Dynamic measurements of low-frequency loudspeakers modeled by Volterra series

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Abstract

This paper describes the application of a general mathematical method to the identification of low-frequency loudspeaker systems. Third-order Volterra series are applied to match the measured harmonic distortion responses of three commercial low-frequency loudspeaker systems with simulation results. This allows to compute the non-linear parameters of the loudspeaker, used within standard physical models. The proposed approach can be used as a general method to define loudspeaker dynamic models, exploiting standard acoustic pressure measurements, instead of displacement variation [1].

1 Loudspeaker modeling

Until now loudspeakers have been modeled, following the well-known Small-Thiele approach [2], that describes their behavior at low frequencies and small input power, and produces a linear transfer function. Unfortunately it does not match our requirements; first of all because it is a liner model and moreover as it does not describe accurately real Sound Pressure Level (SPL) curves.

In the following the Small-Thiele model will be considered, extending it by introducing the displacement dependence in the electrical and mechanical parameters. This allows to obtain an effective relationship between the electrical input and the acoustic output. Therefore we assume that the following equations hold.

$$\begin{aligned} e(t) - R_e i(t) - L(x)\frac{di(t)}{dt} &= Bl(x)\dot{x}(t) \\ U(t) &= S_r \dot{x}(t) \\ F(t) - R_m \dot{x}(t) - M_m \ddot{x} - K(x)x &= p(t)S_r \\ p(t) &= S_r A\frac{d^3x}{dt^3} + S_r B\frac{d^2x}{dt^2} \end{aligned} \tag{1}$$

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Moreover we assume that the dependance on the displacement of the force factor Bl and of the suspension stiffness K can be approximated by a second-order Taylor series.

$$\begin{aligned} Bl(x) &= Bl0/(\alpha x^2 + \beta x + 1) \quad (2) \\ K(x) &= K0(\gamma x^2 + \delta x + 1) \quad (3) \end{aligned}$$

Then the set of equations which characterize loudspeaker behavior becomes a set of non-linear relationships. In order to describe non-linear systems some general structures have been presented in literature [3], [4], [5], [6]. Among them, the Volterra series expansion seems to be a useful tool to describe weakly non-linear systems.

2 Non-linear system theory

A general non-linear system is defined an homogeneous n-order system if its input/output relationship can be expressed as

$$\begin{aligned} y(t) &= \int_{n=-\infty}^{\infty} h(t-\sigma_1, t-\sigma_2, ..., t-\sigma_n) \times \\ & \times u(\sigma_1)u(\sigma_2)...u(\sigma_n)d\sigma_1 d\sigma_2 ... d\sigma_n \quad (4) \end{aligned}$$

where $h(\sigma_1, \sigma_2, ..., \sigma_n)$ is defined kernel of the system and fully identifies the same, $u(t)$ is the input and y(t) is the output of the system.

Given an n-oder system it is possible to describe it in the transformation domain by means of the Laplace transformation $L[]$. In the following capital letters identify the Laplace transformed variables.

It can be easily proofed that

$$Y_n(s_1, s_2, ..., s_n) = H_n(s_1, s_2, ..., s_n)U(s_1)U(s_1)...U(s_1)$$

where

$$H_n(s_1, s_2, ..., s_n) = L[h(\sigma_1, \sigma_2, ..., \sigma_n)]$$

is called system transfer function.

Moreover it can be proofed that by knowing $Y_n(s_1, s_2, ..., s_n)$ the Laplace transformation of the output $Y(s)$ can be computed. This is termed "variables association".

It is well known that the exponential function is an eigenfunction of a linear system, and that the knowledge of the response of the linear system to an exponential function fully characterize the system's behavior.

This propriety can be extended to an homogeneous system, for which it can be proofed that the response to

$$u(t) = Ae^{pt}$$

is

$$y(t) = A^n H_n(p, p, ..., p)e^{npt}.$$

Nevertheless when $n > 1$, in order to characterize the system's behavior is necessary to know the response to a sum of n exponential function

$$u(t) = A_1 e^{p_1 t} + A_2 e^{p_n t} + ... + A_n e^{p_n t}$$

A system transfer function H_n is defined symmetric if

$$H_n(\sigma_1, \sigma_2, ..., \sigma_n) = H_n(\sigma_2, \sigma_1, ..., \sigma_n) = ...$$

for any possible permutation of σ_i.

Another important propriety of the kernels is that it can be proofed that it is always possible to write h_n and H_n in a symmetric way, without affecting the system's behavior.

A kernel h_n is defined symmetric if

$$h_n(t_1, t_2, ..., t_n) = h_n(t_2, t_1, ..., t_n) = ...$$

for any possible permutation of t_i.

Hereafter we will consider 3-order system. In this case the response of the system to the sum of three exponential function can be expressed, exploiting the symmetric propriety, as a combination of H_{3sym}:

$$\begin{aligned} y(t) \quad = \quad & H_{3sym}(p_1, p_2, p_3) e^{3p_1 t} + \\ & H_{3sym}(p_1, p_2, p_3) e^{3p_2 t} + \\ & H_{3sym}(p_1, p_2, p_3) e^{3p_3 t} + \\ & 3H_{3sym}(p_1, p_1, p_2) e^{2p_1 t + p_2 t} + \\ & 3H_{3sym}(p_1, p_1, p_3) e^{2p_1 t + p_3 t} + \\ & 3H_{3sym}(p_2, p_2, p_1) e^{2p_2 t + p_1 t} + \\ & 3H_{3sym}(p_2, p_2, p_3) e^{2p_2 t + p_3 t} + \\ & 3H_{3sym}(p_3, p_3, p_1) e^{2p_3 t + p_1 t} + \\ & 3H_{3sym}(p_3, p_3, p_2) e^{2p_3 t + p_2 t} + \\ & 6H_{3sym}(p_1, p_2, p_3) e^{p_1 t + p_2 t + p_3 t} \end{aligned} \tag{5}$$

2.1 Polynomial systems

A general non-linear system is defined an N-order polynomial system if its input/output relationship can be expressed as

$$\begin{aligned} y(t) \quad = \quad & \sum_{n=1}^{N} \int_{n=-\infty}^{\infty} h_n(t - \sigma_1, t - \sigma_2, ..., t - \sigma_n) \times \\ & \times u(\sigma_1) u(\sigma_2) ... u(\sigma_n) d\sigma_1 d\sigma_2 ... d\sigma_n \end{aligned}$$

where $h_1(\sigma_1), h_2(\sigma_1, \sigma_2), ..., h_N(\sigma_1, \sigma_2, \ldots, \sigma_n)$ are defined respectively 1-order, 2-order, ..., N-order kernels.

Then a polynomial system can be represented by N homogeneous system in parallel (fig. 1), for each of which the propriety previously discussed holds, and it is termed Volterra series.

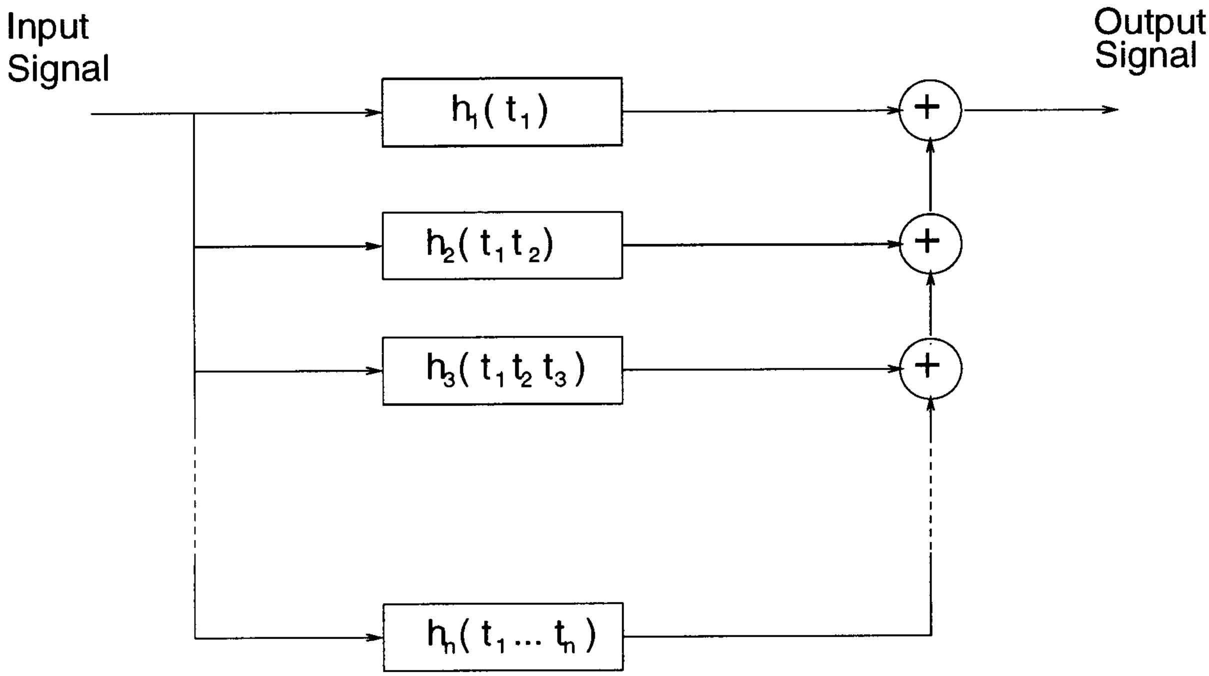

Figure 1: Block diagram of Volterra series expansion.

This mathematical representation combines the modeling issues of the convolution operator and of the Taylor series to provide an effective non-linear model which includes memory effects. As previosly mentioned the knowledge of the kernels is all that is required to characterize the system. The extension of the series over an infinite order is impractical, but a finite Volterra series (often termed Volterra filter) is proved to be a valid approximation for a wide class of systems [7].

Measuring the Volterra kernels of a system is a difficult task, and it has been approached using a number of different test stimuli. Gaussian white noise was first used [8] to give a Volterra series model with a cross-correlation method proposed by Wiener [9]. Unfortunately the cross-correlation method relies upon the statistical averaging properties of the GWN which gives rise to impractically long measurement periods [10]. Typically the model of a loudspeaker may require hundreds of thousands of coefficients to represent it [11], whereas the proposed method relies only on standard SPL measurements and varies the four parameters previously defined in (2).

3 Loudspeaker modeling with Volterra series

Hereafter we assume that the loudspeaker can be characterized with a 3-order polynomial system [9]. Under these hypothesis the displacement $x(t)$ response to the input

$$u(t) = e^{p_1 t} + e^{p_2 t} + ... + e^{p_n t}$$

can be expressed with the sum of the responses of the three homogeneous systems in parallel. Then the following relationship holds:

$$x(t) = \quad H_{1sym}(p_1)e^{p_1 t} + H_{1sym}(p_2)e^{p_2 t} + H_{2sym}(p_1, p_1)e^{2p_1 t} +$$

$$H_{2sym}(p_2,p_2)e^{2p_2t} + 2H_{2sym}(p_1,p_2)e^{p_1t+p_2t} +$$
$$H_{3sym}(p_1,p_1,p_1)e^{3p_1t} + H_{3sym}(p_2,p_2,p_2)e^{3p_2t} +$$
$$H_{3sym}(p_3,p_3,p_3)e^{3p_3t} + 3H_{3sym}(p_1,p_1,p_2)e^{2p_1t+p_2t} +$$
$$3H_{3sym}(p_1,p_1,p_3)e^{2p_1t+p_3t} + 3H_{3sym}(p_2,p_2,p_1)e^{2p_2t+p_1t} +$$
$$3H_{3sym}(p_2,p_2,p_3)e^{2p_2t+p_3t} + 3H_{3sym}(p_3,p_3,p_1)e^{2p_3t+p_1t} +$$
$$3H_{3sym}(p_3,p_3,p_2)e^{2p_3t+p_2t} + 6H_{3sym}(p_1,p_2,p_3)e^{p_1t+p_2t+p_3t}$$

We define $g100$, $g110$ and $g111$, ... such as

$$\begin{aligned} x(t) &= g100e^{p_1t} + g010e^{p_2t} + g001e^{p_3t} + \\ &\quad g110e^{p_1t+p_2t} + g101e^{p_1t+p_3t} + g011e^{p_2t+p_3t} + \\ &\quad g111e^{p_1t+p_2t+p_3t} + \dots \end{aligned} \tag{6}$$

Now we substitute (6) into the Small equations in order to obtain a general expression of $p(t)$ in terms of of $gxxx$. For the sake of simplicity we neglect all the terms of order higher than the third and the intermodulations terms. If we force the Small equations to be satisfied for the terms e^{p_1t}, $e^{p_1t+p_2t}$, $e^{p_1t+p_2t+p_3t}$, we can obtain an expression for $g100$, $g110$ and $g111$ respectively. Namely $g100$ is a function of only linear parameters (Small transfer function), while $g110$ depends also on β and δ (even harmonics), and finally $g111$ is a function of α and γ too (odd harmonics). In appendix are reported the expressions of $g100$, $g110$ and $g111$.

