Dynamic Monopolies of Constant Size

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Abstract

The paper deals with a polling game on a graph. Initially, each vertex is colored white or black. At each round, each vertex is colored by the color shared by the majority of vertices in its neighborhood, at the previous round. (All recolorings are done simultaneously). We say that a set W_0 of vertices is a *dynamic monopoly* or *dynamo* if starting the game with the vertices of W_0 colored white, the entire system is white after a finite number of rounds. Peleg [1] asked how small a dynamic monopoly may be as a function of the number of vertices. We show that the answer is O(1).

1 Introduction

Let $G = (V, E)$ be a simple undirected graph and W_0 a subset of V. Consider the following repetitive polling game. At round 0 the vertices of W_0 are colored white and the other vertices are colored black. At each round, each vertex v is colored according to the following rule. If at round r the vertex v has more than half of its neighbors colored c, then at round $r+1$ the vertex v will be colored c. If at round r the vertex v has exactly half of its neighbors colored white and half of its neighbors colored black, then we say there is a tie. In this case v is colored at round $r+1$ by the same color it had at round r. (Peleg considered other models for dealing with ties. We will refer to these models in section 3. Additional models and further study of this game may be found at [2], [3], [4], [5] and [6].) If there exists a finite r so that at round r all vertices in V are white, then we say that W_0 is a *dynamic monopoly*, abbreviated *dynamo*.

In this paper we prove

Theorem 1 *For every natural number n there exists a graph with more than n vertices and with a dynamic monopoly of 18 vertices.*

We shall use the following notation: If $v \in V$ then $N(v)$ denotes the set of neighbors of v. We call $d(v) = |N(v)|$ the *degree* of v. For every $r = 0, 1 \ldots$ we define C_r as a function from V to $\{\mathcal{B}, \mathcal{W}\}$, so that $C_r(v) = \mathcal{W}$ if v is white at round r and $C_r(v) = \mathcal{B}$ if v is black at this round. We also define $W_r = C_r^{-1}(\mathcal{W})$, $B_r = C_r^{-1}(\mathcal{B})$, $T_r = W_r \cap W_{r-1} (r > 0)$ and $S_r = T_1 \cup \ldots \cup T_r$

2 Proof of Theorem 1

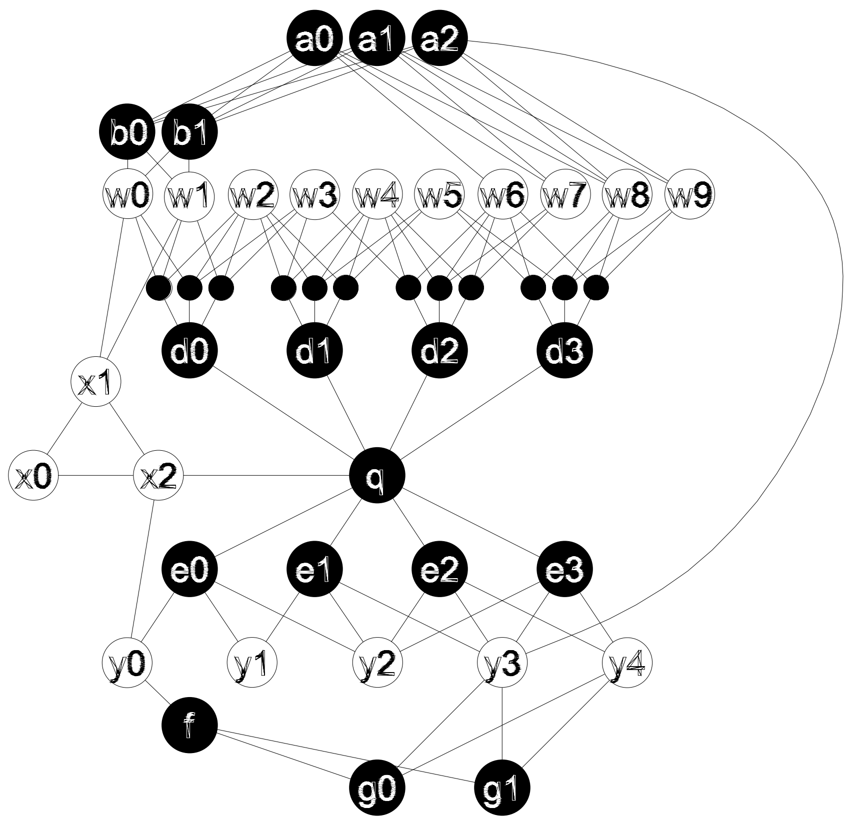


Figure 1: The graph J. (The small black circles are the vertices $c_0 \ldots c_{11}$.)

