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Dynamic Modelling of Differential-Drive Mobile Robots using Lagrange and Newton-Euler Methodologies: A Unified Framework

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Abstract

This paper presents a unified dynamic modeling framework for differential-drive mobile robots (DDMR). Two formulations for mobile robot dynamics are developed; one is based on Lagrangian mechanics, and the other on Newton-Euler mechanics. Major difficulties experienced when modeling non-holonomic systems in both methods are illustrated and design procedures are outlined. It is shown that the two formulations are mathematically equivalent providing a check on their consistency. The presented work leads to an improved understanding of differential-drive mobile robot dynamics, which will assist engineering students and researchers in the modeling and design of suitable controllers for DDMR navigation and trajectory tracking.

Keywords: Differential-drive; Mobile robot; Dynamics; Modeling; Lagrange; Newton-Euler

Introduction

In recent years, there has been a considerable interest in the area of mobile robotics and educational technologies [1-7]. For control engineers and researchers, there is a wealth of literature dealing with wheeled mobile robots (WMR) control and their applications. However, while the subject of kinematic modeling of WMR is well documented and easily understood by students, the subject of dynamic modeling of WMR has not been addressed adequately in the literature. The dynamics of WMR are highly nonlinear and involve non-holonomic constraints which makes difficult their modeling and analysis especially for new engineering students starting their research in this field. Therefore, a detailed and accurate dynamic model describing the WMR motion need to be developed to offer students a general framework for simulation analysis and model based control system design.

In the case of a differential drive mobile robot (DDMR), for example, there is no textbook available that investigates thoroughly the dynamic modeling approach taking into consideration the non-holonomic constraints in a step by step procedure. The analysis is available mainly in journals, conference papers, and technical reports [8]. Moreover, the material presented differs from one paper to another with different variables and reference frames used, and various assumptions. In addition, some papers present different results for the same DDMR used, which adds to the confusion of dynamic modeling.

For the case of DDMR, the methods used are either the Lagrangian approach [9-15] or the Newton-Euler approach [16-19]. Other formalisms such as the Kane's method have been also suggested as viable approaches of modeling DDMR [20]. Therefore, it is not clear for new engineering students and researchers which concept to use and which method offers a better physical insight on the dynamic behavior of the system and the effect of the non-holonomic constraints. Also, it is not clear if both methods will lead to the same final dynamic model.

In the Newton Euler method, one has to take into account two kinds of forces applied to a system: the given forces and the constraint forces. The given forces include the externally impressed forces by the actuators while the constraint forces are the forces of interaction between the robot platform and ground through the wheels. Moreover, in a system with interconnected elements, the components may interact with each other through gears, springs, and frictional elements. Therefore, we need to take into account all of these forces. It is clear that the Newtonian approach includes a few practical difficulties since in most cases these forces are not easily quantifiable.

The methodology developed by Lagrange overcomes these problems by expressing the forces in terms of the energies in the system, i.e., the kinetic energy and the potential energy, which are scalar quantities easily expressible in terms of the system coordinates. The derivation of the Lagrange equations requires also that the generalized coordinates be independent.

The Lagrangian approach usually provides a powerful and versatile method for the formulation of the equations of motion for holonomic systems. However, for non-holonomic systems, the usual method is to introduce the motion constraint equations into the dynamic equations using the additional Lagrange multipliers. These multipliers are not constants and are usually functions of all the generalized coordinates and often of time as well. They represent a set of unknowns whose values should be obtained as a part of the solution. To solve this computational complexity, additional methods have been suggested to remove the presence of the multipliers from the dynamic equations of the given system [21,22].

The focus of this paper is to derive simple and well-structured dynamic equations of the DDMR taking into account the non-holonomic constraints. First, the Lagrange formulation is presented. Coordinates transformation is used to cancel the Lagrange multipliers to obtain well-structured equations. Second, the Newton-Euler method is used to derive the dynamic equations of the DDMR. Major difficulties experienced in using both methods are illustrated and procedures are outlined to offer a systematic approach to the dynamic modeling of

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DDMR with no major mathematical complexity. It is shown that both methods reach equivalent dynamic equations for the mobile robot providing a check on their consistency.

Coordinate Systems

In order to describe the position of the WMR in his environment, two different coordinate systems (frames) need to be defined.

1. Inertial Coordinate System: This coordinate system is a global frame which is fixed in the environment or plane in which the WMR moves in. Moreover, this frame is considered as the reference frame and is denoted as $\{X_I, Y_I\}$.

2. Robot Coordinate System: This coordinate system is a local frame attached to the WMR, and thus, moving with it. This frame is denoted as $\{X_r, Y_r\}$.