Then to obtain the acoustic pressure starting from the displacement $x(t)$, the polynomial system is followed by a linear system which is defined by

$$Z(j\omega) = \frac{SPL}{g100}. \tag{7}$$

where SPL is the sound pressure level curve (module and phase) measured under small signal conditions and normalized to a 2 Volt sinusoidal input.

Thus we obtain the loudspeaker model sketched in figure 2.

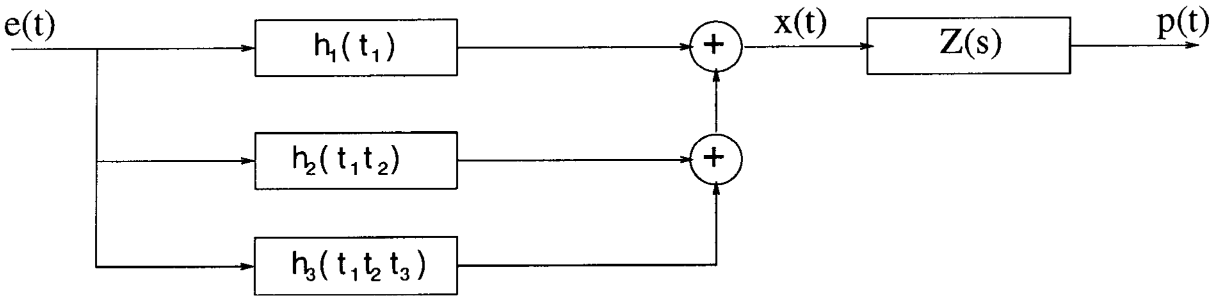

Figure 2: Loudspeaker model through a Volterra series expansion.

It can be easily proofed that this is equivalent to a polynomial system of the same order where the kernels can be expressed by $W_i(s), i = 1, \dots, 3$.

$$\begin{aligned} W_1(s) &= g110(s)Z(s) \\ W_2(s_1,s_2) &= H_1(s_1,s_2)Z(s_1+s_2) \\ W_3(s_2,s_2,s_3) &= H_1(s_2,s_2,s_3)Z(s_2+s_2+s_3) \end{aligned}$$

Cone diameter	130 mm	165 mm	200 mm
R_e (Ω)	1.87	3.16	6.9
M_m (Kg)	8.90*10^{-3}	13.9*10^{-3}	31*10^{-3}
R_m (Kg/s)	1.18	0.43	0.44
S_r (m^2)	84.9*10^{-4}	134.78^{-4}	224*10^{-4}
L_e (H)	0.3*10^{-3}	0.51*10^{-3}	1.77*10^{-3}
Bl0 (T)	2.94	4.2	10.9
C_m $(\frac{m}{N})$	1.4*10^{-3}	0.4*10^{-3}	0.3*10^{-3}
K0 $(\frac{N}{m})$	$\frac{1}{C_m}$	$\frac{1}{C_m}$	$\frac{1}{C_m}$
$x$max (m)	0.008	0.012	0.015
E_{in} (V Vrms)	5	8.9	12.6

If we consider a sinusoidal input to the system of fig 2 the output can be expressed as a sum of three harmonics whose expressions are reported below.

$$G100 = \frac{Ein}{2} g100 + \frac{Ein^3}{8} 3g111Za(j\omega)$$

$$G110 = \frac{Ein^2}{4} g110Za(j2\omega)$$

$$G111 = \frac{Ein^3}{8} g111Za(j3\omega)$$

Then the modeling issue is that of optimizing the response of the defined model to a sinusoidal input with the actual measurements performed in the same conditions. The first three harmonics of the acoustic pressure are computed from the polynomial system characterized before. Then an optimization procedure parametric in α, β, γ and δ is applied to minimize the error between experimental results and the harmonics computed at previous point.

The optimization procedure was applied in order to match the experimental results obtained measuring the SPL harmonics of three commercial loudspeaker in an anechoic room. Loudspeaker physical and acoustic parameters are reported in table 3.

4 Results

The following figures report the comparison between simulation results of the Volterra model of the considered loudspeaker and measurements data in terms of SPL harmonics.

For each kind of considered loudspeaker are reported the first, second and third harmonics, and the $Bl(x)$ and $K(x)$ profiles after the optimization procedure. The measures used for comparison are obtained using an input signal of rms value equal to 5 V for the 130m loudspeaker, to 8.9 V for the 165mm and to 12.6 V for the 200mm. In figs. 18, 19, 20, 21, 22 are reported comparison obtained when the input signal applied to the 200mm loudspeaker has an rms value of 11 V.

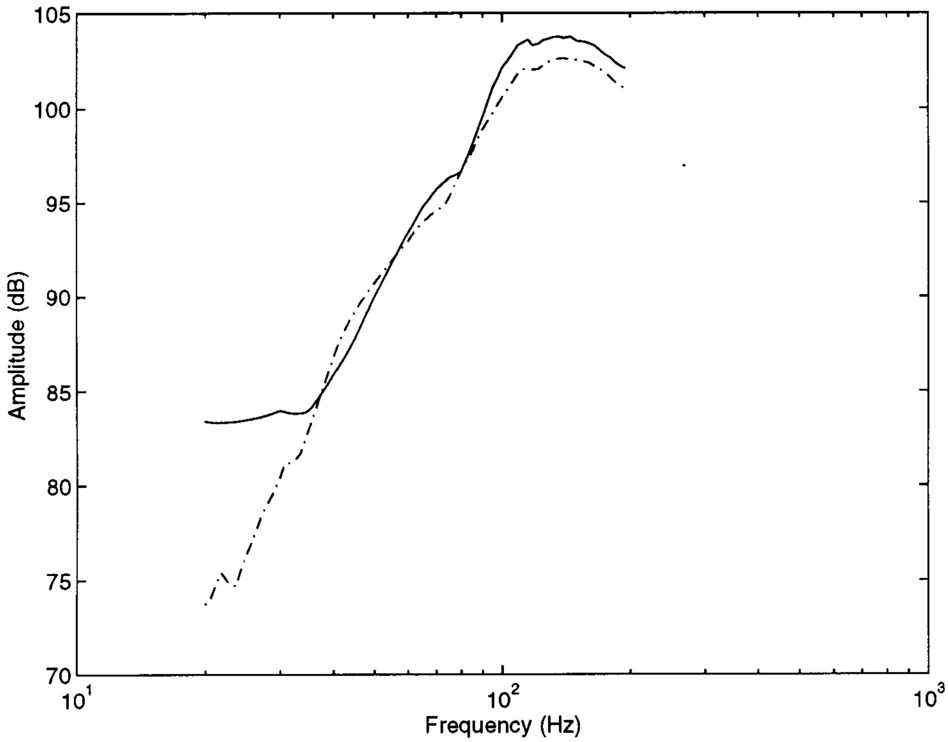

Figure 3: Comparison between measurements and simulation of first harmonics of 130mm.

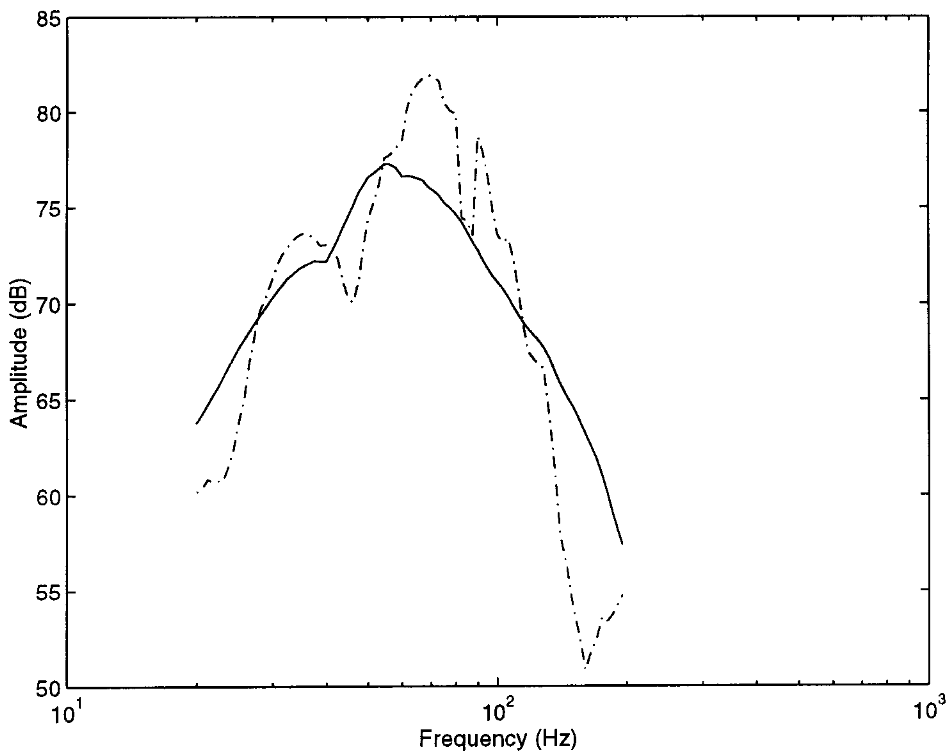

Figure 4: Comparison between measurements and simulation of first harmonics of 130mm.

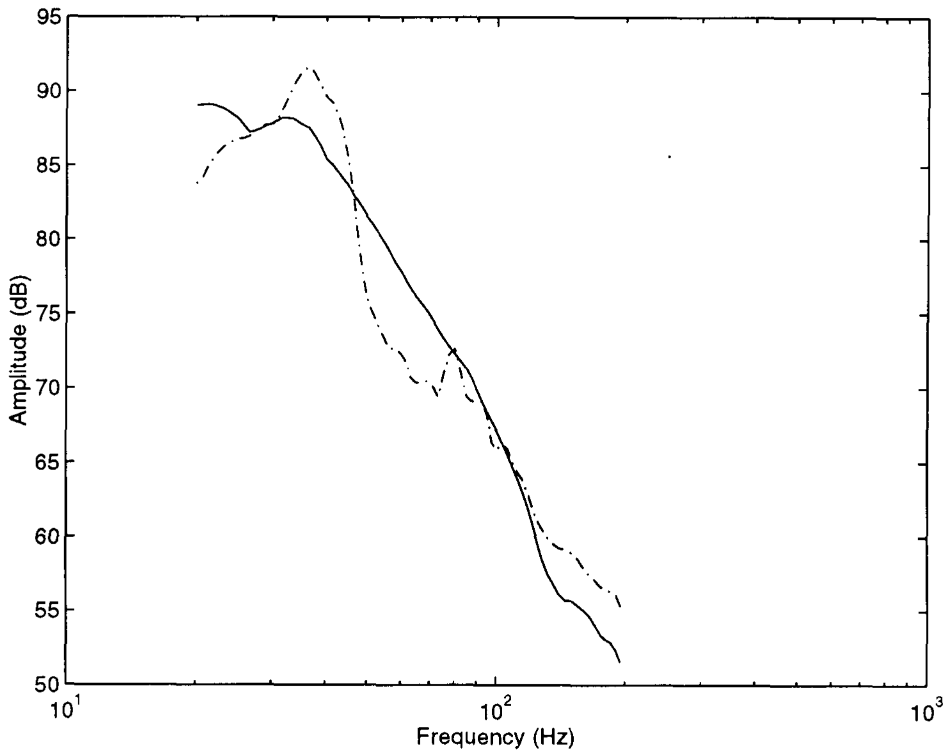

Figure 5: Comparison between measurements and simulation of first harmonics of 130mm.

