QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXII, NUMBER 1
MARCH 2004, PAGES 135-161

DYNAMIC BEHAVIOR OF THE INEXTENSIBLE STRING

By

R. W. DICKEY

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706

Abstract. The two dimensional dynamic behavior of a geometrically exact inextensible string is discussed. A variety of exact solutions are described and various asymptotic theories are derived. The similarity between the motion of the inextensible string and galactic motion is described.

1. Introduction. Problems involving the behavior of the inextensible string have been studied for many years. One of the earliest successes in the calculus of variations was the demonstration that the inextensible string, hanging under gravity, would have the shape of a catenary (cf. [1]). More recently Kolodner [2] solved the problem of finding all shapes of the inextensible string, fixed at one end, hanging under gravity, and rotating with constant angular velocity about a vertical axis. In this paper we discuss the two dimensional dynamic behavior of an inextensible string.

We consider a string whose reference state (the state in which the string is stress-free) is a straight line from $x = 0$ to $x = 1$. A point on the string $(x, 0)$ will have coordinates $(x + u(x,t), w(x,t))$ and will be in a state of stress $T(x,t)$ at some later time t. The equations governing the motion of this string are easily derived (cf. Antman [3]). They consist of a pair of balance of force equations and a constraint that guarantees that the string is inextensible. The equations are

$$u_{tt} - \frac{\partial}{\partial x} T(1 + u_x) = 0 \tag{1.1}$$

$$w_{tt} - \frac{\partial}{\partial x} T w_x = 0 \tag{1.2}$$

$$(1 + u_x)^2 + w_x^2 = 1. \tag{1.3}$$

For simplicity we have assumed the density is 1. The equations (1.1), (1.2), and (1.3) are geometrically exact; i.e., there are no assumptions on the magnitude of the displacements. Indeed, the only assumptions are that the tension (compression) is tangent to the string and the string is inextensible.

Received May 17, 2002.
2000 *Mathematics Subject Classification.* Primary 74-XX.
E-mail address: dickey@math.wisc.edu

Presumably, in addition to the equations (1.1), (1.2), and (1.3), we need to prescribe appropriate initial and boundary conditions. For the linear string and for any extensible string the most obvious boundary condition is fixed ends, i.e., $u(x,t)$ and $w(x,t)$ vanishing at $x = 0$ and $x = 1$. In the case of the inextensible string these boundary conditions would imply the string could not move. Thus at least one end of the string must be free, or if it is attached to a support, that support cannot be rigid. In the latter case there must be some relation between the elastic properties of the support and the motion of the end of the string.

In Sec. 2 we consider the energy associated with equations (1.1), (1.2), and (1.3) and in particular the conditions under which energy is conserved. In Sec. 3 the equations (1.1), (1.2), and (1.3) are rewritten in polar coordinates and we look for solutions in which the radius vector depends only on the polar angle.

In Sec. 4 we introduce a new variable $\psi(x,t)$—the angle the tangent to the string makes with the positive x axis. This new variable reduces equations (1.1), (1.2), and (1.3) to the pair of equations

$$\psi_t^2 - T\psi_x^2 + T_{xx} = 0 \tag{1.4}$$

$$\psi_{tt} - T\psi_{xx} - 2T_x\psi_x = 0. \tag{1.5}$$

The equilibrium problem for equations (1.4) and (1.5) is discussed in Sec. 5 and various exact solutions for the time dependent problem are discussed in Sec. 6.

In Sec. 7 it is shown that, under certain conditions, equations (1.4) and (1.5) can be reduced to a single equation for $\psi(x,t)$. From this equation there follows two distinct asymptotic theories—one in which the amplitude of the angle is small and another in which the amplitude is large. These two theories are discussed in Sections 8 and 9. In Sec. 10 we describe a theory which assumes small angular velocity. In Sec. 11 we compare the similarity between the motions of galaxies and the motion of inextensible strings.

2. Energy considerations. It is of interest to discuss the energy associated with solutions of (1.1), (1.2), and (1.3). If Eq. (1.1) is multiplied by u_t and Eq. (1.2) is multiplied by w_t, the two equations are added together, and then integrated from $x = 0$ to $x = 1$, we find after an integration by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + w_t^2)dx + \frac{1}{2}\int_0^1 T\frac{\partial}{\partial t}((1+u_x)^2 + w_x^2)dx \\ = T((1+u_x)u_t + w_x w_t)|_{x=0}^{x=1}.\end{aligned} \tag{2.1}$$

The constraint (1.3) implies the second integral in (2.1) vanishes, so that the energy identity becomes

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + w_t^2)dx = T(1+u_x, w_x)\cdot(u_t, w_t)|_{x=0}^{x=1}. \tag{2.2}$$

The energy of the inextensible string is entirely kinetic. The string can change shape, but it cannot compress or stretch and hence cannot store energy.

As a special case, consider the situation in which the end is held fixed at $x = 0$. If energy is conserved, (2.2) requires that either $T(1,t) = 0$ or $(1+u_x, w_x)\cdot(u_t, w_t) = 0$ at $x = 1$. Thus either the force vanishes or the velocity is orthogonal to the string at $x = 1$.

The equations (1.1), (1.2), and (1.3) also have a variational formulation. In particular, the problem can be written in the form: Find the stationary values of

$$\frac{d}{dt}\int_0^t\int_0^1((u_t^2+w_t^2)-T(x,t)((1+u_x^2)+w_x^2))dxdt \tag{2.3}$$

where the Lagrange multiplier $T(x,t)$ is to be chosen so that the constraint (1.3) is satisfied. $T(x,t)$ is, of course, the stress in the string.

3. Polar coordinates. It is useful to rewrite equations (1.1), (1.2), and (1.3) in polar coordinates. Introduce the change of variable

$$x+u=r\cos(\theta) \tag{3.1}$$

$$w=r\sin(\theta). \tag{3.2}$$

Equations (1.1) and (1.2) can be written in terms of r and as

$$r_{tt}-r\theta_t^2=\frac{\partial}{\partial x}Tr_x-Tr\theta_x^2 \tag{3.3}$$

$$r\theta_{tt}+2r_t\theta_t=\frac{\partial}{\partial x}Tr\theta_x+Tr_x\theta_x. \tag{3.4}$$

The constraint (1.3) becomes

$$r_x^2+r^2\theta_x^2=1. \tag{3.5}$$

We consider the question of whether the equations (3.3), (3.4), and (3.5) have a solution of the form

$$r=F(\theta). \tag{3.6}$$

If there is a solution of this form, then the constraint (3.5) implies that

$$(F'(\theta)^2+F(\theta)^2)\theta_x^2=1. \tag{3.7}$$

Assuming that θ is an increasing function of x, Eq. (3.7) implies

$$\int H(\theta)d\theta=x+A(t) \tag{3.8}$$

where $A(t)$ is an arbitrary function of t and

$$H(\theta)=\sqrt{F'(\theta)^2+F(\theta)^2}. \tag{3.9}$$

It is a consequence of (3.8) that

$$\theta_x=\frac{1}{H(\theta)} \tag{3.10}$$

$$\theta_{xx}=\frac{-F'(F''+F)}{H^4} \tag{3.11}$$

and

$$\theta_t=\frac{\dot{A}}{H} \tag{3.12}$$

$$\theta_{tt}=\frac{\ddot{A}}{H}-\frac{\dot{A}^2F'(F''+F)}{H^4}. \tag{3.13}$$

In addition, it is a consequence of (3.6), (3.10), (3.11), (3.12), and (3.13) that

$$r_x = \frac{F'}{H} \tag{3.14}$$

$$r_{xx} = \frac{F(F''F - (F')^2)}{H^4} \tag{3.15}$$

$$r_t = \frac{\dot{A}F''}{H} \tag{3.16}$$

$$r_{tt} = \frac{\ddot{A}F''}{H} + \frac{\dot{A}^2 F(F''F - (F')^2)}{H^4}. \tag{3.17}$$

Placing these equations into (3.3) and (3.4), we find

$$\frac{F'(\ddot{A} - T_x)}{H} + \frac{F(FF'' - 2(F')^2 - F^2)(A^2 - T)}{H^4} = 0 \tag{3.18}$$

$$\frac{F(\ddot{A} - T_x)}{H} - \frac{F'(FF'' - 2(F')^2 - F^2)(A^2 - T)}{H^4} = 0. \tag{3.19}$$

Equations (3.18) and (3.19) imply that either

$$T_x = \ddot{A} \tag{3.20}$$

$$T = \dot{A}^2 \tag{3.21}$$

or the determinant of the coefficients in (3.18) and (3.19) vanishes. This determinant is easily computed to show that a necessary and sufficient condition for the determinant to vanish is that

$$FF'' - 2(F')^2 - F^2 = 0. \tag{3.22}$$

If (3.22) is satisfied, equations (3.18) and (3.19) imply that (3.20) must be satisfied. Thus the force on the string is

$$T = \ddot{A}(t)x + B(t). \tag{3.23}$$

Equation (3.22) can be solved explicitly. Introduce the change of variable

$$F = \frac{1}{G}. \tag{3.24}$$

Equation (3.22) reduces to

$$G'' + G = 0. \tag{3.25}$$

Thus the general solution of (3.22) can be written as

$$F = \frac{C_1}{\cos(\theta + C_2)} \tag{3.26}$$

where C_1 and C_2 are arbitrary constants. Given the expression (3.26) for F, there is no difficulty in showing that

$$(1 + u_x, w_x) = (\sin C_2, \cos C_2); \tag{3.27}$$

i.e., the unit tangent is a constant vector. Equivalently, the string is a straight line accelerating through space due to the force $T(x, t)$.