The two defined frames are shown in Figure 1. The origin of the robot frame is defined to be the mid-point A on the axis between the wheels. The center of mass C of the robot is assumed to be on the axis of symmetry, at a distance d from the origin *A*.

As shown in Figure 1, the robot position and orientation in the Inertial Frame can be defined as

$$q^I = \begin{bmatrix} x_a \\ y_a \\ \theta \end{bmatrix} \tag{1}$$

The important issue that needs to be explained at this stage is the mapping between these two frames. The position of any point on the robot can be defined in the robot frame and the inertial frame as follows.

Let $X^r = \begin{bmatrix} x^r \\ y^r \\ \theta^r \end{bmatrix}$, and $X^I = \begin{bmatrix} x^I \\ y^I \\ \theta^I \end{bmatrix}$ and be the coordinates of the given point in the robot frame and inertial frame, respectively.

Then, the two coordinates are related by the following transformation:

$$X^I = R(\theta) X^r \tag{2}$$

Where $R(\theta)$ is the orthogonal rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

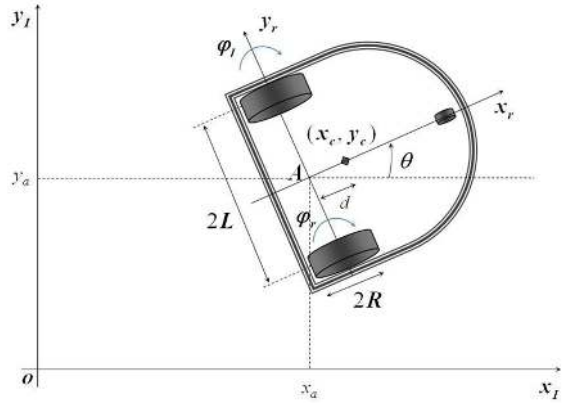

Figure 1: Differential Drive Mobile Robot (DDMR).

This transformation will enable also the handling of motion between frames.

$$\dot{X}^I = R(\theta)\dot{X}^r \tag{4}$$

It will be seen in the next section that equation (4) is very important in deriving the DDMR kinematic and dynamic models as it describes the relationship between the velocities in the Inertial Frame and the Robot Frame.

Kinematic Constraints of the Differential-Drive Robot

The motion of a differential-drive mobile robot is characterized by two non-holonomic constraint equations, which are obtained by two main assumptions:

• No lateral slip motion: This constraint simply means that the robot can move only in a curved motion (forward and backward) but not sideward. In the robot frame, this condition means that the velocity of the center-point A is zero along the lateral axis:

$$\dot{y}_a^r = 0 \tag{5}$$

Using the orthogonal rotation matrix $R(\theta)$, the velocity in the inertial frame gives

$$-\dot{x}_a \sin\theta + \dot{y}_a \cos\theta = 0 \tag{6}$$

• Pure rolling constraint:

The pure rolling constraint represents the fact that each wheel maintains a one contact point P with the ground as shown in Figure 2. There is no slipping of the wheel in its longitudinal axis (x_r) and no skidding in its orthogonal axis (y_r). The velocities of the contact points in the robot frame are related to the wheel velocities by:

$$\begin{cases} v_{pR} = R\dot{\varphi}_R \\ v_{pL} = R\dot{\varphi}_L \end{cases} \tag{7}$$

In the inertial frame, these velocities can be calculated as a function of the velocities of the robot center-point A:

$$\begin{cases} \dot{x}_{pR} = \dot{x}_a + L\dot{\theta}\cos\theta \\ \dot{y}_{pR} = \dot{y}_a + L\dot{\theta}\sin\theta \end{cases} \tag{8}$$

$$\begin{cases} \dot{x}_{pL} = \dot{x}_a + L\dot{\theta}\cos\theta \\ \dot{y}_{pL} = \dot{y}_a + L\dot{\theta}\sin\theta \end{cases} \tag{9}$$

Using the rotation matrix R(θ), the rolling constraint equations are formulated as follows:

$$\begin{aligned} \dot{x}_{pR}\cos\theta + \dot{y}_{pR}\sin\theta &= R\dot{\varphi}_R \\ \dot{x}_{pL}\cos\theta + \dot{y}_{pL}\sin\theta &= R\dot{\varphi}_L \end{aligned} \tag{10}$$

Using the contact points velocities from equation (x,y) and

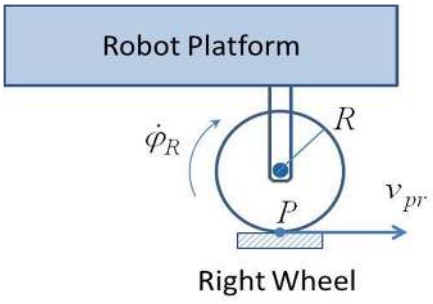

Figure 2: Pure Rolling Motion Constraint.