Let $J = (V_J, E_J)$ be the graph in figure 1. Let

$$W_0 = \{w_0, \ldots, w_9, x_0, \ldots, x_2, y_0, \ldots, y_4\}$$

and let $U = W_0 \cup \{q\}$ and $D = V_J - U$. We construct a graph J_n by duplicating n times the vertices in D. That is,

$$J_n = (V_n, E_n)$$

where

$$V_n = U \cup [n] \times D$$

and

$$E_n = \{(u,v) \in J : u,v \in U\} \cup \{(u,(i,v)) : (u,v) \in J, u \in U, v \in D, i \in [n]\} \\ \cup\{((i,u),(i,v)) : (u,v) \in J,\ u,v \in D, i \in [n]\}$$

(Here, as usual, $[n]$ denotes the set $\{1 \ldots n\}$).

Note that for reasons of symmetry, at a given round, all copies of a vertex in J have the same color. Thus we may write "y_0 is white at round 3" instead of "(i, y_0) is white at round 3 for every $i \in [n]$" etc.

The following table describes the evolution of J_n. The symbol 1 stands for white and 0 stands for black. Note that the table does *not* depend on n. (This property is peculiar to the graph J. In general graphs duplication of vertices may change the pattern of evolution of the graph).

r	a_{012}	b_{01}	$c_0 \dots c_{11}$	d_{0123}	e_{0123}	f	g_{01}	q	$w_0 \dots w_9$	y_{01234}
0	000	00	000000000000	0000	0000	0	00	0	1111111111	11111
1	111	00	111111111111	0000	1111	0	11	0	0000000000	00000
2	000	11	000000000000	1111	0000	1	00	1	1111111111	11111
3	111	00	111111111111	0000	1111	0	11	1	1100000000	10000
4	000	11	100000000000	1111	1000	1	00	1	1111111111	11111
5	111	00	111111111111	1000	1111	0	11	1	1100000000	11000
6	000	11	111000000000	1111	1100	1	00	1	1111111111	11111
7	111	00	111111111111	1000	1111	0	11	1	1111000000	11100
8	000	11	111100000000	1111	1111	1	00	1	1111111111	11111
9	111	00	111111111111	1100	1111	0	11	1	1111000000	11111
10	000	11	111111000000	1111	1111	1	11	1	1111111111	11111
11	111	00	111111111111	1100	1111	1	11	1	1111110000	11111
12	000	11	111111100000	1111	1111	1	11	1	1111111111	11111
13	111	00	111111111111	1110	1111	1	11	1	1111110000	11111
14	000	11	111111111000	1111	1111	1	11	1	1111111111	11111
15	111	00	111111111111	1110	1111	1	11	1	1111111100	11111
16	000	11	111111111100	1111	1111	1	11	1	1111111111	11111
17	111	00	111111111111	1111	1111	1	11	1	1111111100	11111
18	000	11	111111111111	1111	1111	1	11	1	1111111111	11111
19	111	00	111111111111	1111	1111	1	11	1	1111111111	11111
20	111	11	111111111111	1111	1111	1	11	1	1111111111	11111
21	111	11	111111111111	1111	1111	1	11	1	1111111111	11111

The table shows that at round 20 the entire system is white and therefore W_0 is a dynamo. The reader may go through the table by himself, but in order to facilitate the understanding of what happens in the table let us add some explanations as to the mechanism of "conquest" used in this graph.

We say that round j *dominates* round i if $W_i \subseteq W_j$.

We shall make use of the following obvious fact:

Observation 1 *If round j dominates round i $(i, j = 0, 1 \dots)$ then round $j+1$ dominates round $i+1$.*

By applying this observation k times, we find that if round j dominates round i then round $j+k$ dominates round $i+k$ $(i, j, k = 0, 1 \dots)$. By looking at the table one can see that in the graph J_n round 2 dominates round 0 and thus we have

Corollary 1 *Round $k+2$ dominates round k in J_n for every $k = 0, 1 \dots$*

We say that a vertex v *blinks* at round r if $C_{r+2i}(v) = \mathcal{W}$ for every $i = 0, 1 \dots$. We say that a vertex v is *conquered* at round r if $C_{r+i}(v) = \mathcal{W}$

for every $i = 0, 1 \ldots$. Examining rounds 0 to 3 in the table and using Corollary 1 one can see that x_0, x_1 and x_2 are conquered at round 0, and in addition q, w_0, w_1 and y_0 are conquered at round 2. Furthermore, every vertex in J_n blinks either at round 1 or at round 2.

