudo index bond of maturity T .

The component of the loading on r in equation (37) that represents the hedge against changes in the real opportunity set represented by r ,¹⁵ (the ‘‘hedge loading on r ’’), is $\left(\frac{1}{\gamma} - 1\right) B(t, T)$, which depends on three parameters: risk aversion γ , horizon $T - t$, and the intensity of mean reversion of the real interest rate, κ . This hedge loading is monotonically decreasing in γ and

zero for log utility ($\gamma = 1$). Its elasticity with respect to the horizon is equal to the elasticity of $B(t, T)$ with respect to T , which is positive, so that its absolute value is increasing in the horizon. Finally, its elasticity with respect to the mean reversion intensity, κ , is:

$$e^{-\kappa(T-t)} [\kappa(T-t) + 1 - e^{\kappa(T-t)}] / (\kappa^2 B(t, T)) < 0.$$

Thus, the absolute value of the hedge loading *increases* with the horizon and *decreases* with the speed of mean reversion¹⁶ in r . This is reasonable, since the longer the horizon and the slower the mean reversion, the bigger is the effect of a given innovation in r on future investment opportunities.

The optimal portfolio strategy for the interim consumption problem (25)-(26) is the one that yields the optimal consumption program $c^*(s)$, $s \in [t, T]$, given by equation (27). Since the interim consumption problem can be interpreted as a two-stage terminal wealth optimization problem, the optimal portfolio strategy has a similar interpretation: it is a weighted average of the optimal strategies for terminal wealth problems with horizons at s , $s \in [t, T]$.

Theorem 4: optimal portfolio strategy with interim consumption

The vector of optimal portfolio allocations for problem (25,26), $x^ \equiv (x_s^*, x_1^*, x_2^*)'$ is given by expression (34), with $B \equiv B(s, T)$ replaced by $\hat{B} \equiv \hat{B}(s, T)$, where:*

$$\hat{B}(s, T) \equiv \int_s^T \frac{q(s, u) \exp \left\{ \frac{1-\gamma}{\gamma} [B(s, u)r_s + a_1(s, u)] \right\}}{\int_s^T q(s, \nu) \exp \left\{ \frac{1-\gamma}{\gamma} [B(s, \nu)r_s + a_1(s, \nu)] \right\} d\nu} B(s, u) du \quad (42)$$

is a weighted average of $B(s, u)$ ($u \in [s, T]$), and $q(s, u)$ and $a_1(s, u)$ are defined in Theorem

2.

C. Welfare Costs of Unhedgeable Inflation and Myopic Investment Strategy

The effect of market incompleteness on the real terminal wealth allocation, w_T^* , in equation (16) is reflected in the terms $\eta_2^{-1}(t, T)$ and $F_2(t, T)$: the former represents the realization of the unhedgeable component of inflation which directly reduces real wealth, while the latter corresponds to the risk premium for unhedgeable inflation risk. The effect of unhedgeable inflation risk on investor welfare may be assessed by comparing the investor's certainty equivalent wealth¹⁷ in the incomplete market to that in the complete market, using the expressions for expected utility given by (24) and (20). The ratio of the certainty equivalents in the complete and incomplete markets is equal to $\exp[\frac{\gamma}{2}\xi_u^2 - \phi_u\xi_u](T-t)$. Thus, the investor will be made better off by the unhedgeable inflation component dz_u if $\gamma < \frac{2\phi_u}{\xi_u}$, so that he is not too risk averse given the risk premium and risk associated with the unhedgeable inflation.

If the market is complete, indexed bonds are redundant assets and their introduction does not affect investor welfare. However, if the market is incomplete because inflation cannot be hedged with nominal assets ($\xi_u \neq 0$), the introduction of index bonds allows investors to vary their exposure to the previously unhedgeable component of inflation which, given the pricing kernel, will tend to improve welfare. Calculation of optimal state-contingent wealth and consumption then follows the proofs of Theorems 1 and 2 by allowing the investor to vary his exposure to the previously unhedgeable price level innovation $\xi_u dz_u$. Then the expected utility of the terminal wealth investor when there are indexed bonds and $\xi_u \neq 0$, $J^{IB}(W, r, \Pi, t)$, is:

$$J^{IB}(W, r, \Pi, t) = \frac{(W_t/\Pi_t)^{1-\gamma}}{1-\gamma} \psi_1(r, t, T) \exp^{\frac{1-\gamma}{2\gamma}(\phi_u - \gamma\xi_u)^2(T-t)}. \quad (43)$$

Comparing expressions (20) and (43), we see that, for a given pricing kernel,¹⁸ the introduction of index bonds increases the investor's certainty equivalent wealth by the factor $e^{\frac{1}{2\gamma}(\phi_u - \gamma\xi_u)^2(T-t)} \geq 1$. Therefore, except for investors for whom $\gamma = \phi_u/\xi_u$, the introduction of index bonds increases welfare by permitting trade in the previously unhedgeable inflation component $\xi_u dz_u$.

The investor's expected utility depends, not only on the available investment instruments, but also on the investment policy that is followed. For example, the efficiency gain from employing the optimal dynamic strategy rather than a myopic strategy can be measured by the ratio of the certainty equivalent wealth under the optimal dynamic strategy to that under a myopic strategy.¹⁹

This efficiency gain ratio, *EGR*, is:

$$\begin{aligned} EGR &= \exp \left\{ \frac{(1-\gamma)^2}{2\gamma} \text{var} \left(\int_t^T r(s) ds \right) \right\} \geq 1 \\ &= \exp \left\{ \frac{(1-\gamma)^2}{\gamma} \left[\frac{\sigma_r^2}{4\kappa^3} (2\kappa(T-t) - 3 - e^{-2\kappa(T-t)} + 4e^{-\kappa(T-t)}) \right] \right\}. \end{aligned} \quad (44)$$

The efficiency gain depends only on the risk aversion parameter and variance of the cumulative real interest rate; for values of γ close to unity the gain is small; the gain is also small if either σ_r is small, or the mean reversion parameter κ is large.

D. Constrained Optimal Portfolio Strategies

Since many investors are constrained from taking short positions, it is important to also consider constrained strategies. Any portfolio strategy can be characterized in terms of the loadings of the (nominal) portfolio return on the innovations in the stock return, the real interest rate, and inflation rate, $x_S, B_p \equiv -(x_1 B(t, T_1) + x_2 B(t, T_2))$ and $C_p \equiv -(x_1 C(t, T_1) + x_2 C(t, T_2))$,

where x_i is the proportion of wealth invested in bond i , $i = 1, 2$. While Theorem 3 showed that the unconstrained optimal holdings generically involve two bonds, the following proposition establishes that the investor's constrained optimal allocation can be achieved by holding only a single bond in combination with cash and stock.

Proposition 2: an investor who is constrained from borrowing or taking short positions in bonds, but who has available a continuum of possible bond maturities up to a maximum, τ_{max} , can achieve his constrained optimal portfolio allocation by investing in a single bond of an optimally chosen maturity, $\tau^ \leq \tau_{max}$, cash, and stock.*

It is easy to see the intuition behind the above proposition if we recall that the investor is only interested in the optimal loadings, x_S^* , B_p^* and C_p^* , on the stock return, real interest rate, and expected inflation innovations, respectively. To achieve the optimal loadings B_p^* and C_p^* , the investor can invest in several bonds subject to the no short sales constraints. Alternatively, the investor can choose a single bond with the optimal maturity τ^* such that the loadings of this bond adjusted for the cash position are optimal. Figure 1 shows the convex feasible region of constrained portfolio loadings and illustrates that all feasible combinations of loadings can be achieved by a combination of cash and a single bond of the appropriately chosen maturity. It is clear from the figure that it will in general be suboptimal to prespecify the maturity or factor loadings of the bond.

Insert Figure 1 about here

For a CRRA utility function, the indirect utility $J(r, W, \Pi, t)$ is homogeneous in real wealth and can be written as $\frac{(W/\Pi)^{1-\gamma}}{1-\gamma}\psi(r, t)$. The Bellman equation for the constrained portfolio problem is then:

$$\begin{aligned}
& \left\{ \begin{array}{l} \max \\ 0 \leq x_s, x_b \leq 1 \\ 0 \leq \tau \leq \tau_{max} \end{array} \right\} \left\{ \frac{1}{2} \psi_{rr} \sigma_r^2 + \psi_r [(1-\gamma) (x_s \sigma_{rS} - x_b B(t, \tau) \sigma_r^2 - x_b C(t, \tau) \sigma_{r\pi} - \sigma_{r\Pi}) \right. \\
& + \kappa(\bar{r} - r)] + \psi [-\beta - (1-\gamma)^2 (x_s \sigma_{S\Pi} - x_b B(t, \tau) \sigma_{r\Pi} - x_b C(t, \tau) \sigma_{\pi\Pi}) \\
& + (1-\gamma) (r + x_s \sigma_s \lambda_s - x_b B(t, \tau) \sigma_r \lambda_r - x_b C(t, \tau) \sigma_\pi \lambda_\pi - \xi_S \lambda_S - \xi_r \lambda_r - \xi_\pi \lambda_\pi - \xi_u \lambda_u) \\
& - \frac{1}{2} (1-\gamma)(\gamma-2) \sigma_\Pi^2 - \frac{1}{2} \gamma (1-\gamma) [x_s^2 \sigma_s^2 - 2x_s x_b (B(t, \tau) \sigma_{rS} + C(t, \tau) \sigma_{S\Pi}) \\
& \left. + x_b^2 (B(t, \tau)^2 \sigma_r^2 + 2B(t, \tau) C(t, \tau) \sigma_{r\pi} + C(t, \tau)^2 \sigma_\pi^2)] \right\} + \psi_t = 0 \}, \quad (45)
\end{aligned}$$

where x_s and x_b are the proportion of wealth invested in stocks and bonds respectively, and $\tau = T - t$ is the maturity of the bond chosen by the investor. Equation (45) is solved numerically to yield the constrained portfolio strategies that are reported in Section IV below.

