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Dynamic Optimization of a Certain Class of Nonlinear Systems.

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ABSTRACT

In the dynamic optimization of linear systems, it is often easier to represent the system by its transfer function and proceed in the minimization scheme after taking full advantage of the fact that the partial fraction expansion yields a diagonalized system. The elements of the system matrix then represent the poles and the output is expressed as a linear combination of all the state variables. The poles are assummed to be distinct, thus allowing the system to be synthesised and only equality constraints are considered. This method of solution for the linear problem is important in both the formulation of the nonlinear problem and in attempting to minimize a cost function for nonlinear feedback control systems with a nonlinearity in the feedback branch.

Although present numerical techniques are available to obtain an open loop control for nonlinear systems, an exact solution for a closed loop control law is practically impossible. Even for the more simple cases, the Hamilton Jacobi Equation is extremely complex and only approximate solutions have been obtained for the control and the trajectories.

In this study, a certain class of nonlinear systems is considered; one that yields an exact solution for the closed loop control $u^*(\chi,t)$ and limited studies for the open loop control and trajectories. A number of example problems are solved using the methods outlined. These examples show definitely that an exact solution may be obtained for the class of nonlinear systems in

consideration. Finally, the application of these methods of solution to nonlinear feedback systems provides an interesting study, particularly in the case of the nonlinear servomechanism problem.

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NOMENCLATURE

All the required notations and symbols used in this study are as follow:

V	=	Performance or cost function to be minimized.
χ	=	Trajectory variable, with a superscript * indicates optimal trajectory for a one dimensional system.
X	=	Trajectory vector or state vector, with a superscript * indicates optimal trajectory vector.
y	=	Output variable.
$Y(s)$	=	Laplace transform of the output variable.
u	=	Input function or control variable.
$U(s)$	=	Laplace transform of the control variable.
$X(t_o)$	=	Initial state vector.
$X(t_f)$	=	Final state vector.
t_f, t_o	=	Final and initial time.
λ	=	Lagrange multiplier, with a bar ($\bar{\lambda}$) indicates a vector.
H	=	Hamiltonian or Pontryagin Control Function.
$G(s)$	=	Transfer function.
s	=	Complex number.
s_i	=	Poles of the system in consideration.
V	=	Vandermonde matrix defined in terms of the poles of the system.
ϵ_i, α_i	=	Residues of the system in consideration, with a bar ($\bar{\epsilon}$, $\bar{\alpha}$) indicates vectors.

$f^{(i)}(t)$ = The i^{th} derivative of any function $f(t)$.

V^R = Real parts of the Vandermonde matrix.

V^I = Imaginary part of the Vandermonde matrix.

X^R = Real part of the state vector.

X^I = Imaginary part of the state vector.

$\bar{\lambda}^R$ = Real part of Lagrange Multiplier vector.

$\bar{\lambda}^I$ = Imaginary part of Lagrange Multiplier vector.

$F(K,\varphi)$ = Elliptic Integral of the first kind.

g_{n2} = Nonlinear network in feedback branch, a nonlinear function relating the feedback signal to the control effort.

$m(t)$ = Control effort.

$M(s)$ = Laplace transform of $m(t)$.

$d(t)$ = Disturbance function.

$D(s)$ = Laplace transform of $d(t)$.

$b(t)$ = Feedback signal.

$e(t)$ = Laplace transform of $b(t)$.

$E(s)$ = Laplace transform of $e(t)$.

$G_1(s)$ = Transfer function in feed forward branch.

$G_2(s)$ = Transfer function in feedback branch.

$\bar{r}$ = State vector representing the feedback system.

CHAPTER I

BASIC THEORETICAL PRELIMINARIES

Introduction

In the past, most of the procedures and principles used in the study of nonlinear systems have been concentrated on numerical techniques. For example, to get the optimal feedback law, the Hamilton Jacobi equation is num erically solved by assuming a series solution for the optimal return function, thus yielding an approximate solution for the optimal control. Consequently, the work that has been done in the field of nonlinear systems has been limited in nature; thus most attempts toward obtaining an exact solution have failed. However, in this work, an exact solution for the Hamilton Jacobi equation for a certain class of nonlinear systems will be established.

In feedback systems analysis, a nonlinear control system with a nonlinearity in the feedback branch will be considered. One of the primary objectives in this approach is to minimize a cost function using the "Pontryagin Principle" by treating the actuating signal as an input to a reduced linear system. At this point, most of the theories associated with linear systems are conveniently used. For instance, the cost function is minimized to yield the optimal variables. Once determined, these variables will be used to establish the optimal nonlinear control system.

