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Dynamic Motion of a Single Degree of Freedom System Following a Rate and State Dependent Friction Law

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Sequences of dynamic instabilities are analyzed for a single degree of freedom elastic system which slides along a surface having frictional resistance depending on slip rate and slip rate history, in the manner of Dieterich, Ruina and others. The system is represented as a rigid block in contact with a fixed surface and having a spring attached to it whose opposite end is forced to move at a uniform slow speed. The resulting "stick-slip" motions are well understood in the classical case for which there is an abrupt drop from "static" to "sliding" frictional resistance. We analyze them here on the basis of more accurate frictional constitutive models. The problem has two time scales, an inertial scale set by the natural oscillation period T of the analogous frictionless system as $T/2\pi$ and a state relaxation scale L/V occurring in evolution, over a characteristic slip distance L, of frictional stress τ towards a "steady state" value $\tau^{ss}(V)$ associated with slip speed V. We show that $\tau \simeq \tau^{ss}(V)$ during motions for which acceleration a satisfies $aL/V^2 \ll 1$, and that this condition is met during an inertia controlled instability in typical circumstances for which the unstable slip is much greater than L. Since V/a is of order $T/2\pi$ during inertia controlled motion, one has $L/V \ll T/2\pi$, whereas $L/V \gg T/2\pi$ during much of the essentially quasi-static "stick" part of the cycle when there is a sufficiently small imposed velocity at the load point. Thus the physically irrelevant time scale (L/V during inertial controlled motion, $T/2\pi$ during quasi-static motion) is much shorter than the relevant scale, which is troublesome from a numerical point of view as it is the shorter time scale which constrains allowable step size. We propose efficient numerical procedures to deal with such response, in which the full equations with inertia and state relaxation are solved only in a transition regime when both time scales are significant. We show results for several friction laws, all having history dependence based on a single evolving state variable and all having properties that $\partial\tau/\partial V > 0$ for instantaneous changes in V, that τ evolves towards $\tau^{ss}(V)$ as $\exp(-\delta/L)$ with ongoing slip δ when $V = \text{const}$, and that $d\tau^{ss}(V)/dV < 0$ except possibly at high V. During the dynamic instabilities we find that motion continues at a nearly steady state condition, $\tau \simeq \tau^{ss}(V)$, until dynamic overshoot becomes so significant that "arrest" begins. In the arrest stage, V drops rapidly to very much lower values (never zero in our models) under nearly fixed state conditions, and then the long quasi-static "stick" phase of the motion begins again.

INTRODUCTION

A frictional sliding instability between rock surfaces in the laboratory corresponds at least qualitatively to shallow depth earthquake instability along an existing fault [*Brace and Byerlee*, 1966]. The simplest constitutive relation used to describe the frictional interface is the classical "static-kinetic" friction law which is characterized by a static friction before slip initiates and a reduced kinetic friction once slip motion commences. For an initially stationary single degree of freedom elastic system (spring-block arrangement as shown in Figure 1*a*) subjected to constant but slow load point displacement rate $\dot{\delta}_0(=V_0)$, this law predicts that the block will start to slide when the spring force $k\delta_0$ is equal to τ_s. As shown in Figure 1*b*, once sliding begins, the friction resistance instantaneously drops down to the constant kinetic friction τ_k causing acceleration and then deceleration (dynamic overshoot) of the block. The block arrests after it has slipped $2(\tau_s - \tau_k)/k$ and the friction resistance then drops instantaneously from τ_k to the spring force $\tau_k - (\tau_s - \tau_k)$. The resistance starts to increase again slowly as the load point continues to move with constant speed. As the spring force increases to the static friction level, the block once again starts to slide and another cycle repeats. Independently of the spring stiffness, this law always predicts unstable motion once slip starts. However, it is observed in many friction experiments that sliding is stable in sytems of sufficiently great stiffness whereas unstable slip motion may be found in systems of low stiffness. Also, as pointed out by *Ruina* [1984], this law cannot generally predict quasi-static slip motion over a finite slip zone in an elastic continuum. Another constitutive relation commonly used to describe slip motion leading to instability is a slip-weakening law. In this model, once sliding begins, the shear strength decreases from a peak resistance to a residual strength over a characteristic slip distance. Quasi-static analysis shows that slip motion is stable for large enough system stiffness but unstable for low stiffness. This slip-weakening concept has been applied to preinstability fault modeling [*Palmer and Rice*, 1973; *Stuart*, 1979*a*, *b*; *Stuart and Mavko*, 1979; *Li and Rice*, 1983] and to dynamic rupture propagation [*Ida*, 1972; *Andrews*, 1976; *Burridge et al.*, 1979; *Day*, 1982*a*, *b*]. This law is simple to use and able to simulate some seismological phenomena, but because it inherently neglects the time or rate effects, it cannot reproduce more than one cycle without artificially resetting the friction level when the slip stops.

Experimental studies on frictional sliding in rocks as done and interpreted by *Dieterich* [1978, 1979*a*, *b*, 1981] and *Ruina* [1980, 1983] have led to a somewhat more realistic constitutive description. As formalized by Ruina, this description has the shear strength τ depending on normal stress σ_n, slip velocity V and on the prior slip history in the form of dependence on a set of phenomenological parameters called state variables which evolve with ongoing slip. The general mathematical framework for this law has the form [*Ruina*, 1980, 1983]

$$\begin{aligned} \tau &= F(V, \sigma_n, \theta_1, \theta_2, \cdots, \theta_n) \\ d\theta_i/dt &= G_i(V, \sigma_n, \theta_1, \theta_2, \cdots, \theta_n) \qquad i = 1, 2, \cdots, n \end{aligned} \tag{1}$$

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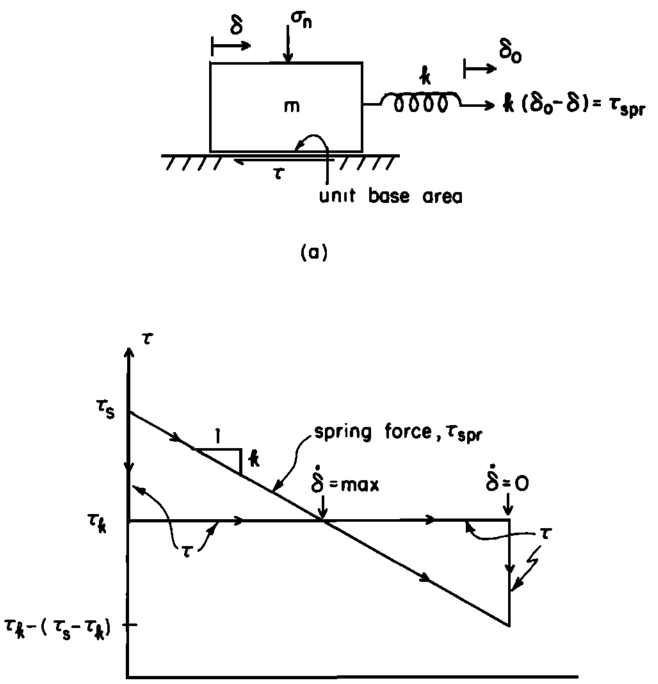

Fig. 1. (*a*) Single degree of freedom elastic system. A block of mass m and unit base area slides distance δ with frictional stress τ. The load point moves a distance δ_0 at an imposed speed V_0, stressing the block through the spring with stiffness k. (*b*) Stress response of the block to a steadily moving load point for the classical static-kinetic friction law. Initially stationary, the block starts to move when the spring force $k\delta_0$ is equal to the static friction τ_s, and it then slides with the constant kinetic friction τ_k. The arrows show the directions of the paths along which the frictional stress and spring force evolve.