If (3.22) is not satisfied, then the only alternative is that (3.20) and (3.21) are satisfied. Equation (3.21) implies that

$$T_x = 0. \tag{3.28}$$

Thus (3.20) requires that

$$\ddot{A}(t) = 0 \tag{3.29}$$

or

$$A(t) = Ct + D \tag{3.30}$$

and from (3.21)

$$T = C^2. \tag{3.31}$$

Given (3.30) and (3.31), equation (3.18) and (3.19) is satisfied regardless of the choice of $F(\theta)$; i.e., $r = F(\theta)$ is a solution when T is given by (3.31) and θ is given by

$$\int H(\theta)d\theta = x + Ct + D. \tag{3.32}$$

It is convenient to denote the solution of (3.32) as

$$\theta = G(x + Ct + D). \tag{3.33}$$

If (3.6) represents any curve, then

$$r = F(G(x + Ct + D)). \tag{3.34}$$

Equations (3.33) and (3.34) satisfy equations (3.3), (3.4), and (3.5) when T is given by (3.31).

At time $t = 0$, the shape of the string is $r = F(G(x + D))$. At some later time the position of points on the string is given by (3.34); i.e., each point on the string moves along the curve (3.6). An elementary example of a string moving along a non-closed curve is the case $r = e^{\lambda\theta}$. Equation (3.32) implies

$$\int \sqrt{\lambda^2 + 1}e^{\lambda\theta}d\theta = x + Ct + D \tag{3.35}$$

or

$$\theta = \frac{1}{\lambda}\ln\left(\frac{\lambda}{\sqrt{\lambda^2 + 1}}(x + Ct + D)\right). \tag{3.36}$$

Every point on the string remains on the curve $r = e^{\lambda\theta}$. The actual position at time t is given by

$$x + u = e^{\lambda\theta}\cos(\theta) \tag{3.37}$$

$$w = e^{\lambda\theta}\sin(\theta) \tag{3.38}$$

where θ is given by (3.36).

If (3.6) describes a closed curve, then points on the string simply move around the curve. A special case occurs when the string itself is closed, i.e., when the length of the closed curve described by (3.6) is 1. The string would appear to be sitting in space without moving since the shape of the string would not change. However, each point on the string could be moving with large angular velocity. The simplest example would be the circle $r = 1/2\pi$. This is a circle of circumference 1. Equation (3.32) implies

$$\theta = 2\pi(x + Ct + D). \tag{3.39}$$

In rectangular coordinates, the position of a point at any time t is

$$x + u = \frac{1}{2\pi}\cos(2\pi(x + Ct + D)) \tag{3.40}$$

$$w = \frac{1}{2\pi}\sin(2\pi(x + Ct + D)). \tag{3.41}$$

Each point on the string moves in a circle at constant angular velocity. In the case of any simple closed curve the situation is similar, except that the angular velocity would not be a constant.

4. Reformulation of the equations. It is possible to remove the constraint (1.3) by introducing a change of variable. Define

$$1 + u_x = \cos(\psi) \tag{4.1}$$

$$w_x = \sin(\psi). \tag{4.2}$$

Since the vector $(1 + u_x, w_x)$ is the unit tangent, $\psi(x,t)$ is the angle the unit tangent makes with the x axis. In order to write the equations (1.1) and (1.2) in terms of ψ, differentiate both equations w.r.t. x, so that

$$u_{xtt} - \frac{\partial^2}{\partial x^2}T(1 + u_x) = 0 \tag{4.3}$$

$$w_{xtt} - \frac{\partial^2}{\partial x^2}Tw_x = 0. \tag{4.4}$$

Thus equations (4.1) and (4.2) imply

$$\frac{\partial^2}{\partial t^2}\cos(\psi) - \frac{\partial^2}{\partial x^2}T\cos(\psi) = 0 \tag{4.5}$$

$$\frac{\partial^2}{\partial t^2}\sin(\psi) - \frac{\partial^2}{\partial x^2}T\sin(\psi) = 0. \tag{4.6}$$

After simplification, equations (4.5) and (4.6) may be rewritten in the form (1.4) and (1.5).

If equations (1.4) and (1.5) (or (4.5) and (4.6)) are solved for $\psi(x,t)$, equations (4.1) and (4.2) require

$$x + u = a(t) + \int_0^x \cos(\psi(\xi,t))d\xi \tag{4.7}$$

$$w = b(t) + \int_0^x \sin(\psi(t))d. \tag{4.8}$$

Given that $\psi(x,t)$ satisfies (4.5) and (4.6), $a(t)$ and $b(t)$ are determined by the requirement that (4.7) and (4.8) satisfy (1.1) and (1.2). It is a consequence of (4.1) that

$$\frac{\partial}{\partial x}T(1 + u_x) = \frac{\partial}{\partial x}T\cos(\psi). \tag{4.9}$$

In addition, differentiation of (4.7) w.r.t. t yields

$$\begin{aligned} u_{tt} &= \ddot{a}(t) + \int_0^x \frac{\partial^2}{\partial t^2} \cos(\psi(\xi, t))d\xi \\ &= \ddot{a}(t) + \int_0^x \frac{\partial^2}{\partial \xi^2} T \cos(\psi(\xi, t))d\xi \\ &= \ddot{a}(t) + \frac{\partial}{\partial \xi} T(\xi, t) \cos(\psi(\xi, t))|_{\xi=0}^{\xi=x} \end{aligned} \tag{4.10}$$

(cf. (4.5)). Subtracting (4.9) from (4.10) yields

$$u_{tt} - \frac{\partial}{\partial x} T(1 + u_x) = \ddot{a}(t) - \frac{\partial}{\partial x} T(x, t) \cos(\psi(x, t))|_{x=0}. \tag{4.11}$$

If equations (1.1) and (1.2) are to be satisfied, we find

$$\ddot{a}(t) = \frac{\partial}{\partial x} T \cos(\psi)|_{x=0}. \tag{4.12}$$

An identical argument of (4.8) implies

$$\ddot{b}(t) = \frac{\partial}{\partial x} T \sin(\psi)|_{x=0}. \tag{4.13}$$

If $u(0, t) = w(0, t) = 0$ so that the end of the string is fixed or if $u(0, t)$ and $w(0, t)$ are linear functions of t, in which case the string is in a reference frame in which the origin is not accelerating, then $\ddot{a}(t)$ and $\ddot{b}(t)$ vanish. In this case equations (4.12) and (4.13) imply that either

$$T(0, t) = T_x(0, t) = 0 \tag{4.14}$$

or

$$T_x(0, t) = \psi_x(0, t). \tag{4.15}$$

If neither of the conditions (4.14) or (4.15) is satisfied, then the string is in a reference frame whose origin is accelerating. If we were looking for solutions of (1.4) and (1.5) which were symmetric about the origin, i.e., solutions which satisfy $\psi(-x, t) = \psi(x, t)$ and $T(-x, t) = T(x, t)$, it follows that $\psi_x(0, t) = T_x(0, t) = 0$. Thus $\ddot{a}(t) = \ddot{b}(t) = 0$. If the string is symmetric about the origin of the reference frame, then the reference frame is either stationary or its origin is moving with constant velocity. It cannot be accelerating.

5. The static problem. Before treating the full dynamic problem, it is of interest to look for solutions which are independent of t. In this case equations (1.4) and (1.5) become

$$\frac{d^2T}{dx^2} - T\left(\frac{d\psi}{dx}\right)^2 = 0 \tag{5.1}$$

$$T\frac{d^2\psi}{dx^2} + 2\frac{dT}{dx}\frac{d\psi}{dx} = 0. \tag{5.2}$$

Equation (5.2) can be rewritten as

$$\frac{d}{dx} T^2 \frac{d\psi}{dx} = 0 \tag{5.3}$$

or equivalently

$$T^2\frac{d\psi}{dx} = C = \text{ constant.} \tag{5.4}$$

If the string is fixed at $x = 0$, then either $T(0) = 0$ or $\psi'(0) = 0$. In either case $C = 0$. Thus either $\psi'(x) = 0$ or $T(x) = 0$. If $\psi'(x) = 0$, then Eq. (5.1) implies $T''(x) = 0$ and hence $T(x)$ is a constant ($T'(0) = 0$). In this case the string is simply a straight line under tension. If $T(x) = 0$, then Eq. (5.1) is satisfied regardless of the choice of ψ. In the absence of tension, every length preserving curve is a solution.

If $C \neq 0$, the end of the string at $x = 0$ is accelerating. Equation (5.4) combined with (5.1) yields

$$T'' - \frac{C^2}{T^3} = 0. \tag{5.5}$$

The general solution of (5.5) is

$$T = \sqrt{\frac{C^2}{A^2} + A^2(x+B)^2} \tag{5.6}$$

where A and B are arbitrary constants. It is a consequence of (5.4) that

$$\psi = \tan^{-1}\left(\frac{A^2}{C}(x+B)\right) + D. \tag{5.7}$$

For simplicity choose $D = 0$ in (5.7). In this case an easy calculation shows that the string is undergoing an acceleration $\ddot{a} = 0$ and $\ddot{b} = A$. It can be directly verified that (5.7) is a catenary. This is what is to be expected. There is no gravity, but the string is undergoing a constant acceleration up to the ordinate.