1. Optimization Via Calculus of Variations

A. Fixed End Point Problem

Consider the problem of minimizing the functional

$$V(\chi) = \int_{t_o}^{t_f} \varphi\,(\chi(t)\;,\;\dot{\chi}(t),\,t)\;dt \tag{1}$$

with the following boundary conditions:

$$\chi(t_o) = \chi_o \tag{2a}$$

and

$$\chi(t_f) = \chi_f \tag{2b}$$

The function φ is assumed to be continuous and have continuous first and second derivatives with respect to all its arguments. By forming the variation of the functional (V) and by using the fundamental theorem of extremum, which states that if a differentiable function takes a minimum at a point, then its variation is zero at this point, one obtains the total variation of V as:

$$\Delta V = \int_{t_o}^{t_f} \varphi\,(\chi + \delta\chi,\,\dot{\chi} + \delta\dot{\chi},\,t)\;dt \;-\; \int_{t_o}^{t_f} \varphi\,(\chi,\,\dot{\chi},\,t)\;dt \tag{3}$$

Note that:

$$\delta\chi(t_o) = \delta\chi(t_f) = o \tag{4}$$

By expanding the integrands of equation (3) by means of a Taylor's series about χ and $\dot{\chi}$, equation (3) becomes:

$$\Delta V = \int_{X_o}^{X_f} \left[\frac{\delta\varphi}{\delta X} \delta X + \frac{\delta\varphi}{\delta \dot{X}} \delta \dot{X} + \frac{1}{2!} \frac{\delta^2\varphi}{\delta X^2} \delta X^2 + \frac{1}{2!} \frac{\delta^2\varphi}{\delta X \delta \dot{X}} \delta X \delta \dot{X} + \right.$$

$$\left. \frac{1}{2!} \frac{\delta^2\varphi}{\delta \dot{X}^2} \delta \dot{X}^2 + \cdots \right] dt \qquad (5)$$

By considering the first order terms only, the first variation of V may be written as:

$$\delta V = \int_{X_o}^{X_f} \left[\frac{\delta\varphi}{\delta X} (X, \dot{X}, t) \, \delta X + \frac{\delta\varphi}{\delta \dot{X}} (X, \dot{X}, t) \, \delta \dot{X} \right] dt \qquad (6)$$

Upon integration by parts the second term of this last equation, the following results:

$$\delta V = \int_{X_o}^{X_f} \left[\frac{\delta\varphi}{\delta X} (X, \dot{X}, t) - \frac{d}{dt} \frac{\delta\varphi}{\delta \dot{X}} (X, \dot{X}, t) \right] \delta X \, dt \qquad (7)$$

Finally, by using the necessary condition for minimum, $\delta V = o$, the following results:

$$\frac{\delta\varphi}{\delta X} (X, \dot{X}, t) - \frac{d}{dt} \left(\frac{\delta\varphi}{\delta X} (X, \dot{X}, t) \right) = o \qquad (8)$$

Equation (8) is referred to as Euler's equation. It represents a necessary condition for an extremum although in general not sufficient.

B. Variable End Point Problem

The variable end point problem is again formulated by the expression:

$$V(\chi) = \int_{t_0}^{t_1} \varphi(\chi, \dot{\chi}, t)\, dt \tag{9}$$

It is obvious that, a function χ that will minimize V for a problem with variable end points, must minimize V for the more restricted case of fixed end points. It should be noted that for the fixed end point problem the two undetermined constants are found by using the two boundary conditions, $\chi(t_0) = \chi_0$ and $\chi(t_f) = \chi_f$. For the case of variable end point problem, additional conditions need to be determined to evaluate the arbitrary constants. Without any loss in generality, assume that the initial time t_0 and initial state χ_0 are fixed, while the end point varies from (t_1, χ_1) to $(t_1 + \delta t_1, \chi_1 + \delta\chi_1)$. Figure (I) illustrates the variable end point problem.

