S. L. Chen
M. Géradin
Aerospace Laboratory (LTAS)
University of Liège
21 Rue E. Solvay
4000 Liège, Belgium

Dynamic Force Identification for Beamlike Structures Using an Improved Dynamic Stiffness Method

In this study a procedure of dynamic force identification for beamlike structures is developed based on an improved dynamic stiffness method. In this procedure, the entire structure is first divided into substructures according to the excitation locations and the measured response sites. Each substructure is then represented by an equivalent element. The resulting model only retains the degree of freedom (DOF) associated with the excitations and the measured responses and the DOF corresponding to the boundaries of the structures. Because the technique partly bypasses the processes of modal parameter extraction, global matrix inversion, and model reduction, it can eliminate many of the approximations and errors that may be introduced during these processes. The principle of the method is described in detail and its efficiency is demonstrated via numerical simulations of three different structures. The sensitivity of the estimated force to random noise is discussed and the limitation of the technique is pointed out. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

The identification of exciting forces using measured responses is not a new concept (see, e.g., Pilkey and Kalinowski, 1972). However, the problem has received considerable attention in recent years, because in many situations the direct measurement of the excitation can be very difficult or even impossible. Many techniques for force identification have been developed. Among others, let us cite the sum of weighted acceleration technique (Gregory et al., 1986; Wang et al., 1987), the deconvolution technique (Hillary and Ewins, 1984), and their engineering applications (Bateman et al., 1991; Carne et al., 1994). Using the wave propagation theory, Park and Park (1994) studied the identification of an arbitrary impact force, including its location and time history. Almost all the force identification algorithms require the measurement of the responses to the unknown forces (displacements, velocities, accelerations, or strains) at several locations on the structure and a model of the structure. While the responses may be easily obtained by direct measurement using transducers, the establishment of the structure model becomes a key issue in the procedure of the force identification. There are three types of structure model that may be used for this purpose: the spatial model (mass, damping, and stiffness matrices), modal model (natural frequencies and mode shapes), and response model (or frequency response function matrix) (Ewins, 1988). If the modal model or the response model is used, the force identification process requires the inversion of the global matrix, which tends to be very ill conditioned. That is, very

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small errors in measurements propagate into large errors in estimated forces, especially at the frequencies close to resonance and antiresonance conditions (Starkey and Merrill, 1989). Progress has been made to overcome this difficulty, e.g., by the pseudoinverse technique (Fabunmi, 1985) or by the singular-value decomposition technique (Elliott et al., 1988).

The use of the spatial model, usually established by the finite element method (FEM), may offer significant advantages in some cases because it avoids any matrix inversion (Dobson and Rider, 1990). However, the FEM model often includes a large number of degrees of freedom (DOF), although most of them are not necessarily needed, especially those to which no forces are applied and those for which no responses are measured. To estimate the excitations, the FEM model has to be reduced by model reduction techniques so that responses need only be measured at a limited number of sites. The reduction process may introduce errors as pointed out by many researchers (i.e., Leung, 1978; Freed and Flanigan, 1990).

Overcoming all the difficulties mentioned above is a hard task. However, we show in this article that some of the difficulties may be overcome or partly overcome for one-dimensional (1-D) beamlike structures by using an improved dynamic stiffness method recently developed by the authors (Chen and Géradin, 1995; Géradin and Chen, 1995).

METHOD PRINCIPLE

Element Dynamic Stiffness Matrix (DSM)

The exact DSM for some basic elements such as beam and shaft elements may be found in many studies, for instance in the work of Richards and Leung (1977), Leung (1985), and Yang and Pilkey (1992). As an example, consider a uniform beam shown in Fig. 1. The exact DSM for a damped beam was obtained by Leung (1985):

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = B \begin{bmatrix} Z_1 & Z_2 & Z_4 & -Z_5 \\ Z_2 & Z_3 & Z_5 & Z_6 \\ Z_4 & Z_5 & Z_1 & -Z_2 \\ -Z_5 & Z_6 & -Z_2 & Z_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ \theta_1 \\ X_2 \\ \theta_2 \end{Bmatrix} \quad (1)$$

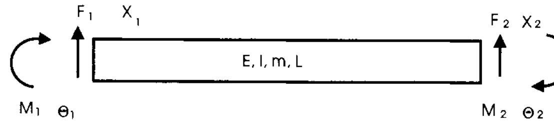

FIGURE 1 A beam element.