6. Time dependent solutions. It is easily verified that equations (1.4) and (1.5) have the solutions

$$T = C^2 \tag{6.1}$$

$$\psi(x,t) = F(x+Ct) \tag{6.2}$$

for every twice differentiable function $F(x)$; i.e., equations (1.4) and (1.5) have travelling wave solutions. None of the solutions correspond to a fixed endpoint solution. In fact, combining (4.9), (4.14), (4.15), (6.1), and (6.2), it is an elementary calculation to show that

$$x + u = At + B + \int_0^{x+Ct} \cos F(\xi)d\xi \tag{6.3}$$

$$w = Dt + E + \int_0^{x+Ct} \sin F(\xi)d\xi \tag{6.4}$$

where A, B, D, and E are arbitrary constants.

It is of interest to give some examples of these solutions. Since the string is accelerating through space, we will plot the solutions in a moving reference frame; i.e., we will plot

(V, W) where

$$V = x + u - a(t) = \int_0^x \cos F(\xi + Ct)d\xi \tag{6.5}$$

$$W = w - b(t) = \int_0^x \sin F(\xi + Ct)d\xi \tag{6.6}$$

for various choices of F and different values of t. In addition, we will plot the motion of the reference frame

$$a(t) = \int_0^{Ct} \cos F(\xi)d\xi \tag{6.7}$$

$$b(t) = \int_0^{Ct} \sin F(\xi)d\xi. \tag{6.8}$$

In the elementary case $F(x) = x$, the integrals in (6.5), (6.6), (6.7), and (6.8) can be evaluated explicitly. The reference frame moves in a circle of radius one and center $(0, 1)$. In the moving reference frame the string rotates periodically about the origin without changing shape. In Fig. 6.1 the case $F(x) = x^3$ $(C = 1)$ is plotted. The string moves into a tighter and tighter spiral. The reference frame also moves in a spiral (Fig. 6.1(e)) and approaches the point

$$a = \int_0^\infty \cos(x^3)dx \tag{6.9}$$

$$b = \int_0^\infty \sin(x^3)dx \tag{6.10}$$

as $t \to \infty$. In Fig. 6.2 we have plotted the solution in the case $F = 1/(x+1)$ $(C = 1)$. In this case the string starts as a spiral and unwinds. As $t \to \infty$, $V \to x$ and $W \to 0$. The reference frame moves in a spiral and runs off to infinity as $t \to \infty$.

Another exact solution of (1.4) and (1.5) is the "whirling string",

$$\psi = Ct \tag{6.11}$$

$$T = T_0 + \frac{C^2}{2}(1 - x^2) \tag{6.12}$$

where C and T_0 are constants. Equations (6.11) and (6.12) satisfy (4.15) so that the end at $x = 0$ is fixed (or moving with constant velocity). The string whirls around the origin with constant angular velocity $\psi_t = C$. Such solutions are well known even in the case of the extensible string (cf. [3]).

This particular solution is a special case. Equations (1.4) and (1.5) have a solution of the form

$$\psi(x, t) = \alpha(t) + \beta(x) \tag{6.13}$$

$$T(x, t) = \gamma(t)\delta(x). \tag{6.14}$$

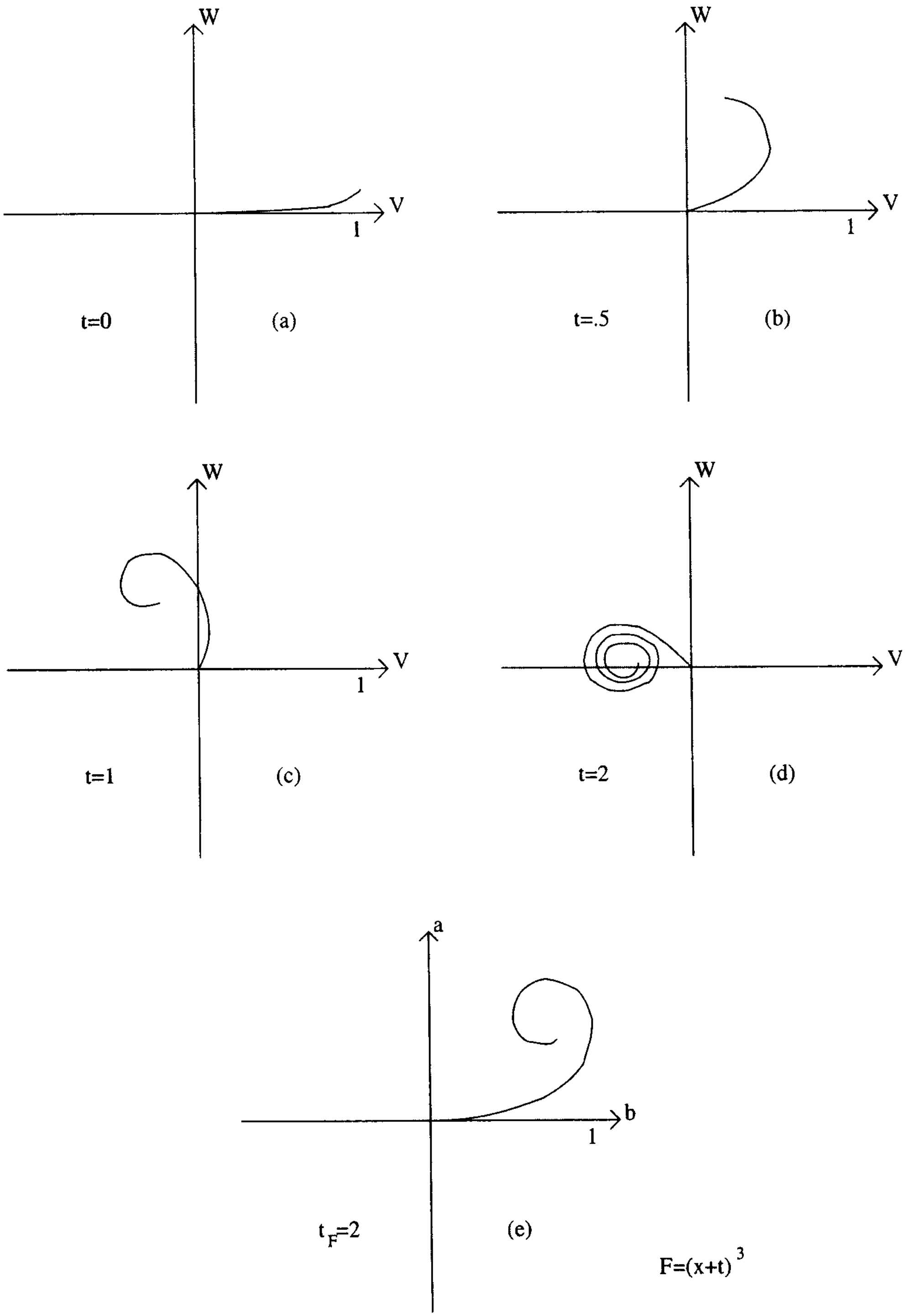

FIG. 6.1

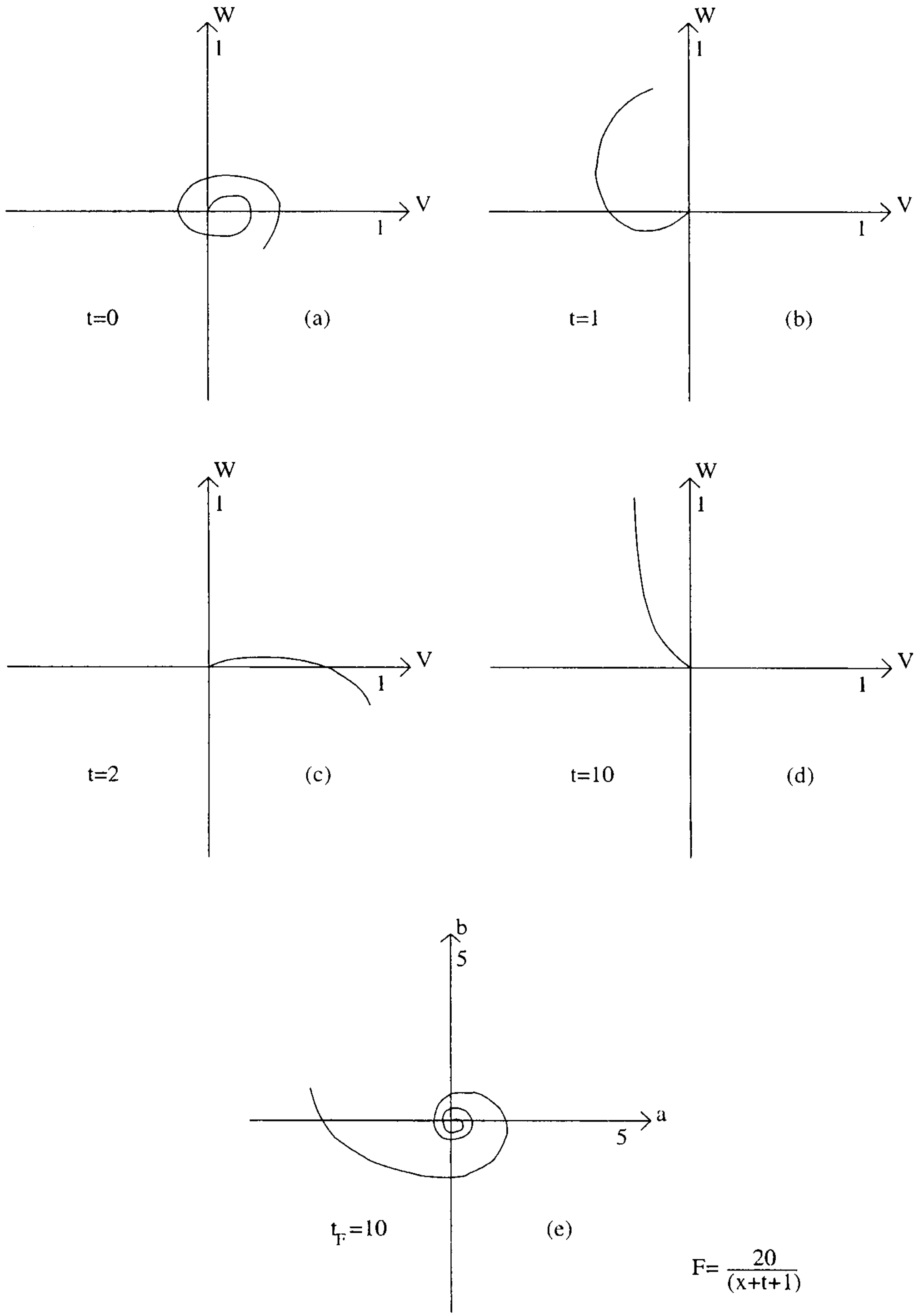

FIG. 6.2

Placing (6.13) and (6.14) into equations (1.4) and (1.5), we find that (6.13) and (6.14) satisfy (1.4) and (1.5) if