In slip motion at a fixed slip rate and normal stress, the state variables evolve towards steady state values θ_i^{ss} satisfying $G_i(V, \sigma_n, \theta_1^{ss}, \theta_2^{ss}, \cdots, \theta_n^{ss}) = 0$ such that the shear strength evolves towards a steady state value $\tau^{ss}(V)$ corresponding to the fixed speed and normal stress (we do not further display σ_n as a variable).

A fairly comprehensive nonlinear analysis of quasi-static slip motion of a driven spring-block system (Figure 1*a*) and its possible instabilities has been given by *Gu et al.* [1984], *Rice and Gu* [1983] and *Blanpied et al.* [1984] for specific one and two state variable laws (as special cases of the class (1)). The one state variable law was proposed by *Ruina* [1980, 1983] as an approximation to a law proposed by *Dieterich* [1979*a*, *b*, 1981] and is cited later here; the two state variable law was of similar form but provided a closer fit to the observed relaxations in Ruina's experiments. Also, extensive numerical simulations of quasi-static motion and its instability are given by *Dieterich* [1980] and G. M. Mavko (unpublished paper, 1984). From a linearized stability analysis that included inertia, *Rice and Ruina* [1983] showed that $d\tau^{ss}(V)/dV < 0$ from the above formulation is a necessary and sufficient condition for steady state sliding to become unstable at sufficiently reduced elastic stiffness for surfaces on which a step change of V, from that of a previous steady state to a new fixed value, is followed by a montonic approach of τ to the new $\tau^{ss}(V)$. Conversely, $d\tau^{ss}(V)/dV > 0$ on surfaces with such monotonic approach at fixed V implies stability of steady sliding to small perturbations no matter what the elastic stiffness. However, as slip motion becomes unstable the slip velocity departs from the steady state value and the linearized analysis breaks down.

The aim of this paper is to examine the full nonlinear description of motion, including the effect of inertia, and to determine when the quasi-static analysis breaks down and how the inclusion of inertia affects the subsequent slip motion for the spring-block with imposed load-point motion. We report numerical simulations of several cycles of slip motion using different forms of a one state variable version of the constitutive law (1), and discuss characteristic time scales, dimensionless measures of their importance, and possible approximations in numerical calculation.

We do not suggest that the single degree of freedom system analyzed can be made to correspond closely to an actual fault. Rather, the analysis is intended to provide an understanding of stress response according to the rate and state dependent friction laws during inertia controlled motions, in the hope that this understanding will be in part transferable to more realsitic fault models involving slip between deformable continua.

RATE AND STATE DEPENDENT LAWS

Specific forms of the general slip rate and slip rate history dependent law (1) have been proposed and discussed by *Dieterich* [1979*a*, 1980, 1981], *Ruina* [1980, 1983], *Gu et al.* [1984], and *Tullis and Weeks* [1985]. Based on the a priori assumption (or approximation) that one state variable θ in a constitutive law as in equations (1) suffices to characterize the surface state, *Rice* [1983] recently outlined a procedure for constructing constitutive relations on the basis of experimental features reported by a variety of workers (see references above). These are as follows:

1. The effect of a suddenly imposed (i.e., at fixed state) increase or decrease of V is to respectively increase or decrease τ. Thus $(\partial\tau/\partial V)_\theta > 0$. We note that when there is a single θ, this derivative can be expressed as a function of τ and V, say, as $(\partial\tau/\partial V)_\theta = A/V$ where $A = A(\tau, V) > 0$ is obtained experimentally from "velocity jump" tests. $A =$ const is suggested in several references cited above.

2. In slip motion at a fixed slip rate V, τ evolves towards a steady state value corresponding to that slip rate V, i.e., $\tau^{ss}(V)$.

3. The evolution of τ towards $\tau^{ss}(V)$ at constant V is characterized by an approximately exponential decay over a characteristic slip distance $L = L(V)$ such that $d\tau/d\delta = -(\tau - \tau^{ss})/L$ in slip δ at constant V. (It is generally understood that a superposition of two or more such decay processes, each with their own L, better fits data and must be included to explain some phenomena, but here for compatibility with the assumed one state variable form we must assume that only a single decay process is active.) Thus, since $d\tau = (\partial\tau/\partial\theta)_V \, d\theta$ in slip at constant V, we are assuming that

$$(\partial\tau/\partial\theta)_V \, d\theta/dt = -[V/L(V)][\tau - \tau^{ss}(V)]$$

As *Rice* [1983] noted, although the last equation is motivated as an idealization of approach to steady state at constant V, the expression that it gives for $d\theta/dt$ in terms of V and θ (τ and $(\partial\tau/\partial\theta)_V$ can be regarded as functions of V and θ) must apply during general motions with variable V. This is because the one state variable version of (1) requires that $d\theta/dt$ be determined only by θ and V and not, e.g., by dV/dt or higher derivatives of V.

A procedure for assigning numerical values to θ has not yet been identified. This can be done with some degree of arbitrariness, associating θ values with members of the one parameter family of curves in a τ, V plane of which each represent constant state response, i.e., instantaneous changes of τ and V [*Ruina*, 1980, 1983]. However, the exercise is unnecessary. We can rewrite the class of constitutive relations just described (or, as recognized also by *Ruina* [1983], all relations that involve a single state variable) without explicit reference to θ. To do so, note that

$$(\partial\tau/\partial\theta)_V\, d\theta = d\tau - (\partial\tau/\partial V)_\theta dV = d\tau - (A/V)\, dV$$

Thus, the constitutive laws discussed in items 1 to 3 above are equivalent to saying that the first order differential equation

$$\frac{d\tau}{dt} - \frac{A(\tau, V)}{V}\frac{dV}{dt} = -\frac{V}{L(V)}[\tau - \tau^{ss}(V)] \quad (2a)$$

links histories of V and τ to one another. For example, given V and τ at $t = 0$ (i.e., given the initial state), the equation associates a stress history $\tau(t)$ with any given velocity history $V(t)$. The experimental quantities or functions required are the instantaneous viscosity parameter $A = \partial\tau/\partial(\ln V)$ at fixed state, the steady state strength τ^{ss}, and the decay slip distance L. Sometimes the system may undergo a motion that is so close to steady state conditions that equation (2*a*) can be replaced by

$$\tau = \tau^{ss}(V) \quad (2b)$$

Various forms for A, τ^{ss} and L can be chosen. The one state variable law of *Ruina* [1980, 1983] mentioned above corresponds to (again, for fixed σ_n)

$$A = \text{const} \qquad L = \text{const} \qquad \tau^{ss} = \tau_* - (B - A)\ln(V/V_*) \quad (3a)$$

where B is a constant and V_* is an arbitrary reference velocity at which the steady state strength would be τ_*. The law is more familiar as Ruina first wrote it,

$$\tau = F(V, \theta) = \tau_* + A\ln(V/V_*) + \theta \quad (3b)$$

$$d\theta/dt = G(V, \theta) = -(V/L)[\theta + B\ln(V/V_*)] \quad (3c)$$

but (2*a*) and (3*a*) are fully equivalent to (3*b*) and (3*c*). Fitting this law to *Dieterich*'s [1981] experiments (under constant normal stress σ_n of 100 bars and at room temperature) on frictional sliding of intact Westerly granite on a gouge layer of the same material with variable layer thickness, gouge particle size and degree of surface roughness, *Gu et al.* [1984] found that for rough surfaces $A/\sigma_n = 0.006$ to 0.008, $B/A = 1.12$ to 1.14, $L = 40$ to 50 μm/s and $\tau_*/\sigma_n = 0.6$ for $V_* = 1$ μm/s. When fitted to the other surface roughness the values are slightly different but they are of the same order of magnitude. One can see that both A and B are small fractions of τ_* (about 1%). This law can therefore be viewed as a small deviation from the classical kinetic friction concept. The state or history dependence gives an effect like "static" friction in certain circumstances [e.g., *Dieterich*, 1980; *Ruina*, 1983].

