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Dynamic Opial diamond-α integral inequalities involving the power of a function

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Abstract

In this paper, we present some new dynamic Opial-type diamond alpha inequalities on time scales. The obtained results are related to the function f^k.

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1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, we say that t is right-scattered, whereas if $\rho(t) < t$, we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The mappings $\mu, \nu : \mathbb{T} \to [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, are called the forward and backward graininess function, respectively. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$.

Theorem 1.1 *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then:*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *The product $fg : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t).$$

Theorem 1.2 *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_k$. Then:*

1. *The sum $f+g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with*

$$(f+g)^{\nabla}(t)=f^{\nabla}(t)+g^{\nabla}(t).$$

2. *For any constant α, $\alpha f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with*

$$(\alpha f)^{\nabla}(t)=\alpha f^{\nabla}(t).$$

3. *The product $fg:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with*

$$(fg)^{\nabla}(t)=f^{\nabla}(t)g(t)+f^{\rho}(t)g^{\nabla}(t)=f(t)g^{\nabla}(t)+f^{\nabla}(t)g^{\rho}(t).$$

The following formulas will be used in our paper:

$$\left(f^{l+1}\right)^{\Delta}=\left\{\sum_{k=0}^{l}f^{k}\left(f^{\sigma}\right)^{l-k}\right\}f^{\Delta},\quad l\in N,$$
$$\left(f^{l+1}\right)^{\nabla}=\left\{\sum_{k=0}^{l}f^{k}\left(f^{\rho}\right)^{l-k}\right\}f^{\nabla},\quad l\in N.$$

Definition 1.3 Let $0\leq\alpha\leq1$ and let f be both delta and nabla differentiable at $t\in\mathbb{T}_k^k$. Then f is diamond-α differentiable at t and $f^{\diamondsuit_\alpha}(t)=\alpha f^{\Delta}(t)+(1-\alpha)f^{\nabla}(t)$.

Definition 1.4 Let $a,b\in\mathbb{T}$, $a<b$, $f:\mathbb{T}\to\mathbb{R}$ and $\alpha\in[0,1]$. The diamond-α integral of t on $[a,b]_{\mathbb{T}}$ is defined by

$$\int_a^b f(t)\diamondsuit_\alpha t=\alpha\int_a^b f(t)\Delta t+(1-\alpha)\int_a^b f(t)\nabla t.$$

Theorem 1.5 *Let $f,g:\mathbb{T}\to\mathbb{R}$ be $\diamondsuit_\alpha$-differentiable at $t\in\mathbb{T}$. Then*
1. *$f+g$ is $\diamondsuit_\alpha$-differentiable $t\in\mathbb{T}$ with $(f+g)^{\diamondsuit_\alpha}=f^{\diamondsuit_\alpha}+g^{\diamondsuit_\alpha}$,*
2. *fg is $\diamondsuit_\alpha$-differentiable at $t\in\mathbb{T}$ with $(fg)^{\diamondsuit_\alpha}=f^{\diamondsuit_\alpha}g+\alpha f^{\sigma}g^{\Delta}+(1-\alpha)f^{\rho}g^{\nabla}$.*

Many authors have studied the theory of integral inequalities on time scales (see, for example, [1–10]). In [3], the following Opial inequality on time scales was established.

Theorem 1.6 ([3]) *For a delta differentiable $f:[0,h]\cap\mathbb{T}\to\mathbb{R}$ with $f(0)=0$, we have*

$$\int_0^h\left|\left(f+f^{\sigma}\right)f^{\Delta}\right|\Delta t\leq h\int_0^h\left|f^{\Delta}\right|^2\Delta t, \tag{1}$$

with equality when $f(t)=ct$.

In [1], the authors established the following theorem.

Theorem 1.7 ([1]) *Let $\omega(t)$ be positive and continuous on $(0,h)$ with $\int_0^h\omega^{1-q}\Delta t<\infty$, $q>1$. For a differentiable $f:[0,h]\to\mathbb{R}$ with $f(0)=0$, we have*

$$\int_0^h\left|\left(f+f^{\sigma}\right)f^{\Delta}\right|\Delta t\leq\left(\int_0^h\omega^{1-q}\Delta t\right)^{\frac{2}{q}}\left(\int_0^h\omega\left|f^{\Delta}\right|^p\Delta t\right)^{\frac{2}{p}},$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, and with equality when $f(t)=c\int_0^t\omega^{1-q}\Delta\tau$ for a constant c.