where $B = EI/(1 - \cos aL \cosh aL)$; $Z_1 = a^3(\cos aL \sinh aL + \sin aL \cosh aL)$; $Z_2 = a^2 \sin aL \sinh aL$; $Z_3 = a(\sin aL \cosh aL - \cos aL \sinh aL)$; $Z_4 = -a^3(\sin aL + \sinh aL)$; $Z_5 = a^2(\cos aL - \cosh aL)$; $Z_6 = a(\sinh aL - \sin aL)$; and $a = ((\omega^2 - 2i\delta\omega)\rho A/EI$; with E being the Young's modulus, L the length, I the area moment of inertia, ρ the mass density, and A the cross-section area. The additional damping constant δ is due to inertia.

Element Transfer Matrix

The element transfer matrix may be obtained either by directly solving the differential equations of motion or by operating a transformation of the DSM. The second possibility is described below.

Equation (1) may be rewritten in submatrix form as follows:

$$\begin{Bmatrix} F_l \\ F_r \end{Bmatrix} = \begin{bmatrix} D_{11}^e & D_{12}^e \\ D_{21}^e & D_{22}^e \end{bmatrix} \begin{Bmatrix} X_l \\ X_r \end{Bmatrix} \quad (2)$$

where $F_l = [F_1, M_1]^t$, $X_l = [X_1, \theta_1]^t$, $F_r = [F_2, M_2]^t$, and $X_r = [X_2, \theta_2]^t$.

Equation (2) may be further modified to yield the following transfer matrix form:

$$\begin{Bmatrix} X_r \\ -F_r \end{Bmatrix} = \begin{bmatrix} T_{11}^e & T_{12}^e \\ T_{21}^e & T_{22}^e \end{bmatrix} \begin{Bmatrix} X_l \\ F_l \end{Bmatrix} \quad (3)$$

with $T_{11}^e = -D_{12}^{e\,-1} D_{11}^e$, $T_{12}^e = D_{12}^{e\,-1}$, $T_{21}^e = -D_{21}^e + D_{22}^e D_{12}^{e\,-1} D_{11}^e$, and $T_{22}^e = -D_{22}^e D_{12}^{e\,-1}$. Note that the FE mass and stiffness matrices may be used to obtain the element transfer matrix as described by Dimarogonas (1975), but the transfer matrix obtained by this approach is not exact.

Substructure Transfer Matrix

The entire structure is divided into several substructures. The number of the substructures depends on the number of the excitations and the number of the responses. For example, let us suppose that the node numbers of the entire structure be ordered from 1 to n, with node 1 being the left-hand node and node n the right-hand node. If the external force is applied to node i and the response is measured at node j (let $i < j$), then the entire structure may be divided into three substructures, with nodes 1, i, j, and n being the boundary nodes of the substructures.

Shown in Fig. 2 is a typical substructure that consists of N_k elastic supports, N_m rigid masses, N_d disks, and N_b beam elements. There are not external forces applied or responses measured at the internal nodes. The global transfer matrix for the substructure is:

$$[\mathbf{T}] = [T_{N_s}][T_{N_s-1}] \cdot \cdot \cdot [T_2][T_1] \quad (4)$$

where N_s is the total number of elements of the substructure.

Substructure DSM

The global transfer matrix $[\mathbf{T}]$ of a substructure relates the forces and displacements at both ends of the substructure in the following way:

$$\begin{Bmatrix} \mathbf{X}_r \\ -\mathbf{F}_r \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{X}_l \\ \mathbf{F}_l \end{Bmatrix} \quad (5)$$

where $\mathbf{F}_l$ and $\mathbf{X}_l$ are the force and displacement vectors at the left end of the substructure and $\mathbf{F}_r$ and $\mathbf{X}_r$ are the same quantities at the right end. Equation (5) may be rewritten in the DSM form:

$$\begin{Bmatrix} \mathbf{F}_l \\ \mathbf{F}_r \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{X}_l \\ \mathbf{X}_r \end{Bmatrix} \quad (6)$$

with

$$D_{11} = -T_{12}^{-1}T_{11},$$
$$D_{12} = T_{12}^{-1},$$
$$D_{21} = -T_{21} + T_{22}T_{12}^{-1}T_{11},$$
$$D_{22} = -T_{22}T_{12}^{-1},$$

where $[\mathbf{D}]$ is the global DSM of the substructure whose coefficients are frequency dependent. Note that the global DSM of the substructure has the same dimension as the element DSM.