$$\frac{\dot{\alpha}^2}{\gamma} = \lambda \tag{6.15}$$

$$\frac{\ddot{\alpha}}{\gamma} = \mu \tag{6.16}$$

$$\delta'' - (\beta')^2\delta = -\lambda \tag{6.17}$$

$$\delta\beta'' + 2\delta'\beta' = \mu \tag{6.18}$$

where λ and μ are separation constants.

The whirling string solution is a special case of (6.15)–(6.18). If $\alpha = Ct$, then equations (6.15) and (6.16) imply that $\mu = 0, \gamma$ is a constant, $\lambda = C^2$, and conversely. There is no loss in assuming $\gamma = 1$. Equation (6.18) reduces to

$$\frac{d}{dx}\delta^2\beta' = 0 \tag{6.19}$$

so that

$$\delta^2\beta' = K. \tag{6.20}$$

Combining (6.17) and (6.20) leads to the equation for the determination of $\delta(x)$:

$$\delta'' - \frac{K^2}{\delta^3} = -C^2. \tag{6.21}$$

If $K = 0$ and $C \neq 0$, then Eq. (6.20) implies that $\beta'(x) = 0$; i.e., the string is a straight line. Thus, up to an additive constant, $\psi = Ct$ and $T = \delta(x)$. Equations (4.12) and (4.13) imply the motion of the end of the string ($x = 0$) is

$$R^2 = a^2 + b^2 = \frac{\delta'(0)^2}{C^4}. \tag{6.22}$$

The end of the string moves in a circle of radius R. The solutions of (6.21) with $K = 0, \delta'(0) = -RC^2$, and $\delta(1) = 0$ are

$$\delta(x) = C^2(1-x)\left(R + \frac{1}{2} + x\right).$$

If $K \neq 0$, then the solutions are not straight lines and there are no solutions satisfying $\delta(1) = 0$.

All solutions of the form (6.13) and (6.14) have the property that they do not change shape as time evolves. In fact, it is clear from (6.13) that the string rotates about the origin with angular velocity $\psi_t = \dot{\alpha}(t)$. However, the shape is determined by $\beta(x)$. These solutions simply represent a string on a turntable turning with angular velocity $\dot{\alpha}(t)$.

If $\mu \neq 0$, then equations (6.15) and (6.16) imply

$$\ddot{\alpha} = \frac{\mu}{\lambda}\dot{\alpha}^2. \tag{6.23}$$

The general solution of (6.23) is

$$\alpha(t) = A - \frac{\lambda}{\mu}\ln\left(1 - \frac{\mu}{\lambda}Bt\right) \tag{6.24}$$

where $A = \alpha(0)$ and $B = \dot{\alpha}(0)$. For simplicity we will choose $A = 0$. It is a consequence of (6.15) that

$$\gamma = \frac{1}{\lambda} \frac{B^2}{(1 - \frac{\mu}{\lambda} Bt)^2}. \tag{6.25}$$

If $\mu B/\lambda > 0$, then the solution blows up in finite time. If $\mu B/\lambda < 0$, then the solution exists forever.

It is possible to write down an explicit solution of (6.17) and (6.18). It is easily verified that

$$\delta = x^2 \tag{6.26}$$

$$\beta = D + E \ln(x) \tag{6.27}$$

satisfies (6.17) and (6.18) if

$$2 - E^2 = -\lambda \tag{6.28}$$

$$3E = \mu. \tag{6.29}$$

We will choose $D = 0$ ($\beta(0) = 0$). If the string is in tension, (6.25) and (6.26) require that $\lambda > 0$.

An exact solution of (4.10) is given by (6.13) and (6.14) where $\alpha(t), \beta(x)$, and $\delta(x)$ are given by (6.24)–(6.27). It may be verified that this solution implies that $a(t) = 0$ and $b(t) = 0$. In rectangular coordinates the solution is (cf. (4.7) and (4.8))

$$x + u = \frac{x}{E^2 + 1}[\cos(\alpha(t) + E \ln x) + E \sin(\alpha(t) + E \ln x)] \tag{6.30}$$

$$w = \frac{x}{E^2 + 1}[\sin(\alpha(t) + E \ln x) - E \cos(\alpha(t) + E \ln x)]. \tag{6.31}$$

The tension in the string is

$$T(x,t) = \frac{1}{\lambda} \frac{B^2 x^2}{(1 - \frac{\mu}{\lambda} Bt)^2} \tag{6.32}$$

and the angular velocity is

$$\psi_t = \dot{\alpha}(t) = \frac{B}{1 - \frac{\mu}{\lambda} Bt}. \tag{6.33}$$

The curve described by (6.30) and (6.31) is a spiral. Note that if $\mu\beta/\lambda > 0$, then both $T(x,t)$ and $\psi_t(x,t)$ blow up as $t \to \lambda/\mu\beta$. If $\mu\beta/\lambda < 0$, then $T(x,t)$ and $\psi_t(x,t)$ approach zero as $t \to \infty$. Even though the angular velocity approaches zeros as $t \to \infty$, the spiral itself does not approach an equilibrium.

7. Asymptotic theories. If $\psi(x,t)$ has three continuous derivatives, then it is possible, in principle, to eliminate the tension T from equations (1.4) and (1.5). Differentiating Eq. (1.5) w.r.t. x, we obtain a third equation involving T, T_x, T_{xx}. In particular we find

$$\psi_{xtt} - T\psi_{xxx} - 3T_x\psi_{xx} - 2T_{xx}\psi_x = 0. \tag{7.1}$$

Equation (7.1) and equations (1.4) and (1.5) are three equations for the determination of T, T_x, T_{xx}.

The solution for T is found to be

$$T = \frac{2\psi_x \psi_{xtt} - 3\psi_{xx}\psi_{tt} + 4\psi_x^2\psi_t^2}{2\psi_x\psi_{xxx} - 3\psi_{xx}^2 + 4\psi_x^4} \tag{7.2}$$

provided the denominator of (7.2) does not vanish. Conversely, if ψ satisfies (1.5) where T is given by (7.2), then ψ also satisfies (1.4). To see this, rewrite (7.2) in the form

$$T(2\psi_x\psi_{xxx} - 3\psi_{xx}^2 + 4\psi_x^4) = 2\psi_x\psi_{xtt} - 3\psi_{xx}\psi_{tt} + 4\psi_x^2\psi_t^2. \tag{7.3}$$

Differentiation of (1.5) w.r.t. x gives (7.1). Using (7.1) and (1.5) to eliminate ψ_{xtt} and ψ_{tt} from (7.3) leads to (1.4).

If the equations (1.4) and (1.5) have solutions for which the denominator of (7.2) vanishes, i.e., if (1.4) and (1.5) have solutions for which

$$2\psi_x\psi_{xxx} - 3\psi_{xx}^2 + 4\psi_x^4 = 0, \tag{7.4}$$

then these solutions may not satisfy equations (1.5) and (7.2). It is of interest to decide which solutions, if any, are eliminated. One obvious case is $\psi_x = 0$, which implies $\psi = Ct$, i.e., the whirling string. In this case T, given by (7.2), is undefined. If $\psi_x \neq 0$, then Eq. (7.4) can still be solved explicitly.

If there is a value of x for which $\psi_x \neq 0$, then Eq. (7.4) can be rewritten as

$$\frac{d}{dx}\frac{\psi_{xx}^2}{\psi_x^3} + 4\psi_{xx} = 0. \tag{7.5}$$

Assuming ψ is an increasing function of x, (7.5) can be written as

$$\frac{d}{dx}\left(\frac{d}{dx}\frac{1}{\sqrt{\psi_x}}\right)^2 + \psi_{xx} = 0. \tag{7.6}$$

For simplicity, introduce the change of variable

$$\phi = \frac{1}{\sqrt{\psi_x}}. \tag{7.7}$$

In terms of ϕ, Eq. (7.6) becomes

$$\frac{d}{dx}\phi_x^2 + \frac{d}{dx}\frac{1}{\phi^2} = 0. \tag{7.8}$$

Integrating (7.8), we find that

$$\phi_x^2 + \frac{1}{\phi^2} = A^2. \tag{7.9}$$

This is an ordinary differential equation for ϕ in which t plays the role of a parameter. Rewriting (7.9) yields

$$\int \frac{\phi d\phi}{\sqrt{A^2\phi^2 - 1}} = x + B. \tag{7.10}$$

After an integration it is possible to solve for explicitness to find

$$\phi^2 = A^2(x + B^2) + 1/A^2 \tag{7.11}$$

so that (cf. (7.8))

$$\psi_x = \frac{A^2}{A^4(x + B)^2 + 1} \tag{7.12}$$

(note that the assumption that $\psi_x \neq 0$ at a point implies that ψ_x never vanishes). Integrating (7.12) leads to

$$\psi = \tan^{-1}(A^2(x+B)) + C. \tag{7.13}$$

The quantities A, B, and C in (7.13) may depend on t. Equation (7.13) defines a one parameter family of catenaries (cf. Sec. 5). In summary, the solutions of (1.4), (1.5), and (7.2) are identical except in the case of the whirling string or solutions of the form (7.13).