substituting in (x ,y), the three constraint equations can be written in the following matrix form:

$$\Lambda(q)\dot{q}=0 \tag{11}$$

Where

$$\Lambda(q)=\begin{bmatrix}-\sin\theta & \cos\theta & 0 & 0 & 0\\ \cos\theta & \sin\theta & L & -R & 0\\ \cos\theta & \sin\theta & -L & 0 & -R\end{bmatrix} \tag{12}$$

and

$$\dot{q}=\begin{bmatrix}\dot{x}_a & \dot{y}_a & \dot{\theta} & \dot{\varphi}_R & \dot{\varphi}_L\end{bmatrix}^T \tag{13}$$

$$\begin{cases}v_R=R\dot{\varphi}_R\\ v_L=R\dot{\varphi}_L\end{cases} \tag{14}$$

The above constraints matrix $\Lambda(q)$ will be used in the next section for the DDMR dynamic modeling.

Kinematic Model

Kinematic modeling is the study of the motion of mechanical systems without considering the forces that affect the motion. For the DDMR, the main purpose of kinematic modeling is to represent the robot velocities as a function of the driving wheels velocities along with the geometric parameters of the robot.

The linear velocity of each driving wheel in the Robot Frame is therefore, the linear velocity of the DDMR in the Robot Frame is the average of the linear velocities of the two wheels

$$v=\frac{v_R+v_L}{2}=R\frac{(\dot{\varphi}_R+\dot{\varphi}_L)}{2} \tag{15}$$

and the angular velocity of the DDMR is

$$\omega=\frac{v_R-v_L}{2L}=R\frac{(\dot{\varphi}_R-\dot{\varphi}_L)}{2} \tag{16}$$

The DDMRs velocities in the robot frame can now be represented in terms of the center-point A velocities in the robot frame as follows:

$$\begin{cases}\dot{x}_a^r=R\dfrac{(\dot{\varphi}_R+\dot{\varphi}_L)}{2}\\ \dot{y}_a^r=0\\ \dot{\theta}=\omega=R\dfrac{(\dot{\varphi}_R+\dot{\varphi}_L)}{2L}\end{cases} \tag{17}$$

Thus

$$\begin{bmatrix}\dot{x}_a^r\\ \dot{y}_a^r\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\frac{R}{2} & \frac{R}{2}\\ 0 & 0\\ \frac{R}{2L} & -\frac{R}{2L}\end{bmatrix}\begin{bmatrix}\dot{\varphi}_R\\ \dot{\varphi}_L\end{bmatrix} \tag{18}$$

The DDMR velocities can be obtained also in the inertial frame as follows:

$$\dot{q}^I=\begin{bmatrix}\dot{x}_a^r\\ \dot{y}_a^r\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\frac{R}{2}\cos\theta & \frac{R}{2}\cos\theta\\ \frac{R}{2}\sin\theta & \frac{R}{2}\sin\theta\\ \frac{R}{2L} & -\frac{R}{2L}\end{bmatrix}\begin{bmatrix}\dot{\varphi}_R\\ \dot{\varphi}_L\end{bmatrix} \tag{19}$$

Equation (19) represents the forward kinematic model of the DDMR. Another alternative form for the kinematic model can be obtained by representing the DDMR velocities in terms of the linear and angular velocities of DDMR in the Robot frame.

$$\dot{q}^I=\begin{bmatrix}\dot{x}_a^r\\ \dot{y}_a^r\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} \tag{20}$$

Dynamic Modeling of the DDMR

Dynamics is the study of the motion of a mechanical system taking into consideration the different forces that affect its motion unlike kinematics where the forces are not taken into consideration. The dynamic model of the DDMR is essential for simulation analysis of the DDMR motion and for the design of various motion control algorithms.

A non-holonomic DDMR with n generalized coordinates (q_1,q_2,… ,q_n) and subject to m constraints can be described by the following equations of motion:

$$M(q)\ddot{q}+V(q,\dot{q})\dot{q}+F(\dot{q})+G(q)+\tau_d=B(q)\tau-\Lambda^T(q)^{\lambda} \tag{21}$$

where:

$M(q)$ an nxn symmetric positive definite inertia matrix, $V(q,\dot{q})$ is the centripetal and coriolis matrix, $F(\dot{q})$ is the surface friction matrix, $G(q)$ is the gravitational vector, τ_d is the vector of bounded unknown disturbances including unstructured unmodeled dynamics, $B(q)$ is the input matrix, τ is the input vector, $\Lambda^T(q)$ is the matrix associated with the kinematic constraints, and λ is the Lagrange multipliers vector [21,22].