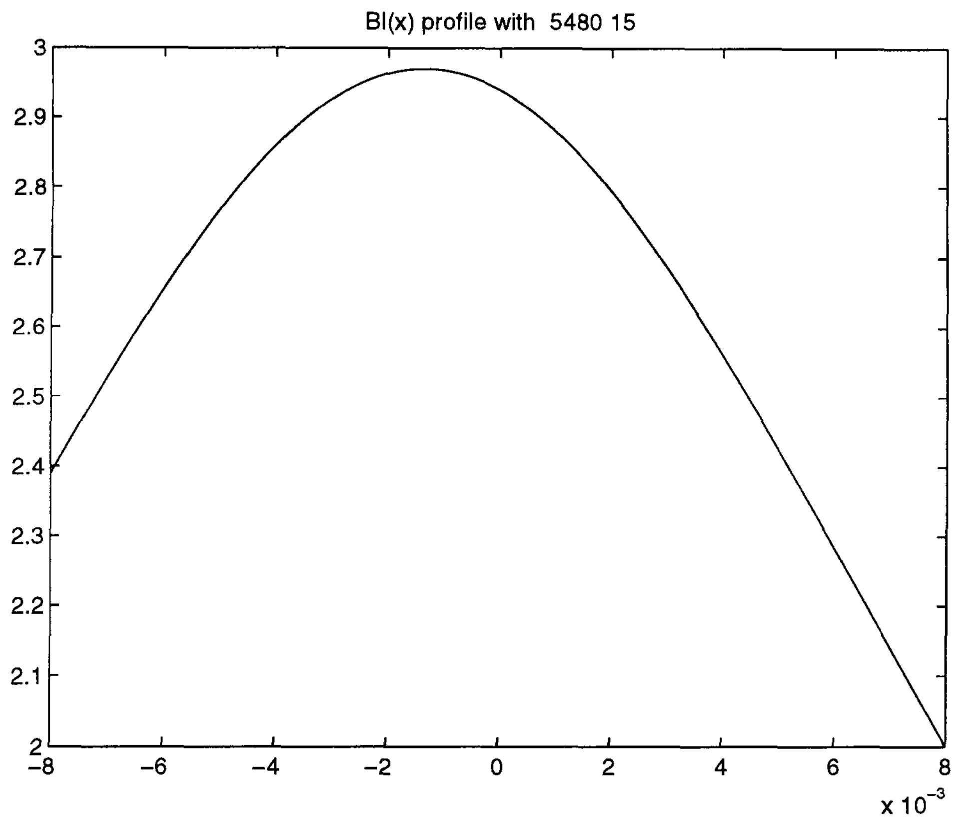

Figure 6: Optimal $Bl(x)$ profile for 130mm, the values of α and β are reported.

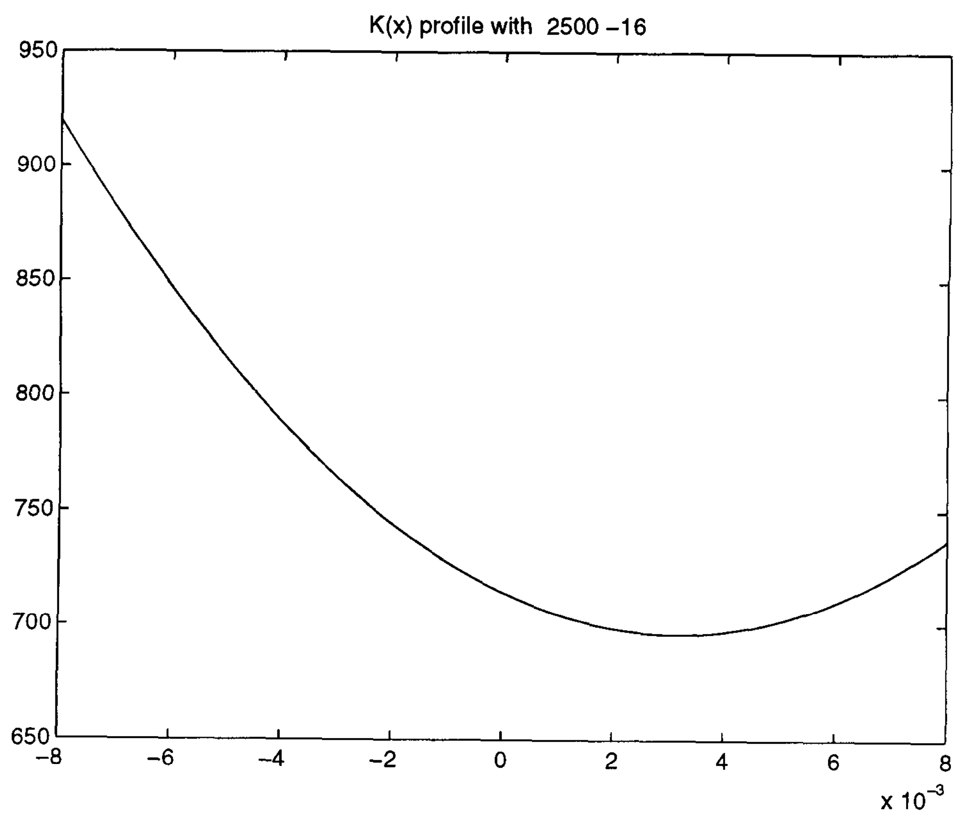

Figure 7: Optimal $K(x)$ profile for 130mm, the values of γ and δ are reported.

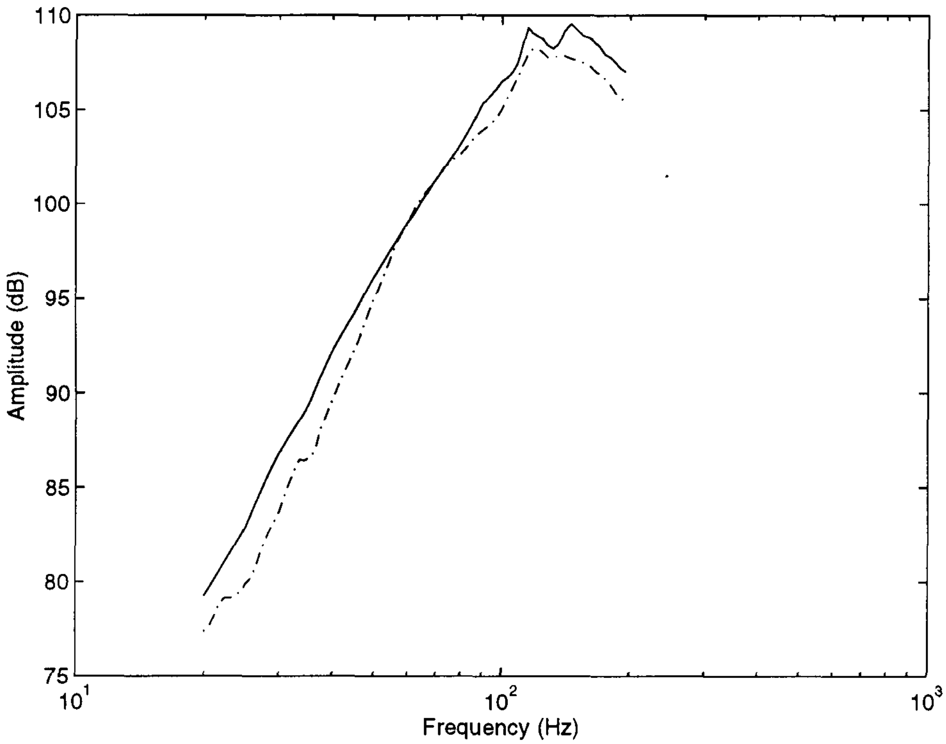

Figure 8: Comparison between measurements and simulation of first harmonics of 165mm.

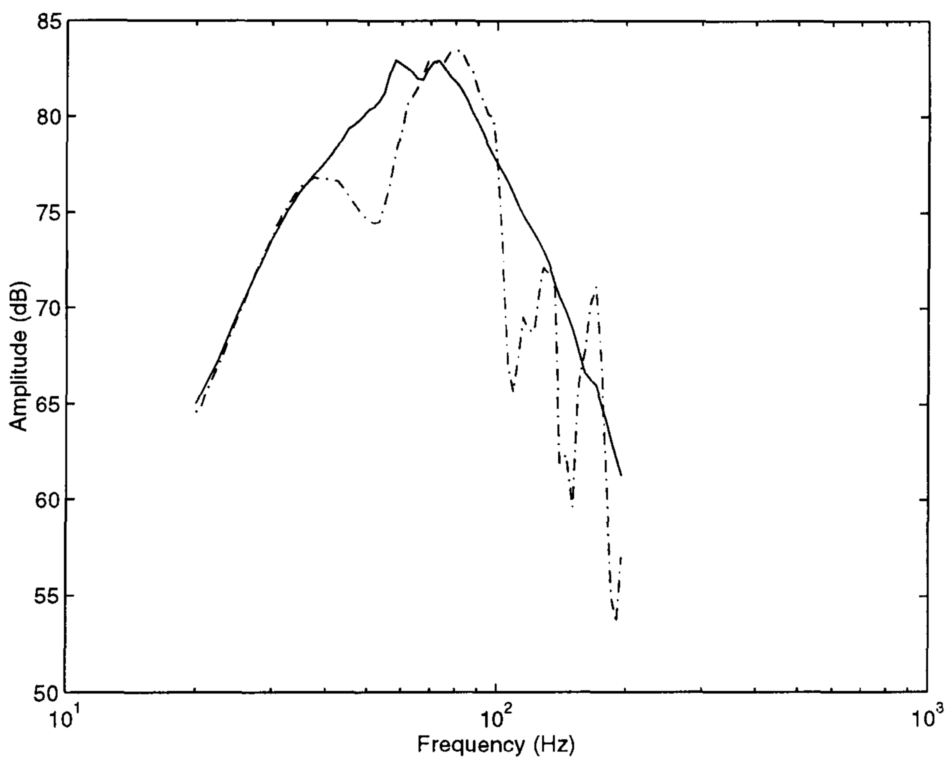

Figure 9: Comparison between measurements and simulation of first harmonics of 165mm.