Although (1.5) and (7.2) constitute a single equation for the determination of ψ, that single equation is quite complicated and furnishes no obvious advantage over the original problem given by (1.4) and (1.5). However, this form of the equation is very useful in developing certain asymptotic theories.

For this purpose we introduce amplitude explicitly as a parameter; i.e., we define $\phi(x,t)$ as

$$\psi(x,t) = A\phi(x,t). \tag{7.14}$$

In this case (1.5) and (7.2) become

$$\phi_{tt} - T\phi_{xx} - 2T_x\phi_x = 0 \tag{7.15}$$

$$T = \frac{2\phi_x\phi_{xtt} - 3\phi_{xx}\phi_{tt} + 4A^2\phi_x^2\phi_t^2}{2\phi_x\phi_{xxx} - 3\phi_{xx}^2 + 4A^2\phi_x^4}. \tag{7.16}$$

We consider two cases:

(a) the small amplitude case $A^2 \ll 1$, and

(b) the large amplitude case $A^2 \gg 1$.

In case (a), Eq. (7.16) implies that to first order

$$T = \frac{2\phi_x\phi_{xtt} - 3\phi_{xx}\phi_{tt}}{2\phi_x\phi_{xxx} - 3\phi_{xx}^2}. \tag{7.17}$$

In the large amplitude case (case (b)), Eq. (7.16) implies that to first order (in $1/A^2$)

$$T = \frac{\phi_t^2}{\phi_x^2}. \tag{7.18}$$

8. Small amplitude theory. The small amplitude theory consists of equations (7.15) and (7.17). This theory is equivalent to a linear theory. In order to see this, rewrite equation (7.17) in the form

$$3\phi_{xx}(\phi_{tt} - T\phi_{xx}) = 2\phi_x(\phi_{xtt} - T\phi_{xxx}). \tag{8.1}$$

Differentiating (7.15) w.r.t. x yields

$$\phi_{xtt} - T\phi_{xxx} = 3T_x\phi_x + 2T_{xx}\phi_x. \tag{8.2}$$

Combining (7.15), (8.1), and (8.2), we find

$$\phi_x T_{xx} = 0. \tag{8.3}$$

Thus, if $\phi_x \neq 0$ (straight line), we find that

$$T(x,t) = T_0(t) + xT_1(t). \tag{8.4}$$

Equation (7.15) becomes

$$\phi_{tt} - (T_0(t) + xT_1(t))\phi_{xx} - 2T_1(t)\phi_x = 0. \tag{8.5}$$

Conversely, if $\phi(x,t)$ satisfies (8.5), then it follows from (7.17) that $T(x,t)$ satisfies (8.4).

The assumption that $|A| \ll 1$ also implies small displacements. Indeed, keeping terms of first order in A in (4.7) and (4.8), we find that

$$u(x,t) = a(t) \tag{8.6}$$

$$w(x,t) = b(t) + A\int_0^x \phi(\xi,t)d\xi \tag{8.7}$$

where $a(t)$ and $b(t)$ must satisfy (to terms of first order in A (cf. (4.12) and (4.13)))

$$\ddot{a}(t) = T_1(t) \tag{8.8}$$

$$\ddot{b}(t) = A(T_1(t)\phi(0,t) + T_0(t)\phi_x(0,t)). \tag{8.9}$$

It follows from (8.7) that

$$w_{xx} = A\phi_x(x,t) \tag{8.10}$$

$$\begin{aligned} w_{tt} &= \ddot{b} + A\int_0^x \phi_{tt}(\xi,t)d\xi \\ &= \ddot{b}(t) + A\int_0^x (T_0(t) + \xi T_1(t))\phi_{\xi\xi}(\xi,t) + 2T_1(t)\phi_\xi(\xi,t)d\xi. \end{aligned} \tag{8.11}$$

Integrating the expression on the right of (8.11) and using (8.9), we find

$$w_{tt} - (T_0(t) + xT_1(t))w_{xx} - 2T_1(t)w_x = 0. \tag{8.12}$$

$w(x,t)$ satisfies the same equation as $\phi(x,t)$, although the boundary conditions will differ.

Equations (8.6) and (8.12) could be obtained directly from equations (1.1), (1.2), and (1.3) by linearizing about the exact solution $u(x,t) = b(t)$ (where $\ddot{b}(t) = T_1(t)$), $w(x,t) = 0$, and $T(x,t) = T_0(t) + xT_1(t)$. If the string is not accelerating along the x axis, then $T_1(t) = 0$ and the problem reduces to

$$w_{tt} - T_0(t)w_{xx} = 0. \tag{8.13}$$

Separations of variables on Eq. (8.13), i.e., $w(x,t) = \alpha(t)\beta(x)$ leads to the pair of equations

$$\beta'' + \omega^2\beta = 0 \tag{8.14}$$

$$\ddot{\alpha} + \omega^2 T_0(t)\alpha = 0. \tag{8.15}$$

The separation constant ω^2 is chosen positive since otherwise, Eq. (8.15) has unbounded solutions ($T_0(t) > 0$). If we require that $w(0,t) = w(1,t) = 0$, then $\omega = n\pi$ and

$$\beta = \sin(n\pi x). \tag{8.16}$$

If $T_0(t) = \lambda + \mu\gamma(t)$ with $\lambda > \mu > 0$ and $\gamma(t)$ is periodic with $|\gamma(t)| \le 1$, then (8.15) are a series of Hill's equations. In the special case $\gamma = \cos(t)$, equations (8.7) are a series of Mathieu equations (cf. [4], [5]). It is well known that the solutions of these equations may be bounded or unbounded depending on the values of λ and μ. Thus, with this choice of tension, the validity of the small amplitude theory depends on the values of λ

and μ. If λ is much larger than μ, then the measure of the set of values of μ which give growing solutions is small (but not zero).

Another choice for $T_0(t)$ is

$$T_0(t) = \frac{\mu^2}{4(\lambda^2 - t)^2}. \tag{8.17}$$

$$\alpha_n(t) = \sqrt{\lambda^2 - t}\cos\left(\frac{1}{2}\sqrt{n^2\pi^2\mu^2 - 1}\ln(\lambda^2 - t)\right) \tag{8.18}$$

if $n^2\pi^2\mu^2 - 1 > 0$. As $t \to \lambda^2 (T_0(t) \to \infty)$, the frequency increases and the amplitude decreases. However, if $\lambda^2 > 1$, then the initial effect of increasing the tension is to decrease the frequency although the frequency will begin to increase when the tension is sufficiently large (i.e., when t is sufficiently close to λ^2). It is also of interest to note that if $n^2\pi^2\mu^2 - 1 < 0$, for example, if $\pi^2\mu - 1 < 0$, then the solution $\alpha_1(t)$ has the form

$$\alpha_1(t) = A(\lambda^2 - t)^{r_1} + B(\lambda^2 - t)^{r_2} \tag{8.19}$$

where r_1 and r_2 are positive. This mode would not vibrate at all, but simply decrease in amplitude. Of course the higher modes would have $n^2\pi^2\mu^2 - 1 > 0$ and these would behave as described above.

Thus, the net effect of increasing the tension as described by (8.17) is to increase the frequency and decrease the amplitude.

9. Large amplitude theory. The only difference between the exact theory (1.4) and (1.5) and the large amplitude theory (7.15) and (7.18) is that the term T_{xx} is missing from the large amplitude theory. It is an immediate consequence that both the exact theory and the large amplitude theory have the same traveling wave solutions (6.1) and (6.2) since the traveling wave solutions have $T_{xx} = 0$.

In order to find other solutions of (7.15) and (7.18), it is convenient to rewrite the equations as a single equation for ϕ. Eliminating T we find

$$\phi_x^2\phi_{tt} - 4\phi_x\phi_t\phi_{xt} + 3\phi_t^2\phi_{xx} = 0. \tag{9.1}$$

Equation (9.1) has the property that if $\phi = S(x,t)$ is a solution, then every function

$$\phi = F(S(x,t)) \tag{9.2}$$

is also a solution. This is easily verified:

$$\phi_x = F'S_x \tag{9.3}$$

$$\phi_t = F'S_t \tag{9.4}$$

$$\phi_{xx} = F''S_x^2 + F'S_{xx} \tag{9.5}$$

$$\phi_{tt} = F''S_t^2 + F'S_{tt} \tag{9.6}$$

$$\phi_{xt} = F''S_xS_t + F'S_{xt}. \tag{9.7}$$

Placing (9.3)–(9.7) into (9.1) we find that the coefficient of F'' vanishes identically, leaving

$$(F')^3(S_x^2S_{tt} - 4S_xS_tS_{xt} + 3S_t^2S_{xx}) = 0 \tag{9.8}$$

since S satisfies (9.1).