The introduction of indexed bonds will generally increase investor welfare when the investor is subject to short-selling or borrowing constraints. With index bonds, the investor's portfolio choice problem involves the nominal bond maturity, the indexed bond maturity, and the proportions of wealth invested in stocks and nominal and indexed bonds.²⁰

III. Model Calibration

To provide illustrative calculations of horizon effects on optimal portfolio choice, the model parameters were estimated using a Kalman filter in which the unobserved state is described by r

and π . There are n observation equations provided by the relation between the yields on n bonds of different maturities and the state variables that follow from the bond pricing equation (10); the final observation equation follows from the discretized relation between the realized inflation rate and the expected inflation rate.²¹

The system was estimated using monthly data on eleven constant maturity U.S. treasury discount bond yields with maturities of 1, 3, 6 and 9 months, and 1, 2, 3, 4, 5, 7, and 10 years, and CPI inflation for the period January 1970 to December 1995.²² Table I reports the parameter estimates, along with their standard errors. The mean reversion coefficients imply half-lives for innovations in the real interest rate and expected inflation of 1.1 and 25.7 years, respectively.²³ The market prices of both interest rate risk and inflation risk are negative and significant.²⁴ Since the loadings of bond returns on innovations in these variables are negative and grow with maturity, estimated bond risk premia are positive and increasing with maturity. The standard deviation of unexpected inflation, σ_{Π} , is about 133 basis points per year, which compares with 411 basis points for the unconditional standard deviation of inflation and with 136 basis points for the standard deviation of innovations in expected inflation, σ_{π} . The standard deviation of innovations in the real interest rate, σ_r , of 260 basis points seems high, but should be considered in conjunction with the strong mean reversion, which implies that only about 53 percent of any innovation remains after one year. The correlation between innovations in the real interest rate and in inflation (expected and realized) is -0.06 , which is consistent with the Mundell-Tobin model and with the empirical findings of Fama and Gibbons (1982) for the period 1953 to 1977.

Insert Table I about here

The standard errors for the bond yield estimates include model error as well as sampling error. The standard errors of these “estimation errors” are quite low, except for maturities up to one year, which vary from 20 basis points for one year to 99 basis points for one month; for longer maturities, the standard errors are in the range of zero to 12 basis points.

Figure 2 plots the time series of the estimated real interest rate, r , and the expected rate of inflation, π . The estimated real interest rate varies between -4 and 7 percent. The series exhibits high short run variability and strong mean reversion. In contrast, the expected rate of inflation series exhibits much less mean reversion. It is possible that the high frequency variability in r is due to the model’s attempts to fit variation in the yields of medium to long-term bonds — this would account also for the relatively poor fit of the model at the short end of the term structure.

Insert Figure 2 about here

The estimate of the real interest rate mean reversion parameter, κ , is likely to be too high in light of the high frequency oscillations in the estimated real interest rate series that it implies, as seen in Figure 2. Figure 3 plots the annual “realized” real interest rates for the period 1890 to 1985.²⁵ It is clear that there is much less mean reversion even in this noisy series than that in the estimated series in Figure 2. Therefore, we re-estimated the parameters of the stochastic processes for r and π with a Kalman filter using only annual data on the nominal interest rate and inflation for the period 1890 to 1985. The results are striking — the new value of κ is only 0.105 in contrast to 0.631, while the estimate of σ_r is only half of its previous estimate.

Insert Figure 3 about here

IV. Unconstrained and Constrained Dynamic Strategies

Optimal investment strategies were determined for the parameter values calibrated from the monthly data set when the whole of the inflation innovation is unhedgeable ($\xi_S = \xi_r = \xi_\pi = 0$).²⁶ Table II summarizes the optimal strategies and the certainty equivalent wealth for different horizons and risk aversion parameters, γ .²⁷ Since $\alpha \neq \kappa$, any two bonds with different maturities are sufficient to span the (nominal or real) returns on all possible bond portfolios, which are characterized by their loadings on dr and $d\pi$; the amounts invested in the bonds depend on which bonds are used to achieve the portfolio loadings. Therefore, in the table, we report the loadings of the nominal portfolio returns, B_p and C_p ; expressions for these loadings follow immediately from equation (B8). Both the optimal stock allocation and inflation loadings are independent of the horizon. While the absolute magnitude of the interest rate loading is increasing in the horizon for $\gamma > 1$, is decreasing in the horizon for $\gamma < 1$, and is independent of the horizon for $\gamma = 1$ (log utility), this loading is relatively insensitive to the horizon for $T > 5$. Thus, in contrast to Xia (2000), who finds strong horizon effects even at long horizons in models with excess return predictability, horizon effects here are limited to about five years. Nevertheless, the hedge component of the optimal bond portfolio is significant. For example, when $\gamma = 3$ the myopic strategy has a loading of -2.29 on dr , while the optimal strategy for a five-year

investment horizon has a loading of -3.25 , so the hedge demand as measured by the loading on dr is about 42 percent of the myopic demand. When the horizon increases from five to twenty years, the absolute value of the hedge demand only increases by a further two percent of the myopic demand. The importance of the hedge demand increases rapidly as risk aversion increases; for example, when $\gamma = 5$ the optimal loading on dr for a five year horizon is 182 percent of the loading for a myopic investor.

Insert Table II about here

Figure 4 plots the optimal portfolio holdings of cash, stock, and one-year and ten-year nominal bonds as a function of the investment horizon for an investor with $\gamma = 3$. The figure confirms the discussion in terms of factor loadings. The optimal bond allocation changes little beyond year four, although there is a big difference between the myopic and the optimal allocation for a long horizon investor: the myopic allocation in the one-year bond is 3.24, while the optimal allocation is 4.94 for $T = 5$, so that the hedge demand is as high as 52 percent of the myopic allocation.

Insert Figure 4 about here

Although not shown here, the optimal portfolio allocations are also quite sensitive to the risk aversion parameter: the stock allocation and the loadings on both dr and $d\pi$ decrease with the

risk aversion. In the limit, as risk aversion becomes infinite, the stock allocation goes to zero and the investor's dynamic strategy in bonds synthesizes the returns on a pseudo inflation indexed bond with maturity equal to the remaining investment horizon. Investment in a bond with positive maturity increases the investor's exposure to inflation risk. If ϕ_π were equal to zero so that there was no reward for bearing inflation risk, the investor would eliminate all inflation risk by taking a short position in at least one of the bonds. For the parameter estimates in Table I, all of the optimal portfolios in Table II involve at least one short bond or cash position.

Perhaps surprisingly, the costs imposed by the unhedgeability of the inflation surprise are quite small. Assuming that the reward for bearing inflation risk is zero ($\phi_u = 0$), the certainty equivalent cost imposed by unhedgeable inflation risk is $\exp\frac{\gamma\xi_u^2}{2}(T-t)$; this amounts to only about one percent of wealth for a twenty-year horizon investor with $\gamma = 5$ because ξ_u , the volatility of unhedgeable inflation, is only 1.3 percent per year.

Table III reports the constrained optimal strategies for the same parameter values as Table II: the portfolio allocation is now shown as the proportion of wealth allocated to bonds and stock together with the optimal maturity of the bond. The constraints modestly reduce the investor's certainty equivalent wealth. For example, an investor $\gamma = 3$ and $T = 20$ years requires only about 13 percent additional wealth to compensate for the constraints. The portfolio allocation is relatively insensitive to the investment horizon: the stock-bond ratio is virtually independent of the horizon, and the maturity of the optimal bond shows little variation beyond year five. However, the optimal portfolio is quite sensitive to the risk aversion parameter. As risk aversion increases, the ratio of bonds to stock increases at all horizons; moreover, the maturity of the optimal bond decreases as shown in Figure 5.

Insert Table III about here

Insert Figure 5 about here

The efficiency gain from following a dynamic strategy is calculated from equation (44). Since the estimate of κ from the first data set is very high, the estimates of efficiency gains reported in Table II are very small except for long horizons and strong risk aversion — for γ equal to 15 the efficiency gain over 20 years is around 21 percent. When the investor faces constraints, EGR is calculated numerically. Estimates of EGR reported in Table III are all close to one, reflecting the fact that the investor’s optimal portfolio strategy is close to the myopic one when there are constraints.

Tables IV and V report the unconstrained and constrained portfolio strategies for the value of κ corresponding to the set of annual data, holding the other parameters unchanged. There are no horizon effects in C_p and x_s , but the unconstrained strategies now exhibit strong horizon effects in B_p even at long horizons: When γ is less than unity, B_p decreases with the horizon, while the reverse is true for γ greater than unity. The strong horizon effect is also evident in Figure 6, which plots the optimal portfolio holdings for an investor with $\gamma = 3$ who can invest in cash, stock, and one-year and ten-year bonds. Most significantly, the efficiency gain over the myopic strategy is now substantial: when the horizon is 20 years, the gain is 119 percent for γ equal 5 and 252 percent for $\gamma = 7$ in the unconstrained case, and 54 percent and 121 percent,

respectively, in the constrained case.