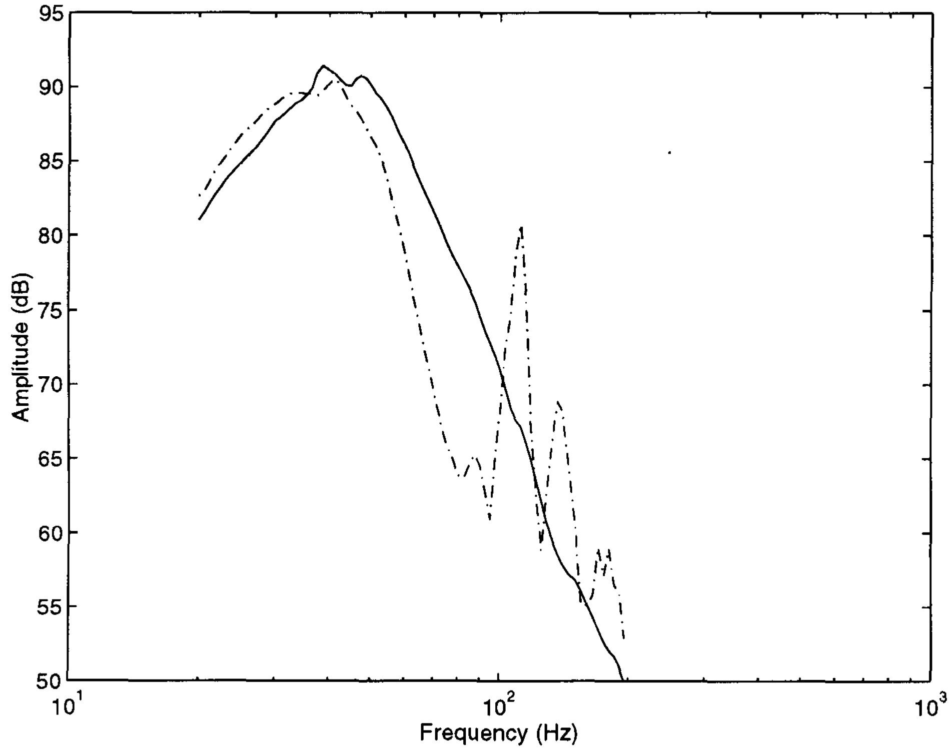

Figure 10: Comparison between measurements and simulation of first harmonics of 165mm.

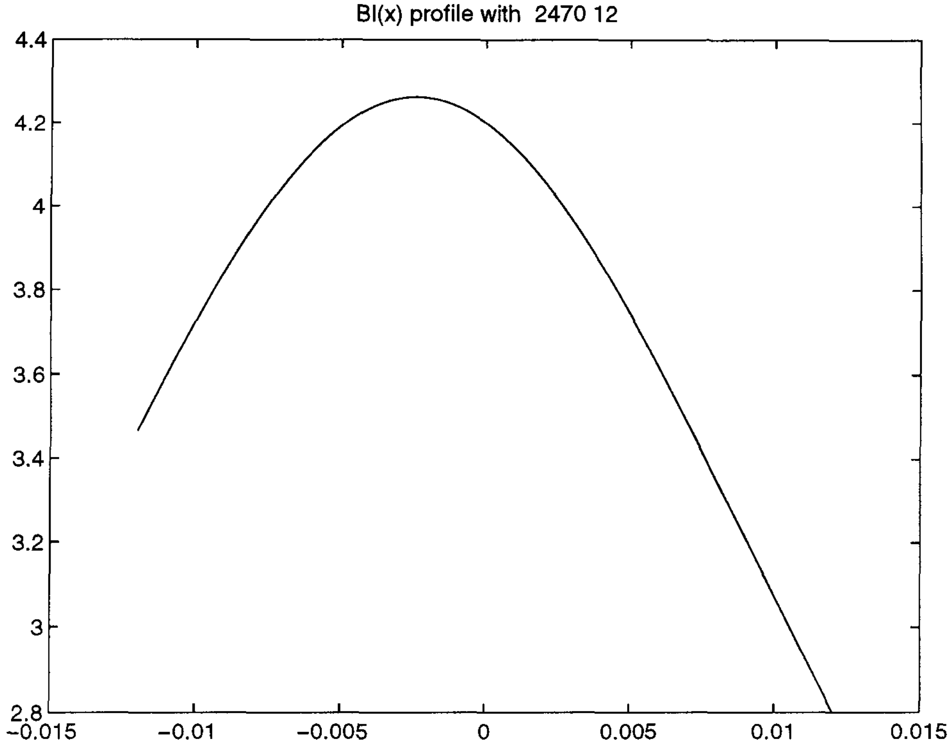

Figure 11: Optimal $Bl(x)$ profile for 165mm, the values of α and β are reported.

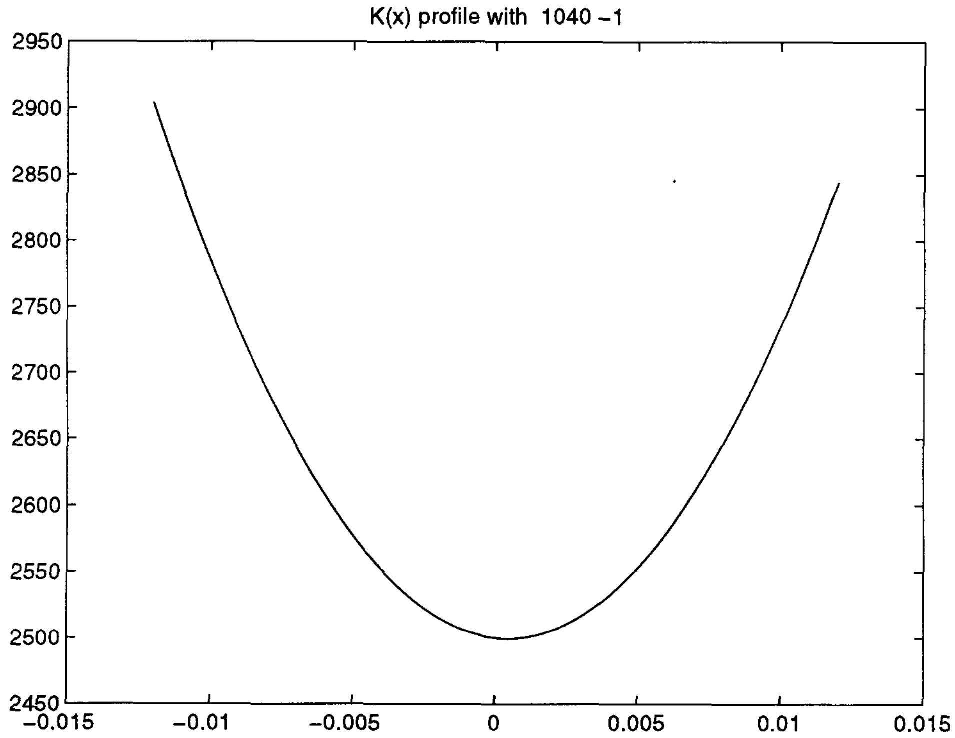

Figure 12: Optimal $K(x)$ profile for 165mm, the values of γ and δ are reported.

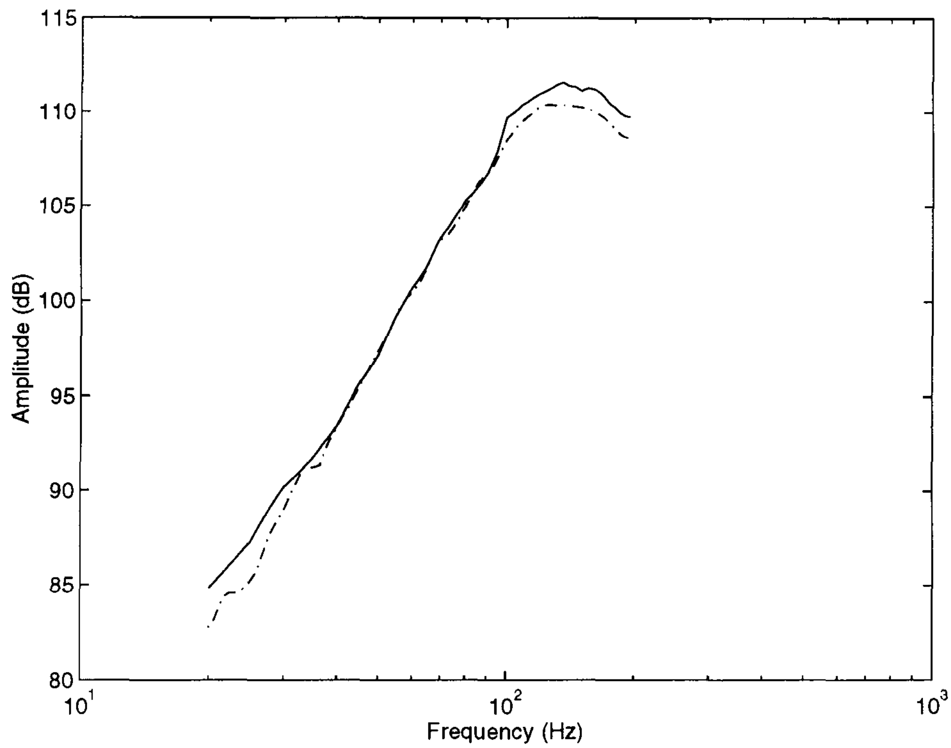

Figure 13: Comparison between measurements and simulation of first harmonics of 200mm.

They confirm that the model can be effective and quite general as far as low-frequency loud-

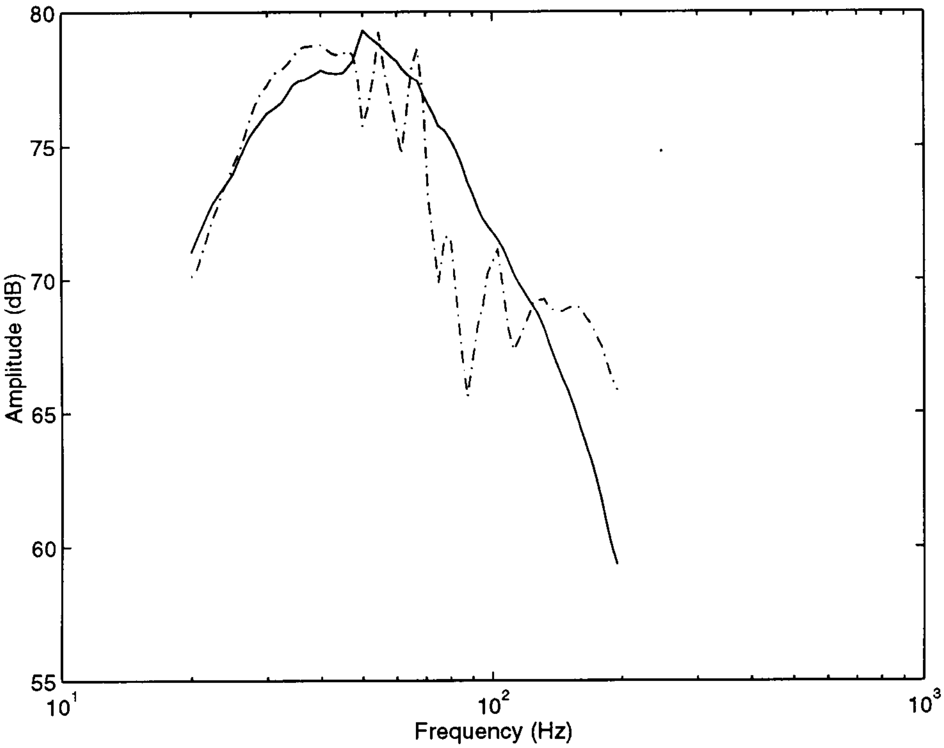

Figure 14: Comparison between measurements and simulation of first harmonics of 200mm.