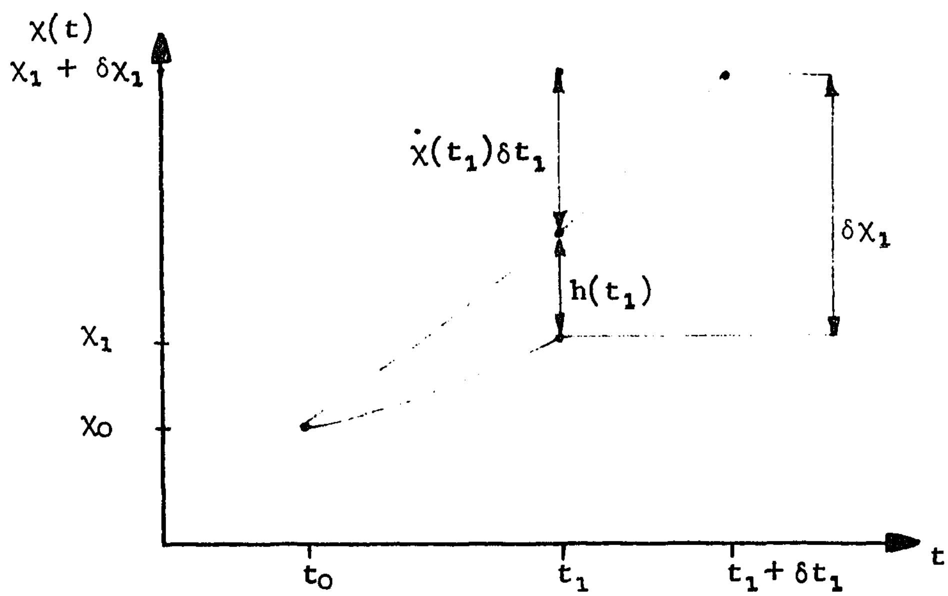


Figure 1. Diagram Illustrating Variable End Point Problem

The total variation in V is given by:

$$\Delta V = \int_{t_0}^{t_1+\delta t_1} \varphi(\chi + h, \dot{\chi} + \dot{h}, t)\, dt - \int_{t_0}^{t_1} \varphi(\chi, \dot{\chi}, t)\, dt \tag{10}$$

or:

$$\Delta V = \int_{t_0}^{t_1} \left[\varphi(\chi + h, \dot{\chi} + \dot{h}, t) - \varphi(\chi, \dot{\chi}, t)\right] dt + \int_{t_1}^{t_1+\delta t_1} \varphi(\chi + h, \dot{\chi} + \dot{h}, t)\, dt \tag{11}$$

The mean value theorem allows us to replace the second integral by:

$$\int_{t_1}^{t_1+\delta t_1} \varphi(\chi + h, \dot{\chi} + \dot{h}, t)\, dt = \varphi(\chi, \dot{\chi}, t)\Big|_{t_1} \delta t_1 + \text{H. O. T} \tag{12}$$

and the expansion into a Taylor's series of the integrand of the first integral yields:

$$\int_{t_0}^{t_1} \left[\varphi(\chi + h, \dot{\chi} + \dot{h}, t) - \varphi(\chi, \dot{\chi}, t)\right] dt =$$

$$\int_{t_0}^{t_1} \left[\frac{\delta\varphi}{\delta\chi}(\chi, \dot{\chi}, t).h + \frac{\delta\varphi}{\delta\dot{\chi}}(\chi, \dot{\chi}, t).\dot{h}\right] dt + \text{H. O. T} \tag{13}$$

Integration by parts of the second term of equation (13) gives:

$$\int_{t_0}^{t_1}\left[\varphi\,(\chi+h\ ,\ \dot{\chi}+\dot{h}\ ,\ t)-\varphi\,(\chi,\ \dot{\chi},\ t)\right]dt\ =$$

$$\int_{t_0}^{t_1}\left[\frac{\delta\varphi}{\delta\chi}\,(\chi,\ \dot{\chi},\ t)-\frac{d}{dt}\,\frac{\delta\varphi}{\delta\chi}\,(\chi,\ \dot{\chi},\ t)\right].hdt\ +\ \frac{\delta\varphi}{\delta\dot{\chi}}\,(\chi,\ \dot{\chi},\ t).h\ \Bigg|_{t_0}^{t_1} \tag{14}$$

Now, the mere assumption that χ is an extremal implies that the integral in equation (14) is zero, thus:

$$\Delta V\ =\ \varphi\,(\chi,\ \dot{\chi},\ t)\Big|_{t_1}\ \delta t_1\ +\ \frac{\delta\varphi}{\delta\dot{\chi}}\,(\chi,\ \dot{\chi},\ t).h\ \Bigg|_{t_0}^{t_1} \tag{15}$$

Note that:

$$h(t_1)\ =\ \delta\chi_1-\dot{\chi}(t_1).\delta t_1 \tag{16}$$

Thus, by neglecting higher order terms, the first variation is:

$$\delta V=\frac{\delta\varphi}{\delta\dot{\chi}}\,(\chi,\ \dot{\chi},\ t)\Big|_{t_1}\ \delta\chi_1+\left[\varphi\,(\chi,\ \dot{\chi},\ t)-\dot{\chi}\ \frac{\delta\varphi}{\delta\dot{\chi}}\,(\chi,\ \dot{\chi},\ t)\right]\Big|\ \delta t_1 \tag{17}$$