2 Main results

In this section, we present our results.

Theorem 2.1 *Let T be a time scale. For $\diamondsuit_\alpha$ differentiable $f:[0,h]\cap T\to R$, with $f(0)=0$ we have*

$$\int_0^h \left|f^k\right|^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t \le h^{k-1}\int_0^h \left|f^{\diamondsuit_\alpha}\right|^k(t)\diamondsuit_\alpha t. \tag{2}$$

Proof Starting with the left side of (2), we obtain

$$\begin{aligned}
\int_0^h \left|f^k\right|^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha(t) &= \int_0^h \left|f\cdot f^{k-1}\right|^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha(t)\\
&= \int_0^h \left|f^{k-1}f^{\diamondsuit_\alpha} + \alpha f^\sigma\left(f^{k-1}\right)^\Delta + (1-\alpha)f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\diamondsuit_\alpha(t)\\
&= \alpha\int_0^h \left|f^{k-1}f^{\diamondsuit_\alpha} + \alpha f^\sigma\left(f^{k-1}\right)^\Delta + (1-\alpha)f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\Delta t\\
&\quad + (1-\alpha)\int_0^h \left|f^{k-1}f^{\diamondsuit_\alpha} + \alpha f^\sigma\left(f^{k-1}\right)^\Delta + (1-\alpha)f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\nabla t\\
&\le \alpha\int_0^h \left|f^{k-1}f^{\diamondsuit_\alpha}\right|(t)\Delta t + \alpha^2\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\Delta t\\
&\quad + \alpha(1-\alpha)\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\Delta t + (1-\alpha)\int_0^h \left|f^{k-1}f^{\diamondsuit_\alpha}\right|(t)\nabla t\\
&\quad + \alpha(1-\alpha)\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\nabla t + (1-\alpha)^2\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\nabla t.
\end{aligned}$$

Using Definition 1.3, we get

$$\begin{aligned}
\int_0^h \left|f^k\right|^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha(t) &\le \alpha\int_0^h \left|\alpha f^{k-1}f^\Delta + (1-\alpha)f^{k-1}f^\nabla\right|(t)\Delta t\\
&\quad + \alpha^2\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\Delta t + \alpha(1-\alpha)\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\Delta t\\
&\quad + (1-\alpha)\int_0^h \left|\alpha f^{k-1}f^\Delta + (1-\alpha)f^{k-1}f^\nabla\right|(t)\nabla t\\
&\quad + \alpha(1-\alpha)\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\nabla t + (1-\alpha)^2\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\nabla t\\
&\le \alpha^2\int_0^h \left|f^{k-1}f^\Delta\right|(t)\Delta t + \alpha(1-\alpha)\int_0^h \left|f^{k-1}f^\nabla\right|(t)\Delta t\\
&\quad + \alpha^2\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\Delta t + \alpha(1-\alpha)\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\Delta t\\
&\quad + \alpha(1-\alpha)\int_0^h \left|f^{k-1}f^\Delta\right|(t)\nabla t + (1-\alpha)^2\int_0^h \left|f^{k-1}f^\nabla\right|(t)\nabla t\\
&\quad + \alpha(1-\alpha)\int_0^h \left|f^\sigma\left(f^{k-1}\right)^\Delta\right|(t)\nabla t + (1-\alpha)^2\int_0^h \left|f^\rho\left(f^{k-1}\right)^\nabla\right|(t)\nabla t.
\end{aligned}$$

We find that

$$\begin{aligned}
\int_0^h \left|f^\sigma \left(f^{k-1}\right)^\Delta\right|(t)\Delta t &= \int_0^h \left|f^\sigma \left(f\cdot f^{k-2}\right)^\Delta\right|(t)\Delta t \\
&= \int_0^h \left|f^\sigma \left(f^\Delta f^{k-2} + f^\sigma \left(f\cdot f^{k-3}\right)^\Delta\right)\right|(t)\Delta t \\
&\ \ \vdots \\
&= \int_0^h \left|f^\sigma \left(f^\Delta f^{k-2} + f^\sigma f^\Delta f^{k-3} + \cdots + \left(f^\sigma\right)^2 f^\Delta\right)\right|(t)\Delta t \\
&= \int_0^h \left|f^\sigma f^{k-2} + \left(f^\sigma\right)^2 f^{k-3} + \cdots + \left(f^\sigma\right)\right|\left|f^\Delta\right|(t)\Delta t \\
&= \int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\sigma\right)^{k-1-n}\right|\left|f^\Delta\right|(t)\Delta t.
\end{aligned}$$