Experimental observation over a wide range of V suggests that the steady state stress $\tau^{ss}(V)$ tends to level out to a residual level at higher slip speeds (above 0.1 mm/s in the results of *Dieterich* [1978] on Westerly granite at room temperature). This was also observed by *Scholz and Engelder* [1976]. It is found that $\tau^{ss}(V)$ is, approximately, inversely proportional to the logarithm of slip velocity from about 10^{-4} mm/s to 10^{-1} mm/s. Hence, one can replace the last entry in equations (3*a*) by

$$\tau^{ss}(V) = \tau_* + (B - A)\ln(V_*/V + e^{-n}) \quad (4)$$

and use this in (2*a*). Here, n is a positive number chosen (depending on the choice of reference speed V_*) in such a way that $\tau^{ss}(V)$ reduces to a residual level for slip velocity higher than a certain value. For example, if V_* is equal to 30 mm/yr (typical plate velocity) and $\tau^{ss}(V)$ begins to level off after 10^{-1} mm/s, n can be chosen as 10.

The function $\tau^{ss}(V)$ in equations (3*a*) and (4) has the property that $d\tau^{ss}(V)/dV < 0$ for the values of B and A quoted. Reversal of the inequality has been observed in experiments at high temperatures [*Stesky et al.*, 1974; *Stesky*, 1975, 1978] and on surfaces that have undergone relatively little total slip [*Dieterich*, 1981]. Based on the stability analyses mentioned in the introduction, for $d\tau^{ss}(V)/dV < 0$ steady state sliding is unstable for small enough stiffness or large enough perturbations while for $d\tau^{ss}(V)/dV > 0$ steady state sliding is always stable. *Tse and Rice* [1984] suggested the depth cut-off of crustal earthquake activity (e.g., between approximately 10 and 15 km along the San Andreas fault) can be understood in terms of the variation of the friction response with depth (increasing temperature) from a regime with $d\tau^{ss}(V)/dV < 0$ to one with $d\tau^{ss}(V)/dV > 0$. *Mavko* [1980, unpublished paper, 1984] showed that a change in sign of $d\tau^{ss}(V)/dV$ at a certain depth tends to concentrate instability, in quasi-static strike slip sliding between two elastic plates, into the shallower region with $d\tau^{ss}/dV < 0$.

Rice [1983] also discussed a case for which $V(\partial\tau/\partial V)_\theta \equiv A = \alpha\tau$, with $\alpha = \text{const}$. In this case integration at fixed θ shows that the variation of τ with V is $\tau = \Psi V^\alpha$ where the integration "constant" Ψ is a function of state θ and, for that matter, could be used itself as the variable to be identified as θ. Since A is typically of order 0.01τ, α would have to be of order 0.01. In that case experimental data fitted over several orders of magnitude in V to $V(\partial\tau/\partial V)_\theta = A = \text{const} \simeq 0.01\tau_*$, leading to the logarithmic direct velocity dependence in (3*a*), could be fitted comparably well to $V(\partial\tau/\partial V)_\theta = A = \alpha\tau \simeq 0.01\tau$, leading to a power form $\tau \propto V^\alpha$ for the direct velocity dependence, since τ differs only modestly from τ_* over a wide range of V. The latter form remains well defined at $V = 0$, although such simulations as we report here never involve V so close to zero that the awkward behavior of $\ln(V/V_*)$ very near that point shows up.

The formulation in equation (2*a*) can also be applied with the quotient form

$$\tau = \sigma_n C(\theta)/f(V)$$

for $F(V, \theta)$ favored by *Dieterich* [1979*a*, 1980, 1981]. In his case,

$$C(\theta) = c_1 + c_2\log_{10}(1 + c_3\theta)$$

$$f(V) = 1 + f_2\log_{10}(1 + f_3/V)$$

where c_1, c_2, c_3, f_2, f_3 are constants and θ is interpreted as an effective contact lifetime whose steady state value is d_c/V. For this case,

$$A = -\tau V[df(V)/dV]/f(V) \qquad \tau^{ss}(V) = \sigma_n C(d_c/V)/f(V)$$

and $L = d_c$ in equation (2*a*), although the presumed exponential evolution, in the form $\exp(-\delta/L)$, of τ towards $\tau^{ss}(V)$ at fixed V, implicit in equation (2*a*), is different in detail from results of the different evolution laws for θ adopted by Dieterich at different times in his work [see *Dieterich*, 1981].

Elastic Load System Interaction, Inertial Effects, and Characteristic Time Scales

For the spring-block arrangement of Figure 1*a* with unit base area, let k be the spring constant and m the mass per unit base area. We write $m = k(T/2\pi)^2$ where T is the vibration period of the analogous freely slipping system. Since the forces acting on the block are the spring force $k(\delta_0 - \delta)$ and the resistance τ, the equation of motion is

$$(T/2\pi)^2 \frac{d^2\delta}{dt^2} = (\delta_0 - \delta) - \tau/k \qquad (5a)$$

with $\delta_0 = V_0 t$, $V_0 = \text{const}$, whereas the quasi-static analyses discussed earlier began with

$$\tau = k(\delta_0 - \delta) \qquad (5b)$$

The complete description of the motion requires simultaneous solution of equations (2*a*) and (5*a*), the variables of which are linked by $V = d\delta/dt$. We see that each of these equations contains a characteristic time scale. In equation (2*a*), L/V is a characteristic time for state relaxation and in equation (5*a*) $T/2\pi$ is a characteristic vibrational or inertial time. The state relaxation time scale varies throughout the motion and hence, as will be seen, in cases as we consider here the ratio of inertial to relaxation times, $VT/2\pi L$, varies from $\ll 1$ to $\gg 1$.

We are concerned here with loading speeds V_0 that are sufficiently slow that the time between unstable inertia-limited motions, analogous to that depicted in Figure 1*b*, is very much larger than T. Thus, although constitutive relations of the type we use predict that the block is never truly stationary, we expect that the quasi-static equation (5*b*) together with equation (2*a*) is a quite accurate description over most of the long time interval that is analogous to the "stick" phase for the classical stick-slip model. Also, the nature of parameters is such that the slip during the inertia limited instabilities is very much larger than the relaxational slip distance L. As will be discussed, this implies that the steady state equation (2*b*) together with equation (5*a*) give an accurate description of the motion over most of the inertia limited phase.