Lagrange dynamic approach

Lagrange dynamic approach is a very powerful method for formulating the equations of motion of mechanical systems. This method, which was introduced by Lagrange, is used to systematically derive the equations of motion by considering the kinetic and potential energies of the given system.

The Lagrange equation can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)+\frac{\partial L}{\partial q_i}=F-\Lambda^T(q)^{\lambda} \tag{22}$$

Where $L=T-V$ is the Lagrangian function, T, is the kinetic energy of the system, V is the potential energy of the system, q_i are the generalized coordinates, F is the generalized force vector, Λ is the constraints matrix, and λ is the vector of Lagrange multipliers associated with the constraints.

The first step in deriving the dynamic model using the Lagrange approach is to find the kinetic and potential energies that govern the motion of the DDMR. Furthermore, since the DDMR is moving in the X_I-Y_I plane, the potential energy of the DDMR is considered to be zero.

For the DDMR, the generalized coordinates are selected as

$$q=\begin{bmatrix}x_a & y_a & \theta & \varphi_R & \varphi_L\end{bmatrix}^T \tag{23}$$

The kinetic energies of the DDMR is the sum of the kinetic energy of the robot platform without wheels plus the kinetic energies of the wheels and actuators.

The kinetic energy of the robot platform is

$$T_c=\frac{1}{2}m_c v_c^2+\frac{1}{2}I_c\dot{\theta}^2 \tag{24}$$

While the kinetic energy of the right and left wheel is

$$T_{wR} = \frac{1}{2} m_w v_{wR}^2 + \frac{1}{2} I_m \dot{\theta}^2 + \frac{1}{2} I_w \dot{\varphi}_R^2 \tag{25}$$

$$T_{wL} = \frac{1}{2} m_w v_{wL}^2 + \frac{1}{2} I_m \dot{\theta}^2 + \frac{1}{2} I_w \dot{\varphi}_L^2 \tag{26}$$

where, m_c is the mass of the DDMR without the driving wheels and actuators (DC motors), m_w is the mass of each driving wheel (with actuator), I_c is the moment of inertia of the DDMR about the vertical axis through the center of mass, I_w is the moment of inertia of each driving wheel with a motor about the wheel axis, and I_m is the moment of inertia of each driving wheel with a motor about the wheel diameter.

All velocities will be first expressed as a function of the generalized coordinates using the general velocity equation in the inertial frame.

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 \tag{27}$$

The X_i and Y_i components of the center of mass and wheels can be obtained in terms of the generalized coordinates as follow

$$\begin{cases} x_c = x_a + d\cos\theta \\ y_c = y_a + d\sin\theta \end{cases} \tag{28}$$

$$\begin{cases} x_{wR} = x_a + L\sin\theta \\ y_{wR} = y_a + L\cos\theta \end{cases} \tag{29}$$

$$\begin{cases} x_{wL} = x_a - L\sin\theta \\ y_{wL} = y_a + L\cos\theta \end{cases} \tag{30}$$

Using equations (24)-(26) along with equations (27- (30), the total kinetic energy of the DDMR is

$$T = \frac{1}{2} m\left(\dot{x}_a^2 + \dot{y}_a^2\right) - m_c d\dot{\theta}\left(\dot{y}_a\cos\theta - \dot{x}_a\sin\theta\right) + \frac{1}{2} I_w\left(\dot{\varphi}_R^2 + \dot{\varphi}_L^2\right) + \frac{1}{2} I\dot{\theta}^2 \tag{31}$$

where the following new parameters are introduced

$m = m_c + 2m_w$ is the total mass of the robot, $I = I_c + m_c d^2 + 2m_w L^2 + 2I_m$ and is the total equivalent inertia.