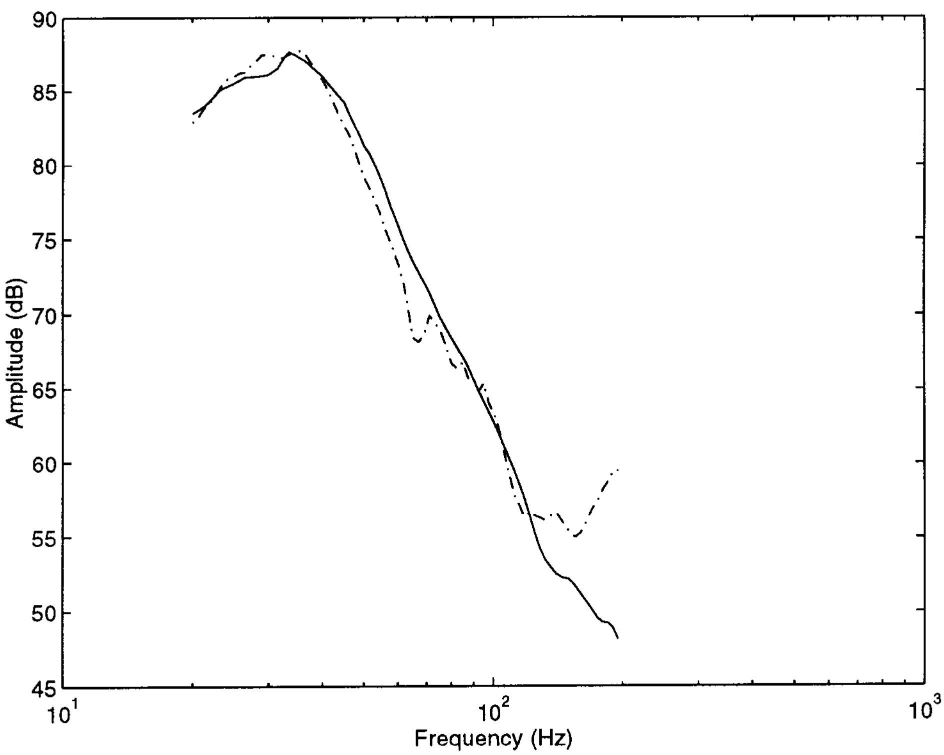

Figure 15: Comparison between measurements and simulation of first harmonics of 200mm.

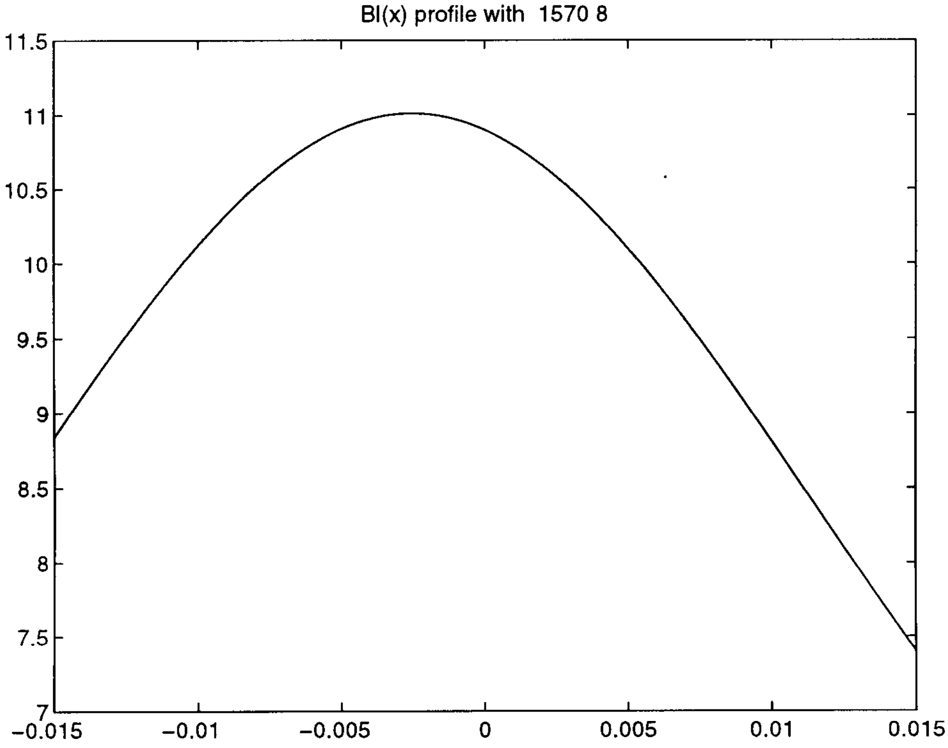

Figure 16: Optimal $Bl(x)$ profile for 200mm, the values of α and β are reported.

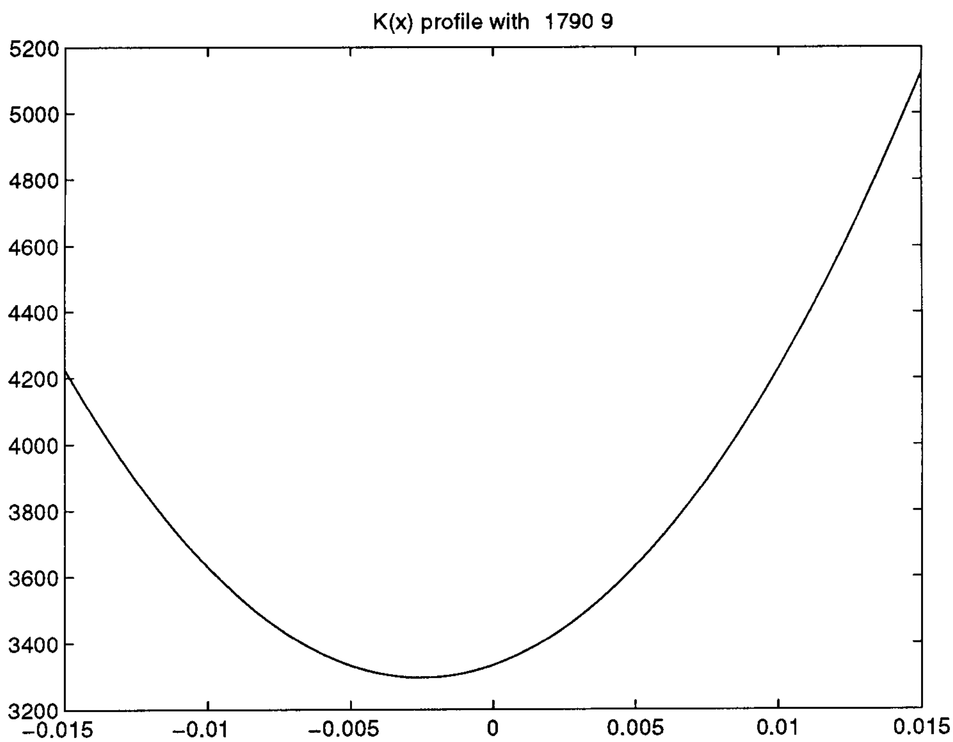

Figure 17: Optimal $K(x)$ profile for 200mm, the values of γ and δ are reported.

speakers are concerned. Moreover the amplitude of the driving signal is so that we are not in weakly non-linear conditions, and we could expect that the proposed model would not be effec-

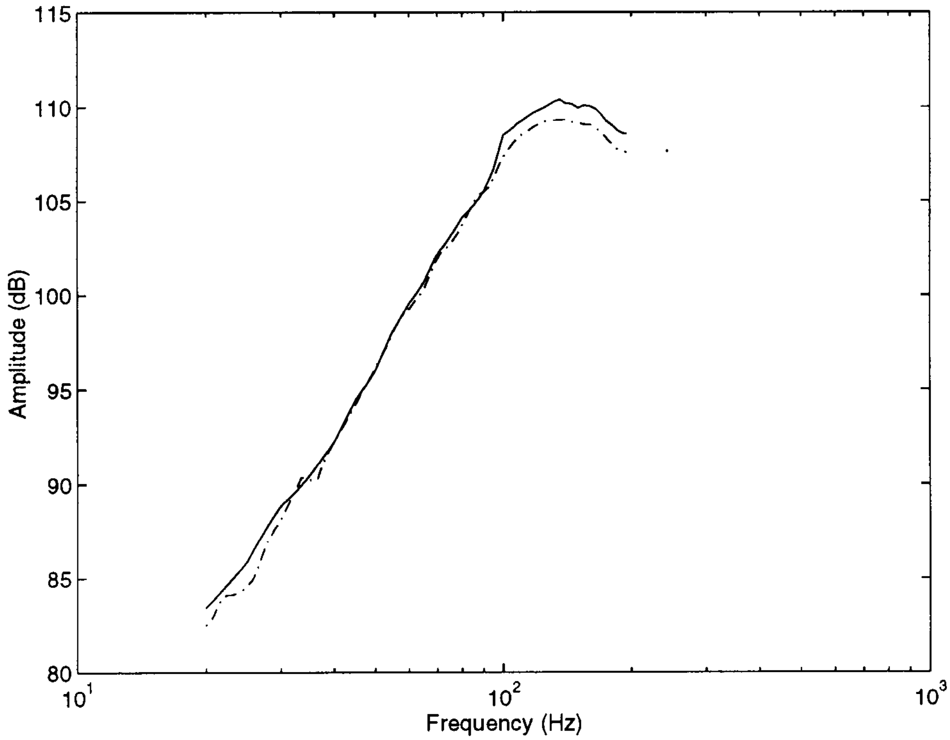

Figure 18: Comparison between measurements and simulation of first harmonics of 200mm (input rms amplitude 11V).

tive. In fact with a low amplitude input signal the error between simulation and measurements is lower, but the global results is fine even in worse conditions, validating the effectiveness of the proposed structure.

The main interesting characteristics is that it is possible to describe the dynamic behavior of loudspeakers only through standard acoustic measurements The parameters obtained this way can be successfully used within commercial DSP systems to obtain an effective sound equalization and distortion compensation, as it was performed in [12].

5 Appendix

Volterra Kernel expressions in the `Matlab` [13] syntax.

alfa = α, beta = β, gamma = γ, delta = δ, Rm = R_m, Re = Re, ...

$$l1 = j \times 2 \times \pi \times f1$$

$$l2 = j \times 2 \times \pi \times f2$$

$$l3 = j \times 2 \times \pi \times f3$$

```
First Kernel ( linear transfer function )


g100 = - ( Bl0 ./ ...
       ( (-(Bl0.^2.*l1)) - K0.*L.*l1 - L.*l1.^3.*Mm -
```

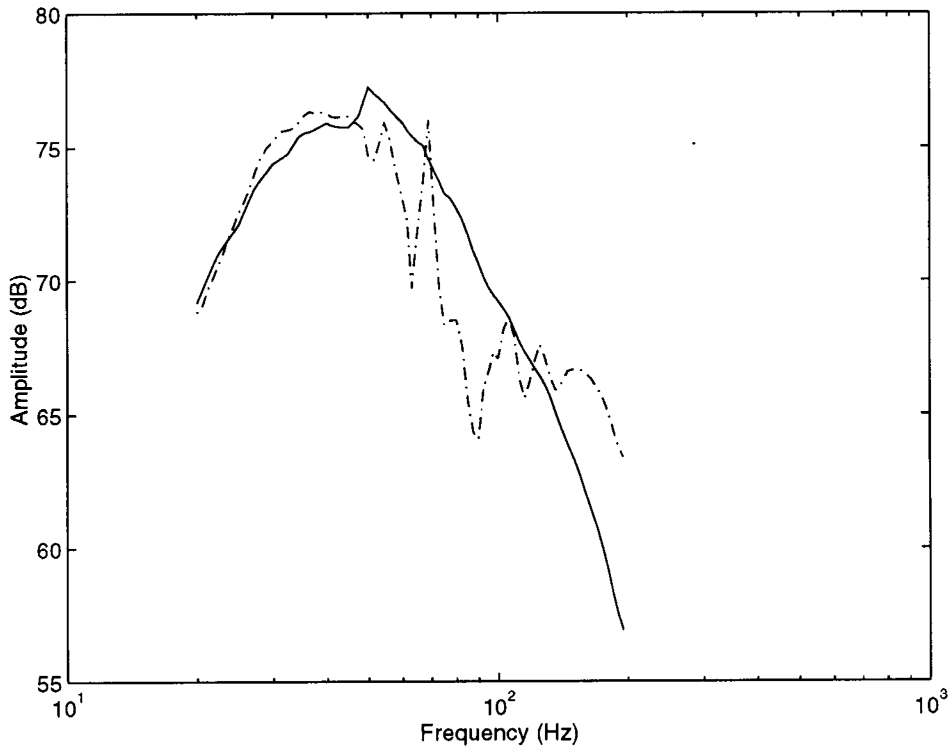

Figure 19: Comparison between measurements and simulation of first harmonics of 200mm (input rms amplitude 11V).

```
        K0.*Re - l1.^2.*Mm.*Re - ...  L.*l1.^2.*Rm - l1.*Re.*Rm -
        Bs.*L.*l1.^3.*Sr.^2 - As.*L.*l1.^4.*Sr.^2 - ...
        Bs.*l1.^2.*Re.*Sr.^2 - As.*l1.^3.*Re.*Sr.^2));
```