Let us note the difficulties that occur if one tries to solve the complete equations, (2*a*) and (5*a*), for all phases of the motion. In terms of the characteristic inertial and relaxational times, $T/2\pi \ll L/V$ during most of what is the analogue of the stick part of the cycle. We know that $T/2\pi$ is irrelevant then, and that solutions of equation (5*a*) effectively reduce to those (5*b*). However, if one attempts to solve equations (2*a*) and (5*a*) numerically, it is the much shorter but irrelevant time scale $T/2\pi$ which controls numerical step size, and hence greatly increases the time required for numerical solution. On the other hand, during much of the inertia controlled instability, $L/V \ll T/2\pi$. Solutions of equation (2*a*) then effectively reduce to equation (2*b*) and the relaxational time L/V is then irrelevant. However, there again occurs the problem that if we try to solve the full system of equations (2*a*) and (5*a*) numerically, the much shorter but irrelevant time scale, L/V, controls numerical step size and greatly increases solution time.

Thus in numerical simulations one may simplify the equation set to equations (2*a*) and (5*b*) in phases of the stick-slip cycle when $VT/2\pi L \ll 1$ and simplify to equations (2*b*) and (5*a*) when $VT/2\pi L \gg 1$, but one must contend with the full set of equations (2*a*) and (5*a*) in an intermediate range whose limits can be set for given surface, system and loading parameters after some trial computations.

The condition for steady state resistance, i.e., that equation (2*a*) reduces to (2*b*) can be understood as follows. Let

$$u = [\tau - \tau^{ss}(V)]/A$$

and, for simplicity, assume $A = \text{const}$ as in equation (3*a*). Equation (2*a*) can then be rewritten, using $dt = d\delta/V$ where δ = slip and writing acceleration $a = dV/dt$, as

$$du/d(\delta/L) + u = [1 - V(d\tau^{ss}/dV)/A](aL/V^2)$$

The bracketed term is of order unity for the constitutive laws discussed, and hence if $|a|L/V^2 \ll 1$ throughout some phase of a motion for which slip δ of order of a few times L occurs, then $u \simeq 0$ in that motion. More precisely, $|u|$ is of order $|a|L/V^2$ which is $\ll 1$. Thus we have in those conditions $\tau = \tau^{ss}(V)$ plus A times a very small number, which means that $\tau \simeq \tau^{ss}(V)$. Therefore the condition for solutions to (2*a*) to reduce to those of (2*b*) is that

$$|a|L/V^2 \ll 1$$

during slips of a few times L and more. Related to this, we note that *Gu* [1984] showed that solutions to (3*b*), (3*c*) for uniformly accelerated motion, $a = \text{const}$, have the property that τ approaches $\tau^{ss}(V)$ as V increases such that the above condition is met. The condition can also be understood in the following way: $|dV|/V$ is a fractional change of V, associated with activation of the instantaneous viscosity effect and with change of the $\tau^{ss}(V)$ towards which τ evolves, whereas $d\delta/L$ is a fractional slip towards steady state. The latter will dominate, so that $\tau \simeq \tau^{ss}(V)$, in sustained slip with

$$\frac{|dV|/V}{d\delta/L} = \frac{|a|/V}{V/L} = \frac{|a|L}{V^2} \ll 1$$

The inequality is satisfied during inertia controlled instabilities for the conditions considered here. To see why we note that $\tau^{ss}(V)$ is either constant or a slowly (logarithmic) varying function of V at large V, and may be treated as effectively constant over at least a modest range of V. Thus making the assumption, subject to verification, that equation (2*b*) applies during much of the inertia controlled range, we will have the harmonic oscillation form

$$V \simeq V_{max} \sin(2\pi t/T) \qquad a \simeq (2\pi/T)V_{max} \cos(2\pi t/T)$$

where V_{max} is the maximum slip velocity. (These equations describe exactly the case in Figure 1*b*.) Thus $|a|$ is of order $(2\pi/T)V$ during such motion, and hence $|a|L/V^2$ is of order $(L/V)/(T/2\pi)$. But this is, apart from a numerical factor, the ratio of the characteristic state relaxation slip L to the total slip during the inertia controlled instability. For the cases considered here, the latter slip is many times L, and we confirm that $|a|L/V^2 \ll 1$, which means that the condition for equation (2*a*) to reduce to (2*b*) is met.

We do not consider slip between deformable elastic continua in detail here. However, we may note that if a uniformly stressed fault grows in circular shape with rupture speed V_r, and has a uniform stress drop $\Delta\tau$, then the *Kostrov* [1964] solution applies. The slip at distance $r < V_r t$ from the center is

$$\delta = \beta(\Delta\tau/\mu)(V_r^2 t^2 - r^2)^{1/2}$$

where μ is the shear modulus of the elastic medium and β is a function of V_r which is of order unity for all speeds less than the shear wave speed. Elementary calculation with $V = \partial\delta/\partial t$, $a = \partial V/\partial t$ shows then that

$$aL/V^2 = (L/\delta)(r^2/V_r^2 t^2) \le L/\delta$$

Thus at all points of the dynamic rupture surface for which a slip δ much greater than L has occurred, we have $aL/V^2 \ll 1$ and hence meet conditions for equation (2*a*) to reduce to (2*b*). This should be useful in dynamic rupture analysis for realistic fault models with slip between elastic continua.

The simple spring-block model that we analyze here cannot be made to correspond closely to such realistic fault models.

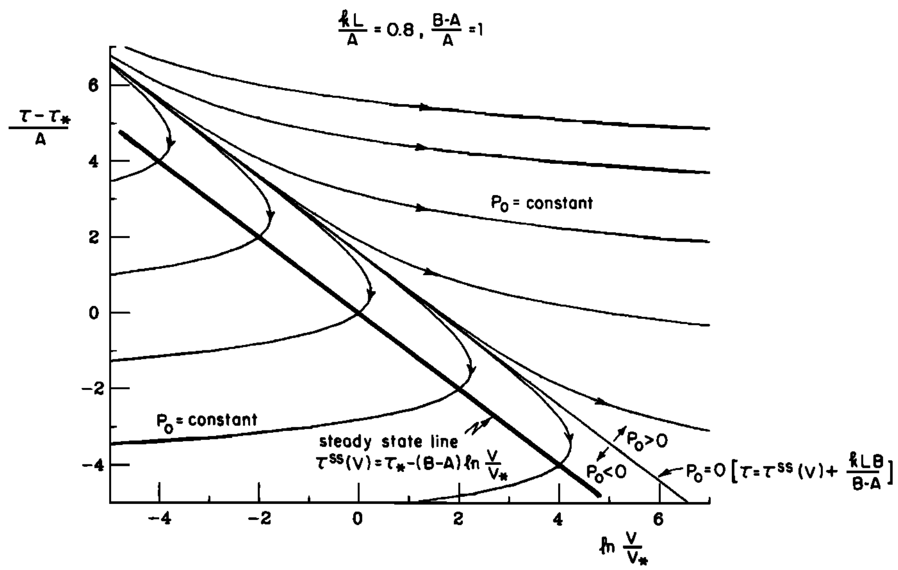

Fig. 2. Trajectories of the block motion according to quasi-static analysis for a stationary load point in the normalized stress, $(\tau - \tau_*)/A$, versus logarithm of velocity, $\ln (V/V_*)$, phase diagram for the one state variable law (3) for the case $kL/A = 0.8$ and $(B - A)/A = 1$. The thick solid line is the steady state line. The curve $P_0 = 0$ separates unstable orbits ($P_0 > 0$) from stable orbits ($P_0 < 0$).