Using equation (22) along with the Lagrangian function, L=T the equations of motion of the DDMR are given by

$$m\ddot{x}_a - md\ddot{\theta}\sin\theta - md\dot{\theta}^2\cos\theta = C_1 \tag{32}$$

$$m\ddot{y}_a - md\ddot{\theta}\cos\theta - md\dot{\theta}^2\sin\theta = C_2 \tag{33}$$

$$I\ddot{\theta} - md\ddot{x}_a\sin\theta + md\ddot{y}_a c\,\mathrm{os}\,\theta = C_3 \tag{34}$$

$$I_w\ddot{\varphi}_R = \tau_R + C_4 \tag{35}$$

$$I_w\ddot{\varphi}_L = \tau_L + C_5 \tag{36}$$

where (C_1, C_2, C_3, C_4, C_5), are coefficients related to the kinematic constraints, which can be written in terms of the Lagrange multipliers vector λ and the kinematic constraints matrix Λ introduced in section 3.

$$\Lambda^T(q) = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \tag{37}$$

Now, the obtained equations of motion (32)-(36) can be represented in the general form given by equation (21) as

$$M(q)\ddot{q} + V(q,\dot{q})\dot{q} = B(q)\tau - \Lambda^T(q)^{\lambda} \tag{38}$$

Where

$$M(q) = \begin{bmatrix} m & 0 & -md\sin\theta & 0 & 0 \\ 0 & m & md\cos\theta & 0 & 0 \\ -md\sin\theta & md\cos\theta & I & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & I_w \end{bmatrix},$$

$$V(q,\dot{q}) = \begin{bmatrix} 0 & -md\dot{\theta}\cos\theta & 0 & 0 & 0 \\ 0 & -md\dot{\theta}\sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B(q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \Lambda^T(q)\lambda = \begin{bmatrix} -\sin\theta & \cos\theta & \cos\theta \\ \cos\theta & \sin\theta & \sin\theta \\ 0 & L & -L \\ 0 & -R & 0 \\ 0 & 0 & -R \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}$$

Next, the system described by equation (38) is transformed into an alternative form which is more convenient for the purpose of control and simulation. The main aim is to eliminate the constraint term $\Lambda^T(q)\lambda$ in equation (88) since the Lagrange multipliers λ_i are unknown. This is done first by defining the reduced vector

$$\dot{\eta} = \begin{bmatrix} \dot{\varphi}_R \\ \dot{\varphi}_L \end{bmatrix} \tag{39}$$

Next, by expressing the generalized coordinates velocities using the forward kinematic model (19). Then we have

$$\begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{\theta} \\ \dot{\varphi}_R \\ \dot{\varphi}_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} R\cos\theta & R\cos\theta \\ R\sin\theta & R\sin\theta \\ \frac{R}{L} & -\frac{R}{L} \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \dot{\varphi}_R \\ \dot{\varphi}_L \end{bmatrix} \tag{40}$$

This can be written in the form

$$\dot{q} = S(q)\eta \tag{41}$$

It can be verified that the transformation matrix $S(q)$ is in the null space of the constraint matrix $\Lambda(q)$. Therefore we have

$$S^T(q)\Lambda^T(q) = 0 \tag{42}$$

Next, taking the time derivative of equation (41) gives

$$\ddot{q} = \dot{S}(q)\eta + S(q)\dot{\eta} \tag{43}$$

Substituting equations (41) and (43) in the main equation (38) we obtain

$$M(q)\left[\dot{S}(q)\eta + S(q)\dot{\eta}\right] + V(q,\dot{q})\left[S(q)\eta\right] = B(q)\tau - \Lambda^T(q)\lambda \tag{44}$$

Next, rearranging the equation and multiplying both sides by leads to

$$S^T(q)M(q)S(q)\dot{\eta}+S^T(q)\left[M(q)\dot{S}(q)+V(q,\dot{q})S(q)\right]\eta = S^T(q)B(q)\tau - S^T(q)\Lambda^T(q)\lambda \quad (45)$$

where the last term is identically zero. Now defining the new matrices

$$\overline{M}(q)=S^T(q)M(q)S(q)$$

$$\overline{V}=S^T(q)M(q)\dot{S}(q)+S^T(q)V(q,\dot{q})S(q),$$

$$\overline{B}=S^T(q)B(q)$$

The dynamic equations are reduced to the form

$$\overline{M}(q)\dot{\eta}+\overline{V}(q,\dot{q})\eta=\overline{B}(q)\tau \quad (46)$$

Where

$$\overline{M}(q)=\begin{bmatrix} I_w+\frac{R^2}{4L^2}\left(mL^2+I\right) & \frac{R^2}{4L^2}\left(mL^2-I\right) \\ \frac{R^2}{4L^2}\left(mL^2-I\right) & I_w+\frac{R^2}{4L^2}\left(mL^2+I\right)\end{bmatrix}$$