Upon application of the fundamental condition for a minimum, we get:

$$\frac{\delta\varphi}{\delta\dot{\chi}}\ (\chi,\ \dot{\chi},\ t)\Big|_{t_1}.\delta\chi_1\ +\ \left[\varphi(\chi,\ \dot{\chi},\ t)-\dot{\chi}\ \frac{\delta\varphi}{\delta\dot{\chi}}\,(\chi,\ \dot{\chi},\ t)\right]\Big|\ \delta t_1=0 \tag{18}$$

The same is true if the initial point is free, thus:

$$\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\bigg|_{t_0} \cdot \delta X_0 + \left[\varphi(X, \dot{X}, t) - \dot{X}\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\right]\bigg|_{t_0} \cdot \delta t_0 = 0 \tag{19}$$

Note that if the final time t_1 is fixed ($\delta t_1 = o$), equation (18) reduces to:

$$\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\bigg|_{t_1} = 0 \tag{20}$$

On the other hand, if the final state is fixed ($\delta X_1 = 0$), then:

$$\left[\varphi(X, \dot{X}, t) - \dot{X}\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\right]\bigg|_{t_1} = 0 \tag{21}$$

This set of equations represents the transversality conditions. These conditions complement the other known conditions to completely determine the solution to the optimization problem.

To generalize the above results for a multi-dimensional problem, we consider the n vector **X**, where

$$X = [X_1\ X_2\ \cdots\ X_n]^T \tag{22}$$

The general transversality conditions may be written in matrix form as:

$$\delta X^T\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\bigg|_{t_1} + \left[\varphi(X, \dot{X}, t) - \dot{X}^T\frac{\delta\varphi}{\delta\dot{X}}(X, \dot{X}, t)\right]\bigg|_{t_1}\delta t_1 = 0 \tag{23}$$

or in scalar form as:

$$\sum_{i=1}^{n} \frac{\delta\varphi}{\delta\dot{X}_i} (X, \dot{X}, t).\delta\chi_i \Bigg|_{t_1} + \left[\varphi(X, \dot{X}, t) - \sum_{i=1}^{n} \dot{X}_i \frac{\delta\varphi}{\delta\dot{X}_i} (X, \dot{X}, t)\right]\Bigg|_{t_1} .\delta t_1 = 0 \tag{24}$$

Note that:

$$\frac{\delta\varphi}{\delta\dot{X}} = \begin{bmatrix} \frac{\delta\varphi}{\delta\dot{X}_1} & \frac{\delta\varphi}{\delta\dot{X}_2} & \cdots & \frac{\delta\varphi}{\delta\dot{X}_n} \end{bmatrix}^T \tag{25}$$

and Euler's equation becomes

$$\frac{\delta\varphi}{\delta X} (X, \dot{X}, t) - \frac{d}{dt} \frac{\delta\varphi}{\delta\dot{X}} (X, \dot{X}, t) = 0 \tag{26}$$

This section then essentially sets the governing equations for the fixed and variable end point problem to establish the optimal control and trajectories.

2. Differential Constraints Considerations: Pontryagin Principle

In the preceeding section no considerations were made for differential constraints. In this section, the Euler's equations will be modified to yield the Pontryagin principle. We consider the functional

$$V(X) = \int_{t_0}^{t_f} \varphi(X, U, t)\, dt \tag{27}$$

subject to the constraints:

$$\dot{X} = F(X, U, t) \tag{28}$$

where:

$$U = [u_1\ u_2\ \cdots\ u_r]^T \tag{29a}$$

and

$$F = [f_1\ f_2\ \cdots\ f_n]^T \tag{29b}$$

By defining an augmented functional:

$$V'(X, U) = \int_{t_0}^{t_f} \varphi'(X, \dot{X}, U, t)\, dt \tag{30}$$

where:

$$\varphi' = \varphi(X, U, t) + \sum_{i=1}^{n} \lambda_i(t)\left[f_i(X, U, t) - \dot{X}_i\right] \tag{31}$$

and by using Euler's equation, we get:

$$\frac{\delta}{\delta X_i}\left\{\varphi(X, U, t) + \sum_{i=1}^{n} \lambda_i(f_i(X, U, t) - \dot{X}_i)\right\} - \frac{d}{dt}\frac{\delta}{\delta \dot{X}_i}\left\{\varphi(X, U, t) + \sum_{i=1}^{n} \lambda_i(f_i(X, U, t) - \dot{X}_i)\right\} = 0 \tag{32}$$