However, for a rupture involving slip over a region which grows at the shear wave speed V_s to diameter D and then stops growing, we might approximately associate the period T of the oscillator with $2D/V_s$. That is, $T/2$, the portion of a complete cycle with positive velocity, is the sum of the time $D/2V_s$ over which rupture grows plus the same time $D/2V_s$ for arrest phases to spread backwards from the stopped edge of the rupture. Thus, for $D = 9$ km and $V_s = 3.5$ km/s, $T \simeq 5$ s.

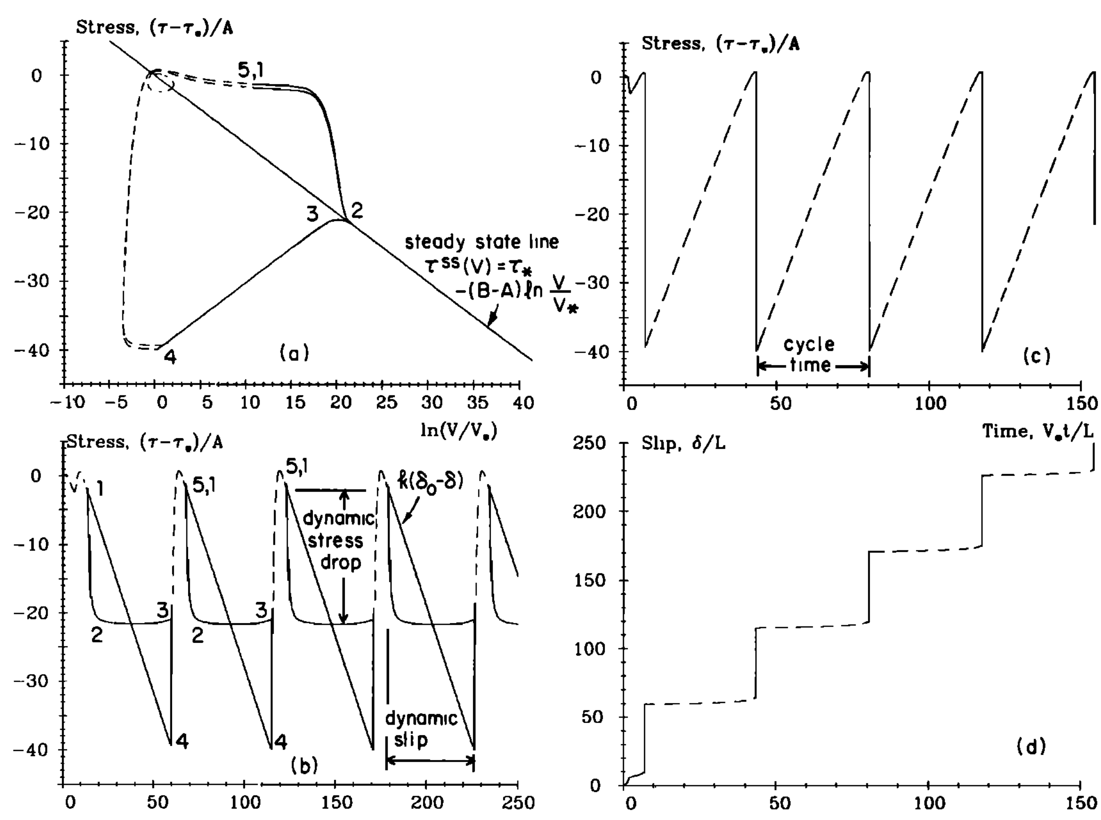

Fig. 3. Numerical simulation of slip motion for the one state variable law (3) ($\tau^{ss}(V) = \tau_* - (B - A) \ln (V/V_*)$) with $kL/A = 0.8$, $(B - A)/A = 1$, $V_0 = V_* = 30$ mm/yr, $L/V_0 = 2.7$ yr and $T = 5$ s. The block and load point initially slide in steady state at velocity $V = V_0 = V_*$. The load point speed is then suddenly increased to $1.5V_0$ and subsequently held constant. Dashed and solid lines are results from quasi-static and dynamic calculations, respectively. (*a*) Plot of dimensionless stress $(\tau - \tau_*)/A$ versus logarithm of velocity, $\ln (V/V_*)$. (*b*) Plot of dimensionless stress versus dimensionless slip, δ/L. (*c*) Plot of dimensionless stress versus dimensionless time, V_*t/L. (*d*) Plot of dimensionless slip versus dimensionless time.

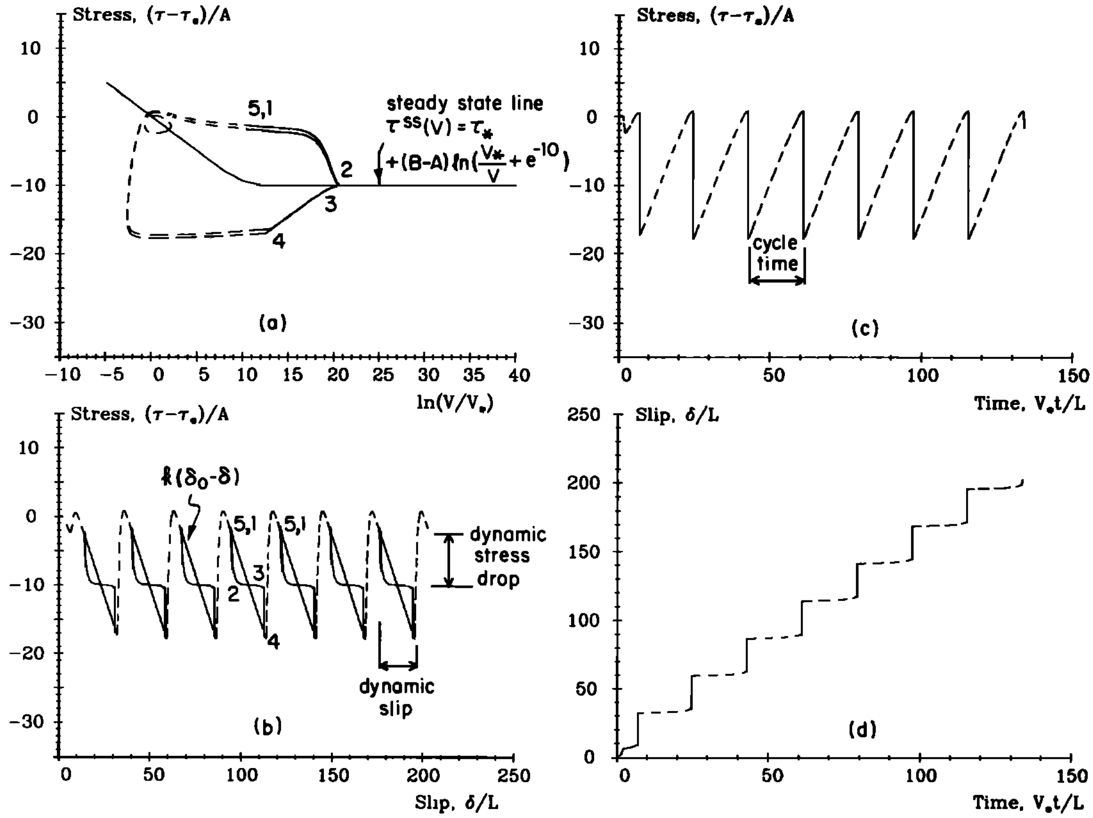

Fig. 4. Numerical simulation of the slip motion for the one state variable law (4) with $\tau^{ss} = \tau_* + (B - A) \ln (V_*/V + e^{-10})$. The parameters and initial condition are identical to the case in Figure 3. Dashed and solid lines are results from quasi-static and dynamic analyses, respectively. (*a*) Plot of dimensionless stress versus dimensionless slip, δ/L. (*c*) Plot of dimensionless stress versus dimensionless time, V_*t/L. (*d*) Plot of dimensionless slip versus dimensionless time.