Equation (6) tells us that the substructure shown in Fig. 2 is reduced to an equivalent element in an exact manner. The internal coordinates of the substructure are not used.

Global DSM

The global DSM for the entire structure can be assembled using the above DSMs of all substructures. After introducing the boundary conditions, the dimension of the global stiffness matrix can further be decreased. The unwanted DOF such as rotational DOF can be removed in an exact manner (Leung, 1978).

The dynamic equilibrium equation in the frequency domain is finally obtained:

$$F(\omega) = [D_g(\omega)]X(\omega) \quad (7)$$

where $\mathbf{D}_g$ is the global DSM.

Using the matrix $\mathbf{D}_g$, the other frequency-dependent matrices, model parameters, and dynamic responses may be obtained by the usual dynamic stiffness method (Leung and Fergusson, 1993) or finite dynamic element method (Fergusson and Pilkey, 1994).

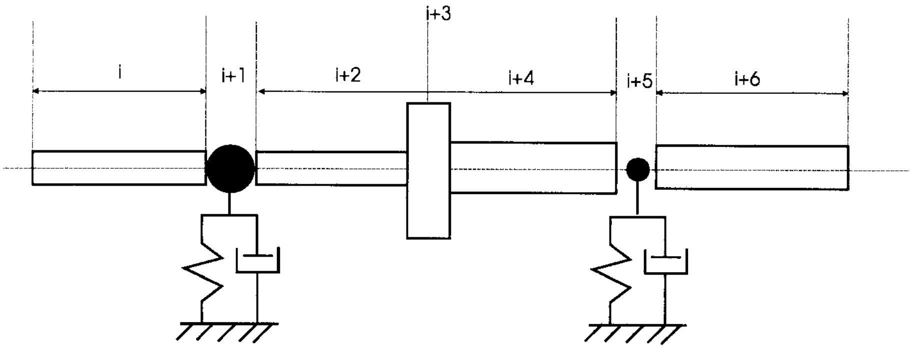

FIGURE 2 A substructure.

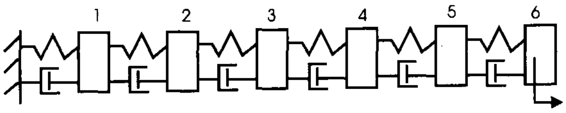

FIGURE 3 A 6-DOF lumped system.

Dynamic Force Identification

The dynamic force in the frequency domain may be directly estimated from Eq. (7). The corresponding time history can be easily obtained by performing an inverse fast Fourier transform (IFFT) as follows:

$$f(t) = \text{IFFT}([\mathbf{D}_g(\omega)]X(\omega)). \tag{8}$$

It is noted that the matrix $\mathbf{D}_g(\omega)$ depends neither on the external forces nor on the responses. Therefore, once the frequency parameter Δf is determined for the IFFT, the matrix $\mathbf{D}_g(\omega)$ can be computed once for all and stored for subsequent identifications of different excitations.

NUMERICAL EXAMPLES

To demonstrate the efficiency of the procedure described above, three numerical examples are presented in this section. In each example, four different types of excitation forces are applied to the structures. The excitations are monoharmonic, multiharmonic, impulsion, and harmonic with exponential envelope. The responses are first calculated by directly integrating the equations of motion using the Newmark method and the excitation forces are then recovered based on the principle described here.

Example 1

Consider the 6-DOF lumped system shown in Fig. 3. It is assumed that an external force is applied at station 6 and that the response at the same point can be measured. Let us estimate the magnitude of the external force.

Because the external force is applied to the right boundary, the system can be treated as one substructure. The global transfer matrix is:

$$\mathbf{T} = T_m^6 T_{kc}^6 T_m^5 T_{kc}^5 \cdot \cdot \cdot T_m^1 T_{kc}^1 \tag{9}$$

where T_m^j and T_{kc}^j are defined as

$$T_m^j = \begin{bmatrix} 1 & 0 \\ -m_j\omega^2 & 0 \end{bmatrix} \tag{10}$$

$$T_{kc}^j = \begin{bmatrix} 1 & 1/(k_j + i\omega c_j) \\ 0 & 1 \end{bmatrix} \tag{11}$$

with $i = \sqrt{-1}$.