Equation (32) may be simplified to yield:

$$\frac{\delta}{\delta X_i}\left[\varphi(X, U, t) + \sum_{i=1}^{n} \lambda_i f_i(X, U, t)\right] - \frac{d}{dt}\frac{\delta}{\delta \dot{X}_i}(-\lambda_i \dot{X}_i) = 0 \tag{33}$$

or:

$$\dot{\lambda}_i = - \frac{\delta}{\delta X_i} \left[\varphi\,(X,\ U,\ t) + \sum_{i=1}^{n} \lambda_i f_i\,(X,\ U,\ t) \right] \tag{34}$$

Next, we can use Euler's equation in terms of U, thus:

$$\frac{\delta}{\delta u_j} \left[\varphi(X,\ U,\ t) + \sum_{i=1}^{n} \lambda_i f_i\,(X,\ U,\ t) \right] = 0\ ;\ j = 1,\ \cdots,\ r \tag{35}$$

If we define a function H called the state function of Pontryagin or Hamiltonian as:

$$H\,(X,\ U,\ \bar{\lambda},\ t) = \varphi\,(X,\ U,\ t) + \sum_{i=1}^{n} \lambda_i f_i\,(X,\ U,\ t) \tag{36}$$

equation (34) may be written as:

$$\dot{\lambda}_i = - \frac{\delta H}{\delta X_i}\,(X,\ U,\ \bar{\lambda},\ t)\ ;\ i = 1,\ \cdots,\ n \tag{37}$$

and (35) as:

$$\frac{\delta H}{\delta u_j} = 0 \qquad ;\ j = 1,\ \cdots,\ r \tag{38}$$

Note that:

$$\bar{\lambda} = [\lambda_1\ \lambda_2\ \cdots\ \lambda_n]^T \tag{39}$$

In general, these equations are complicated to solve for high order systems because of the tedious nature of two point boundary value problems. Later, in succeeding chapters, these equations will be solved by a proper selection of the state variables, and a method will be developed to dispense of any backward integration that is necessary to solve the problem.

3. General Optimal Control Problem

We will assume in this section that the performance function V is given by:

$$V = \int_{t_o}^{t_f} \varphi[X(\zeta), U(\zeta), \zeta] \, d\zeta + g\,[X(t_f), t_f] \tag{40}$$

which may also be written as:

$$V = \int_{t_o}^{t_f} \left\{ \varphi[X(\zeta), U(\zeta), \zeta] \, d\zeta + \left[\frac{\delta g}{\delta X}(X, \zeta)\right] . \dot{X} + \frac{\delta g}{\delta \zeta}(X, \zeta) \right\} d\zeta \tag{41}$$

By augmenting the performance function V and by adding the Lagrange multiplier's contribution, the following results:

$$V'(X, U) = \int_{t_o}^{t_f} \varphi'(X, \dot{X}, U, \dot{U}, \zeta) \, d\zeta \tag{42}$$

where;

$$\varphi' = \varphi(X, U, t) + \left[\frac{\delta g}{\delta X}(X, t)\right]^T . \dot{X} + \frac{\delta g}{\delta t}(X, t) + \bar{\lambda}^T F(X, U, t) - \bar{\lambda} \dot{X} \tag{43}$$

Next, the Calculus of Variation is used and the Euler's equation applied to X and U yields:

$$\frac{\delta}{\delta X} \left\{ \varphi(X, U, t) + \left[\frac{\delta g}{\delta X}(X, t)\right]^T . \dot{X} + \frac{\delta g}{\delta t}(X,t) + \bar{\lambda}^T F(X, U, t) \right\} - \frac{d}{dt}\left[\frac{\delta g}{\delta X}(X, t) - \bar{\lambda}\right] = 0 \tag{44}$$

Note that all zero terms are already removed (i.e., after differentiation).