Insert Table IV about here

Insert Table V about here

Insert Figure 6 about here

Horizon effects are now also evident for the constrained strategies shown in Table V. First, in contrast to the popular view that long-horizon investors should hold more stock than short horizon investors,²⁸ the optimal stock holding decreases with the horizon; for example, for $\gamma = 3$, the optimal holding of stock is 66 percent for a one-month horizon and only 40 percent for a twenty-year horizon. Similarly, the stock-bond ratio also decreases with the horizon beyond year one. These results are obviously sensitive to the assumption of the model that the equity risk premium is constant. In contrast to the previous example, a myopic investor with a large enough risk aversion parameter may want to hold cash, and, in such circumstances, the constraints are not binding. However, when the investment horizon is longer than one year, a mix of stocks and bonds (no cash) dominates portfolios with cash. The second horizon effect, which is shown in Figure 7, is that the maturity of the optimally chosen bond increases with the horizon — the

effect is much more pronounced with the reduced degree of mean reversion in r , which generally leads the long-term investor to hold a much longer maturity bond: the maturity of the optimally chosen bond for a twenty-year investor with $\gamma = 3$ rises from seven years when κ is 0.63 to 13.5 years when κ is 0.11.

Insert Figure 7 about here

V. Conclusion

In this paper, we have derived the optimal dynamic strategies for an investor with power utility in an economy with stochastic inflation and real interest rates, as well as a constant equity premium, when there exists no riskless security. Closed form expressions were obtained for the optimal portfolio when the investor is free to take unconstrained portfolio positions, and it was shown that the optimal portfolio position can be achieved by investments in stocks, cash, and two nominal bonds. In this setting, the optimal allocation to stocks and the optimal portfolio loading on the innovation in inflation are independent of the horizon, while the optimal loading on the innovation in the shadow real interest rate is increasing in the horizon for investors more risk averse than the log. The efficiency gain of the optimal dynamic strategy over the myopic strategy was shown to be a function of both the investor's risk aversion and the variance of the cumulative real interest rate over the investor's horizon: the gain is small for risk aversion close to the log, as well as when either the variance of innovations in the real interest rate is small

or the mean reversion in the real interest rate is large. When the investor is constrained from holding short positions or borrowing, the optimal portfolio was shown to be achievable with an investment in stocks, cash, and a single bond with an optimally chosen maturity.

The model was calibrated to monthly data on U.S. Treasury bond yields and inflation for the period 1970 to 1995. The resulting parameter estimates implied an unreasonably high degree of mean reversion in the real interest rate and yielded very small estimates of the efficiency gain of the dynamic strategy and only limited horizon effects in optimal portfolios. A striking characteristic of the optimal constrained portfolios is that, not only does the allocation to bonds increase with risk aversion, but the maturity of the optimal bond decreases as risk aversion increases.

When the real interest rate mean reversion parameter is calibrated to a long history of annual data, it falls from 0.63 to 0.11. With this parameterization, both horizon effects and the efficiency gains of the optimal dynamic strategy become large. Thus, the importance of dynamic considerations in optimal asset allocation depends critically on the stochastic characteristics of the investment opportunity set. Further work is required to assess more precisely the dynamics of the real interest rate in the United States.

Table I
Estimates of Model Parameters

Maximum Likelihood parameter estimates for the joint process of real interest rate, expected rate of inflation, and stock returns estimated by implementing Kalman Filter using monthly yields of eleven U.S. constant maturity treasury bonds, CPI data, and CRSP value-weighted stock returns for the period from January 1970 to December 1995. Parameter estimates using annual nominal interest rate and CPI data from 1890 to 1985 are reported in parentheses.

Parameter	Estimate	Standard Error	
Stock Return Process: $\frac{dS}{S} = (R_f + \lambda_S \sigma_S)dt + \sigma_S dz_S$			
σ_S	0.158		
λ_S	0.343	0.057	
Real Interest Rate: $dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r$			
\bar{r}	0.012 (0.017)	0.002	
κ	0.631 (0.105)	0.003	
σ_r	0.026 (0.013)	0.004	
λ_r	-0.209	0.077	
Expected Inflation: $d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi$			
$\bar{\pi}$	0.054	0.023	
α	0.027	0.009	
σ_π	0.014	0.005	
λ_π	-0.105	0.005	
Realized Inflation: $d\Pi/\Pi = \pi dt + \sigma_\Pi dz_\Pi$			
σ_Π	0.013	0.005	
Parameters of the Pricing Kernel Process: Φ			
ϕ_S	-0.333		
ϕ_r	0.170		
ϕ_π	0.120		
ρ_{Sr}	-0.129		
$\rho_{S\pi}$	-0.024		
$\rho_{r\pi}$	-0.061	0.002	
Standard Errors of Bond Yield Errors by Bond Maturities			
1 month	0.0099	3 month	0.0066
6 month	0.0043	9 month	0.0030
1 year	0.0020	2 year	0.0004
3 year	0.0003	4 year	0.0002
5 year	< 0.0001	7 year	0.0005
10 year	0.0012		

Table II
An Unconstrained Optimal Portfolio Strategy for Investors with the Terminal Wealth Objective
($\kappa = 0.631$)

This table reports the unconstrained optimal strategy for an investor with different values of the risk aversion parameter, γ , and the investment horizon, T . B_p (C_p) is the sensitivity of the optimal portfolio to innovations in r (π); \mathfrak{g} is the proportional portfolio allocation to the stock. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table V for a current interest rate of three percent. The variable EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated from equation (44). The variable κ is the mean reversion coefficient for the shadow real interest rate process.

Horizon	Risk Aversion Parameter, γ						
	0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month							
CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.37	-4.50	-2.29	-1.41	-1.03	-0.74	-0.52
C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
1 year							
CE	1.13	1.08	1.05	1.04	1.04	1.03	1.03
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.21	-4.72	-2.73	-1.94	-1.59	-1.34	-1.14
C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
5 years							
CE	1.77	1.43	1.27	1.21	1.18	1.16	1.14
EGR	1.00	1.00	1.00	1.01	1.01	1.02	1.03
x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.01	-4.98	-3.25	-2.56	-2.26	-2.04	-1.86
C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
10 years							
CE	3.05	2.00	1.58	1.43	1.37	1.32	1.29
EGR	1.00	1.00	1.01	1.02	1.03	1.05	1.09
x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.00	-5.00	-3.29	-2.61	-2.32	-2.10	-1.92
C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
20 years							
CE	9.02	3.92	2.43	2.01	1.84	1.72	1.63
EGR	1.00	1.00	1.02	1.05	1.08	1.13	1.21
x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.00	-5.00	-3.29	-2.61	-2.32	-2.10	-1.93
C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51

Table III
A Constrained Optimal Portfolio Strategy for Investors with the Terminal Wealth Objective
($\kappa = 0.631$)

This table reports the constrained optimal strategy for an investor with different values of the risk aversion parameter, γ , and the investment horizon, T . The variables x_S and x_B denote the proportional allocations to the stock and the optimal bond. The variable τ represents the maturity of the optimally chosen bond. The variable CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table V for a current interest rate of three percent. The variable EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated numerically. The variable κ is the mean reversion coefficient for the shadow real interest rate process.

Horizon	Risk Aversion Parameter, γ						
	0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month							
CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	1.00	1.00	0.64	0.40	0.29	0.20	0.13
x_B	0.00	0.00	0.36	0.60	0.71	0.80	0.87
τ	n.a.	n.a.	7.49	3.08	1.86	1.09	0.62
1 year							
CE	1.07	1.06	1.05	1.04	1.04	1.03	1.03
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	1.00	1.00	0.63	0.40	0.29	0.20	0.14
x_B	0.00	0.00	0.37	0.60	0.71	0.80	0.86
τ	n.a.	n.a.	7.24	3.34	2.35	1.71	1.27
5 years							
CE	1.36	1.31	1.23	1.18	1.16	1.14	1.12
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.01
x_S	1.00	1.00	0.62	0.39	0.29	0.20	0.14
x_B	0.00	0.00	0.38	0.61	0.71	0.80	0.86
τ	n.a.	n.a.	7.00	3.54	2.71	2.18	1.81
10 years							
CE	1.81	1.68	1.47	1.37	1.31	1.27	1.22
EGR	1.00	1.01	1.00	1.00	1.00	1.01	1.03
x_S	1.00	1.00	0.62	0.39	0.28	0.20	0.14
x_B	0.00	0.00	0.38	0.61	0.72	0.80	0.86
τ	n.a.	n.a.	7.00	3.54	2.73	2.21	1.85
20 years							
CE	3.22	2.77	2.16	1.82	1.65	1.49	1.31
EGR	1.00	1.00	1.00	1.00	1.00	1.01	1.02
x_S	1.00	1.00	0.63	0.40	0.29	0.21	0.14
x_B	0.00	0.00	0.37	0.60	0.71	0.79	0.86
τ	n.a.	n.a.	7.00	3.55	2.74	2.22	1.85

Table IV
An Unconstrained Optimal Portfolio Strategy for Investors with the Terminal Wealth Objective
($\kappa = 0.105$)

This table reports the unconstrained optimal strategy for an investor with different values of the risk aversion parameter, γ , and the investment horizon, T . The variable B_p (C_p) is the sensitivity of the optimal portfolio to innovations in r (π); x_S is the proportional portfolio allocation to the stock. The variable CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table for a current interest rate of three percent. The variable EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated from equation (44). The variable κ is the mean reversion coefficient for the shadow real interest rate process.