The global DSM can be obtained by rearranging the global transfer matrix $\mathbf{T}$ using Eq. (6). Introducing the boundary condition at the left end, the global DSM is reduced to a scalar function of the excitation frequency, $d(\omega)$.

Figures 4 and 5 depict the comparison between the true forces and the estimated forces under the excitations mentioned above. In IFFT calculations, the number of the sampling points is $N = 4096$ and the frequency step is $\Delta f = 0.01$ Hz. It is seen that the estimated force histories agree well with the true values.

Example 2

A nonuniform clamped beam shown in Fig. 6 is analyzed in this example. It is assumed that the beam is excited at its center by an external force.

The beam is divided into two substructures, each of which consists of two beam elements. According to the principle of the procedure, the structure is equivalent to two elements. After introducing the boundary conditions, the dynamic

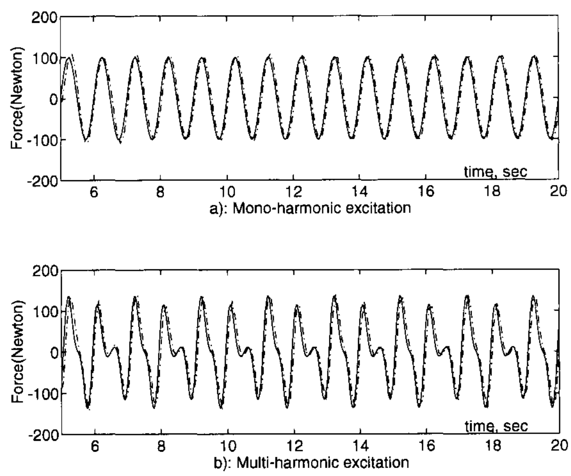

FIGURE 4 Estimated force and true force history for example 1: (——) true value and (– · – · –) estimated.

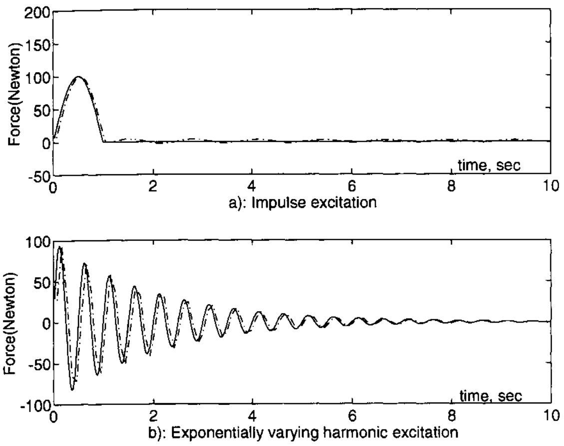

FIGURE 5 Estimated force and true force history for example 1: (——) true value and (– · – · –) estimated.

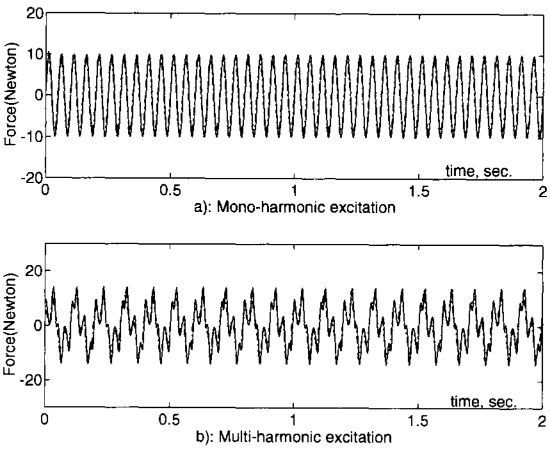

FIGURE 7 Estimated force and true force history for example 2: (——) true value and (– · – · –) estimated.

equation has the form

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} X \\ \theta \end{Bmatrix}, \tag{12}$$

where F and M are, respectively, the external force and the moment at the center of the beam and X and θ are, respectively, the displacement and the rotation angle at the same node.