It is easy to find a particular solution of (9.1). We look for a solution of the form

$$\phi = (A_1 x + A_2)^{\lambda}(C_1 t + C_2)^{\mu}. \tag{9.9}$$

It is easily seen that (9.9) satisfies (9.1) if $\lambda = -3\mu$. For simplicity choose $\mu = -1$ so that $\lambda = 3$. Thus equation (9.1) has the solution

$$\phi = F\left(\frac{(A_1 x + A_2)^3}{C_1 t + C_2}\right) \tag{9.10}$$

for every twice differentiable function F. Given the solution (9.10), the corresponding tension is (cf. (7.18))

$$T = \frac{C_1^2 (A_1 x + A_2)^2}{9A_1^2 (C_1 t + C_2)^2}. \tag{9.11}$$

There are other methods for solving (9.1). For example, variables can be separated in the form $\phi = \alpha(t)\beta(x)$ or $\phi = \alpha(t) + \beta(x)$. Both of these methods lead to solutions which are simply different functions of the variable

$$S(x,t) = \frac{(A_1 x + A_2)^3}{C_1 t + C_2}. \tag{9.12}$$

It is of interest to look for various special solutions of the form (9.1). In particular we will look for solutions for which $T(1,t) = 0$. We will choose $A_1 = A_2$ and $C_1 = C_2 = A_2$ so that (cf. (9.11))

$$T = \frac{(1-x)^2}{9(t+1)^2}. \tag{9.13}$$

The solution of (9.1) is of the form

$$\phi = F\left(\frac{(1-x)^3}{t+1}\right). \tag{9.14}$$

Thus,

$$\psi(x,t) = AF\left(\frac{(1-x)^3}{t+1}\right). \tag{9.15}$$

In order to graph the solution represented by (9.15) for various functions $F(x)$ we will, as in Sec. 6, plot the results in the moving reference frame $V = x + u - a(t)$, $W = w - b(t)$ (cf. (9.9)). We find

$$V = \int_0^x \cos\left(AF\left(\frac{(1-\xi)}{t+1}\right)^3\right) d\xi \tag{9.16}$$

$$W = \int_0^x \sin\left(AF\left(\frac{(1-\xi)^3}{t+1}\right)\right) d\xi. \tag{9.17}$$

Equations (9.16) and (9.17) were evaluated numerically for various choices of F—including powers, exponential functions, trigonometric functions, and logarithms. The results are qualitatively the same for all of these choices. In particular the results are spirals, behaving in one of two ways—either approaching a straight line as $t \to \infty$ (if $\lim_{x\to 0} F(x)$ exists) or spiraling into the origin (if $\lim_{x\to 0} F(x) = \infty$).

In Figs. (9.1) and (9.2), we have plotted (9.16) and (9.17) for the function $F(x) = x^k$. In Fig. 9.1, we have plotted the case $A = 10$ and $k = 1/2$ for increasing t. Different choices of k ($k > 0$) and different choices of A affect the overall shape of the spiral and

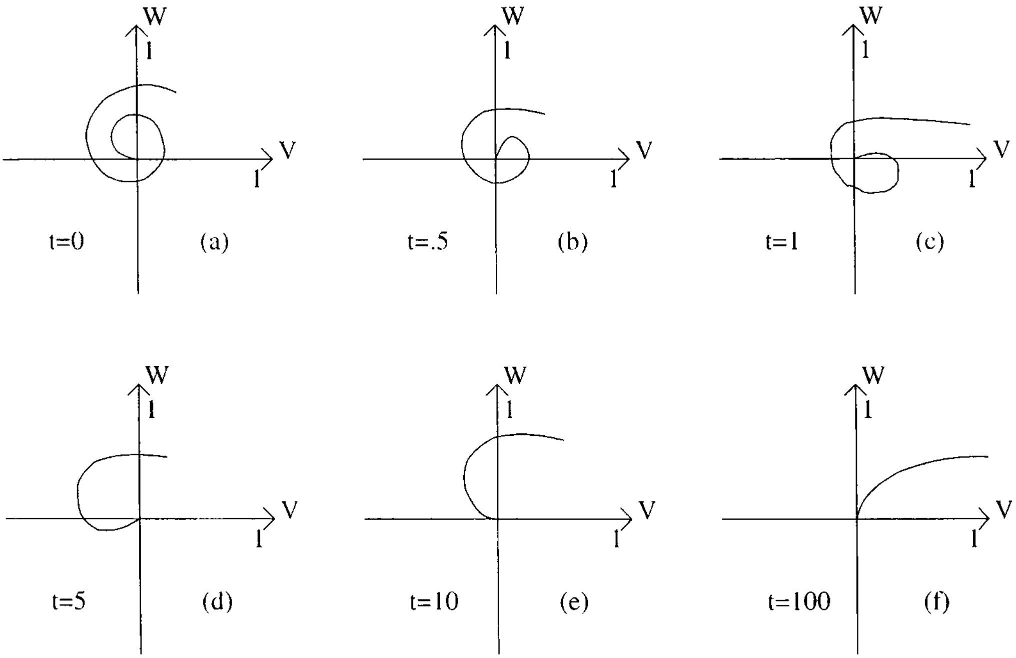

FIG. 9.1

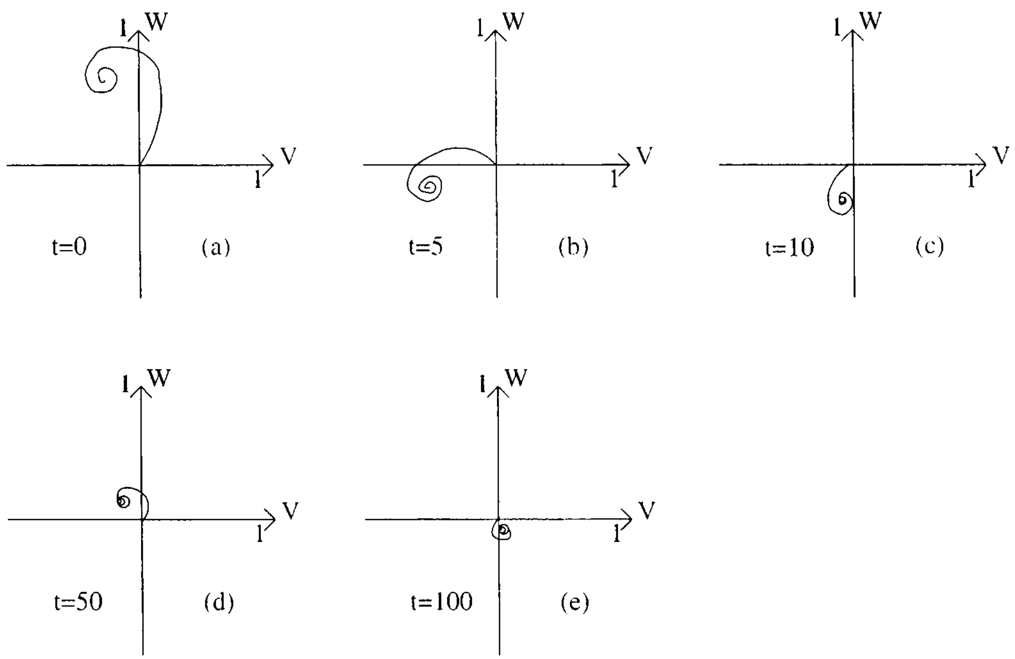

FIG. 9.2

the number of loops for each value of t, but qualitatively the results are similar. In this case $V(x,t) \to x$ and $W(x,t) \to 0$ as $t \to \infty$. In Fig. 9.2, we have plotted the case $A = 1$ and $k = -1/2$ for increasing t. For this data the spiral is wrapping up on itself. Eventually, the spiral becomes so tight that it appears to be a point. As above, different choices of k ($k < 0$) and A lead to similar results.

We have not plotted the motion of the reference frame as we did in Sec. 6. However, we note that all of the solutions of the form (9.15) with T given by (9.13) are in accelerating reference frames ($T_x(0,t) \neq 0$). We cannot piece together these solutions to obtain a solution which is symmetric about the origin.

10. Small angular velocity theory. An exact solution of (1.4) and (1.5) is given by (6.11) and (6.12) with $T = 0$—the "whirling string". It is convenient to introduce new dependent variables

$$\psi(x,t) = Ct + \phi(x,t) \tag{10.1}$$

$$T(x,t) = \frac{C^2}{2}(1-x^2) + S(x,t). \tag{10.2}$$

The boundary conditions on $\phi(x,t)$ and $S(x,t)$ are $\phi(1,t)$ is bounded

$$\phi_x(0,t) = 0 \tag{10.3}$$

$$S_x(0,t) = 0 \tag{10.4}$$

$$S(1,t) = 0. \tag{10.5}$$

Any solution satisfying these conditions will be in a non-accelerating reference frame and have no force on the end ($x = 1$). Rewriting (1.4) and (1.5) in terms of $\phi(x,t)$ and $S(x,t)$, we find

$$2C\phi_t + \phi_t^2 - \left(\frac{C^2}{2}(1-x^2) + S\right)\phi_x^2 + S_{xx} = 0 \tag{10.6}$$

$$\phi_{tt} - \left(\frac{C^2}{2}(1-x^2) + S\right)\phi_{xx} + 2(C^2x - S_x)\phi_x = 0. \tag{10.7}$$

It is convenient to begin by looking at the linearized theory

$$2C\phi_t + S_{xx} = 0 \tag{10.8}$$

$$\phi_{tt} - \frac{C^2}{2}(1-x^2)\phi_{xx} + 2C^2x\phi_x = 0. \tag{10.9}$$

Equation (10.9) can be solved by separation of variables. Let

$$\phi(x,t) = \alpha(t)\beta(x). \tag{10.10}$$

The usual arguments lead to the pair of equations

$$\ddot{\alpha} + \lambda\alpha = 0 \tag{10.11}$$

$$(1-x^2)\beta'' - 4x\beta' + \frac{2\lambda}{C^2}\beta = 0. \tag{10.12}$$

Equation (10.12) can be written in the self-adjoint form

$$\frac{d}{dx}(1-x^2)\beta' + \frac{2\lambda}{C^2}(1-x^2)\beta = 0. \tag{10.13}$$

The boundary conditions on (10.12) (or (10.13)) are

$$\beta'(0) = 0 \tag{10.14}$$

and $\beta(1)$ should be finite.