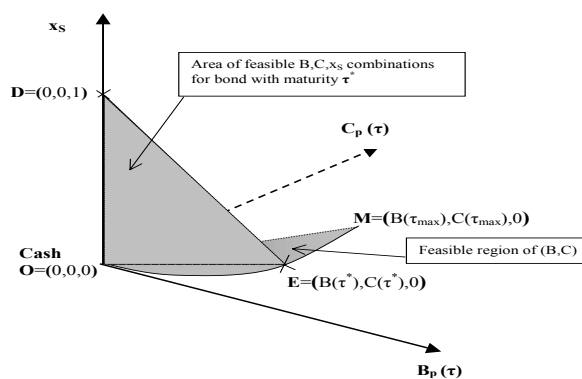
Horizon	Risk Aversion Parameter, γ						
	0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month							
CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.37	-4.50	-2.29	-1.41	-1.03	-0.75	-0.52
C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
1 year							
CE	1.14	1.09	1.06	1.05	1.04	1.04	1.04
EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-8.15	-4.79	-2.87	-2.10	-1.77	-1.53	-1.33
C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
5 years							
CE	1.85	1.51	1.35	1.28	1.26	1.24	1.22
EGR	1.00	1.00	1.01	1.03	1.05	1.08	1.13
x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-7.42	-5.77	-4.83	-4.45	-4.29	-4.17	-4.08
C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
10 years							
CE	3.31	2.27	1.83	1.68	1.61	1.57	1.53
EGR	1.00	1.01	1.08	1.19	1.33	1.56	2.05
x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-6.84	-6.54	-6.36	-6.29	-6.27	-6.24	-6.23
C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
20 years							
CE	10.19	5.02	3.35	2.84	2.64	2.49	2.37
EGR	1.01	1.04	1.39	2.19	3.52	7.24	24.41
x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
B_p	-6.30	-7.26	-7.81	-8.03	-8.12	-8.19	-8.25
C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51

Table V
A Constrained Optimal Portfolio Strategy for Investors with the Terminal Wealth Objective
($\kappa = 0.105$)

This table reports the constrained optimal strategy for an investor with different values of the risk aversion parameter, γ , and investment horizon, T . x_S and x_B are the proportional allocations to the stock and the optimal bond. τ is the maturity of the optimally chosen bond. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table for $\sigma = 3\%$. EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated numerically. κ is the mean reversion coefficient for the shadow real interest rate process.

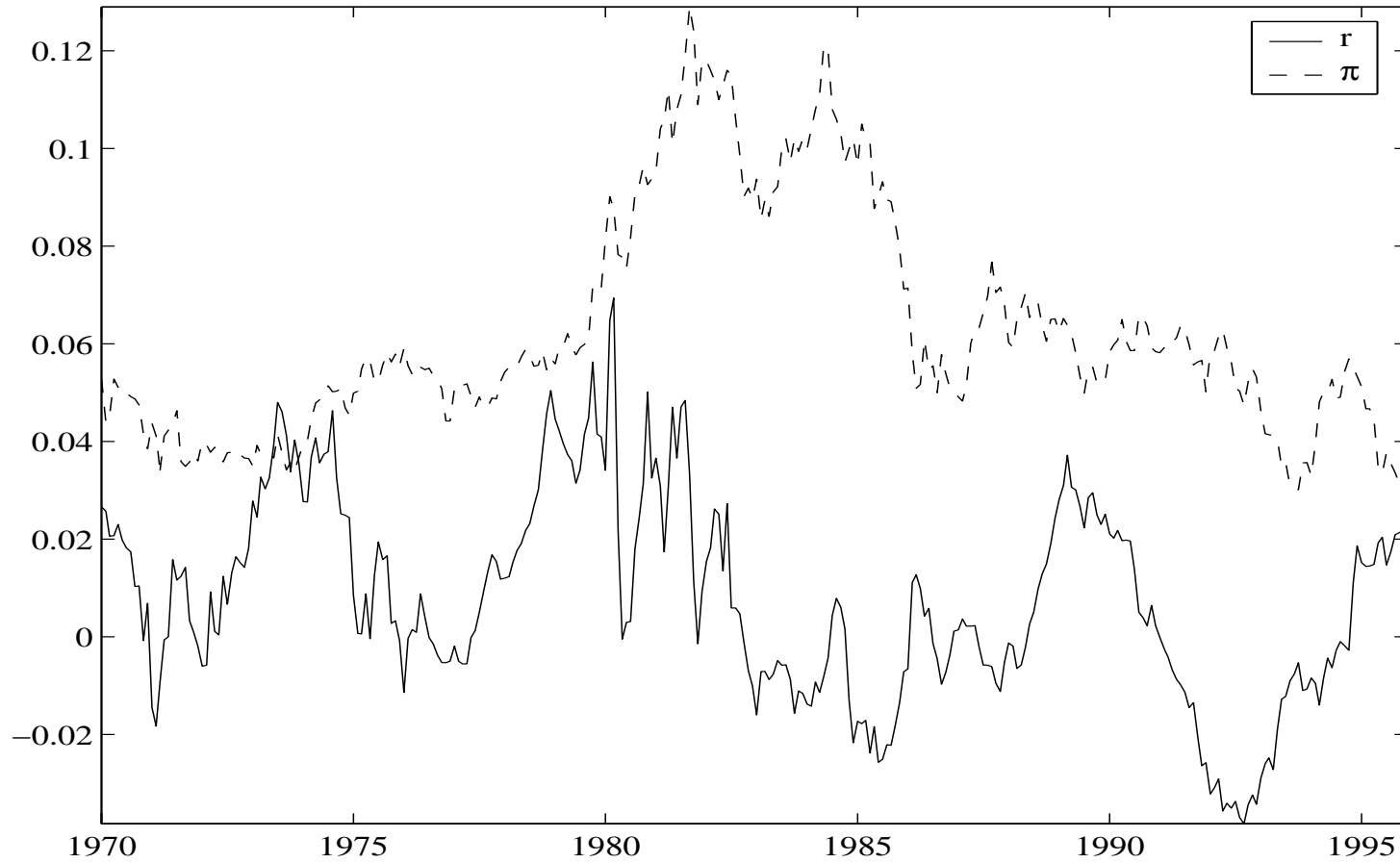
Horizon		Risk Aversion Parameter, γ						
		0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month								
	CE	1.01	1.01	1.01	1.00	1.00	1.00	1.00
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.66	0.40	0.29	0.20	0.13
	x_B	0.00	0.00	0.34	0.44	0.31	0.22	0.15
	τ	n.a.	n.a.	9.71	3.72	3.72	3.72	3.72
1 year								
	CE	1.08	1.07	1.06	1.05	1.04	1.04	1.04
	EGR	1.00	1.00	1.01	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.64	0.40	0.29	0.20	0.14
	x_B	0.00	0.00	0.36	0.60	0.71	0.80	0.86
	τ	n.a.	n.a.	10.81	3.69	2.40	1.72	1.30
5 years								
	CE	1.48	1.42	1.33	1.27	1.24	1.21	1.18
	EGR	1.04	1.04	1.01	1.01	1.04	1.06	1.10
	x_S	1.00	1.00	0.55	0.38	0.28	0.20	0.13
	x_B	0.00	0.00	0.45	0.62	0.72	0.80	0.87
	τ	n.a.	n.a.	12.60	7.15	5.61	4.72	4.21
10 years								
	CE	2.14	1.97	1.73	1.56	1.46	1.27	1.28
	EGR	1.06	1.10	1.03	1.11	1.20	1.36	1.72
	x_S	1.00	0.66	0.47	0.32	0.23	0.16	0.09
	x_B	0.00	0.34	0.53	0.68	0.77	0.84	0.91
	τ	n.a.	27.77	13.20	9.07	7.72	6.94	6.58
20 years								
	CE	4.39	3.61	2.63	2.00	1.66	1.37	n.a.
	EGR	1.11	1.24	1.12	1.54	2.21	3.97	n.a.
	x_S	1.00	0.63	0.40	0.24	0.16	0.09	n.a.
	x_B	0.00	0.37	0.60	0.76	0.84	0.91	n.a.
	τ	n.a.	25.98	13.53	10.26	9.15	8.50	n.a.

Figure 1. Illustration of the feasible region for a portfolio strategy with short sales constraint.



The curve OEM is the locus of $(B(t), C(t))$ combinations as t is varied from zero to τ_{\max} . The point M corresponds to $(B(\tau_{\max}), C(\tau_{\max}))$. The area OEM is the feasible region of (B, C) combinations that are attainable with cash and bonds. Given a selected bond maturity τ^* , the area DOE is the feasible region of (B, C, x_S) combinations.