Equation (12) can be further reduced by setting the moment M to zero. Finally we have:

$$F(\omega) = \left(D_{11} - \frac{D_{12}D_{21}}{D_{22}} \right) X(\omega). \tag{13}$$

Figures 7 and 8 show the estimated exciting forces under four types of excitations. For comparison, the true values are also depicted in the figures. The responses are first calculated by the Newmark integration method based on an FEM (eight beam elements). In the IFFT, the number of sampling points is $N = 4096$ with the frequency step $\Delta f = 0.1$ Hz. From these figures, it is seen that the present method can correctly identify the external forces.

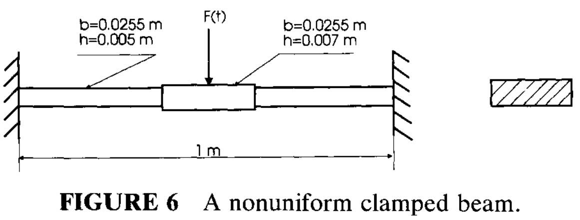

FIGURE 6 A nonuniform clamped beam.

Example 3

In the above two examples, the excitation and the response were colocated. It is not always the case in practice. In this example, we show that the developed procedure can also deal with the cases where the excitation and the responses are not colocated.

Consider a lumped system with 30 DOFs [Fig. 9(a)]. An external force is applied at station 20, while the associated response is measured at station 10. The entire structure is therefore divided into three substructures, each having 10 DOFs. Using the developed procedure, the original system is reduced to a 2-DOFs system for a given frequency, as shown in Fig. 9(b). We have the

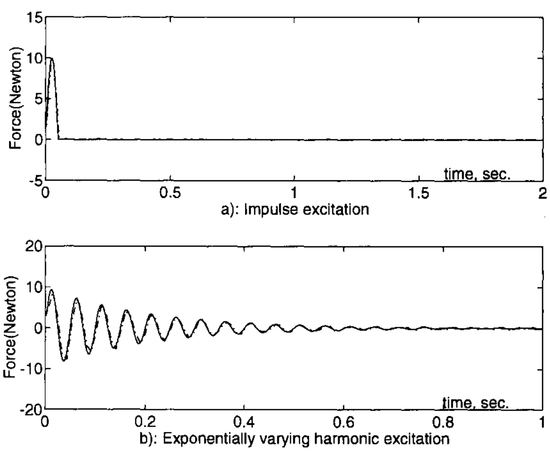

FIGURE 8 Estimated force and true force history for example 2: (——) true value and (– · – · –) estimated.

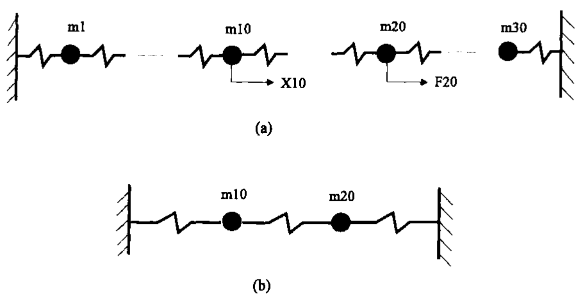

FIGURE 9 (a) A lumped system with 30 DOFs and (b) reduced system with 2 DOFs.

following relationships:

$$\begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} X_{10} \\ X_{20} \end{Bmatrix}. \quad (14)$$

By setting $F_{10} = 0$ and solving Eq. (14), we finally have

$$F_{20}(\omega) = \left(D_{21} - \frac{D_{22}D_{11}}{D_{12}} \right) X_{10}(\omega). \quad (15)$$

Figures 10 and 11 depict the calculated forces using Eq. (15). In the IFFT, the number of sampling points is $N = 4096$ with the frequency step $\Delta f = 0.01$ Hz. We see that the identified force histories agree well with the applied excitations.

SENSITIVITY ANALYSIS

The responses and the DSMs used in the above examples are theoretically pure values. However, in the actual measurements, the noise is usually unavoidable. Therefore, it is necessary to analyze the sensitivity of the estimated force to the noise level. To this end, example 2 is reanalyzed. Three cases are considered.

Case 1: Only Response Is Contaminated

The calculated responses by Newmark direct integration are contaminated by adding ±10% of normal random noise and then they are used as the inputs of the present method. Figure 12 depicts the comparison between estimated and true force histories.

Case 2: Only DSM Is Contaminated

The theoretical DSM is contamined by adding ±10% of random noise. The estimated forces are shown in Fig. 13. It is seen that the estimated forces agree well with the true values.