Similarly,

$$\int_0^h \left|f^\rho \left(f^{k-1}\right)^\nabla\right|(t)\Delta t = \int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\rho\right)^{k-1-n}\right|\left|f^\nabla\right|(t)\nabla t.$$

Therefore,

$$\begin{aligned}
\int_0^h \left|f^k\right|^{\diamond_\alpha}(t)\diamond_\alpha t \le{}& \alpha^2 \int_0^h \left|f^{k-1} f^\Delta\right|(t)\Delta t + \alpha(1-\alpha)\int_0^h \left|f^{k-1} f^\nabla\right|(t)\Delta t \\
&+ \alpha^2 \int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\sigma\right)^{k-1-n}\right|\left|f^\Delta\right|(t)\Delta t \\
&+ \alpha(1-\alpha)\int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\rho\right)^{k-1-n}\right|\left|f^\nabla\right|(t)\Delta t \\
&+ \alpha(1-\alpha)\int_0^h \left|f^{k-1} f^\Delta\right|(t)\nabla t + (1-\alpha)^2 \int_0^h \left|f^{k-1} f^\nabla\right|(t)\nabla t \\
&+ \alpha(1-\alpha)\int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\sigma\right)^{k-1-n}\right|\left|f^\Delta\right|\nabla t \\
&+ (1-\alpha)^2 \int_0^h \left|\sum_{n=0}^{k-2} f^n \left(f^\rho\right)^{k-1-n}\right|\left|f^\nabla\right|(t)\nabla t \\
={}& \alpha^2 \int_0^h \left(\left|f^{k-1}\right| + \left|\sum_{n=0}^{k-2} f^n \left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\Delta t \\
&+ \alpha(1-\alpha)\int_0^h \left(\left|f^{k-1}\right| + \left|\sum_{n=0}^{k-2} f^n \left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\Delta t \\
&+ \alpha(1-\alpha)\int_0^h \left(\left|f^{k-1}\right| + \left|\sum_{n=0}^{k-2} f^n \left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\nabla t
\end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha)^2\int_0^h\left(\left|f^{k-1}\right|+\left|\sum_{n=0}^{k-2}f^n\left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\nabla t\\
&\le\alpha^2\int_0^h\left(\left|f^{k-1}\right|+\sum_{n=0}^{k-2}\left|f^n\left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\left|f^{k-1}\right|+\sum_{n=0}^{k-2}\left|f^n\left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\left|f^{k-1}\right|+\sum_{n=0}^{k-2}\left|f^n\left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\nabla t\\
&+(1-\alpha)^2\int_0^h\left(\left|f^{k-1}\right|+\sum_{n=0}^{k-2}\left|f^n\left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\nabla t\\
&=\alpha^2\int_0^h\left(\sum_{n=0}^{k-1}\left|f^n\right|\left|\left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\sum_{n=0}^{k-1}\left|f^n\right|\left|\left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\sum_{n=0}^{k-1}\left|f^n\right|\left|\left(f^\sigma\right)^{k-1-n}\right|\right)\left|f^\Delta(t)\right|\nabla t\\
&+(1-\alpha)^2\int_0^h\left(\sum_{n=0}^{k-1}\left|f^n\right|\left|\left(f^\rho\right)^{k-1-n}\right|\right)\left|f^\nabla(t)\right|\nabla t.
\end{aligned}
$$

Consider $g(t)=\int_0^t|f^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s$. Then we have $g^\Delta(t)=|f^\Delta(t)|$, $g^\nabla(t)=|f^\nabla(t)|$, and $|f|\le g$, so that $g(t)=\int_0^t|f^{\diamondsuit_\alpha}(s)|\diamondsuit_\alpha s\ge|\int_0^t f^{\diamondsuit_\alpha}(s)\diamondsuit_\alpha s|=|f(t)-f(0)|=|f(t)|$.