$$\overline{V}(q,\dot{q})=\begin{bmatrix} 0 & \frac{R^2}{2L}m_c d\dot{\theta} \\ -\frac{R^2}{2L}m_c d\dot{\theta} & 0\end{bmatrix}, \quad \overline{B}(q)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Equation (46) shows that the DDMR dynamics are expressed only as a function of the right and left wheel angular velocities $(\dot{\varphi}_R,\dot{\varphi}_L)$, the robot angular velocity $\dot{\theta}$ and the driving motor torques (τ_R,τ_L). The equations of motion (46) can be also transformed into an alternative form which is represented by the linear and angular velocities (v,w)of the DDMR. Using the kinematic model equations (15) and (16), it can be easily shown that the model equations (46) can be rearranged in the following compact form

$$\begin{cases}\left(m+\frac{2I_w}{R^2}\right)\dot{v}-m_c d\omega^2=\frac{1}{R}\left(\tau_R+\tau_L\right)\\ \left(I+\frac{2L^2}{R^2}I_w\right)\dot{\omega}+m_c d\omega v=\frac{L}{R}\left(\tau_R-\tau_L\right)\end{cases} \quad (47)$$

Newton-Euler approach

The first and most important step in Newton-Euler dynamic modeling is to draw the free body diagram of the system and to analyze the forces acting on it. The free body diagram of the differential drive mobile robot is shown in Figure 3. Using the robot local frame {xr, yr}, the following notations are introduced.

(v_u,v_w) represents the velocity of the vehicle center of mass C in the local frame; v_u is the longitudinal velocity and v_w is the lateral velocity; (a_u,a_w) represent the acceleration of the vehicle's center of mass C; (F_{u_R},F_{u_L}) are the longitudinal forces exerted on the vehicle by the left and right wheels; (F_{w_R},F_{w_L}) are the lateral forces exerted on the vehicle by the left and right wheels; θ is the orientation of the robot; ω is the angular velocity; *m* is the mass of the robot; and is the yaw moment of inertia with respect to the center of mass.

As it can be seen from the above free body diagram, the only forces acting on the robot are actuator forces acting on the robot wheels.

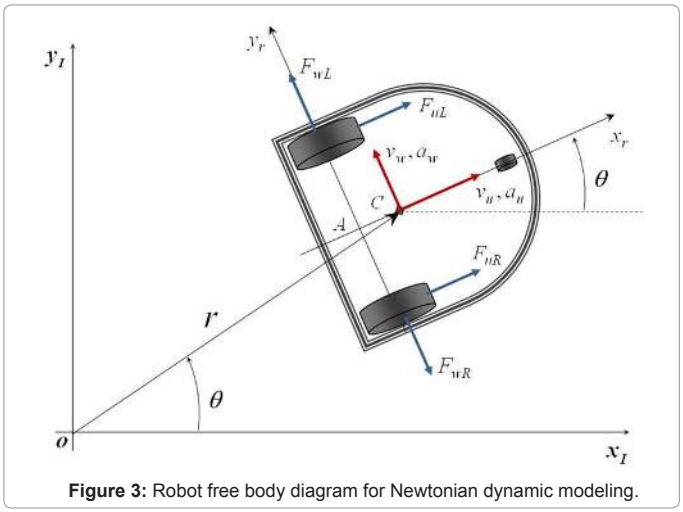

Figure 3: Robot free body diagram for Newtonian dynamic modeling.

We start the derivation by representing the robot position using polar coordinates. Assuming that the robot is a rigid body, its polar coordinates in the inertial frame can be represented using a complex vector

$$\hat{r}=re^{j\theta} \quad (48)$$

Differentiating the above position vector with respect to time will give us the velocity and acceleration of the robot in the inertial frame.

$$\dot{\hat{r}}=\dot{r}e^{j\theta}+jr\dot{\theta}e^{j\theta} \quad (49)$$

$$\ddot{\hat{r}}=\ddot{r}e^{j\theta}+2j\dot{r}\dot{\theta}e^{j\theta}+jr\ddot{\theta}e^{j\theta}-r\dot{\theta}^2e^{j\theta} \quad (50)$$

Simplifying and writing the velocity and acceleration terms in radial and tangential terms, we have

$$\dot{\hat{r}}=\left[\dot{r}\right]e^{j\theta}+\left[r\dot{\theta}\right]e^{j\left(\theta+\frac{\pi}{2}\right)} \quad (51)$$

$$\ddot{\hat{r}}=\left[\ddot{r}-r\dot{\theta}^2\right]e^{j\theta}+\left[2\dot{r}\dot{\theta}+r\ddot{\theta}\right]e^{j\left(\theta+\frac{\pi}{2}\right)} \quad (52)$$

The radial and tangential velocity and acceleration terms are defined as

$$v_u=\dot{r} \quad (53)$$

$$v_w=r\dot{\theta} \quad (54)$$

$$a_u=\ddot{r}-r\dot{\theta}^2 \quad (55)$$

$$a_w=2\dot{r}\dot{\theta}+r\ddot{\theta} \quad (56)$$

From the above four equations, we can write the following relations between the radial and tangential components of the robot velocity and acceleration

$$a_u=\dot{v}_u-v_w\dot{\theta} \quad (57)$$

$$a_w=\dot{v}_w-v_u\dot{\theta} \quad (58)$$

The above equations (57) and (58) are the fundamental acceleration equations that can be also obtained using the theorem of motion of a rigid body in a rotating reference frame [21,22].