Case 3: Response and DSM Are Contaminated Simultaneously

In this case, ±10% of random noise is added simultaneously to the calculated response and to the coefficients of the DSM. The identified forces are shown in Fig. 14.

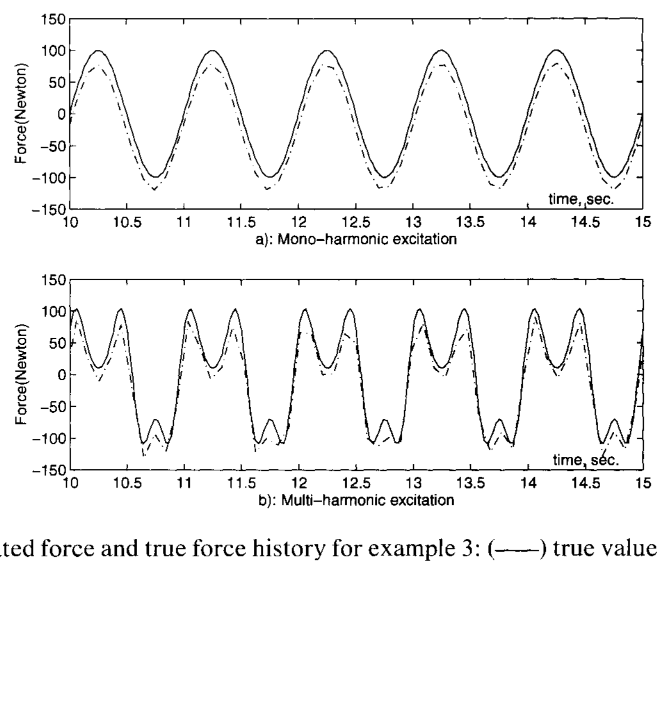

FIGURE 10 Estimated force and true force history for example 3: (——) true value and (– · – · –) estimated.

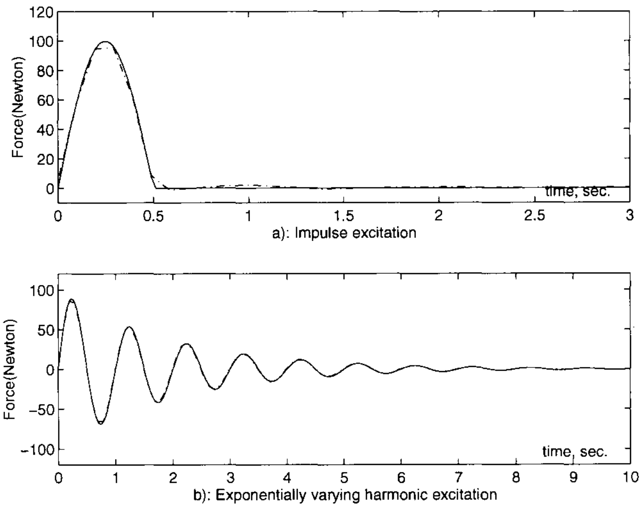

FIGURE 11 Estimated force and true force history for example 3: (——) true value and (– · – · –) estimated.

From the above analysis, it is seen that the estimated force is not very sensitive to the noise inherent to the response and to the DSM. This is because the force identification developed here is not an inverse procedure, i.e., no global matrix inversion is necessary.

DISCUSSION

The identification method needs an FFT and an IFFT process to obtain the force histories in the time domain. Therefore, the signal processing parameters Δt or Δf and the number of sampling points N have to be selected carefully. The parameter Δt has a significant influence on the accuracy of the responses, while the parameter Δf determines the accuracy of the DSM. If the frequency contained in the response is high, a small Δt is usually needed. In this case, the sampling number N must be large enough.

The damping value has a direct influence on the accuracy of the DSM. In the calculations of both DSM and responses, the same damping value is used in the above numerical simulations. However, it is usually a difficult task to accurately identify the damping value. Therefore, if the measured response is used as an input, the damping value has to be determined experimentally before the calculation of the DSM.