Equation (10.12) has orthogonal polynomials as solutions. If we look for solutions of the form

$$\beta = \sum_{\ell=0}^{\infty} b_\ell x^\ell \tag{10.15}$$

we find that this is a formal solution if b_ℓ satisfies the recursion relations

$$b_2 = -\frac{\lambda}{C^2} b_0 \tag{10.16}$$

$$b_3 = -\frac{1}{3}\left(1 - \frac{\lambda}{C^2}\right) b_1 \tag{10.17}$$

$$b_{\ell+2} = -\frac{(\ell(\ell+3) - \frac{2\lambda}{C^2})}{(\ell+2)(\ell+1)} b_\ell, \quad \ell \geq 2. \tag{10.18}$$

The boundary condition (10.14) requires that $b_1 = 0$ and hence all the odd coefficients vanish. The eigenvalues are

$$\lambda_\ell = \frac{C^2}{2}\ell(\ell+3), \quad \ell = 0, 2, 4, \ldots. \tag{10.19}$$

The corresponding eigenfunctions can be written

$$\beta_0 = 1 \tag{10.20}$$

$$\beta_2 = 1 - 5x^2 \tag{10.21}$$

$$\beta_4 = 1 - 14x^2 + 21x^4 \tag{10.22}$$

$$\beta_6 = 1 - 27x^2 + 99x^4 - \frac{429}{5}x^6 \tag{10.23}$$

$$\beta_8 = 1 - 44x^2 + 286x^4 - 572x^6 + \frac{2431}{7}x^8. \tag{10.24}$$

The higher eigenfunctions are found recursively. It is a consequence of (10.13) that these eigenfunctions satisfy the orthogonality relation

$$\int_0^1 (1-x^2)\beta_{2j}\beta_{2k}\,dx = 0, \quad j \neq k. \tag{10.25}$$

In order to simplify the notation, let $\omega_\ell^2 = \ell(\ell+3)/2$ so that

$$\lambda_\ell = C^2\omega_\ell^2. \tag{10.26}$$

If the initial data for (10.9) are sufficiently smooth, then we can write the solution in terms of the Fourier series

$$\phi(x,t) = (A_0 + B_0 t) + \sum_{\ell=1}^{\infty}(A_{2\ell}\cos(C\omega_{2\ell}t) + B_{2\ell}\sin(C\omega_{2\ell}t))\beta_{2\ell}(x). \tag{10.27}$$

Assume the initial conditions have the form

$$\phi(x,0) = f(x) \tag{10.28}$$

$$\phi_t(x,0) = Cg(x). \tag{10.29}$$

Combining (10.27), (10.28), and (10.29), we find that the coefficients in (10.27) must satisfy

$$A_{2\ell} = \frac{1}{\mu_{2\ell}^2}\int_0^1 (1-x^2)f(x)\beta_{2\ell}(x)dx \tag{10.30}$$

$$B_{2\ell} = \frac{1}{\omega_{2\ell}\mu_{2\ell}^2}\int_0^1 (1-x^2)g(x)\beta_{2\ell}(x)dx \tag{10.31}$$

where

$$\mu_{2\ell}^2 = \int_0^1 (1-x^2)\beta_{2\ell}(x)^2 dx. \tag{10.32}$$

In addition, since we will require the solution to be bounded for all time, we will require that $B_0 = 0$; i.e., $g(x)$ must satisfy the condition

$$\int_0^1 (1-x^2)g(x)dx = 0. \tag{10.33}$$

We will assume that (10.27) can be twice differentiated term by term and that the resulting functions are bounded for all time. This would certainly be the case, for example, if the initial data (10.28) and (10.29) generated a finite series. Differentiation of (10.27) yields

$$\begin{aligned}\phi_x(x,t) &= \sum_{\ell=1}^{\infty}(A_{2\ell}\cos(C\omega_{2\ell}t) + B_{2\ell}\sin(C\omega_{2\ell}t))\beta'_{2\ell}(x) \\ &= F_1(x,t)\end{aligned} \tag{10.34}$$

$$\begin{aligned}\phi_{xx}(x,t) &= \sum_{\ell=1}^{\infty}(A_{2\ell}\cos(C\omega_{2\ell}t) + B_{2\ell}\sin(C\omega_{2\ell}t))\beta''_{2\ell}(x) \\ &= F_2(x,t)\end{aligned} \tag{10.35}$$

$$\begin{aligned}\phi_t(x,t) &= C\sum_{\ell=1}^{\infty}(-A_{2\ell}\sin(C\omega_{2\ell}t) + \beta_{2\ell}\cos(C\omega_{2\ell}t))\omega_{2\ell}\beta_{2\ell}(x) \\ &= CG_1(x,t).\end{aligned} \tag{10.36}$$

It follows from (10.8) that

$$S(x,t) = 2C\int_0^1 H(x,\xi)\phi_t(\xi,t)d\xi \tag{10.37}$$

where the Green's function $H(x,\xi)$ is

$$H(x,\xi) = \begin{cases} 1-x & 0 \le \xi \le x \\ 1-\xi & x \le \xi \le 1. \end{cases} \tag{10.38}$$

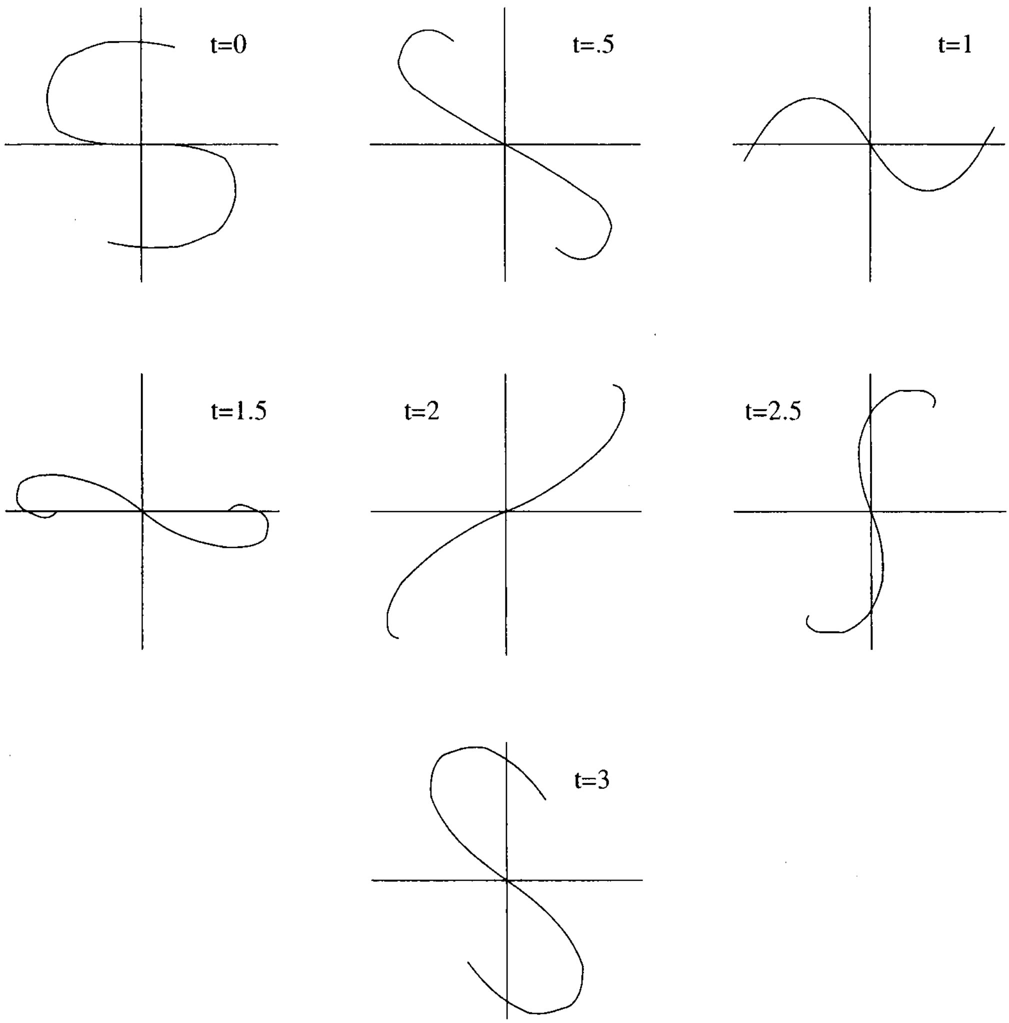

FIG. 10.1

Thus (10.36) implies

$$S(x,t) = 2C^2 \int_0^1 H(x,\xi)G_1(\xi,t)d\xi. \tag{10.39}$$

We wish to consider the question of how close the solution of the linearized problem (10.27) (with $B_0 = 0$) and (10.39) is to an exact solution of (10.6) and (10.7). Alternatively, we consider the question of what body forces must be applied to the string in order to make the solution of the linearized problem an exact solution. Define $F(x,t)$

and $G(x,t)$ by

$$F(x,t) = \phi_x^2 - \frac{C^2}{2}(1-x^2)\phi_x^2 - S\phi_x^2 \qquad (10.40)$$

$$= C^2\left(G_1^2 - \frac{1}{2}(1-x^2)F_1^2 + 2F_1^2\int_0^1 H(x,\xi)G_1(\xi,t)d\xi\right)$$

$$G(x,t) = S\phi_{xx} - 2S_x\phi_x \qquad (10.41)$$

$$= -2C^2\left(F_2 + 2F_1\int_0^x G_1(\xi,t)d\xi\right).$$

The functions $F_1(x,t), F_2(x,t)$, and $G_1(x,t)$ are bounded for all time. We assume that $|C| \ll 1$ (small angular velocity). In this case the "body forces" $F(x,t)$ and $G(x,t)$ in (10.40) and (10.41) are $\mathcal{O}(C^2)$. Thus $F(x,t)$ and $G(x,t)$ will be small if the rate of rotation is sufficiently slow.