The above inequality becomes

$$
\begin{aligned}
\int_0^h\left|f^k\right|^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha&\le\alpha^2\int_0^h\left(\sum_{n=0}^{k-1}g^n\left(g^\sigma\right)^{k-1-n}\right)\left(g^\Delta\right)(t)\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\sum_{n=0}^{k-1}g^n\left(g^\rho\right)^{k-1-n}\right)\left(g^\nabla\right)(t)\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(\sum_{n=0}^{k-1}g^n\left(g^\sigma\right)^{k-1-n}\right)\left(g^\Delta\right)(t)\nabla t\\
&+(1-\alpha)^2\int_0^h\left(\sum_{n=0}^{k-1}g^n\left(g^\rho\right)^{k-1-n}\right)\left(g^\nabla\right)(t)\nabla t\\
&=\alpha^2\int_0^h\left(g^k\right)^\Delta(t)\Delta t+\alpha(1-\alpha)\int_0^h\left(g^k\right)^\nabla(t)\Delta t\\
&+\alpha(1-\alpha)\int_0^h\left(g^k\right)^\Delta(t)\nabla t+(1-\alpha)^2\int_0^h\left(g^k\right)^\nabla(t)\nabla t\\
&=\alpha\left[\alpha\int_0^h\left(g^k\right)^\Delta\Delta t+(1-\alpha)\int_0^h\left(g^k\right)^\Delta\nabla t\right]
\end{aligned}
$$

$$+ (1-\alpha)\left[\int_0^h \alpha\left(g^k\right)^{\nabla}\Delta t + (1-\alpha)\int_0^h \left(g^k\right)^{\nabla}\nabla t\right]$$

$$= \alpha\int_0^h \left(g^k\right)^{\Delta}\diamondsuit_\alpha + (1-\alpha)\int_0^h \left(g^k\right)^{\nabla}\diamondsuit_\alpha = \int_0^h \left(g^k\right)(t)^{\diamondsuit_\alpha}\diamondsuit_\alpha$$

$$= g^k(t)\big|_0^h = g^k(h) - g^k(0) = \left[g(h)\right]^k = \left[\int_0^h \left|f^{\diamondsuit_\alpha}(s)\right|\diamondsuit_\alpha s\right]^k.$$

By using Hölder's inequality with indices $p = \frac{k}{k-1}$ and $q = k$, we obtain

$$\left[\int_0^h 1\cdot\left|f^{\diamondsuit_\alpha}(s)\right|\diamondsuit_\alpha s\right]^k \le \left[\left(\int_0^h 1^{\frac{k}{k-1}}\diamondsuit_\alpha s\right)^{\frac{k-1}{k}}\left(\int_0^h \left|f^{\diamondsuit_\alpha}(s)\right|^k\diamondsuit_\alpha s\right)^{\frac{1}{k}}\right]^k$$

$$= \left(\int_0^h \diamondsuit_\alpha s\right)^{k-1}\left(\int_0^h \left|f^{\diamondsuit_\alpha}(s)\right|^k\diamondsuit_\alpha s\right)$$

$$= \left(s\big|_0^h\right)^{k-1}\int_0^h \left|f^{\diamondsuit_\alpha}(s)\right|^k\diamondsuit_\alpha s$$

$$= h^{k-1}\int_0^h \left|f^{\diamondsuit_\alpha}(s)\right|^k\diamondsuit_\alpha s,$$

hence the proof is complete. □

Theorem 2.2 *Let $\omega(t)$ be positive and continuous on $(0,h)$, with $\int_0^h \omega^{1-q}(t)\Delta t < \infty$, $q > 1$. For differentiable $f : [0,h] \to \mathbb{R}$ with $f(0) = 0$ we have*

$$\int_0^h \left|f^k\right|^{\Delta}\Delta t \le \left(\int_0^h \omega^{1-q}\Delta t\right)^{\frac{k}{q}}\left(\int_0^h \omega\left|f^{\Delta}\right|^p\Delta t\right)^{\frac{k}{q}}, \tag{3}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof We take $g(t) = \int_0^t |f^{\Delta}(s)|\Delta s$. Then $|f(t)| \le g(t)$, $g^{\Delta}(t) = |f^{\Delta}(t)|$, so we have

$$\int_0^h \left|f^k\right|^{\Delta}\Delta t = \int_0^h \left|\sum_{k=0}^{n-1} f^k\left(f^{\sigma}\right)^{n-1-k}\right|\left|f^{\Delta}\right|(t)\Delta t$$