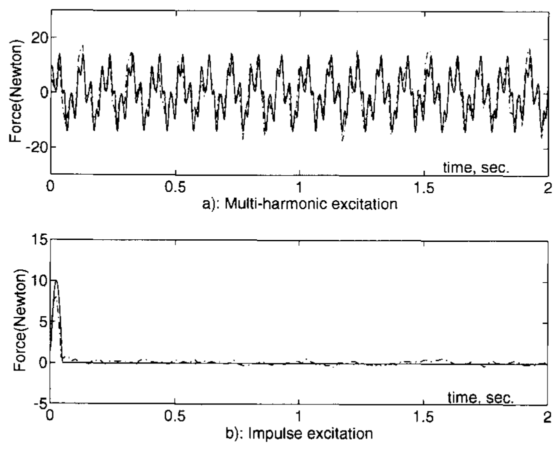

FIGURE 12 True and estimated force history; response is contaminated by ±10% random noise: (——) true value and (– · – · –) estimated.

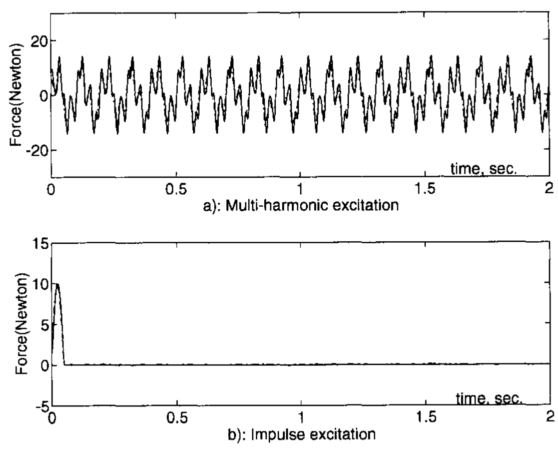

FIGURE 13 True and estimated force history; dynamic stiffness matrix is contaminated by ±10% random noise: (——) true value and (– · – · –) estimated.

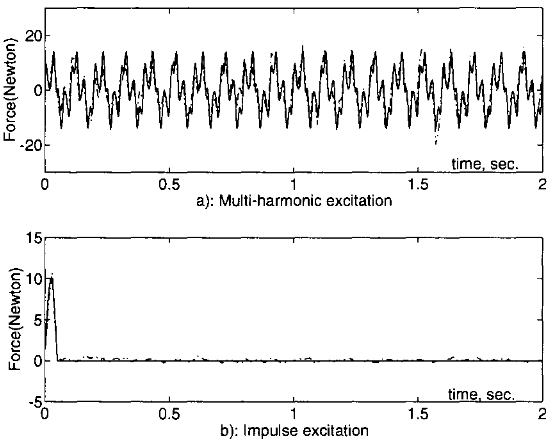

FIGURE 14 True and estimated force history; both response and dynamic stiffness matrix are contaminated by ±10% random noise: (——) true value and (– · – · –) estimated.

To get the DSM of a substructure, the method developed here requires the inversion of a submatrix of the global transfer matrix [T_{12}^{-1} in Eq. (6)]. Because the global transfer matrix **T** has the same dimension as the element transfer matrix $\mathbf{T}_e$, with its maximal dimension being 12 × 12 for 1-D structures, the maximal dimension of the submatrix T_{12} is 6 × 6. Obviously, the inversion of such a low dimension matrix will be much easier and more accurate than that of the global system matrix that is usually higher in dimension if the FEM method is used. Note that in example 3 the dimension of T_{12} is only 1 × 1.

It should be noted that the successive multiplications of matrices in Eq. (4) may introduce numerical instability when N_s is very large. Fortunately, the number of the elements of the substructure N_s is usually much smaller than that of the original structure. This fact may be found in example 3 where the original system has 30 DOFs while each substructure has only 10 DOFs.

To apply the developed procedure one needs to specify the locations and the number of the excitation forces. In addition the method can only be used presently for 1-D structures.

CONCLUSIONS

A procedure of force identification for beamlike structures (or more generally for 1-D structures) is presented. According to the procedure, the structures can be modeled directly with the DOF at which external forces are applied and/or responses are measured. In this way, the force identification process becomes a straightforward problem. The processes of global matrix inversions, modal parameter extractions, as well as model reductions are not necessary in the proposed procedure. Therefore, the sensitivity of the estimated force to the noise level is lower and the solution is characterized by a good accuracy. The developed procedure can presently only be used for 1-D structures. The experimental validation of the method will be reported in a forthcoming article.

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