The next step is to write the Newton's second law of motion in the robot frame and find the relationship between the forces, torques, and accelerations. The DDMR exhibits two types of motion: translations in the radial and tangential directions, and rotation around the vertical axis at the center of mass. Let M be the total mass of the robot including

the wheels and actuators and J the moment of inertia with respect to the center of mass. Then the dynamic equations are

$$Ma_u = F_{uL} + F_{uR} \tag{59}$$

$$Ma_w = F_{wL} - F_{wR} \tag{60}$$

$$J\ddot{\theta} = (F_{uR} - F_{uL})L + (F_{wR} - F_{wL})d \tag{61}$$

Substituting the acceleration terms from (57) and (58) we get

$$\dot{v}_u = v_w\dot{\theta} + \frac{F_{uL} + F_{uR}}{M} \tag{62}$$

$$\dot{v}_w = -v_u\dot{\theta} + \frac{F_{wL} - F_{wR}}{M} \tag{63}$$

$$\ddot{\theta} = \frac{L}{J}(F_{uR} - F_{uL}) + \frac{d}{J}(F_{wR} - F_{wL}) \tag{64}$$

The absence of slipping (pure rolling) in the longitudinal direction and no sliding in the lateral direction creates independence between the longitudinal, lateral and angular velocities and simplifies the dynamic equations. These non-holonomic constraints are taken into account by defining the velocity of the center-point A in the local frame and forcing it to be zero. Using the transformation matrix $R(\theta)$, we first find the velocity of the center of mass C in the inertial frame as

$$\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} v_u \\ v_w \end{bmatrix} \tag{65}$$

Next, using equation (28), we can find the velocity of the center-point A in the inertial frame. It can then be shown that the lateral velocity of point A in the local frame is $v_w - d\dot{\theta}$. Therefore, in the absence of lateral slippage we have $v_w = d\dot{\theta}$ (66)

Next, substituting (66) in (62), (63), and combining with (64) we obtain

$$\dot{v}_u = d\dot{\theta}^2 + \frac{1}{M}(F_{uL} + F_{uR}) \tag{67}$$

$$\ddot{\theta} = \frac{L}{Md^2 + J}(F_{uR} - F_{uL}) - \frac{Mdv_u}{Md^2 + J}\dot{\theta} \tag{68}$$

The above two equations are the dynamic equations of the robot considering the non-holonomic constraints. These equations can now easily be transformed to show the actuator torques applied to the wheels similar to the notations used in the Lagrangian approach.

$$M\dot{v}_u\dot{\theta} - Md\dot{\theta}^2 = \frac{1}{R}(\tau_R + \tau_L) \tag{69}$$

$$(Md^2 + J)\ddot{\theta} + Mdv_u\dot{\theta} = \frac{L}{R}(\tau_R - \tau_L) \tag{70}$$

Next, these two equations can be written in matrix form as follows

$$\begin{bmatrix} M & 0 \\ 0 & Md^2 + J \end{bmatrix}\begin{bmatrix} \dot{v}_u \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -Md\dot{\theta} \\ Md\dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{v}_u \\ \ddot{\theta} \end{bmatrix} = \frac{1}{R}\begin{bmatrix} 1 & 1 \\ L & -L \end{bmatrix}\begin{bmatrix} \tau_R \\ \tau_L \end{bmatrix} \tag{71}$$

As it can be observed, equation (62) is similar to equation (47), which was obtained using the Lagrangian approach. Note that in the Newton-Euler approach the mass and inertias of the wheels were not taken into consideration, and the robot is considered as one rigid body. Therefore both formulations are equivalent if the inertia and mass parameters are defined as

$$M = m_c \tag{72}$$

$$J = I_c \tag{73}$$

Next, using the forward kinematics equations (15) and (16), we can easily rewrite the general dynamic equations (71) in terms of the wheels rotational velocities and actuator torques. The leads to the following formulation