Second kernel

```
g110 = -(beta.*Bl0.*g010 + beta.*Bl0.*g100 - ...
         3.*beta.*g010.*g100.*K0.*L.*l1 - ...
         2.*delta.*g010.*g100.*K0.*L.*l1 - ...
         3.*beta.*g010.*g100.*K0.*L.*l2 - ...
         2.*delta.*g010.*g100.*K0.*L.*l2 - ...
         2.*beta.*g010.*g100.*L.*l1.^3.*Mm - ...
         beta.*g010.*g100.*L.*l1.^2.*l2.*Mm - ...
         beta.*g010.*g100.*L.*l1.*l2.^2.*Mm - ...
         2.*beta.*g010.*g100.*L.*l2.^3.*Mm - ...
         4.*beta.*g010.*g100.*K0.*Re - ...
         2.*delta.*g010.*g100.*K0.*Re - ...
         2.*beta.*g010.*g100.*l1.^2.*Mm.*Re - ...
         2.*beta.*g010.*g100.*l2.^2.*Mm.*Re - ...
         2.*beta.*g010.*g100.*L.*l1.^2.*Rm - ...
         2.*beta.*g010.*g100.*L.*l1.*l2.*Rm - ...
         2.*beta.*g010.*g100.*L.*l2.^2.*Rm - ...
         2.*beta.*g010.*g100.*l1.*Re.*Rm - ...
         2.*beta.*g010.*g100.*l2.*Re.*Rm - ...
         2.*beta.*Bs.*g010.*g100.*L.*l1.^3.*Sr.^2 - ...
```

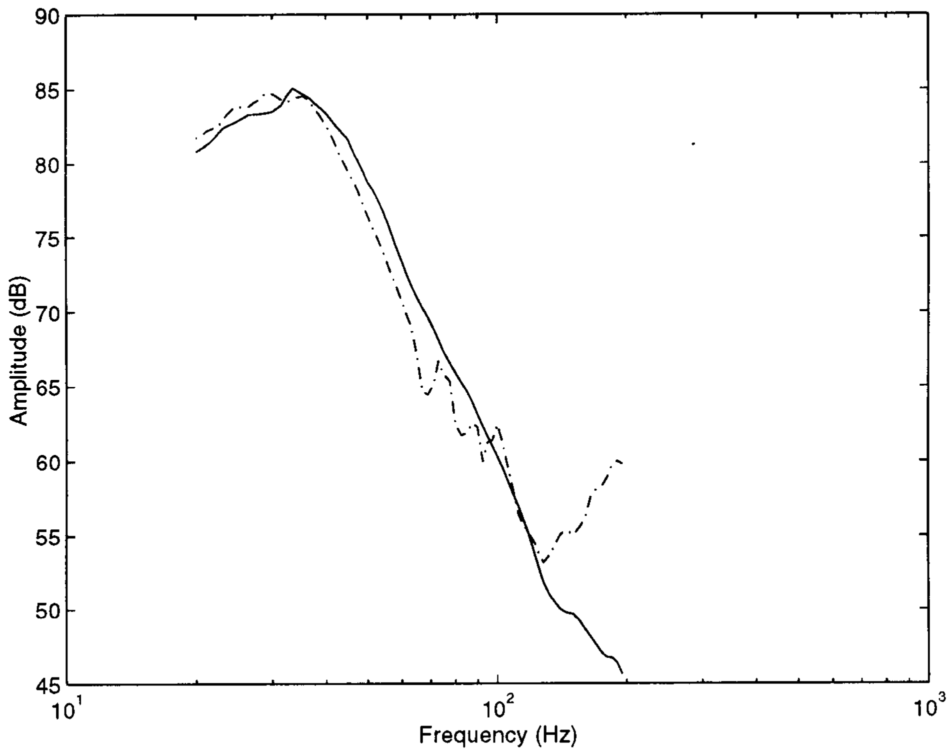

Figure 20: Comparison between measurements and simulation of first harmonics of 200mm (input rms amplitude 11V).

```
        2.*As.*beta.*g010.*g100.*L.*l1.^4.*Sr.^2 - ...
        beta.*Bs.*g010.*g100.*L.*l1.^2.*l2.*Sr.^2 - ...
        As.*beta.*g010.*g100.*L.*l1.^3.*l2.*Sr.^2 - ...
        beta.*Bs.*g010.*g100.*L.*l1.*l2.^2.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g100.*L.*l2.^3.*Sr.^2 - ...
        As.*beta.*g010.*g100.*L.*l1.*l2.^3.*Sr.^2 - ...
        2.*As.*beta.*g010.*g100.*L.*l2.^4.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g100.*l1.^2.*Re.*Sr.^2 - ...
        2.*As.*beta.*g010.*g100.*l1.^3.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g100.*l2.^2.*Re.*Sr.^2 - ...
        2.*As.*beta.*g010.*g100.*l2.^3.*Re.*Sr.^2)./ ...
(-(Bl0 ./ g010) - Bl0 ./g100 + K0.*Re - ...
        4.*As.*L.*l1.^3.*l2.*Sr.^2 + ...
l1.^2.*(-6.*As.*L.*l2.^2.*Sr.^2 + l2.*(-3.*L.*Mm - ...
3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2)) + ...
        l1.*(-4.*As.*L.*l2.^3.*Sr.^2 + ...
l2.^2.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2) + ...
l2.*(-2.*L.*Rm + Re.*(-2.*Mm - 2.*Bs.*Sr.^2))));
```

```
Third Kernel
```

```
g111 = - ( g111num1 + g111num2 + g111num3 + g111num4 ) ./ g111den;
```

where

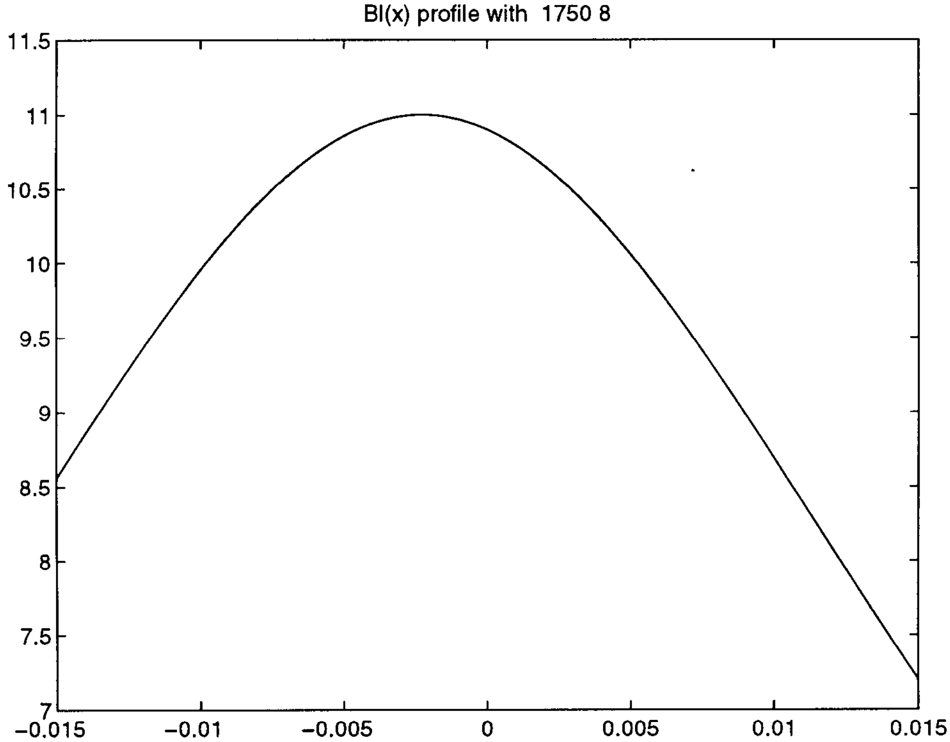

Figure 21: Optimal $Bl(x)$ profile for 200mm, the values of α and β are reported (input rms amplitude 11V).