QUASI-STATIC MOTIONS

In this limit $T/2\pi \ll L/V$ and equation (5*a*) reduces to (5*b*). Then equations (5*b*) and (2*a*) show that motions with $d\delta_0/dt = V_0 = \text{const}$ are described by trajectories in a stress versus velocity plane which are the integral curves of

$$kA(\tau, V)L(V)(V - V_0)\, dV/V^2 + [\tau - \tau^{ss}(V) - kL(V)(V - V_0)/V]\, d\tau = 0$$

Gu et al. [1984] and *Rice and Gu* [1983] show examples of these trajectories for the particular constitutive law described by equations (3). In that case *Gu et al.* [1984] show that for motions with $V \gg V_0$ (precisely, for $V_0 = 0$), the trajectories are given by

$$P_0 = \left(\frac{\tau - \tau^{ss}(V) - kLB/(B - A)}{A}\right) \exp\left[\left(\frac{B - A}{A}\right)\left(\frac{\tau}{kL}\right)\right] = \text{const} \qquad (6)$$

where the constant P_0 is determined from initial conditions. The trajectories for the case $kL/A = 0.8$, $(B - A)/A = 1$ are drawn in Figure 2. $P_0 > 0$ gives unbounded slip velocity in finite time; $P_0 < 0$ gives decaying slip speed and $P_0 = 0$ is a trajectory dividing stable ($P_0 < 0$) and unstable ($P_0 > 0$) regions. The predicted motions involve a nonrelaxed state as can be seen by calculating aL/V^2 using $a = V\, dV/d\delta = -kV\, dV/d\tau$. Thus one finds

$$aL/V^2 = kL/(B - A) + P_0 \exp\left[\left(\frac{B - A}{A}\right)\left(\frac{\delta - \delta_0}{L}\right)\right] \qquad (7)$$

and it is seen that along the unstable trajectories (positive factor before the exponential) this becomes much larger than unity, implying nonrelaxation of state. This quasi-static analysis neglects the inertia effect for the whole slip history and this must ultimately become inappropriate along the unstable trajectories as V increases (towards infinity in finite time) and we no longer meet the condition $T/2\pi \gg L/V$.

DYNAMIC MOTION, NUMERICAL SIMULATIONS

As such conditions are approached it is necessary to revert from (5*b*) to (5*a*), that is, to solve the full equations with inertia. However, from what has been discussed earlier we know that once inertia exerts its control of acceleration, at relatively high slip velocity, we will typically be in the regime for which state relaxation can be neglected $T/2\pi \gg L/V$, and equation (2*a*) can be reduced to (2*b*). Then the motion as governed by equations (5*a*) and (2*b*) can again be described by a trajectory in a spring force $F[F = k(\delta_0 - \delta)]$ per unit area versus velocity plane. (Note that $\tau = F$ in the quasi-static case discussed above.) These trajectories satisfy

$$(T/2\pi)^2(V - V_0)\, dV + [F - \tau^{ss}(V)]\, dF/k^2 = 0$$

and integrate to simple elliptical curves when V is sufficiently high that τ^{ss} is independent of V.

Because of the high slip speed involved during dynamic instability, the nearly steady state sliding is expected to continue through at least the beginning of the dynamic overshoot stage (the stage at which the block decelerates due to the compression of the spring). Towards the ends of the dynamic overshoot when further shortening of the spring becomes too

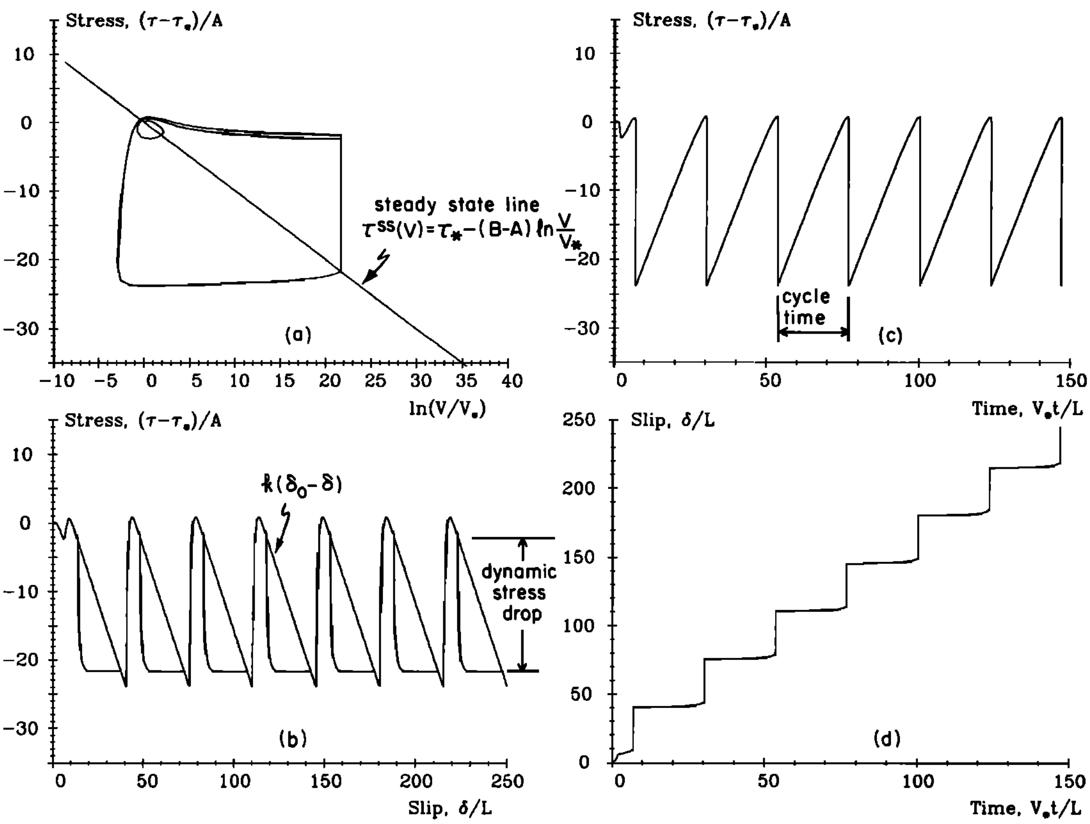

Fig. 5. Numerical simulation of the slip motion for the one state variable law (3) with limiting speed approximation (see text for details). The parameters and initial condition are identical to the case in Figure 3. For slip speed lower than limiting speed V_L (i.e., $\ln(V_L/V_*) = 21.6$), the solution is found from quasi-static analysis. When the slip speed reaches the limiting speed, the block is allowed to slide at this speed until the sliding stress is balanced by the spring force. Then, quasi-static analysis is resumed. (*a*) Plot of dimensionless stress $(\tau - \tau_*)/A$ versus logarithm of velocity, $\ln(V/V_*)$. (*b*) Plot of dimensionless stress versus dimensionless slip, δ/L. (*c*) Plot of dimensionless stress versus dimensionless time, V_*t/L. (*d*) Plot of dimensionless slip versus dimensionless time.

difficult, the full equations (5*a*) and (2*a*) must again be used. We find in numerical solutions that the high block speed is slowed down substantially or "arrested" within a short slip distance. This implies that the "arrest" process is achieved under a nearly constant or "frozen" state. After the block is arrested, the slip speed becomes low enough so that quasi-static analysis applies again.