$$\left[\frac{R(Md^2 + J)}{4L^2} + \frac{MR}{4}\right]\ddot{\varphi}_R + \left[-\frac{R(Md^2 + J)}{4L^2} + \frac{MR}{4}\right]\ddot{\varphi}_L - \left(\frac{MdR^2}{4L^2}\right)\dot{\varphi}_L^2 + \left(\frac{MdR^2}{4L^2}\right)\dot{\varphi}_R\dot{\varphi}_L = \frac{1}{R}\tau_R \tag{74}$$

$$\left[\frac{R(Md^2 + J)}{4L^2} + \frac{MR}{4}\right]\ddot{\varphi}_L + \left[-\frac{R(Md^2 + J)}{4L^2} + \frac{MR}{4}\right]\ddot{\varphi}_R - \left(\frac{MdR^2}{4L^2}\right)\dot{\varphi}_R^2 + \left(\frac{MdR^2}{4L^2}\right)\dot{\varphi}_R\dot{\varphi}_L = \frac{1}{R}\tau_L \tag{75}$$

The above equations are also equivalent to those derived using the Lagrangian approach as given by equation (46).

Figure 4 shows the dynamic model of the DDMR representing the equations of motion (69) and (70). This model shows clearly the coupling between the motor torques, the linear and angular velocities of the robot, and the wheels velocities. This model can be adequately used for DDMR simulation and analysis.

Actuator Modeling

The dc motors which are generally used to drive the wheels of a differential drive mobile robot system are considered to be the servo actuators. In an armature-controlled dc motor which is the case for our DDMR system, the armature voltage v_a is used as the control input while keeping the conditions in the field circuit constant. In particular, for a permanent-magnet dc motor, we have the following equations for the armature circuit

$$\begin{cases} v_a = R_a i_a + L_a\dfrac{di_a}{dt} + e_a \\ e_a = K_b\omega_m \\ \tau_m = K_t i_a \\ \tau = N\tau_m \end{cases} \tag{76}$$

where, i_a is the armature current,(R_a,L_a) is the resistance and inductance of the armature winding respectively, e_a is the back emf, w_m is the rotor angular speed , τ_m is the motor torque, (K_t,K_b) are the torque constant and back emf constant respectively, N is the gear ratio, and is τ the output torque applied to the wheel.

Since in the DDMR the motors are mechanically coupled to the robot wheels through the gears, the mechanical equations of motion of the motors are linked directly with the mechanical dynamics of the DDMR. Therefore each dc motor will have

$$\begin{cases} \omega_{mR} = N\dot{\varphi}_R \\ \omega_{mL} = N\dot{\varphi}_L \end{cases} \tag{77}$$

The total dynamic equations of the DDMR with the actuators

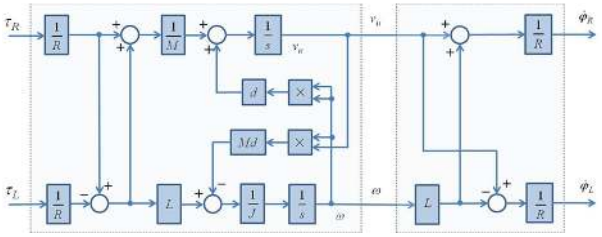

Figure 4: DDMR Dynamic Model.

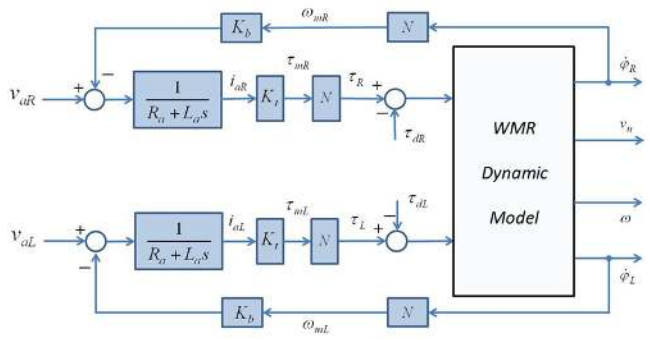

Figure 5: DDMR dynamic model with actuators.

are obtained by combining equation (76) for each motor with the mechanical dynamics of the DDMR. Additional torque disturbances acting on the wheels can be included as additive terms to the motor torques. Figure 5 shows a block diagram representation of the overall system. The forward kinematic model (19) can be added as a cascade to the dynamic model to form a complete model for simulation and analysis of the DDMR.

Conclusion

We have presented a detailed derivation of the dynamic model of a differential-drive mobile robot using the Lagrange and Newton-Euler methods. They were shown to be mathematically equivalent providing a check on their consistency. The equations of motion of DC motors actuators were also added to form the complete dynamic model of the DDMR. The insight gained in this study will assist engineering students and researchers in the modeling and design of suitable controllers for DDMR navigation and trajectory tracking.

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