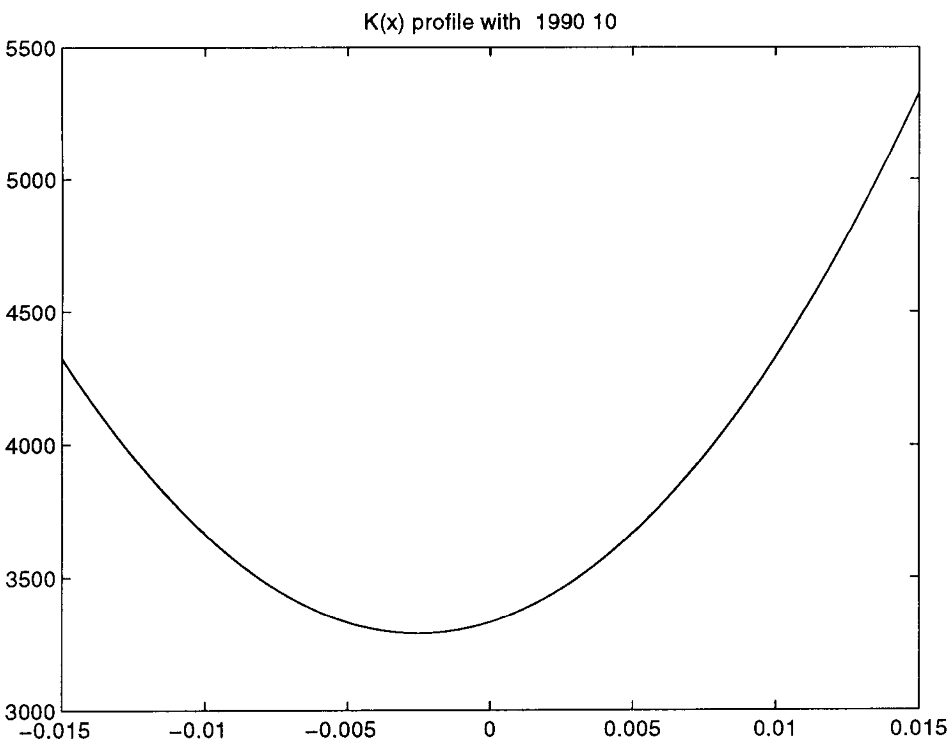

Figure 22: Optimal $K(x)$ profile for 200mm, the values of γ and δ are reported (input rms amplitude 11V).

```
g111num1 = 2.*alpha.*B10.*g001.*g010 + beta.*B10.*g011 + ...
        2.*alpha.*B10.*g001.*g100 + 2.*alpha.*B10.*g010.*g100 + ...
        beta.*B10.*g101 + beta.*B10.*g110 - ...
        8.*alpha.*g001.*g010.*g100.*K0.*L.*l1 - ...
        4.*beta.^2.*g001.*g010.*g100.*K0.*L.*l1 - ...
        10.*beta.*delta.*g001.*g010.*g100.*K0.*L.*l1 - ...
        6.*gamma.*g001.*g010.*g100.*K0.*L.*l1 - ...
 3.*beta.*g011.*g100.*K0.*L.*l1 - ...
        2.*delta.*g011.*g100.*K0.*L.*l1 - ...
        3.*beta.*g010.*g101.*K0.*L.*l1 - ...
        2.*delta.*g010.*g101.*K0.*L.*l1 - ...
        3.*beta.*g001.*g110.*K0.*L.*l1 - ...
        2.*delta.*g001.*g110.*K0.*L.*l1 - ...
        8.*alpha.*g001.*g010.*g100.*K0.*L.*l2 - ...
        4.*beta.^2.*g001.*g010.*g100.*K0.*L.*l2 - ...
        10.*beta.*delta.*g001.*g010.*g100.*K0.*L.*l2 - ...
        6.*gamma.*g001.*g010.*g100.*K0.*L.*l2 - ...
        3.*beta.*g011.*g100.*K0.*L.*l2 - ...
        2.*delta.*g011.*g100.*K0.*L.*l2 - ...
        3.*beta.*g010.*g101.*K0.*L.*l2 - ...
        2.*delta.*g010.*g101.*K0.*L.*l2 - ...
        3.*beta.*g001.*g110.*K0.*L.*l2 - ...
        2.*delta.*g001.*g110.*K0.*L.*l2 - ...
        8.*alpha.*g001.*g010.*g100.*K0.*L.*l3 - ...
        4.*beta.^2.*g001.*g010.*g100.*K0.*L.*l3 - ...
        10.*beta.*delta.*g001.*g010.*g100.*K0.*L.*l3 - ...
        6.*gamma.*g001.*g010.*g100.*K0.*L.*l3 - ...
        3.*beta.*g011.*g100.*K0.*L.*l3 - ...
        2.*delta.*g011.*g100.*K0.*L.*l3 - ...
        3.*beta.*g010.*g101.*K0.*L.*l3 - ...
        2.*delta.*g010.*g101.*K0.*L.*l3 - ...
        3.*beta.*g001.*g110.*K0.*L.*l3 - ...
        2.*delta.*g001.*g110.*K0.*L.*l3 - ...
        4.*alpha.*g001.*g010.*g100.*L.*l1.^3.*Mm - ...
        2.*beta.^2.*g001.*g010.*g100.*L.*l1.^3.*Mm - ...
        2.*beta.*g011.*g100.*L.*l1.^3.*Mm - ...
        2.*beta.*g010.*g101.*L.*l1.^3.*Mm - ...
        2.*beta.*g001.*g110.*L.*l1.^3.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l1.^2.*l2.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l1.^2.*l2.*Mm - ...
        beta.*g011.*g100.*L.*l1.^2.*l2.*Mm - ...
        beta.*g010.*g101.*L.*l1.^2.*l2.*Mm - ...
        6.*beta.*g001.*g110.*L.*l1.^2.*l2.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l1.*l2.^2.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l1.*l2.^2.*Mm - ...
        beta.*g011.*g100.*L.*l1.*l2.^2.*Mm - ...
        beta.*g010.*g101.*L.*l1.*l2.^2.*Mm - ...
        6.*beta.*g001.*g110.*L.*l1.*l2.^2.*Mm - ...
        4.*alpha.*g001.*g010.*g100.*L.*l2.^3.*Mm - ...
        2.*beta.^2.*g001.*g010.*g100.*L.*l2.^3.*Mm - ...
```

```
        2.*beta.*g011.*g100.*L.*l2.^3.*Mm - ...
        2.*beta.*g010.*g101.*L.*l2.^3.*Mm - ...
        2.*beta.*g001.*g110.*L.*l2.^3.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l1.^2.*l3.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l1.^2.*l3.*Mm - ...
        beta.*g011.*g100.*L.*l1.^2.*l3.*Mm - ...
        6.*beta.*g010.*g101.*L.*l1.^2.*l3.*Mm - ...
        beta.*g001.*g110.*L.*l1.^2.*l3.*Mm - ...
        2.*beta.*g011.*g100.*L.*l1.*l2.*l3.*Mm - ...
        2.*beta.*g010.*g101.*L.*l1.*l2.*l3.*Mm - ...
        2.*beta.*g001.*g110.*L.*l1.*l2.*l3.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l2.^2.*l3.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l2.^2.*l3.*Mm - ...
        6.*beta.*g011.*g100.*L.*l2.^2.*l3.*Mm - ...
        beta.*g010.*g101.*L.*l2.^2.*l3.*Mm - ...
        beta.*g001.*g110.*L.*l2.^2.*l3.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l1.*l3.^2.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l1.*l3.^2.*Mm - ...
        beta.*g011.*g100.*L.*l1.*l3.^2.*Mm - ...
        6.*beta.*g010.*g101.*L.*l1.*l3.^2.*Mm - ...
        beta.*g001.*g110.*L.*l1.*l3.^2.*Mm - ...
        2.*alpha.*g001.*g010.*g100.*L.*l2.*l3.^2.*Mm - ...
        beta.^2.*g001.*g010.*g100.*L.*l2.*l3.^2.*Mm - ...
        6.*beta.*g011.*g100.*L.*l2.*l3.^2.*Mm - ...
        beta.*g010.*g101.*L.*l2.*l3.^2.*Mm - ...
        beta.*g001.*g110.*L.*l2.*l3.^2.*Mm - ...
        4.*alpha.*g001.*g010.*g100.*L.*l3.^3.*Mm - ...
        2.*beta.^2.*g001.*g010.*g100.*L.*l3.^3.*Mm - ...
        2.*beta.*g011.*g100.*L.*l3.^3.*Mm - ...
        2.*beta.*g010.*g101.*L.*l3.^3.*Mm - ...
        2.*beta.*g001.*g110.*L.*l3.^3.*Mm - ...
        12.*alpha.*g001.*g010.*g100.*K0.*Re - ...
        6.*beta.^2.*g001.*g010.*g100.*K0.*Re - ...
        12.*beta.*delta.*g001.*g010.*g100.*K0.*Re ;

g111num2 = - 6.*gamma.*g001.*g010.*g100.*K0.*Re - ...
        4.*beta.*g011.*g100.*K0.*Re - ...
        2.*delta.*g011.*g100.*K0.*Re - ...
        4.*beta.*g010.*g101.*K0.*Re - ...
        2.*delta.*g010.*g101.*K0.*Re - ...
        4.*beta.*g001.*g110.*K0.*Re - ...
        2.*delta.*g001.*g110.*K0.*Re - ...
        4.*alpha.*g001.*g010.*g100.*l1.^2.*Mm.*Re - ...
        2.*beta.^2.*g001.*g010.*g100.*l1.^2.*Mm.*Re - ...
        2.*beta.*g011.*g100.*l1.^2.*Mm.*Re - ...
        2.*beta.*g010.*g101.*l1.^2.*Mm.*Re - ...
        2.*beta.*g001.*g110.*l1.^2.*Mm.*Re - ...
        4.*beta.*g001.*g110.*l1.*l2.*Mm.*Re - ...
        4.*alpha.*g001.*g010.*g100.*l2.^2.*Mm.*Re - ...
        2.*beta.^2.*g001.*g010.*g100.*l2.^2.*Mm.*Re - ...
```

```
2.*beta.*g011.*g100.*l2.^2.*Mm.*Re - ...
2.*beta.*g010.*g101.*l2.^2.*Mm.*Re - ...
2.*beta.*g001.*g110.*l2.^2.*Mm.*Re - ...
4.*beta.*g010.*g101.*l1.*l3.*Mm.*Re - ...
4.*beta.*g011.*g100.*l2.*l3.*Mm.*Re - ...
4.*alpha.*g001.*g010.*g100.*l3.^2.*Mm.*Re - ...
2.*beta.^2.*g001.*g010.*g100.*l3.^2.*Mm.*Re - ...
2.*beta.*g011.*g100.*l3.^2.*Mm.*Re - ...
2.*beta.*g010.*g101.*l3.^2.*Mm.*Re - ...
2.*beta.*g001.*g110.*l3.^2.*Mm.*Re - ...
4.*alpha.*g001.*g010.*g100.*L.*l1.^2.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l1.^2.*Rm - ...
2.*beta.*g011.*g100.*L.*l1.^2.*Rm - ...
2.*beta.*g010.*g101.*L.*l1.^2.*Rm - ...
2.*beta.*g001.*g110.*L.*l1.^2.*Rm - ...
4.*alpha.*g001.*g010.*g100.*L.*l1.*l2.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l1.*l2.*Rm - ...
2.*beta.*g011.*g100.*L.*l1.*l2.*Rm - ...
2.*beta.*g010.*g101.*L.*l1.*l2.*Rm - ...
4.*beta.*g001.*g110.*L.*l1.*l2.*Rm - ...
4.*alpha.*g001.*g010.*g100.*L.*l2.^2.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l2.^2.*Rm - ...
2.*beta.*g011.*g100.*L.*l2.^2.*Rm - ...
2.*beta.*g010.*g101.*L.*l2.^2.*Rm - ...
2.*beta.*g001.*g110.*L.*l2.^2.*Rm - ...
4.*alpha.*g001.*g010.*g100.*L.*l1.*l3.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l1.*l3.*Rm - ...
2.*beta.*g011.*g100.*L.*l1.*l3.*Rm - ...
4.*beta.*g010.*g101.*L.*l1.*l3.*Rm - ...
2.*beta.*g001.*g110.*L.*l1.*l3.*Rm - ...
4.*alpha.*g001.*g010.*g100.*L.*l2.*l3.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l2.*l3.*Rm - ...
4.*beta.*g011.*g100.*L.*l2.*l3.*Rm - ...
2.*beta.*g010.*g101.*L.*l2.*l3.*Rm - ...
2.*beta.*g001.*g110.*L.*l2.*l3.*Rm - ...
4.*alpha.*g001.*g010.*g100.*L.*l3.^2.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*L.*l3.^2.*Rm - ...
2.*beta.*g011.*g100.*L.*l3.^2.*Rm - ...
2.*beta.*g010.*g101.*L.*l3.^2.*Rm - ...
2.*beta.*g001.*g110.*L.*l3.^2.*Rm - ...
4.*alpha.*g001.*g010.*g100.*l1.*Re.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*l1.*Re.*Rm - ...
2.*beta.*g011.*g100.*l1.*Re.*Rm - ...
2.*beta.*g010.*g101.*l1.*Re.*Rm - ...
2.*beta.*g001.*g110.*l1.*Re.*Rm - ...
4.*alpha.*g001.*g010.*g100.*l2.*Re.*Rm - ...
2.*beta.^2.*g001.*g010.*g100.*l2.*Re.*Rm - ...
2.*beta.*g011.*g100.*l2.*Re.*Rm - ...
2.*beta.*g010.*g101.*l2.*Re.*Rm - ...
2.*beta.*g001.*g110.*l2.*Re.*Rm - ...
```

```
        4.*alpha.*g001.*g010.*g100.*l3.*Re.*Rm - ...
        2.*beta.^2.*g001.*g010.*g100.*l3.*Re.*Rm - ...
        2.*beta.*g011.*g100.*l3.*Re.*Rm - ...
        2.*beta.*g010.*g101.*l3.*Re.*Rm - ...
        2.*beta.*g001.*g110.*l3.*Re.*Rm - ... .
        4.*alpha.*Bs.*g001.*g010.*g100.*L.*l1.^3.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*L.*l1.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*L.*l1.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*L.*l1.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*L.*l1.^3.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*L.*l1.^4.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*L.*l1.^4.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*L.*l1.^4.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*L.*l1.^4.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*L.*l1.^4.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l1.^2.*l2.*Sr.^2 - ...
        beta.^2.*Bs.*g001.*g010.*g100.*L.*l1.^2.*l2.*Sr.^2 - ...
        beta.*Bs.*g011.*g100.*L.*l1.^2.*l2.*Sr.^2 - ...
        beta.*Bs.*g010.*g101.*L.*l1.^2.*l2.*Sr.^2 - ...
        6.*beta.*Bs.*g001.*g110.*L.*l1.^2.*l2.*Sr.^2;

g111num3 = - 2.*alpha.*As.*g001.*g010.*g100.*L.*l1.^3.*l2.*Sr.^2 - ...