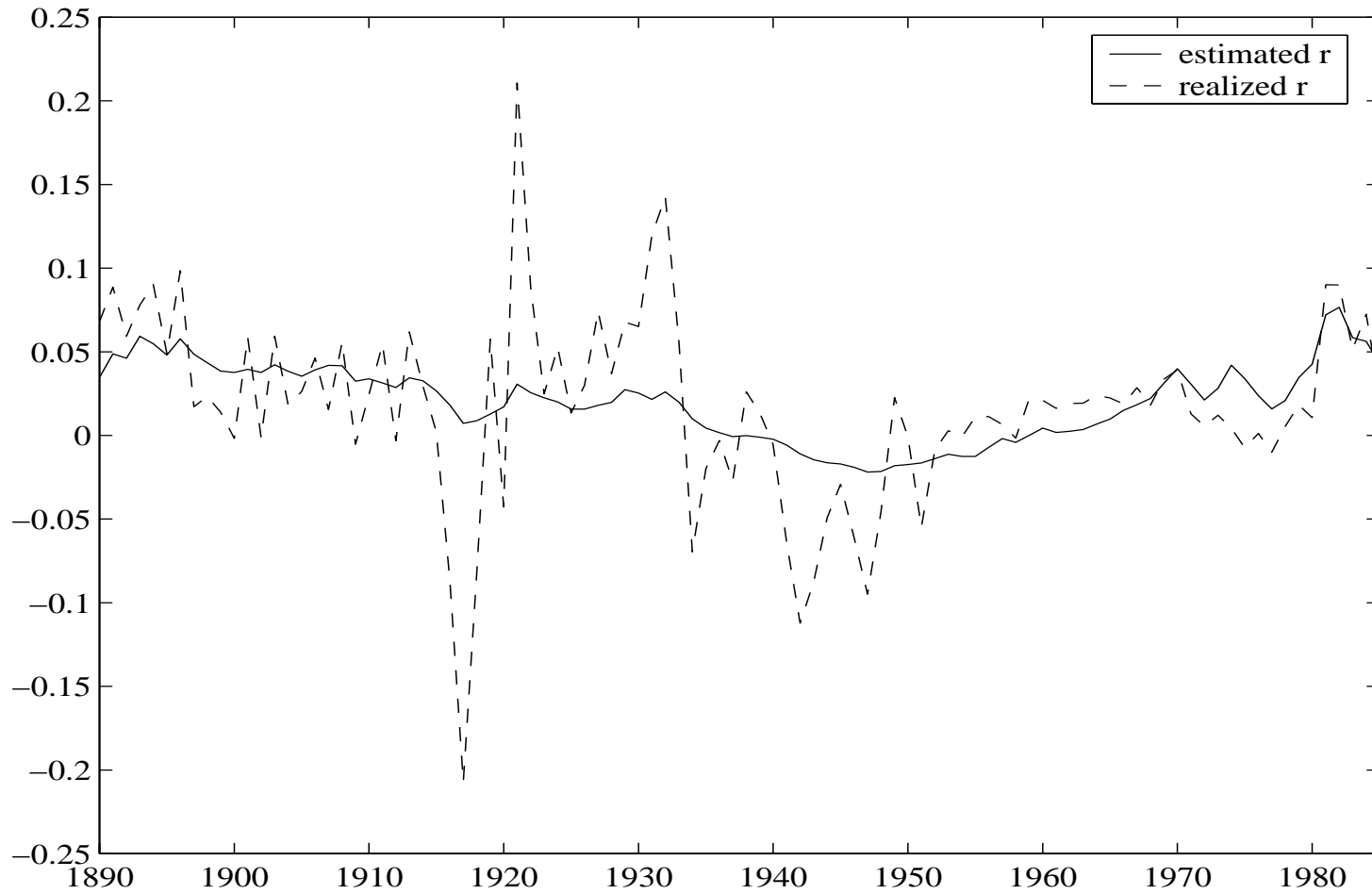
Figure 2. Estimated state variables, r and π .



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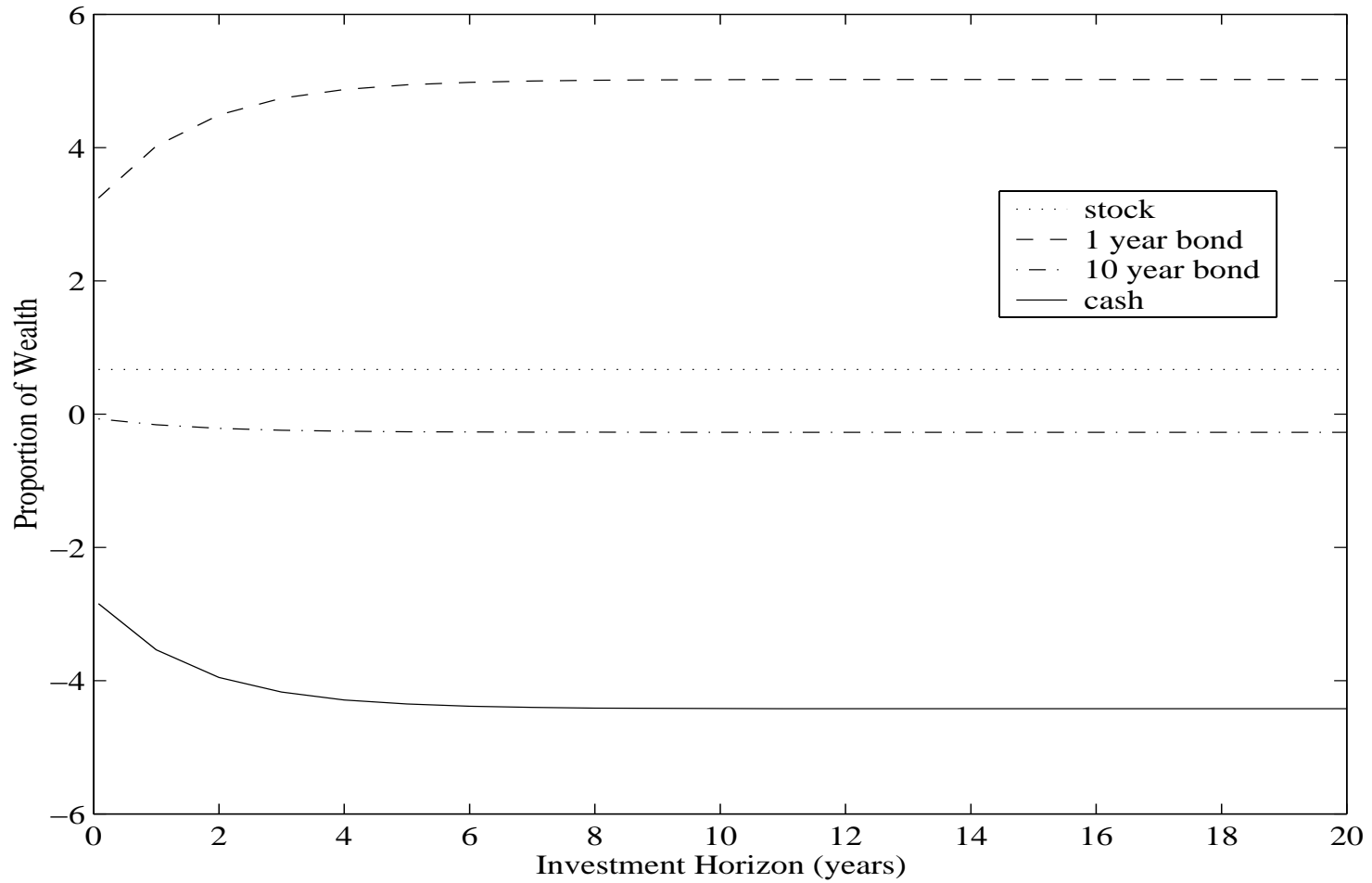
This figure shows the estimated state variables, r , the real interest rate, and π , the expected rate of inflation, derived from the Kalman filter. The sample period is January 1970 to December 1995. The series are estimated using monthly observations on eleven U.S. treasury constant maturity bond yields and U.S. CPI data.

Figure 3. Estimated state variables, r , and its realized counterpart.



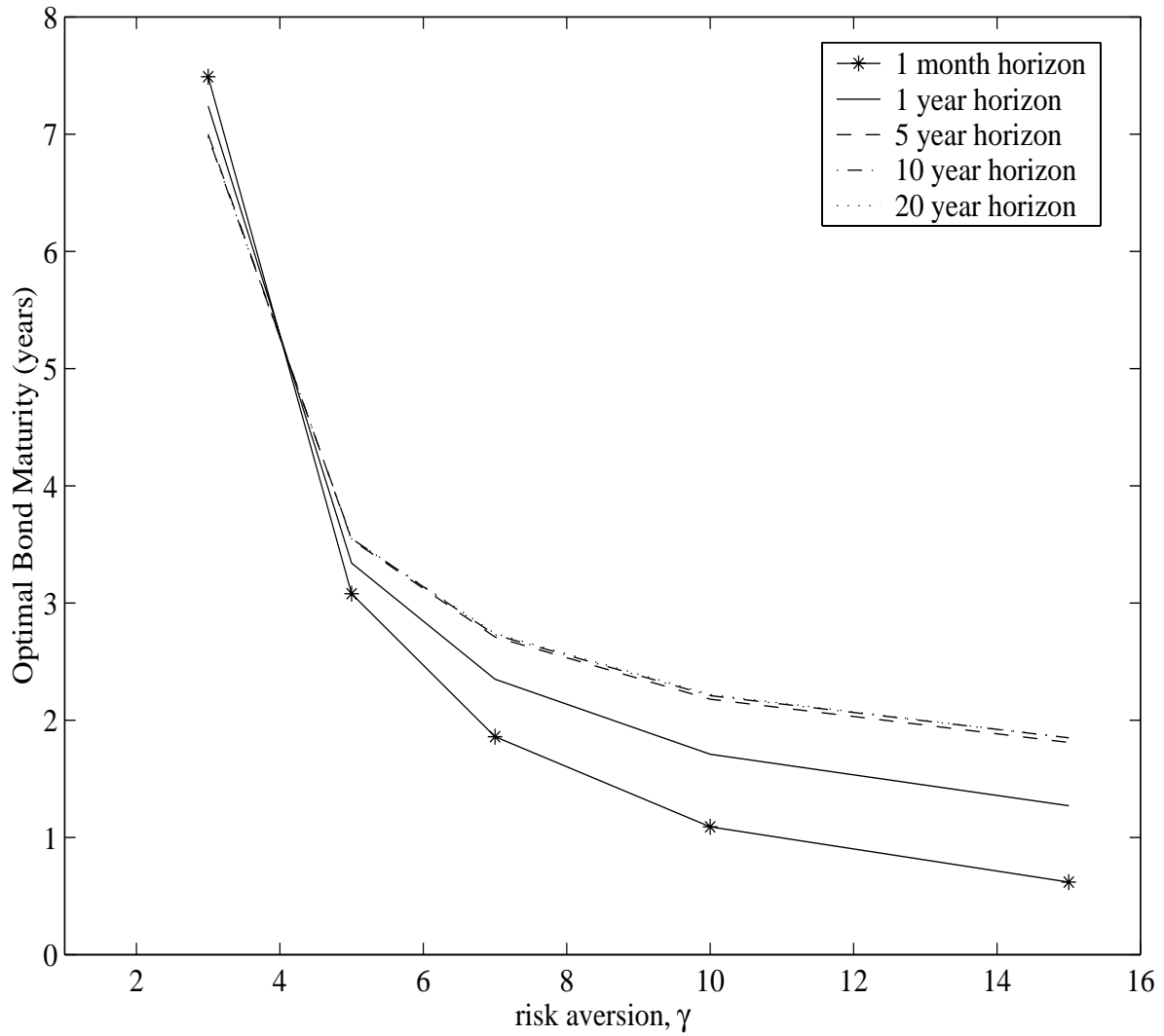
This figure shows the estimated state variables, r , the real interest rate derived from the Kalman filter, and the realized real interest rate derived by subtracting actual inflation rate from the nominal interest rate. The sample period is 1890 to 1985. The series are estimated using observations on annual U.S. nominal interest rate and U.S. CPI data.