Finally, we have

Lemma 1 *If at round r a vertex v in J_n has at least half of its neighbors conquered then v is conquered at round $r + 2$.*

Proof: Every vertex in J_n blinks either at round 1 or at round 2, and hence v is white either at round $r + 1$ or at round $r + 2$. From this round on, at least half of the neighbors of v are white, so v will stay white.

□

Now the vertices will be conquered in the following order:

$x_0, x_1, x_2, q, w_0, w_1, y_0, c_0, e_0, d_0, y_1, c_1, c_2, e_1, w_2 w_3, y_2, c_3, e_2, e_3, d_1, y_3, y_4,$
$c_4, c_5, g_0, g_1, f, w_4, w_5, c_6, d_2, c_7, c_8, w_6, w_7, c_9, d_3, c_{10}, c_{11}, w_8, w_9,$
$a_0, a_1, a_2, b_0, b_1.$

Eventually, the entire graph is colored white. J_n is a graph with $19 + 27n > n$ vertices and W_0 is a dynamo of size 18, proving Theorem 1.

3 Questions and Remarks

The result of Section 2 gives rise to the following questions:

Question 1 *Does there exist an infinite graph with a finite dynamo?*

The answer is *no*. This follows from the following theorem:

Theorem 2 *If W_0 is finite then T_r is finite for all $r = 1, 2 \ldots$. Moreover, every vertex in T_r has a finite degree.*

Proof: The proof is by induction on r. For $r = 1$ the theorem is true because every vertex $v \in W_0$ with an infinite degree becomes black at round 1. For $r > 1$, if $C_{r-1}(v) = \mathcal{W}$ and v has an infinite degree λ then by the induction hypotheses $C_{r-2}(v) = \mathcal{B}$ and $|N(v) \cap B_{r-2}| < \lambda$. Hence $|N(v) \cap W_{r-1}| \leq |N(v) \cap B_{r-2}| + |T_{r-1}| < \lambda$ and $C_r(v) = \mathcal{B}$.

If $v \in T_r$ has a finite degree then v has a neighbor in T_{r-1}. By the induction hypotheses only finitely many vertices have such a neighbor, and thus T_r is finite.

□

The next question deals with other models considered by Peleg:

Question 2 *Do we still have a dynamo of size O(1) if we change the rules of dealing with ties? (e.g. if a vertex becomes black whenever there is a tie.)*

The answer here is *yes*. If $G = (V, E)$ is a graph, introduce a new vertex v' for every $v \in V$ and consider the graph $\hat{G} = (\hat{V}, \hat{E})$ where

$$\hat{V} = \{v, v' : v \in V\}$$

and

$$\hat{E} = E \cup \{(u', v') : (u, v) \in E\} \cup \{(v, v') : 2|d(v)\}$$

If W_0 is a dynamo of G according to the model in Theorem 1, then it is easy to prove that $\hat{W}_0 = \{v, v' : v \in W_0\}$ is a dynamo of $\hat{G}$. But all vertices of $\hat{G}$ have odd degrees, and thus ties are not possible and $\hat{W}_0$ is a dynamo of $\hat{G}$ according to *any* rule of dealing with ties.

Therefore, for every $n = 1, 2 \ldots$ the graph $\hat{J}_n$ has a dynamo of size 36.

4 Another Model

Let $\rho > 1$ be a real number. Consider the following model, which will henceforth be called *the ρ-model*. At every round, for every vertex v with b neighbors colored black and w neighbors colored white, if $w > \rho b$ then v is colored white at the next round, otherwise it is black. For the sake of simplicity we will assume that ρ is irrational and that there are no isolated vertices, so that $w = \rho b$ is impossble.

The most interesting question regarding this model is whether there exist graphs with O(1) dynamo like in Theorem 1. This question is as yet open. We only have some partial results, which can be summarized as follows:

i. If ρ is big enough then the size of a dynamo is $\Omega(\sqrt{n})$.

ii. If ρ is small enough then there exist graphs in which the size of a dynamo is $O(\log n)$.

iii. If there exist graphs with O(1) dynamo then the number of rounds needed until the entire system becomes white is $\Omega(\log n)$.