$$\le \int_0^h \left(\sum_{k=0}^{n-1} g^k\left(g^{\sigma}\right)^{n-1-k}\right)\left(g^{\Delta}\right)(t)\Delta t = \int_0^h \left(g^k\right)^{\Delta}\Delta t$$

$$= g^k(h) - g^k(0) = g^k(h) = \left(\int_0^h \left|f^{\Delta}\right|(t)\Delta t\right)^k$$

$$= \left(\int_0^h \omega^{-\frac{1}{p}}\omega^{\frac{1}{p}}\left|f^{\Delta}\right|(t)\Delta t\right)^k$$

$$\le \left[\left(\int_0^h \left(\omega^{-\frac{1}{p}}\right)^q\Delta t\right)^{\frac{1}{q}}\left(\int_0^h \left(\omega^{\frac{1}{p}}\left|f^{\Delta}\right|\right)^p(t)\Delta t\right)^{\frac{1}{p}}\right]^k$$

$$= \left(\int_0^h \omega^{1-q}\Delta t\right)^{\frac{k}{q}}\left(\int_0^h \left(\omega\left|f^{\Delta}\right|\right)^p(t)\Delta t\right)^{\frac{k}{p}}.$$ □

Theorem 2.3 *Let $\omega(t)$ be positive and continuous on $(0,h)$, with $\int_0^h \omega^{1-q}(t)\nabla t < \infty$, $q > 1$. For differentiable $f : [0,h] \to \mathbb{R}$ with $f(0) = 0$ we have*

$$\int_0^h |f^k|^\nabla \nabla t \le \left(\int_0^h \omega^{1-q}\nabla t\right)^{\frac{k}{q}} \left(\int_0^h \omega|f^\nabla|^p \nabla t\right)^{\frac{k}{q}}, \tag{4}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.4 *Assume that $p > 1$, $q = \frac{p}{p-1}$, $\alpha \in [0,1]$, $h \in (0,\infty)_\mathbb{T}$, $\omega \in \mathbb{C}([0,h]_\mathbb{T},(0,\infty))$ and $f \in \mathbb{C}^1_{\diamond_\alpha}([0,h]_\mathbb{T},\mathbb{R})$. If $\alpha f^\Delta \ge 0$, $(1-\alpha)f^\nabla \ge 0$ and $f(0) = 0$ then*

$$\begin{aligned}
&\alpha^k \int_0^h \left|\left(f^k\right)^\Delta(t)\right|\Delta t + (1-\alpha)^k \int_0^h \left|\left(f^k\right)^\nabla(t)\right|\nabla t \\
&\quad \le \left(\int_0^h \omega^{1-q}(t)\diamond_\alpha t\right)^{\frac{k}{q}} \left(\int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right)^{\frac{k}{p}}.
\end{aligned} \tag{5}$$

Proof By Theorems 2.2, 2.3, Hölder's inequality and $k = \frac{k}{q} + (1+p)\frac{k}{p}$, we get