Figure 4. Unconstrained optimal asset allocation for different horizons ($\gamma = 3$, $\kappa = 0.631$ and $r_0 = 3\%$).



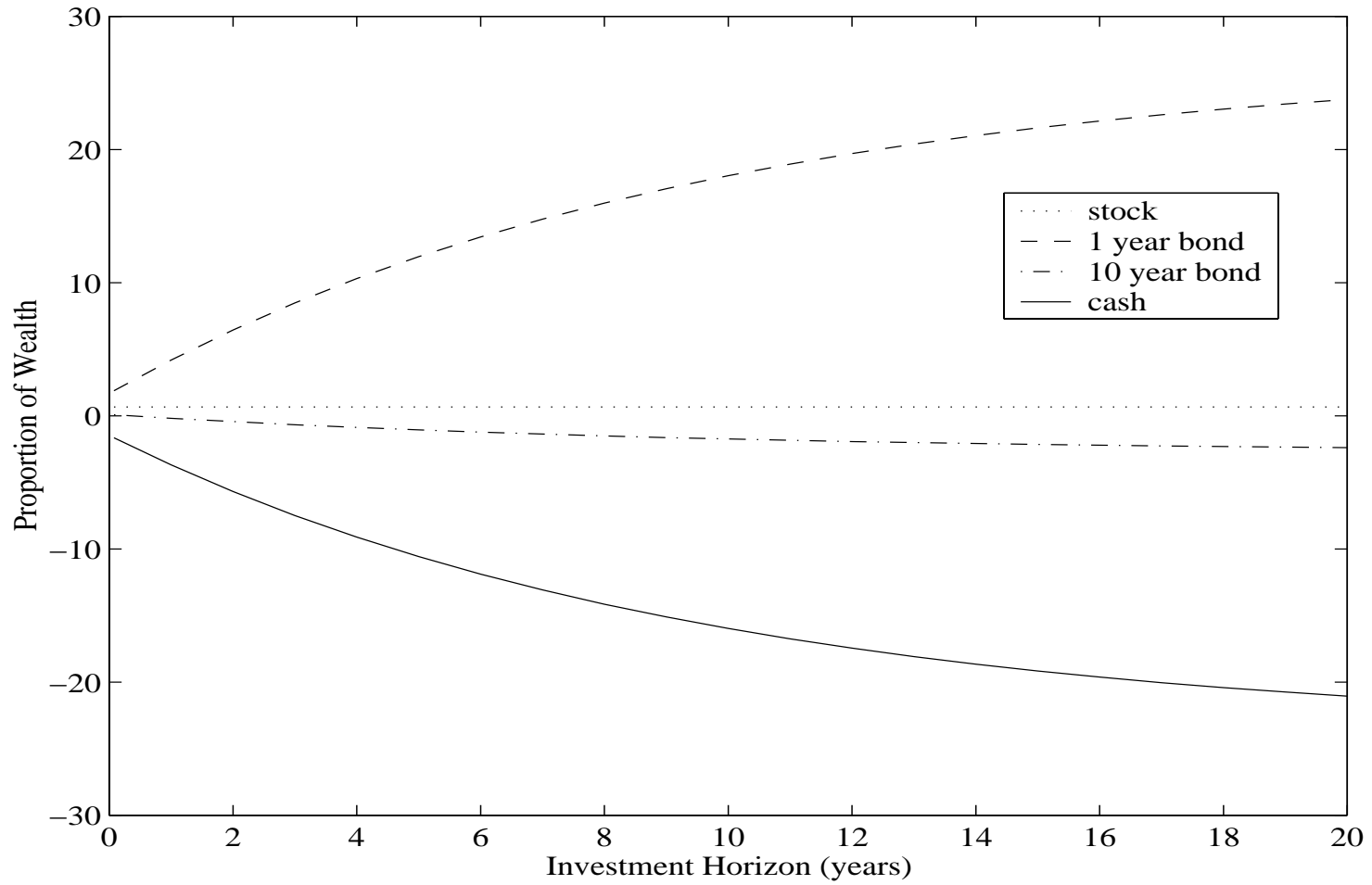
The unconstrained optimal proportion of wealth invested in the stock index, the one-year bond, the ten-year bond, and cash. The optimal allocations are derived using the parameter estimates in Table V for a current interest rate of three percent.

Figure 5. Optimal bond maturity and risk aversion for different horizons ($r_0 = 3\%$, $\kappa = 0.631$).



This figure plots the optimal bond maturity chosen by an investor who faces borrowing and short sales constraints for different values of the risk aversion parameter, γ , and investment horizon.

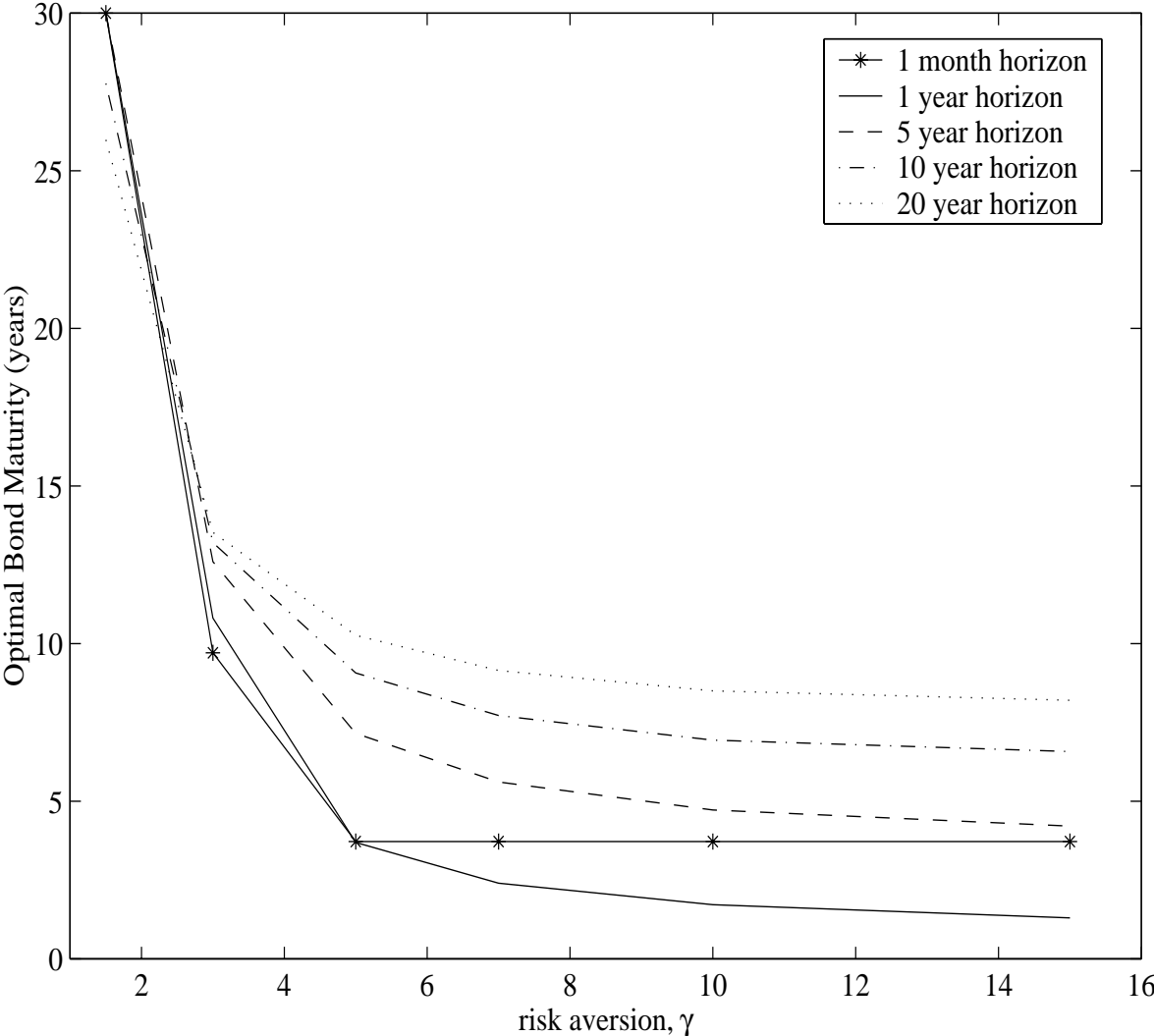
Figure 6. Unconstrained optimal asset allocation for different horizons ($\gamma = 10$, $\kappa = 0.105$ and $r_0 = 3\%$).



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This figure plots the unconstrained optimal proportion of wealth invested in the stock, the one-year bond, the ten-year bond, and cash. The optimal allocations are derived using the parameter estimates in Table V.