In Fig. 10.1 we give an example of the motion of a string whose initial position is a (barred) spiral which is symmetric about the origin and whose initial angular velocity $g(x) = 0$. This solution is discussed in Sec. 11.

11. A string universe. Galaxies appear in a variety of shapes. There are three-dimensional structures such as elliptical galaxies and two-dimensional structures such as lenticular galaxies. However, many galaxies appear to have an essentially one-dimensional structure, e.g., spiral galaxies (cf. Fig. 11.1) and barred spiral galaxies (cf. Fig. 11.2). There are galaxies which have the form of a simple closed curve (cf. Fig. 11.3) and galaxies which have no geometrically simple shape (irregular galaxies).

All of these one-dimensional shapes have their analog in the theory of the inextensible string. It is of interest to compare a universe consisting of inextensible strings to the one-dimensional objects actually observed in the real universe.

It is convenient to begin with irregular galaxies. It is possible that an irregular shape is transitory and will evolve with time. However, we have seen (Sec. 5) that in a string universe these irregular shapes may correspond to solutions that do not change with time. It is quite possible to have strings which are moving with constant velocity but unchanging shape. A second possibility is a string in the shape of a simple closed curve (cf. Sec. 8 and Fig. 11.3). These strings do not change shape with time. However, it is possible that each point on the string is moving with a non-zero angular velocity that depends on the position of the point on the string. Thus even though the string does not change shape, it is a time-dependent solution.

The most familiar galactic shapes are the spiral and barred spiral galaxies. These galaxies are symmetric. We note that in a universe consisting of inextensible strings, these shapes cannot be accelerating; i.e., the center of symmetry is either at rest or is moving with constant velocity. In Sec. 10 we described an approximate theory of string which is applicable in the case of small angular velocity. In the case of a string the size of a galaxy rotating with the angular velocity of a galaxy, the linear theory described in Sec. 10 should be appropriate. We would like to compare the motion of such a string to the actual motion of a spiral or barred spiral galaxy. For this purpose we need initial

FIG. 11.1

conditions (cf. equations 10.1, 10.28, 10.29)

$$\psi(x,0) = \varphi(x,0) = f(x) \tag{11.1}$$

$$\psi_t(x,0) = h(x) = c + \varphi_t(x,0) = c + cg(x). \tag{11.2}$$

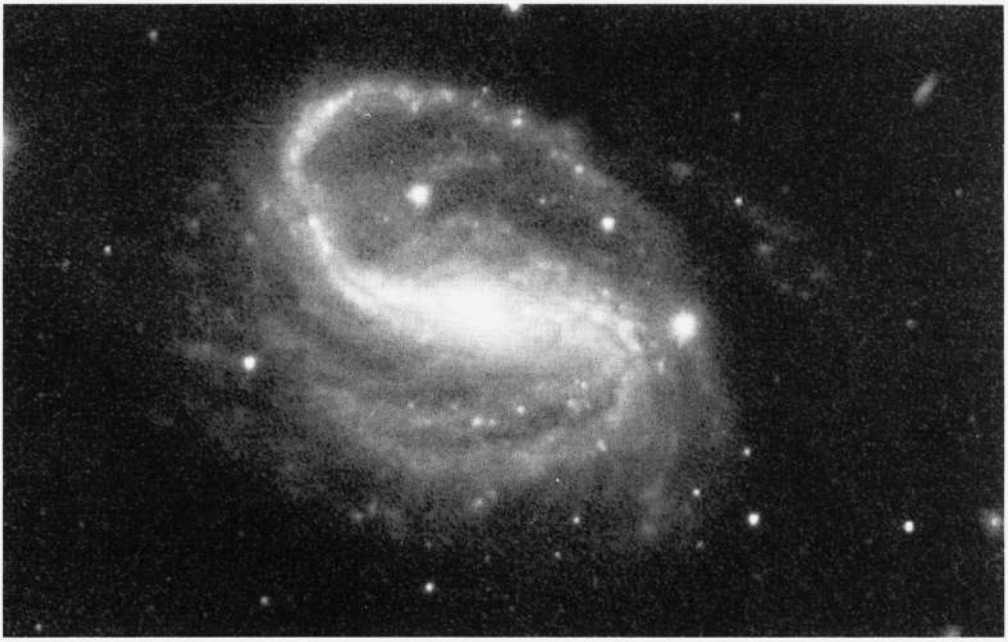

FIG. 11.2

In addition $g(x)$ must satisfy condition (10.33) so that

$$C = \frac{3}{2}\int_0^1 (1 - x^2)h(x)dx. \tag{11.3}$$

Thus, all of the quantities in the small angular velocity theory are known if the initial shape and angular velocity are known.

In Fig. 10.1 we have plotted the evolution of a string whose initial shape is that of a barred spiral. In the absence of information on $h(x)$, we have chosen $g(x) = 0$. Fig. 10.1 depicts the motion of the string in units of C. The motion is not periodic (the frequencies are irrational) but are almost periodic. In the situation shown in Fig. 10.1, the string returns to a shape similar to its initial position after three units of time, although the orientation is different.

Computations were made with other choices of $g(x)$. In many of these cases the string actually crosses itself. Although this is not possible for a string, it is possible for a galaxy and could have the appearance of two galaxies colliding. There is a question of whether there is an initial angular velocity such that a string initially in the shape of a barred spiral would evolve into a spiral. While a series of numerical experiments are not definitive, we note that in none of the cases that were run did this happen. The string changed shape in various ways, but never became a spiral.

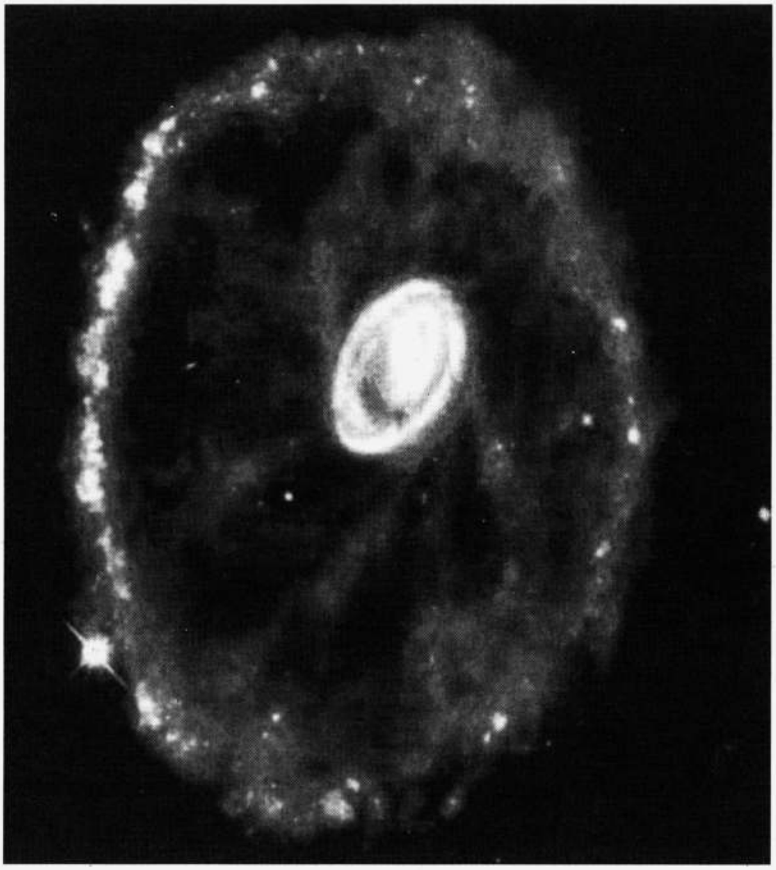

FIG. 11.3

REFERENCES

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, VI. Interscience Publishers, New York, 1962

[2] I. I. Kolodner, *Heavy rotating string—a nonlinear eigenvalue problem*, Comm. on Pure and App. Math. **8**, 395–408 (1955)

[3] S. S. Antman, *Nonlinear Problems of Elasticity*, Springer-Verlag, New York, 1995

[4] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Co., New York, 1955

[5] J. Stoker, *Nonlinear Vibrations*, Interscience Publishers, New York, 1950