A numerical simulation of the slip motion described above is performed for the case $kL/A = 0.8$, $(B - A)/A = 1$, imposed load point velocity $V_0 = V_* = 30$ mm/yr, $L/V_0 = 2.7$ years and $T = 5$ s. The block is initially sliding in a steady state condition with constant load point velocity V_0 such that $V = V_0 = V_*$ and $\tau^{ss} = \tau_*$. The slip motion is then made unstable quasi-statically by suddenly increasing the load point velocity to a new constant value $1.5V_0$. The results are shown in Figure 3. In the early stage of the slip motion, the block responds to the sudden increase of the load point velocity by a growing oscillation. However, the slip speed is still small so that quasi-static analysis remains valid. Because of this and computational efficiency, the system of equations is solved quasi-statically in the low speed range and dynamically in the high speed range.

In the calculations performed here, the quasi-static calculation is switched over to a dynamic calculation when $VT/2\tau L$ is greater than a prescribed small number (5×10^{-4} is used here) and vice versa. The dashed and solid lines in Figure 3 show results from quasi-static and dynamic analyses, respectively. As shown in the plot of shear stress versus displacement in Figure 3*b*, during the initial stage of the dynamic range, the excess of spring force $k(\delta_0 - \delta)$ over the sliding friction τ accelerates the block and the sliding stress evolves rapidly towards a steady state value corresponding to the current slip rate V (line 1–2). The block then continues to slide under a nearly steady state condition until further shortening of spring is prohibited (line 2–3). At this point, the block is "arrested" within an extremely short slip distance (line 3–4). It can be seen from Figure 3*a* that this arrest occurs at a frozen state. The response should be compared to Figure 1*b* for the classical friction law. At the end of the arrest, the spring force and the sliding friction are equal and the speed is low enough so that quasi-static calculation is resumed. During this quasi-static stage (line 4–5), the sliding stress restrengthens to a peak value at a nominally stationary contact. After the peak, the friction stress decreases rapidly so that the block is accelerated and once again the system enters the dynamic range at high speed. The result shows that the system exhibits a limit cycle. The stress variations with slip and time during the cycle are shown in Figures 3*b* and 3*c*, respectively. Although the cycles are repeated exactly, the stress build up after a slip event is not strictly linear with time (Figure 3*c*) and the system is nominally stationary (effectively locked) for most of the cycle time (Figure 3*d*) with accelerating slip preceding the next event.

Experiments on rocks by *Tullis and Weeks* [1985] confirm that arrest after instability occurs at nearly frozen state, represented by a measured τ versus $\ln V$ line analogous to line 3–4 in Figures 3 and 4.

An identical numerical simulation as described above has been carried out with the friction law (4). In this case, the

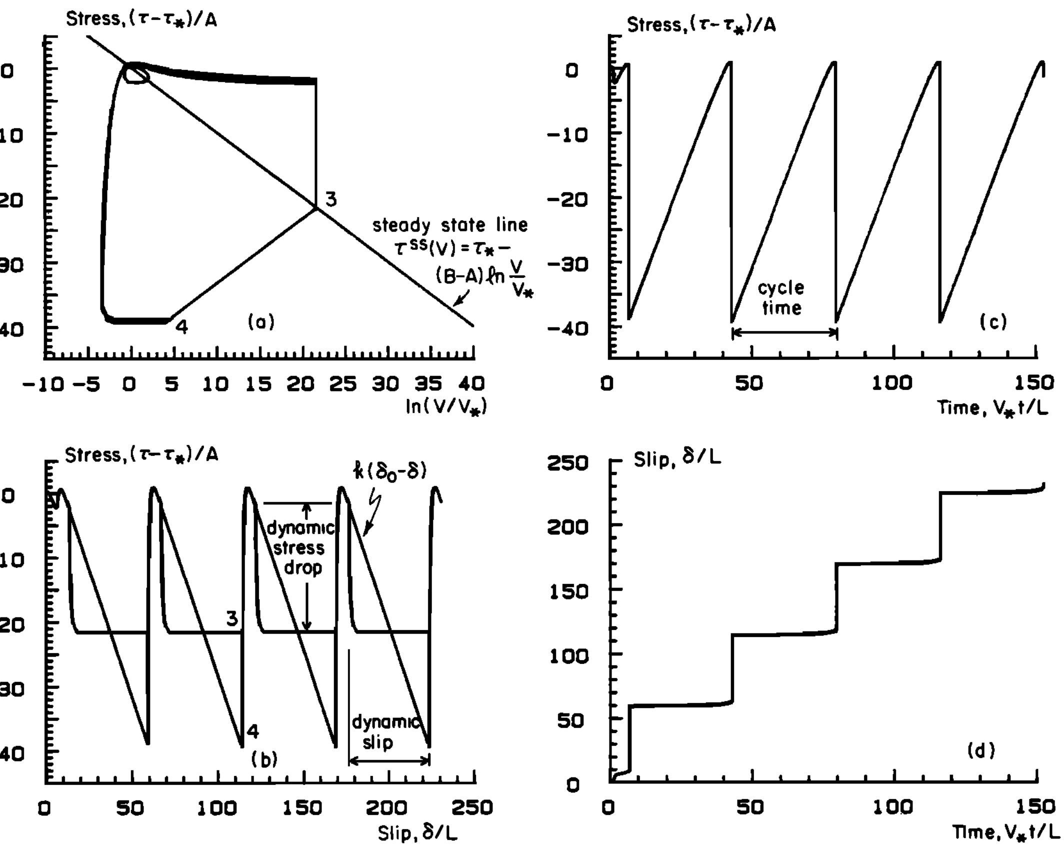

Fig. 6. Numerical simulation of the slip motion for the one state variable law (3) with limiting speed approximation including dynamic overshoot (see text for details). The parameters and initial condition are identical to the case in Figure 3. For slip speed lower than limiting speed V_L (i.e., ln (V/V_*) = 21.6), the solution is found from quasi-static analysis. When the slip speed reaches the limiting speed, the block is allowed to slide at this speed until the spring force is lowered to the desired level. Then, quasi-static analysis is resumed. (*a*) Plot of dimensionless stress $(\tau - \tau_*)/A$ versus logarithm of velocity, ln (V/V_*). (*b*) Plot of dimensionless stress versus dimensionless slip, δ/L. (*c*) Plot of dimensionless stress versus dimensionless time, V_*t/L. (*d*) Plot of dimensionless slip versus dimensionless time.

steady state stress $\tau^{ss}(V)$ decreases linearly with ln (V/V_*) at low speeds and reduces to residual level at high speeds (here, n in equations (4) is taken to be 10 with $V_* = 30$ mm/yr so that the steady state stress becomes relatively constant for speeds higher than 10^{-1} mm/s). The results are plotted in Figure 4 which show identical features as described above except that the dynamic stress drop, dynamic slip, total strength drop and cycle time are expectedly smaller.