Figure 7. Optimal bond maturity and risk aversion for different horizons ($r_0 = 3\%$, $\kappa = 0.105$).



This figure plots the optimal bond maturity chosen by an investor who faces borrowing and short sales constraints for different values of the risk aversion parameter, γ , and investment horizon.

Appendix A. Nominal Bond Price

The nominal bond price, $P(t, T)$, is given by:

$$P(t, T) = \exp \left\{ E_t \left[\ln \left(\frac{M_T}{M_t} \right) - \ln \left(\frac{\Pi_T}{\Pi_t} \right) \right] + \frac{1}{2} Var_t \left[\ln \left(\frac{M_T}{M_t} \right) - \ln \left(\frac{\Pi_T}{\Pi_t} \right) \right] \right\}. \quad (A1)$$

Letting $\phi_1^2 \equiv \phi' \rho \phi$, $\xi_1^2 \equiv \xi' \rho \xi$, $V_M = \phi_1^2 + \phi_u^2$, $V_\Pi = \xi_1^2 + \xi_u^2$,

$$\ln (M_T/M_t) = \int_t^T \left(-r(s) - \frac{1}{2} V_M \right) ds + \int_t^T \phi' dz + \int_t^T \phi_u dz_u \quad (A2)$$

$$\ln (\Pi_T/\Pi_t) = \int_t^T \left(\pi(s) - \frac{1}{2} \sigma_\Pi^2 \right) ds + \int_t^T \xi' dz + \xi_u \int_t^T dz_u \quad (A3)$$

Then define

$$B(t, T) = \kappa^{-1} (1 - e^{\kappa(t-T)}), \quad (A4)$$

$$C(t, T) = \alpha^{-1} (1 - e^{\alpha(t-T)}). \quad (A5)$$

Using equation (A2) and (A3),

$$E_t [\ln (M_T/M_t)] = -\bar{r}(T-t) + (\bar{r} - r_t)B(t, T) - \frac{1}{2} V_M(T-t) \quad (A6)$$

$$\begin{aligned} Var_t [\ln (M_T/M_t)] &= -\frac{\sigma_r^2}{2\kappa^3} [2\kappa (B(t, T) - (T-t)) + \kappa^2 B^2(t, T)] + V_M(T-t) \\ &\quad - \frac{2\sigma_r}{\kappa} (\phi_S \rho_{Sr} + \phi_r + \phi_\pi \rho_{r\pi}) [T-t-B(t, T)]; \end{aligned} \quad (A7)$$

and

$$E_t [\ln (\Pi_T/\Pi_t)] = \bar{\pi}(T-t) - (\bar{\pi} - \pi_t)C(t, T) - \frac{1}{2}(\xi_1^2 + \xi_u^2)(T-t) \quad (\text{A8})$$

$$\begin{aligned} Var_t [\ln (\Pi_T/\Pi_t)] &= -\frac{\sigma_\pi^2}{2\alpha^3} [2\alpha (C(t, T) - (T-t)) + \alpha^2 C^2(t, T)] + (\xi_1^2 + \xi_u^2)(T-t) \\ &+ \frac{2\sigma_\pi}{\alpha} (\xi_S \rho_{S\pi} + \xi_r \rho_{r\pi} + \xi_\pi) [T-t - C(t, T)]. \end{aligned} \quad (\text{A9})$$

In addition, we have

$$\begin{aligned} CV &\equiv cov \left[\ln \left(\frac{M_T}{M_t} \right), \ln \left(\frac{\Pi_T}{\Pi_t} \right) \right] \\ &= -\frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\alpha \kappa} \left[(T-t) - B(t, T), -C(t, T) + \frac{1 - e^{(\alpha+\kappa)(t-T)}}{\alpha + \kappa} \right] \\ &- \frac{\sigma_r}{\kappa} (\xi_S \rho_{Sr} + \xi_r + \xi_\pi \rho_{r\pi}) [(T-t) - B(t, T)] + \phi_u \xi_u (T-t) \\ &+ \frac{\sigma_\pi}{\alpha} (\phi_S \rho_{S\pi} + \phi_r \rho_{r\pi} + \phi_\pi) [(T-t) - C(t, T)] + \phi' \rho \xi (T-t). \end{aligned} \quad (\text{A10})$$

Therefore,

$$P(t, T) = \exp \{ A(t, T) - B(t, T)r_t - C(t, T)\pi_t \}, \quad (\text{A11})$$

where

$$\begin{aligned}
A(t, T) &= [B(t, T) - (T - t)] \bar{r}^* + [C(t, T) - (T - t)] \bar{\pi}^* \\
&\quad - \frac{\sigma_r^2}{4\kappa^3} [2\kappa(B(t, T) - (T - t)) + \kappa^2 B^2(t, T)] \\
&\quad - \frac{\sigma_\pi^2}{4\alpha^3} [2\alpha(C(t, T) - (T - t)) + \alpha^2 C^2(t, T)] \\
&\quad + \frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\kappa \alpha} \left[(T - t) - C(t, T) - B(t, T) + \frac{1 - e^{(\alpha + \kappa)(t - T)}}{\alpha + \kappa} \right] \\
&\quad + (\xi_S \lambda_S + \xi_r \lambda_r + \xi_\pi \lambda_\pi + \xi_u \lambda_u)(T - t). \tag{A12}
\end{aligned}$$

In the above equation, $\bar{r}^* = \bar{r} - \lambda_r \frac{\sigma_r}{\kappa}$, and $\bar{\pi}^* = \bar{\pi} - \lambda_\pi \frac{\sigma_\pi}{\alpha}$ where

$$\lambda_S \equiv (\xi_S + \xi_r \rho_{Sr} + \xi_\pi \rho_{S\pi}) - (\phi_S + \phi_r \rho_{Sr} + \phi_\pi \rho_{S\pi}), \tag{A13}$$

$$\lambda_r \equiv (\xi_S \rho_{Sr} + \xi_r + \xi_\pi \rho_{r\pi}) - (\phi_S \rho_{Sr} + \phi_r + \phi_\pi \rho_{r\pi}), \tag{A14}$$

$$\lambda_\pi \equiv (\xi_S \rho_{S\pi} + \xi_r \rho_{r\pi} + \xi_\pi) - (\phi_S \rho_{S\pi} + \phi_r \rho_{r\pi} + \phi_\pi), \tag{A15}$$

$$\lambda_u \equiv \xi_u - \phi_u. \tag{A16}$$

The variable \bar{r}^* can be interpreted as the long run mean of the real interest rate under the risk neutral measure and $\bar{\pi}^*$ as the long run mean of the expected inflation rate.

Appendix B. Proof of Theorems

1. Proof of Theorem 1

Since η_1 , and ζ_1 are orthogonal to η_2 , and ζ_2 , and $\Pi_T = \Pi_t \eta_1(t, T) \eta_2(t, T)$, we can write the

Lagrangian for *feasible* wealth processes as:

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_t \left[\frac{(W_T/\eta_1(t, T))^{1-\gamma}}{1-\gamma} \right] \mathbb{E}_t \left[\frac{1}{\eta_2(t, T)^{1-\gamma}} \right] \frac{1}{\Pi_t^{1-\gamma}} \\ & - \delta \left\{ \mathbb{E}_t \left[\zeta_1(t, T) \frac{W_T}{\eta_1(t, T)} \right] \mathbb{E}_t \left[\zeta_2(t, T) \frac{1}{\eta_2(t, T)} \right] \frac{1}{\Pi_t} - \frac{W_t}{\Pi_t} \right\} \end{aligned} \quad (\text{B1})$$

because the investor can vary nominal wealth across states in \mathcal{O}_1 , but not across states in \mathcal{O}_2 .

The first order conditions are:

$$\left(\frac{W_T}{\eta_1(t, T)\Pi_t} \right)^{-\gamma} \mathbb{E}_t \left[\frac{1}{\eta_2(t, T)^{1-\gamma}} \right] = \delta \zeta_1(t, T) \mathbb{E}_t \left[\zeta_2(t, T) \frac{1}{\eta_2(t, T)} \right] \quad (\text{B2})$$

$$\mathbb{E}_t \left[\zeta_1(t, T) \frac{W_T}{\eta_1(t, T)} \right] \mathbb{E}_t \left[\zeta_2(t, T) \frac{1}{\eta_2(t, T)} \right] = W_t. \quad (\text{B3})$$

Equation (B2) can be rewritten as

$$W_T = (\hat{\delta})^{-\frac{1}{\gamma}} (\zeta_1(t, T))^{-\frac{1}{\gamma}} \eta_1(t, T) \Pi_t. \quad (\text{B4})$$

Substitute equation (B4) into (B3) and solve for $\hat{\delta}$, then eliminate $\hat{\delta}$ from (B4) by substituting the value of $\hat{\delta}$, to obtain

$$w_T^* \equiv W_T^*/\Pi_T = \eta_2^{-1}(t, T) w_t \zeta_1(t, T)^{-\frac{1}{\gamma}} F_1(t, T)^{-1} F_2(t, T)^{-1}, \quad (\text{B5})$$

where $F_1(t, T) = \mathbb{E}_t \left[\zeta_1(t, T)^{1-\frac{1}{\gamma}} \right]$ and $F_2(t, T) = \mathbb{E}_t \left[\zeta_2(t, T)/\eta_2(t, T) \right] = \exp^{(\xi_u^2 - \xi_u \phi_u)(T-t)}$.