If we assume that g(X, t) has continuous second partial derivatives with respect to X and t, equation (44) is simplified to yield:

$$\dot{\bar{\lambda}} = -\frac{\delta}{\delta X}\left[\varphi(X, U, t) + \bar{\lambda}^T F(X, U, t)\right] \tag{45}$$

This is an identical result to the one obtained in equation (34). Similarly, the Euler's equations in terms of U also reduces to equation (35), thus

$$\frac{\delta}{\delta U}\left[\varphi(X, U, t) + \bar{\lambda}^T F(X, U, t)\right] = 0 \tag{46}$$

Equation (23) or (24) is next used to establish the generalized remaining boundary conditions. For instance,

$$\left[\frac{\delta\varphi'}{\delta\dot{X}}(X, \dot{X}, U, \dot{U}, t)\right]^T \cdot dX \Bigg|_{t_f} + \left[\varphi'(X, \dot{X}, U, \dot{U}, t) - \left[\frac{\delta\varphi'}{\delta\dot{X}}(X, \dot{X}, U, \dot{U}, t)\right]^T \cdot\dot{X}\right] \cdot dt \Bigg|_{t_f} = 0 \tag{47}$$

Upon substitution of $\frac{\delta\varphi'}{\delta\dot{X}}(X, \dot{X}, U, \dot{U}, t)$, equation (47) becomes:

$$\left[\frac{\delta g}{\delta X}(X, t) - \bar{\lambda}\right]^T \cdot dX \Bigg|_{t_f} + \left\{\varphi(X, U, t) + \left[\frac{\delta g}{\delta X}(X, t)\right]^T \cdot\dot{X} + \frac{\delta g}{\delta t}(X, t) + \bar{\lambda}^T F(X, U, t) - \bar{\lambda}^T \dot{X} - \left[\frac{\delta g}{\delta X}(X, t) - \bar{\lambda}\right]^T \cdot\dot{X}\right\} dt \Bigg|_{t_f} = 0 \tag{48}$$

By introducing the Hamiltonian function, equation (48) takes the form:

$$\left[\frac{\delta g}{\delta X}(X,\ t)\ -\bar{\lambda}\right]^T .dX \Bigg|_{t_f} + \left[H^*(X,\ \bar{\lambda},\ t) + \frac{\delta g}{\delta t}(X,\ t)\right] .dt \Bigg|_{t_f} = 0 \tag{49}$$

where the * indicates that the Hamiltonian is evaluated at the optimal conditions.

Note that for t_f fixed and final state unspecified, $dt|_{t_f} = 0$, and equation (49) reduces to:

$$\left(\frac{\delta g}{\delta X}(X,\ t) - \bar{\lambda}\right)\Bigg|_{t_f} = 0 \tag{50}$$

On the other hand, if the final state is fixed and final time free (i.e., $dX|_{t_f} = 0$) , then:

$$H^*\ (X,\ \bar{\lambda},\ t)\ +\ \frac{\delta g}{\delta t}(X,\ t)\ \Bigg|_{t_f} = 0 \tag{51}$$

Of course, when both final state and final time are free, then both (dX) and dt) are arbitrary and equation (49) must be used.

As it was shown, Euler's equation and the state function of Pontryagin are used to determine the optimal control function $U^*(t)$ that will minimize the performance index V. It should be noted that, finding a control vector $U^*(t)$ depends on the initial conditions. In general, because of errors or disturbances, the system will not stay on the nominal open loop trajectory $X^*(t)$ but will be in some neighborhood δX about it. Then, it is necessary to determine the control perturbation $\delta U^*(t)$ to be added to $U^*(t)$ so as to reach the desired terminal state in an optimal manner. This determination of the control law $U^*(X, t)$ is independent of any initial state and will be more useful for practical feedback systems. In the next chapter, dynamic programming and the Hamilton Jacobi equation will be discussed since it covers a method of determining the control law $U^*(X, t)$.

CHAPTER II

OPTIMIZATION VIA DYNAMIC PROGRAMMING

1. The Hamilton Jacobi Equation

In the first chapter, the necessary tools to solve an optimization problem were presented. These tools allow one to determine the optimal open loop control function as a function of time, and the optimal trajectories for some specified initial conditions. In this chapter, a closed loop control will be found by using the dynamic programming approach.

Consider the system

$$\dot{X} = F(X, U, t) \tag{1}$$

for which we wish to minimize the performance index

$$V = g(X_f, t_f) + \int_{t_0}^{t_f} \varphi[X(\zeta), U(\zeta), \zeta] \, d\zeta \tag{2}$$

subject to the differential constraints given in equation (1) where

$$F = [f_1 \ f_2 \ \cdots \ f_n]^T \tag{3a}$$

$$X = [x_1 \ x_2 \ \cdots \ x_n]^T \tag{3b}$$

$$U = [u_1 \ u_2 \ \cdots \ u_r]^T \tag{3c}$$

The initial conditions t_o and $\chi_i(t_o)$ $(i=1,\cdots,n)$ are specified. The final conditions are specified the following way.