$$\begin{aligned}
&\alpha^k \int_0^h \left|\left(f^k\right)^\Delta(t)\right|\Delta t + (1-\alpha)^k \int_0^h \left|\left(f^k\right)^\nabla(t)\right|\nabla t \\
&= \alpha^{\frac{k}{q}+(1+p)\frac{k}{p}} \int_0^h \left|\left(f^k\right)^\Delta(t)\right|\Delta t + (1-\alpha)^{\frac{k}{q}+(1+p)\frac{k}{p}} \int_0^h \left|\left(f^k\right)^\nabla(t)\right|\nabla t \\
&\le \alpha^{\frac{k}{q}+(1+p)\frac{k}{p}} \left(\int_0^h \omega^{1-q}(t)\Delta t\right)^{\frac{k}{q}} \left(\int_0^h \omega(t)\left|f^\Delta(t)\right|^p \Delta t\right)^{\frac{k}{p}} \\
&\quad + (1-\alpha)^{\frac{k}{q}+(1+p)\frac{k}{p}} \left(\int_0^h \omega^{1-q}(t)\nabla t\right)^{\frac{k}{q}} \left(\int_0^h \omega(t)\left|f^\nabla(t)\right|^p \nabla t\right)^{\frac{k}{p}} \\
&\le \left(\alpha \int_0^h \omega^{1-q}(t)\Delta t\right)^{\frac{k}{q}} \left(\alpha \int_0^h \omega(t)\left|\alpha f^\Delta(t) + (1-\alpha)f^\nabla(t)\right|^p \Delta t\right)^{\frac{k}{p}} \\
&\quad + \left((1-\alpha) \int_0^h \omega^{1-q}(t)\nabla t\right)^{\frac{k}{q}} \\
&\quad \cdot \left((1-\alpha) \int_0^h \omega(t)\left|\alpha f^\Delta(t) + (1-\alpha)f^\nabla(t)\right|^p \nabla t\right)^{\frac{k}{p}} \\
&= \left(\alpha \int_0^h \omega^{1-q}(t)\Delta t\right)^{\frac{k}{q}} \left(\alpha \int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p \Delta t\right)^{\frac{k}{p}} \\
&\quad + \left((1-\alpha) \int_0^h \omega^{1-q}(t)\nabla t\right)^{\frac{k}{q}} \left((1-\alpha) \int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p \nabla t\right)^{\frac{k}{p}} \\
&\le \left[\left(\alpha \int_0^h \omega^{1-q}(t)\Delta t\right)^k + \left((1-\alpha) \int_0^h \omega^{1-q}(t)\nabla t\right)^k\right]^{\frac{1}{q}} \\
&\quad \cdot \left[\left(\alpha \int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p \Delta t\right)^k + \left((1-\alpha) \int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p \nabla t\right)^k\right]^{\frac{1}{q}} \\
&\le \left(\alpha \int_0^h \omega^{1-q}(t)\Delta t + (1-\alpha) \int_0^h \omega^{1-q}(t)\nabla t\right)^{\frac{k}{q}}
\end{aligned}$$

$$\cdot\left(\alpha\int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p\Delta t+(1-\alpha)\int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p\nabla t\right)^{\frac{k}{p}}$$

$$=\left(\int_0^h \omega^{1-q}(t)\diamond_\alpha t\right)^{\frac{k}{q}}\left(\int_0^h \omega(t)\left|f^{\diamond_\alpha}(t)\right|^p\diamond_\alpha t\right)^{\frac{k}{p}}. \qquad \square$$

Theorem 2.5 *Assume that* $1 < p \le 2$, $q = \frac{p}{p-1}$, $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$, $\omega \in \mathbb{C}([0,h]_{\mathbb{T}},(0,\infty))$ *and* $f \in \mathbb{C}^1_{\diamond_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$. *If* $\alpha f^{\Delta} \ge 0$, $(1-\alpha)f^{\nabla} \ge 0$ *and* $f(0) = 0$, *then*

$$\begin{aligned}
&\alpha^k\int_0^u \left|\left(f^k\right)^{\Delta}(t)\right|\Delta t+(1-\alpha)^k\int_0^u \left|\left(f^k\right)^{\nabla}(t)\right|\nabla t\\
&\quad\le \sum_{j=0}^{k-2}\alpha^j\binom{k}{j}\gamma^j\beta^{\frac{k-j}{q}}\left[\int_0^h \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p\diamond_\alpha t\right]^{\frac{k-1}{p}}\\
&\qquad+\alpha^{k-1}\binom{k}{k-1}\gamma^{k-1}\big(f(h)-f(0)\big),
\end{aligned}\tag{6}$$

where $\beta := \min_{u\in[0,h]_T}\nu(u)$, $\nu(u) = \max\{\int_0^u \omega^{1-q}(t)\diamond_\alpha t, \int_u^h \omega^{1-q}(t)\diamond_\alpha t\}$, $\gamma := \max\{|f(0)|, |f(h)|\}$.

Proof We let $u \in [0,h]_{\mathbb{T}}$ be arbitrary. By applying Theorem 2.4 to the function $g(t) = f(t) - f(0)$, we obtain