Substituting w_T^* given in equation (B5) into the investor's J function (20) yields

$$J = \frac{(w_t)^{1-\gamma}}{1-\gamma} F_1(t, T)^\gamma F_2(t, T)^{\gamma-1} F_3(t, T), \quad (\text{B6})$$

where $F_3(t, T) = E_t [\eta_2(t, T)^{\gamma-1}] = \exp\left[\frac{(1-\gamma)(2-\gamma)}{2} \xi_u^2(T-t)\right]$. The expression for $\psi_1(t, T)$ is derived by substituting for $F_1(t, T)$, $F_2(t, T)$ and $F_3(t, T)$.

The proof of Theorem 2, which is similar to that of Theorem 1, is omitted and is available from the authors on request.

2. Proof of Theorem 3

Define $G_s \equiv G[M_s, r, s] \equiv E_s \left[\frac{W_T^* M_T}{\Pi_T M_s} \right]$ as the (real) value at time s of the optimally chosen terminal payoff W_T^* . Then, using the definition of W_T^* , equation (16),

$$G_s = \frac{W_t F_1(s, T) F_2(s, T) \zeta_1(t, s)^{-\frac{1}{\gamma}}}{\Pi_t F_1(t, T) F_2(t, T) \eta_2(t, s)}, \quad (\text{B7})$$

where $F_1(t, T)$ and $F_2(t, T)$ are given in equations (17) - (18). Note that $F_2(t, T)$ and $F_2(s, T)$ are functions of horizon only, while $F_1(t, T)$ is a function of the horizon and the realization of state variable at time t , r_t . In contrast, $F_1(s, T)$ is a function of the current state variable r_s . More specifically, $\ln F_1(s, T)$ is a summation of a function of $T - s$ and $-\left(1 - \frac{1}{\gamma}\right) B(t, T)r_s$. Therefore, the stochastic terms of G_s come from $\zeta_1(t, s)^{-\frac{1}{\gamma}}$, $\eta_2(t, s)$, and r_s , while the other terms are deterministic functions of the horizon $T - s$. Using Ito's Lemma, we know that the stochastic terms of $d \ln G_s$ will be from those of $-d \ln \eta_2$, $-\frac{1}{\gamma} d \ln \zeta_1$ and $-\left(1 - \frac{1}{\gamma}\right) B(t, T)dr$.

Thus, the (log) instantaneous real return on optimally invested wealth is:

$$d \ln G_s = g_1(r, T - s)dt - \frac{\phi_S}{\gamma} dz_s - \left[\frac{\phi_r}{\gamma} + \left(1 - \frac{1}{\gamma}\right) B(s, T) \sigma_r \right] dz_r - \frac{\phi_\pi}{\gamma} dz_\pi - \xi_u dz_u, \quad (\text{B8})$$

where the drift term $g_1(r, s)$ is a function of current real interest rate r_s and the remaining investment horizon $T - s$.

Now consider the log real return on portfolio \mathbf{x}^* , which is a vector of optimal proportion of wealth invested in stock, a bond with maturity T_1 and a bond with maturity T_2 . The remaining wealth, $\mathbf{1} - \mathbf{i}'\mathbf{x}^*$, is invested in cash (a nominal instantaneous risk free asset). The nominal wealth process is given by:

$$\frac{dW}{W} = (R_f + \mathbf{x}'\Lambda) dt + \mathbf{x}'\sigma dz. \quad (\text{B9})$$

The log real wealth process, $\ln w = \ln W - \ln \Pi$, is derived by using the inflation process (7) and Ito's Lemma:

$$\begin{aligned} d \ln w &= g_2(r, T - s)dt + (x_1\sigma_s - \xi_S) dz_s - [(x_2B_1 + x_3B_2)\sigma_r + \xi_r] dz_r \\ &\quad - [(x_2C_1 + x_3C_2)\sigma_\pi + \xi_\pi] dz_\pi - \xi_u dz_u. \end{aligned} \quad (\text{B10})$$

Since strategy x^* yields the terminal payoff w_T^* , it follows that the coefficients in equations (B8) and (B10) are identical. Equating coefficients yields (34).

The proof of Theorem 4, which is similar to that of Theorem 3, is omitted and is available from the authors on request.

3. Proof of Proposition 2

Suppose without loss of generality that $\kappa > \alpha$. Then, allowing bond maturity, τ , to vary, a bond's return loading on the real interest rate, B , is an increasing convex function of its loading on inflation, C . Then the set of achievable factor loading combinations is defined by $\mathcal{S} = \left\{ (B, C) \mid \frac{C_{\tau_{max}}}{B_{\tau_{max}}} B \geq C \geq \frac{1 - (1 - \kappa B)^{\frac{\alpha}{\kappa}}}{\alpha} \right\}$. Any point in this set, (B_p, C_p) , can be achieved by a convex combination of cash $(0, 0)$ and the loadings of a single bond with maturity $\tau^* \leq \tau_{max}$, such that $\frac{C_{\tau^*}}{B_{\tau^*}} = \frac{C_p}{B_p}$, where the weight on the bond is $\frac{B_p}{B_{\tau^*}}$.

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Footnotes

¹Wachter (1999) assumes that the innovation in the equity premium is perfectly correlated with the stock return.

²Brennan and Xia (1999) allow for a generalized Vasicek process.

³For completeness, we analyze both a utility of lifetime consumption model and a utility of final wealth model. These are essentially equivalent.

⁴While the assumption of constant risk premia may seem restrictive, Bossaerts and Hillion (1999) find no evidence of *out-of-sample* excess return predictability in fourteen countries using as potential predictors lagged excess returns, January dummies, bond and bill yields, dividend yields, etc.

⁵Perhaps because monetary policy has a short run impact on the real interest rate.

⁶The linear approximation of CV does not become exact even in a continuous time setting except for the special case of *unit* intertemporal substitution, which corresponds to log utility of consumption, and, in this case, there is no closed form solution for the indirect utility function (Giovannini and Weil (1989)).

⁷By the shadow real interest rate, we mean the instantaneous real return that would prevail for an asset whose instantaneous real return was non-stochastic given the pricing kernel, if such an asset were to exist. In our model, there is no instantaneous real riskless asset. However, as we shall show below, when the rate of inflation is spanned by the nominal returns on assets, as will

be the case if the expected rate of inflation is unobservable and must be inferred from the price level realization, the model allows for the construction of a portfolio of nominal assets whose instantaneous real return is riskless.

⁸Even countries such as Canada, the United States and the United Kingdom, which have inflation indexed bonds, have them for only a few (long) maturities.

⁹We are simplifying by ignoring labor income and the consumption-investment decision.

¹⁰Formally, the *martingale multiplicity* of the investor's information structure is equal to two (Duffie and Huang (1985)). This condition will be satisfied if, for example, the only information that is available about expected inflation is derived from the historical inflation series.

¹¹See Cox and Huang (1989) for a formal analysis.

¹²See Pagés (1987) and Karatzas et. al. (1991) for a rigorous proof. In particular, see Pagés (1987) Proposition 2.15.

¹³Wachter (1999) provides a similar interpretation in a complete market setting where there is constant interest rate and stochastic equity risk premium and where the investor only invests in stock and cash.

¹⁴Where the tangency is from the nominal riskless rate R_f to the nominal risky efficient set.

¹⁵The other component of the loading, ξ_r/σ_r , is part of the portfolio hedge against changes in the price level, Π .

¹⁶Note that $e^{\kappa(T-t)}$ can be written as $\sum_{n=0}^{\infty} \frac{(\kappa(T-t))^n}{n!}$, which is $1 + \kappa(T-t) + \frac{(\kappa(T-t))^2}{2} + \dots$. Since $\kappa(T-t)$ is positive, the omitted terms in the series expansion of $e^{\kappa(T-t)}$ are all positive. Thus, $[\kappa(T-t) + 1 - e^{\kappa(T-t)}] < 0$.

¹⁷The certainty equivalent wealth is that amount of wealth at the horizon that would leave the investor indifferent between receiving it for sure and having \$1 today to invest in the stock and bonds up to the horizon.

¹⁸It is important to remember that the introduction of a new security may change the prices of existing securities.

¹⁹The expected utility under the myopic strategy can be simply calculated by inserting the myopic portfolio allocation in the process for real wealth and calculating $E[w^{1-\gamma}/\gamma]$.

²⁰The constrained optimization problem can only be solved numerically and the availability of indexed bonds significantly increases the dimension of the problem. Therefore, we do not consider indexed bonds in our calculations in Section IV. Campbell and Viceira (1999) offer a welfare analysis of indexed bonds for constrained investors but do not allow for maturity choice.

²¹See Harvey (1989) for the discussion of the Kalman filter and its estimation. See, for example, de Jong (1998) for a detailed discussion of estimating term structure parameters using the Kalman filter technique.

²²The data on yields were kindly provided by David Backus. The CPI data are from the Bureau of Labor Statistics.

²³These estimates are comparable to those of Campbell and Viceira (1999) for the period of 1952 to 1996. The slow mean reversion in the estimated expected rate of inflation is consistent with Fama and Gibbons (1982, p. 305) who report that “the short term expected inflation rate is close to a random walk.”

²⁴Note that by ignoring the standard errors of $\bar{\pi}$ and \bar{r} , the standard errors of all the parameters are understated.

²⁵The “realized” real interest rate for a year is the difference between the nominal interest rate and the realized rate of inflation. All data are from Shiller (1989).

²⁶No significant relation was found in the data between unexpected inflation and innovations in the state variables or stock return.

²⁷We analyze the optimal strategies only for the terminal wealth problem and not for the lifetime consumption problem since, as shown in Theorem 4, the optimal stock holding is the same for the two problems and the bond allocation for the second problem with horizon T is a weighted average of the allocations for the terminal wealth problem for horizons from 0 to T .

²⁸See, for example, Siegel (1998, p 283), “Stocks should constitute the overwhelming proportion of all long-term financial portfolios.”