As is seen in Figures 3*a* and 4*a*, the sliding stress τ changes rapidly with the slip rate to the steady state value during the acceleration stage. This seems to take place over a narrow slip rate range (on the logarithmic scale) as compared to the whole slip rate range calculated. One may ask if this could be approximated by the following procedure: As the slip rate of an unstable slip increases to a limiting speed V_L, the quasi-static assumption becomes invalid. The block is then artificially constrained to slide at this limiting speed such that the stress evolves towards the steady state value. The assumption of instability continues until the spring force is reduced to equal to the sliding stress, after which quasi-static equilibrium is reasserted. This approximation procedure has been adopted by *Dieterich* [1981] and G. M. Mavko (unpublished paper, 1984) in their numerical simulations of experiments and tectonic earthquake instability models respectively. The limiting speed V_L can be estimated from an energy balance condition such that the potential energy released from the unloading spring is used to increase the kinetic energy of the sliding block, i.e.,

$$\Delta\tau^2/2k = mV_L^2/2 \tag{8a}$$

or

$$V_L = (2\pi/T)(\Delta\tau/k) \tag{8b}$$

where T is the period of natural vibration and $\Delta\tau$ is the dynamic stress drop. This dynamic stress drop can be estimated as

$$\Delta\tau = \tau_q(V_L) - \tau^{ss}(V_L) \tag{9}$$

where $\tau_q(V_L)$ denotes the stress predicted along the quasi-static trajectory at speed V_L. Hence equations (8*b*) and (9) become an implicit equation for V_L but are easy to solve iteratively since $\Delta\tau$ is a slowly varying function of V_L. When the case in Figure 3 is analyzed in this way (see Figure 5a for the full quasi-static trajectories), we find V_L = 2.40 m/s. Hence ln (V_L/V_*) = 21.6 and this is very close to the maximum speed attained in the full dynamic calculation (Figure 3*a* shows ln (V_L/V_*) = 21.6). A quasi-static numerical calculation for the same elastic system with the limiting speed taken to be such that ln (V_L/V_*) = 21.6 has been performed. The results are plotted in Figure 5. Comparing Figure 5 with Figure 3 shows that this procedure approximates the response reasonably well in the acceleration stage but not in the dynamic overshoot stage. The latter is expected because the procedure implicitly neglects the dynamic overshoot phenomenon. Therefore, in the limiting speed approximation, the total strength drop, dynamic slip and cycle time are generally underestimated (at least for this one degree of freedom system) while the dynamic stress drop is the same.

It is also possible to incorporate dynamic overshoot effects directly into quasi-static calculations with the limiting speed procedure. Once the quasi-static analysis suggests that V has reached V_L, calculated as above, we suspend use of equation (5*b*) and let slip continue ar rate V_L until the spring force F has reduced to whatever level below $\tau^{ss}(V_L)$ is desired (for example, to make the overshoot $\tau^{ss}(V_L) - F$ a certain fraction of the "dynamic" stress drop $\tau_q(V_L) - \tau^{ss}(V_L)$). During this rapid slip at V_L, τ evolves rapidly to $\tau^{ss}(V_L)$ and the state evolves to the corresponding steady state value. Once the spring force has been lowered to the desired level, we instantaneously reduce velocity (i.e., at constant state, along line 3–4 in Figure 6*a*), to make τ fall from $\tau^{ss}(V_L)$ to the spring force, and then resume quasi-static calculations using equation (5*b*). Figure 6 shows the result of this procedure when the overshoot is chosen as 90% of the dynamic stress drop. Comparing Figure 6 and Figure 3, this modification of the quasi-static limiting speed procedure predicts the dynamic motion remarkably well. Under crustal conditions where some energy is lost to seismic radiation, the dynamic overshoot is expected to be smaller than the virtually complete overshoot analyzed here.

CONCLUSIONS AND DISCUSSION

In summary, we have shown that two different time scales exist for single degree of freedom elastic systems sliding according to rate and state dependent friction. For low slip speeds, the state relaxation time scale L/V dominates and a quasi-static analysis is valid. For higher slip speeds occurring along unstable trajectories of the quasi-static analysis, the time scale characterized by the period T of natural vibration becomes essential and full dynamic effects have to be considered. Slow quasi-static slip motion that is developing towards instability is characterized by an unrelaxed state which continuously diverges from steady state. During unstable slip, the frictional stress decreases with slip faster than the spring force does, and the excess spring force accelerates the block to higher speeds, which allows the state to evolve rapidly towards a steady state condition. The unstable inertia limited slip motion occurs in nearly steady state conditions, i.e., $\tau \simeq \tau^{ss}(V)$, until the block overshoots dynamically. Motion is arrested rapidly such that final velocity reduction occurs under a nearly frozen state as the friction force reduces to the spring force. Quasi-static analysis is found to be appropriate when $T/2\pi \ll L/V$, whereas dynamic analysis under the assumption of instantaneously steady state resistance ($\tau = \tau^{ss}(V)$) is appropriate for large slip δ/L with $T/2\pi \gg L/V$.

Two numerical examples of the one state variable friction law are examined. In one the steady state stress $\tau^{ss}(V)$ decreases linearly with ln (slip rate) and in the other it decreases linearly with ln (slip rate) at low speeds but becomes relatively constant at high speeds. The two show identical features as described above but the latter has less dynamic stress drop, slip and cycle time. Comparing the stress versus slip curves in Figure 1*b* with those in Figures 3*b* and 4*b*, one finds a close resemblance. In fact, due to the "smallness" of the parameters A and B, the slip rate and state dependent laws (3) or (4) can be viewed as a small deviation from the classical law. However, this small deviation contributes important features affecting the stability of the slip motion. One difference between the rate and state dependent law and the classical law is that the first predicts a gradual change in friction stress with ongoing slip whereas the latter predicts an instantaneous drop in stress as motion begins. Also, just before the "arrest" process, slip speed predicted by the classical law is zero while the slip speed predicted by the rate and state dependent law begins high and falls at essentially constant state to very small, but nonzero, values.

The quasi-static limiting speed procedure seems to approximate the slip motion reasonably well in the acceleration stage but to under estimate the total strength drop, dynamic slip and cycle time because dynamic overshoot is implicitly ignored. However, we show a modification of the procedure which seems to duplicate dynamic results quite closely. The dynamic overshoot is expected to be less pronounced in a continuous system in which seismic radiation occurs.

The full numerical simulations demonstrate cycles of the slick-slip phenomenon. In dynamic slipping, the block slides rapidly with a reduced shear stress over a large slip distance. Seismologically, this is analogous to an earthquake rupture. After the dynamic motion, the slip velocity becomes so low that the slip surface is essentially stationary. During this nearly stationary contact, slowly increasing loads cause shear stress to increase and, because of the concomitant evolution of state, a preinstability peak level must be overcome as the system prepares for the next event. This slow "stick" process is an analog to the formation of a "locked" zone in seismology. The stick-slip cycle is complete when the shear stress passes a peak value and decreases so that velocity grows rapidly to values controlled by inertia.

Our work also shows a new representation for rate and state dependent friction with a single evolving state variable, and notes that constitutive rate relations between stress and velocity can be written without explicit reference to the state variable.

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