$$\begin{aligned}
&\alpha^k\int_0^u \left|\left(f^k\right)^{\Delta}(t)\right|\Delta t+(1-\alpha)^k\int_0^u \left|\left(f^k\right)^{\nabla}(t)\right|\nabla t\\
&\quad=\alpha^k\int_0^u\left|\sum_{j=0}^{k-1}\binom{k}{j}\left(g^{k-j}\right)^{\Delta}f^j(0)\right|\Delta t\\
&\qquad+(1-\alpha)^k\int_0^u\left|\sum_{j=0}^{k-1}\binom{k}{j}\left(g^{k-j}\right)^{\nabla}f^j(0)\right|\nabla t\\
&\quad\le\binom{k}{0}\left[\alpha^k\int_0^u\left|g^k\right|^{\Delta}\Delta t+(1-\alpha)^k\int_0^u\left|g^k\right|^{\nabla}\nabla t\right]\\
&\qquad+\alpha\binom{k}{1}\left|f(0)\right|\left[\alpha^{k-1}\int_0^u\left|g^{k-1}\right|^{\Delta}\Delta t+(1-\alpha)^{k-1}\int_0^u\left|g^{k-1}\right|^{\nabla}\nabla t\right]\\
&\qquad+\alpha^2\binom{k}{2}\left|f^2(0)\right|\left[\alpha^{k-2}\int_0^u\left|g^{k-2}\right|^{\Delta}\Delta t+(1-\alpha)^{k-2}\int_0^u\left|g^{k-2}\right|^{\nabla}\nabla t\right]\\
&\qquad\vdots\\
&\qquad+\alpha^{k-2}\binom{k}{k-2}\left|f^{k-2}(0)\right|\left[\alpha^2\int_0^u\left|g^2\right|^{\Delta}\Delta t+(1-\alpha)^2\int_0^u\left|g^2\right|^{\nabla}\nabla t\right]\\
&\qquad+\alpha^{k-1}\binom{k}{k-1}\left|f^{k-1}(0)\right|\left[\alpha\int_0^u|f|^{\Delta}\Delta t+(1-\alpha)\int_0^u|f|^{\nabla}\nabla t\right]\\
&\quad\le\binom{k}{0}\left[\int_0^u\omega^{1-q}(t)\diamond_\alpha t\right]^{\frac{k}{q}}\left[\int_0^u\omega(t)\left|g^{\diamond_\alpha}(t)\right|^p\diamond_\alpha t\right]^{\frac{k}{p}}
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha \binom{k}{1} \left|f(0)\right| \left[\int_0^u \omega^{1-q}(t) \diamond_\alpha t\right]^{\frac{k-1}{q}} \left[\int_0^u \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right]^{\frac{k-1}{p}} \\
&+ \alpha^2 \binom{k}{2} \left|f^2(0)\right| \left[\int_0^u \omega^{1-q}(t) \diamond_\alpha t\right]^{\frac{k-2}{q}} \left[\int_0^u \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right]^{\frac{k-2}{p}} \\
&\vdots \\
&+ \alpha^{k-2} \binom{k}{k-2} \left|f^{k-2}(0)\right| \left[\int_0^u \omega^{1-q}(t) \diamond_\alpha t\right]^{\frac{2}{q}} \\
&\cdot \left[\int_0^u \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right]^{\frac{2}{p}} \\
&+ \alpha^{k-1} \binom{k}{k-1} \left|f^{k-1}(0)\right| \alpha \int_0^u \left|f^{\Delta}(t)\right| \Delta t \\
&+ \alpha^{k-1} \binom{k}{k-1} \left|f^{k-1}(0)\right| (1-\alpha) \int_0^u \left|f^{\nabla}(t)\right| \nabla t \\
\leq{}& \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} \left|f^j(0)\right| \left[\int_0^u \omega^{1-q}(t) \diamond_\alpha t\right]^{\frac{k-j}{q}} \left[\int_0^u \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right]^{\frac{k-j}{p}} \\
&+ \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1}\bigl(f(u)-f(0)\bigr).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\alpha^k \int_u^h \left|\left(f^k\right)^{\Delta}(t)\right| \Delta t + (1-\alpha)^k \int_u^h \left|\left(f^k\right)^{\nabla}(t)\right| \nabla t \\
&\leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} \left|f^j(0)\right| \left[\int_u^h \omega^{1-q}(t) \diamond_\alpha t\right]^{\frac{k-j}{q}} \left[\int_u^h \omega(t)\left|g^{\diamond_\alpha}(t)\right|^p \diamond_\alpha t\right]^{\frac{k-j}{p}} \\
&\quad + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1}\bigl(f(h)-f(u)\bigr).
\end{aligned}
$$

Adding these two inequalities and taking into account that $a^r + b^r \leq (a+b)^r$ holds, for $a, b \geq 0$ and $r \geq 1$, yield the desired inequality. □

3 Conclusion

In this paper, we have obtained several Opial-type integral inequalities on time scales via the notion of the diamond-alpha derivative. These inequalities are related to the function f^k.

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Author's contributions

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