        As.*beta.^2.*g001.*g010.*g100.*L.*l1.^3.*l2.*Sr.^2 - ...
        As.*beta.*g011.*g100.*L.*l1.^3.*l2.*Sr.^2 - ...
        As.*beta.*g010.*g101.*L.*l1.^3.*l2.*Sr.^2 - ...
        8.*As.*beta.*g001.*g110.*L.*l1.^3.*l2.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l1.*l2.^2.*Sr.^2 - ...
        beta.^2.*Bs.*g001.*g010.*g100.*L.*l1.*l2.^2.*Sr.^2 - ...
        beta.*Bs.*g011.*g100.*L.*l1.*l2.^2.*Sr.^2 - ...
        beta.*Bs.*g010.*g101.*L.*l1.*l2.^2.*Sr.^2 - ...
        6.*beta.*Bs.*g001.*g110.*L.*l1.*l2.^2.*Sr.^2 - ...
        12.*As.*beta.*g001.*g110.*L.*l1.^2.*l2.^2.*Sr.^2 - ...
        4.*alpha.*Bs.*g001.*g010.*g100.*L.*l2.^3.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*L.*l2.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*L.*l2.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*L.*l2.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*L.*l2.^3.*Sr.^2 - ...
        2.*alpha.*As.*g001.*g010.*g100.*L.*l1.*l2.^3.*Sr.^2 - ...
        As.*beta.^2.*g001.*g010.*g100.*L.*l1.*l2.^3.*Sr.^2 - ...
        As.*beta.*g011.*g100.*L.*l1.*l2.^3.*Sr.^2 - ...
        As.*beta.*g010.*g101.*L.*l1.*l2.^3.*Sr.^2 - ...
        8.*As.*beta.*g001.*g110.*L.*l1.*l2.^3.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*L.*l2.^4.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*L.*l2.^4.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*L.*l2.^4.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*L.*l2.^4.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*L.*l2.^4.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l1.^2.*l3.*Sr.^2 - ...
        beta.^2.*Bs.*g001.*g010.*g100.*L.*l1.^2.*l3.*Sr.^2 - ...
        beta.*Bs.*g011.*g100.*L.*l1.^2.*l3.*Sr.^2 - ...
```

```
        6.*beta.*Bs.*g010.*g101.*L.*l1.^2.*l3.*Sr.^2 - ...
        beta.*Bs.*g001.*g110.*L.*l1.^2.*l3.*Sr.^2 - ...
        2.*alpha.*As.*g001.*g010.*g100.*L.*l1.^3.*l3.*Sr.^2 - ...
        As.*beta.^2.*g001.*g010.*g100.*L.*l1.^3.*l3.*Sr.^2 - ...
        As.*beta.*g011.*g100.*L.*l1.^3.*l3.*Sr.^2 - ...
        8.*As.*beta.*g010.*g101.*L.*l1.^3.*l3.*Sr.^2 - ...
        As.*beta.*g001.*g110.*L.*l1.^3.*l3.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*L.*l1.*l2.*l3.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*L.*l1.*l2.*l3.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*L.*l1.*l2.*l3.*Sr.^2 - ...
        3.*As.*beta.*g010.*g101.*L.*l1.^2.*l2.*l3.*Sr.^2 - ...
        3.*As.*beta.*g001.*g110.*L.*l1.^2.*l2.*l3.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l2.^2.*l3.*Sr.^2 - ...
        beta.^2.*Bs.*g001.*g010.*g100.*L.*l2.^2.*l3.*Sr.^2 - ...
        6.*beta.*Bs.*g011.*g100.*L.*l2.^2.*l3.*Sr.^2 - ...
        beta.*Bs.*g010.*g101.*L.*l2.^2.*l3.*Sr.^2 - ...
        beta.*Bs.*g001.*g110.*L.*l2.^2.*l3.*Sr.^2 - ...
        3.*As.*beta.*g011.*g100.*L.*l1.*l2.^2.*l3.*Sr.^2 - ...
        3.*As.*beta.*g001.*g110.*L.*l1.*l2.^2.*l3.*Sr.^2 - ...
        2.*alpha.*As.*g001.*g010.*g100.*L.*l2.^3.*l3.*Sr.^2 - ...
        As.*beta.^2.*g001.*g010.*g100.*L.*l2.^3.*l3.*Sr.^2 - ...
        8.*As.*beta.*g011.*g100.*L.*l2.^3.*l3.*Sr.^2 - ...
        As.*beta.*g010.*g101.*L.*l2.^3.*l3.*Sr.^2 - ...
        As.*beta.*g001.*g110.*L.*l2.^3.*l3.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l1.*l3.^2.*Sr.^2 - ...
        beta.^2.*Bs.*g001.*g010.*g100.*L.*l1.*l3.^2.*Sr.^2 - ...
        beta.*Bs.*g011.*g100.*L.*l1.*l3.^2.*Sr.^2 - ...
        6.*beta.*Bs.*g010.*g101.*L.*l1.*l3.^2.*Sr.^2 - ...
        beta.*Bs.*g001.*g110.*L.*l1.*l3.^2.*Sr.^2 - ...
        12.*As.*beta.*g010.*g101.*L.*l1.^2.*l3.^2.*Sr.^2 - ...
        2.*alpha.*Bs.*g001.*g010.*g100.*L.*l2.*l3.^2.*Sr.^2;

g111num4 = - beta.^2.*Bs.*g001.*g010.*g100.*L.*l2.*l3.^2.*Sr.^2 - ...
        6.*beta.*Bs.*g011.*g100.*L.*l2.*l3.^2.*Sr.^2 - ...
        beta.*Bs.*g010.*g101.*L.*l2.*l3.^2.*Sr.^2 - ...
        beta.*Bs.*g001.*g110.*L.*l2.*l3.^2.*Sr.^2 - ...
        3.*As.*beta.*g011.*g100.*L.*l1.*l2.*l3.^2.*Sr.^2 - ...
        3.*As.*beta.*g010.*g101.*L.*l1.*l2.*l3.^2.*Sr.^2 - ...
        12.*As.*beta.*g011.*g100.*L.*l2.^2.*l3.^2.*Sr.^2 - ...
        4.*alpha.*Bs.*g001.*g010.*g100.*L.*l3.^3.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*L.*l3.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*L.*l3.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*L.*l3.^3.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*L.*l3.^3.*Sr.^2 - ...
        2.*alpha.*As.*g001.*g010.*g100.*L.*l1.*l3.^3.*Sr.^2 - ...
        As.*beta.^2.*g001.*g010.*g100.*L.*l1.*l3.^3.*Sr.^2 - ...
        As.*beta.*g011.*g100.*L.*l1.*l3.^3.*Sr.^2 - ...
        8.*As.*beta.*g010.*g101.*L.*l1.*l3.^3.*Sr.^2 - ...
        As.*beta.*g001.*g110.*L.*l1.*l3.^3.*Sr.^2 - ...
        2.*alpha.*As.*g001.*g010.*g100.*L.*l2.*l3.^3.*Sr.^2 - ...
```

```
        As.*beta.^2.*g001.*g010.*g100.*L.*l2.*l3.^3.*Sr.^2 - ...
        8.*As.*beta.*g011.*g100.*L.*l2.*l3.^3.*Sr.^2 - ...
        As.*beta.*g010.*g101.*L.*l2.*l3.^3.*Sr.^2 - ...
        As.*beta.*g001.*g110.*L.*l2.*l3.^3.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*L.*l3.^4.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*L.*l3.^4.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*L.*l3.^4.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*L.*l3.^4.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*L.*l3.^4.*Sr.^2 - ...
        4.*alpha.*Bs.*g001.*g010.*g100.*l1.^2.*Re.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*l1.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*l1.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*l1.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*l1.^2.*Re.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*l1.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*l1.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*l1.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*l1.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*l1.^3.*Re.*Sr.^2 - ...
        4.*beta.*Bs.*g001.*g110.*l1.*l2.*Re.*Sr.^2 - ...
        6.*As.*beta.*g001.*g110.*l1.^2.*l2.*Re.*Sr.^2 - ...
        4.*alpha.*Bs.*g001.*g010.*g100.*l2.^2.*Re.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*l2.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*l2.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*l2.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*l2.^2.*Re.*Sr.^2 - ...
        6.*As.*beta.*g001.*g110.*l1.*l2.^2.*Re.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*l2.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*l2.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*l2.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*l2.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*l2.^3.*Re.*Sr.^2 - ...
        4.*beta.*Bs.*g010.*g101.*l1.*l3.*Re.*Sr.^2 - ...
        6.*As.*beta.*g010.*g101.*l1.^2.*l3.*Re.*Sr.^2 - ...
        4.*beta.*Bs.*g011.*g100.*l2.*l3.*Re.*Sr.^2 - ...
        6.*As.*beta.*g011.*g100.*l2.^2.*l3.*Re.*Sr.^2 - ...
        4.*alpha.*Bs.*g001.*g010.*g100.*l3.^2.*Re.*Sr.^2 - ...
        2.*beta.^2.*Bs.*g001.*g010.*g100.*l3.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g011.*g100.*l3.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g010.*g101.*l3.^2.*Re.*Sr.^2 - ...
        2.*beta.*Bs.*g001.*g110.*l3.^2.*Re.*Sr.^2 - ...
        6.*As.*beta.*g010.*g101.*l1.*l3.^2.*Re.*Sr.^2 - ...
        6.*As.*beta.*g011.*g100.*l2.*l3.^2.*Re.*Sr.^2 - ...
        4.*alpha.*As.*g001.*g010.*g100.*l3.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.^2.*g001.*g010.*g100.*l3.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g011.*g100.*l3.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g010.*g101.*l3.^3.*Re.*Sr.^2 - ...
        2.*As.*beta.*g001.*g110.*l3.^3.*Re.*Sr.^2;

g111den = -(B10./g001) - B10./g010 - B10./g100 + 2.*K0.*Re - ...
```

```
4.*As.*L.*12.^3.*13.*Sr.^2 + ...
11.^3.*(-4.*As.*L.*12.*Sr.^2 - 4.*As.*L.*13.*Sr.^2) + ...
12.^2.*(-6.*As.*L.*13.^2.*Sr.^2 + ...
13.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2)) + ...
11.^2.*(-6.*As.*L.*12.^2.*Sr.^2 - 6.*As.*L.*13.^2.*Sr.^2 + ...
13.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2) + ...
12.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 12.*As.*L.*13.*Sr.^2 - ...
3.*As.*Re.*Sr.^2)) + ...
12.*(-4.*As.*L.*13.^3.*Sr.^2 + ...
13.^2.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2) + ...
13.*(-2.*L.*Rm + Re.*(-2.*Mm - 2.*Bs.*Sr.^2))) + ...
11.*(-4.*As.*L.*12.^3.*Sr.^2 - 4.*As.*L.*13.^3.*Sr.^2 + ...
13.^2.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - 3.*As.*Re.*Sr.^2) + ...
12.^2.*(-3.*L.*Mm - 3.*Bs.*L.*Sr.^2 - ...
12.*As.*L.*13.*Sr.^2 - 3.*As.*Re.*Sr.^2) + ...
13.*(-2.*L.*Rm + Re.*(-2.*Mm - 2.*Bs.*Sr.^2)) + ...
12.*(-2.*L.*Rm - 12.*As.*L.*13.^2.*Sr.^2 + ...
Re.*(-2.*Mm - 2.*Bs.*Sr.^2) + ...
13.*(-6.*L.*Mm - 6.*Bs.*L.*Sr.^2 - 6.*As.*Re.*Sr.^2)));
```

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