More explicitly:

Theorem 3 *Let $\rho > 3$. If a graph with n vertices has a dynamo of size k in the ρ-model then*

$$n < k^2$$

proof:

For every $r = 1, 2, \ldots$, let $(S_r, \overline{S}_r)$ be the set of edges with one vertex in S_r and the other not in S_r. Call $s_r = |S_r| + |(S_r, \overline{S}_r)|$. Note that S_1 is the set of vertices which are white at both round 0 and round 1. Every $v \in S_1$ is connected to at most $k - |S_1|$ vertices in $W_0 \setminus S_1$ and at most $\frac{k-1}{\rho} < k - 1$ vertices outside of W_0. Therefore we have

$$s_1 < |S_1| + |S_1|(k - |S_1| + k - 1) = k^2 - (k - |S_1|)^2 \leq k^2$$

Thus all we need is to show $s_{r+1} \leq s_r$ and we are done.

Let r be fixed. By definition $S_r \subseteq S_{r+1}$. Let $\Delta = S_{r+1} \setminus S_r$, and let $v \in \Delta$. More than $\frac{3}{4}$ of the neighbors of v are white at round r and more than $\frac{3}{4}$ of the neighbors of v are white at round $r - 1$. Thus more than $\frac{1}{2}$ of the neighbors of v belong to S_r. We therefore have

$$|(S_r, \overline{S}_r) \setminus (S_{r+1}, \overline{S}_{r+1})| - |(S_{r+1}, \overline{S}_{r+1}) \setminus (S_r, \overline{S}_r)| \geq |\Delta|$$

which implies $s_{r+1} \leq s_r$. By induction $s_r < k^2$ for all r. If we begin with a dynamo then for some finite m we have $S_m = V$ and $n = s_m < k^2$ $\square$

Theorem 4 *Let $\rho > 1$. If $|W_0| = k$ and $W_m = V$ (the set of all vertices), then the number e of edges in the graph satisfies*

$$e < k^2 (\frac{2\rho}{\rho - 1})^m$$

proof:

Let d_r denote the sum of the degrees of the vertices in S_r. Recall that every $v \in S_1$ is white at both round 0 and round 1, and thus $|N(v) \cap B_0| < k$ and $d(v) < k$. Therefore, $d_1 < 2k^2$. Again, let r be fixed, let Δ be as in the proof of Theorem 3 and let $v \in \Delta$. More than $\frac{\rho}{\rho+1}$ of the neighbors of v are white at round r and more than $\frac{\rho}{\rho+1}$ of the neighbors of v are white at round $r - 1$. Thus more than $\frac{\rho-1}{rho+1}$ of the neighbors of v belong to S_r. Therefore, we have

$$d_{r+1} < d_r + \frac{\rho + 1}{\rho - 1} d_r = \frac{2\rho}{\rho - 1} d_r$$

By induction $d_r < 2k^2 (\frac{2\rho}{\rho-1})^{r-1}$. If the entire system is white at round m then $d_{m+1} = 2e$ and thus we have