$$\chi_i(t_f) \qquad i=1,\cdots,p \le n \qquad \text{specified, while}$$

t_f may or may not be specified. The problem may be formulated as

$$V[X(t),t] = \min_{(U)} \left\{ g[X(t_f),t_f] + \int_t^{t_f} \varphi[X(\zeta),U(\zeta),\zeta]\, d\zeta \right\} \tag{4}$$

where $V[X(t),t]$ is the minimum performance index or optimal return function. The symbol (U) indicates that the minimization is to be carried out over the range of admissible controls in the interval $[t_o,t_f]$. Rewriting the integral in equation (4) as the sum of two integrals, one gets

$$V[X(t),t] = \min \left\{ g[X(t_f),t_f] + \int_t^{t+\Delta t} \varphi[X(\zeta),U(\zeta),\zeta]\, d\zeta + \int_{t+\Delta t}^{t_f} \varphi[X(\zeta),U(\zeta),\zeta]\, d\zeta \right\} \tag{5}$$

Because of the small range of the first integral, equation (5) may be rewritten to yield:

$$V[X(t),t] = \min \left\{ \varphi[X(t),U(t),t]\,\Delta t + O(\Delta t^2) + g[X(t_f),\ t_f] + \int_{t+\Delta t}^{t_f} \varphi[X(\zeta),U(\zeta),\zeta]\, d\zeta \right\} \tag{6}$$

Equation (1) is written as

$$\Delta X = F[X(t), U(t), t]\ \Delta t + O(\Delta t^2) \tag{7}$$

By using the "principle of optimality", which states that, whatever the control decision U(t) at time t, the control policy over the remaining interval must be optimal if the policy over the entire interval is to be optimal. Thus, by starting at the state $(X(t) + \Delta X)$ at time $(t+\Delta t)$, the return from the optimal policy is obtained after substitution of ΔX given in equation (7) in equation (6). Hence,

$$V[X(t),t] = \min_{U(t)} \Big\{ \varphi[X(t),U(t),t] \cdot \Delta t + O(\Delta t^2) + V[X(t) + F[X(t),U(t),t] \cdot \Delta t + O(\Delta t^2),\ t+\Delta t] \Big\} \tag{8}$$

Note that in equation (8), the minimization is performed over the control vector U(t) to be applied at time t rather than, as in equation (6), over the control in the interval t to t_f.

Next, we expand $V[X(t) + \Delta X,\ t+\Delta t]$ about X and t to get

$$V[X(t),t] = \min_{U(t)} \Bigg\{ \varphi[X(t),U(t),t] \cdot \Delta t + V[X(t),t]$$

$$+ \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot f_i[X(t),U(t),t] \cdot \Delta t$$

$$+ \frac{\delta V}{\delta t}[X(t),t] \cdot \Delta t + O(\Delta t^2) \Bigg\} \tag{9}$$

Note that both $V[X(t),t]$ and $\frac{\delta V}{\delta t}[X(t),t]$ are not functions of $U(t)$, thus they may be taken out from under the minimization on the right side of equation (9). Upon dividing by Δt and taking the limit as $\Delta t \to 0$, the following is obtained

$$0 = \frac{\delta V}{\delta t}[X(t),t] + \min_{U(t)} \Bigg\{ \varphi[X(t),U(t),t] + \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot f_i[X(t),U(t),t] \Bigg\} \tag{10}$$

By realizing that the minimization of the brackedted term in equation (10) requires that its partial derivatives with respect to each of the control variables vanish, one can remove the minimization notation and replace equation (10) by this set of equations.

$$0 = \frac{\delta V}{\delta t}[X(t),t] + \varphi[X(t),t] + \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot f_i[X(t),U(t),t] \tag{11}$$

$$0 = \frac{\delta \varphi}{\delta u_j}[X(t),U(t),t] + \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot \frac{\delta f_i}{\delta u_j}[X(t),U(t),t]$$

$$\text{For } j = 1,\cdots, r \tag{12}$$

Obviously, the boundary condition to be satisfied is

$$V[X(t_f),t_f] = g[X(t_f),t_f] \tag{13}$$

Next, we will derive the Hamilton Jacobi Equation the following way.