$$e < k^2 (\frac{2\rho}{\rho - 1})^m$$

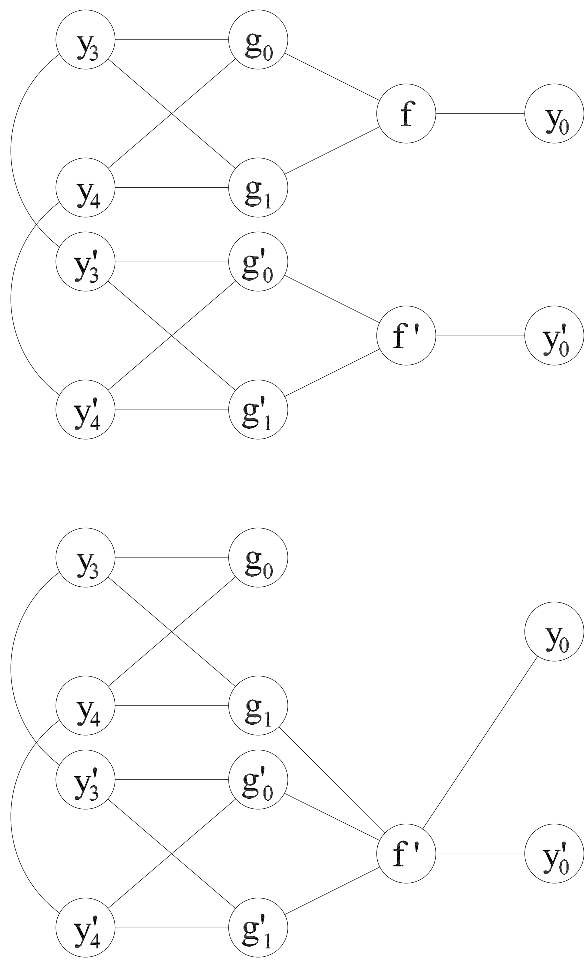
$\square$

Theorem 5 *Let $1 < \rho < \frac{257}{256}$. For every integer $n > 5$ there exists in the ρ model a graph with more than 2^n vertices and with a dynamo of size* $30(n-5)+36$.

Outline of proof:

Let $\hat{J}$ be as defined in the answer to Question 2. Construct $\tilde{J}$ by eliminating f from $\hat{J}$ and connecting f' to y_0 and g_1 (but *not* to g_0). Note that in $\tilde{J}$ the vertex g_0 is connected only to y_3 and to y_4.

In figure 2, the upper graph is a part of $\hat{J}$. The lower graph is the corresponding part in $\tilde{J}$. The rest of $\tilde{J}$ is identical to the rest of $\hat{J}$.



Construct $\tilde{J}_{32}, \tilde{J}_{64}, \ldots \tilde{J}_{2^n}$ as in the construction of J_n, where the duplicated vertices are all black vertices except for q and q'. (Note that the graphs are constructed separately, namely, the sets of vertices of $\tilde{J}_{2^i}$ and $\tilde{J}_{2^j}$ are disjoint for $i \neq j$.) Now connect the graphs in the following way. First, eliminate the copies of x_0, x_1, x_2 from all graphs except for $\tilde{J}_{32}$. Note that in $\tilde{J}_{2^i}$ there are 2^i copies of g_0 (when $i = 5, \ldots n-1$). Divide them into 32 disjoint sets $P_0, \ldots P_{31}$, of size 2^{i-5} each. Now connect the vertices in P_0 to the copy of q in $\tilde{J}_{2^{i+1}}$, connect P_1 to the copy of q', and connect each one of $P_2 \ldots P_{31}$ to a respective white vertex in $\tilde{J}_{2^{i+1}}$ (see in figure 3).

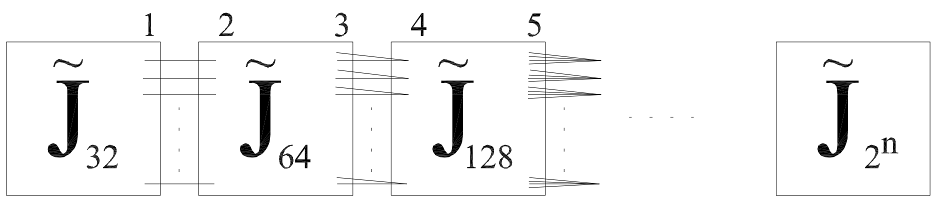


Figure 2: This figure illustrates the graph used in the proof of Theorm 5. The vertices under the numeral 1 are the 32 copies of g_0 in $\tilde{J}_{32}$. Under the numeral 2 are the 32 unduplicated vertices in $\tilde{J}_{64}$ (q, q' and the initiallly white vertices). Under the numeral 3 are the 64 copies of g_0 in $\tilde{J}_{64}$, under the numeral 4 are the 32 unduplicated vertices in $\tilde{J}_{128}$, under the numeral 5 are the 128 copies of g_0 in $\tilde{J}_{128}$, and so on.

It is possible to verify the following:

i. All vertices of the obtained graph blink either at round 1 or at round 2.

ii. All vertices of K_{32} are eventually conquered. (The evolution of this conquest is similar to the one in Theorem 1.)

iii. If all copies of g_0 in $\tilde{J}_{2^i}$ are conquered at a certain round, then all vertices of $\tilde{J}_{2^{i+1}}$ are eventually conquered. (Again, the evolution is similar to the one in Theorem 1. Note that we need the bound $\rho < \frac{257}{256}$ in order to have q and q' conquered.)

Thus all vertices are eventually conquered. The theorem follows upon noticing that our graph has more than 2^n vertices, and the size of the dynamo is $30(n-5)+36$. □

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