Let $\tilde{R}$ be the admissible region of the control space. The minimization over U must be carried out for U within $\tilde{R}$. If we rewrite equation (10) with the boundary conditions associated with it, we get:

$$0 = \frac{\delta V}{\delta t}[X(t),t] + \min_{U(t)} \left\{ \varphi[X(t),U(t),t] + \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot f_i[X(t), U(t),t] \right\} \tag{14}$$

with

$$V[X(t_f),t_f] = g[X(t_f),t_f] \tag{15}$$

By defining a new function $\overline{H}$ given by

$$\overline{H}[X(t),U(t),\frac{\delta V}{\delta X}[X(t),t],t] = \varphi[X(t),U(t),t] + \sum_{i=1}^{n} \frac{\delta V}{\delta X_i}[X(t),t] \cdot f_i[X(t),U(t),t] \tag{16}$$

Then, equation (14) becomes

$$\frac{\delta V}{\delta t}[X(t),t] + \min_{U(t)} \overline{H}[X(t),U(t), \frac{\delta V}{\delta X}, t] = 0 \tag{17}$$

Now if U^* is the optimal control vector within $\tilde{R}$, then we can write

$$U^* = U^* [\frac{\delta V}{\delta X}, X, t] \tag{18}$$

and by defining

$$\overline{H}^* [\frac{\delta V}{\delta X}, x, t] = \overline{H} [\frac{\delta V}{\delta X}, X, U, t] \Big|_{U=U^*} \tag{19}$$

where $\overline{H}^*$ is the minimum value of the Hamiltonian with respect to U within $\tilde{R}$. Equation (17) now takes the form

$$\frac{\delta V}{\delta t}[X(t), t] + \overline{H}^* [\frac{\delta V}{\delta X}, X, t] = 0 \tag{20}$$

Equation (20) is the "Hamilton Jacobi Equation" for the control problem. Its solution yields $V[X,t]$ after using the boundary condition given in equation (15). The optimal control vector $U^*(X,t)$ is obtained using equation (18). It represents the control law or closed loop optimal solution to the control problem. In practice, the analytic solution of the "Hamilton Jacobi Equation" is complex and virtually impossible. That is why the optimal control law $U^*(X,t)$ is determined by computational methods. Nevertheless, in succeeding chapters, a method will be devised to give the exact solution to the "Hamilton Jacobi Equation" for certain classes of nonlinear systems. Also, it should be noted that when the final time is extended to infinity ($t_f \to \infty$), the dependence of V on time is eliminated ($\frac{\delta V}{\delta t} = 0$), thus reducing the equation to

$$\bar{H}^*(\frac{\delta V}{\delta X}, X) = 0 \tag{21}$$

2. Linear Systems With Quadratic Performance (Ricatti Equation)

Consider the problem of minimizing the performance index

$$V = \frac{1}{2} X^T(t_f)\, \bar{g}\, X(t_f) + \frac{1}{2}\int_{t_o}^{t_f} (X^T Q_1 X + 2X^T Q_2 U + U^T P U)\,dt \tag{22}$$

where

$\bar{g}$ = nxn symmetric matrix of known constants.
Q_1 = nxn symmetric matrix of known function of time.
Q_2 = nxr matrix of known function of time.
P = rxr symmetric matrix of known function of time. Also a positive definite matrix.

The differential constraint is represented by

$$\dot{X} = A(t) \cdot X + B(t) \cdot U \tag{23}$$

where

X = n vector of state variables.
U = r vector of control variables.
A(t) = nxn matrix of known function of time.
B(t) = nxr matrix of known function of time.

The system is to be transferred from the known initial state $(t_o, X(t_o))$, to the final manifold

$$\bar{\Psi} \cdot [X(t_f)] = \Psi_o \quad (t_f \text{ specified}) \tag{24}$$

where

$\bar{\Psi}$ = rxn matrix of known constants.

Ψ_o = rxl matrix of known constants.

Upon constructing the Hamiltonian and using the necessary conditions for minimum, one gets

$$\dot{\bar{\lambda}} = -A^T\bar{\lambda} + Q_1 X + Q_2 U \tag{25a}$$

$$0 = B^T\bar{\lambda} - Q_2{}^T X - PU \tag{25b}$$

where $\bar{\lambda}$ is the n Lagrange multiplier vector or costate vector. The transversality condition is written as

$$\bar{\lambda}(t_f) = -[\bar{g}\, X(t_f) + \bar{\Psi}^T\, \bar{\mu}] \tag{26}$$

$\bar{\mu}$ Is a constant multiplier vector (pxl matrix). Solving for the control U in terms of X and $\bar{\lambda}$ yields

$$U = P^{-1}(B^T\,\bar{\lambda} - Q_2{}^T X) \tag{27}$$

Note that since P is positive definite, then its inverse always exists. Upon substituting U in equation (23) and (25a), the following results

$$\dot{X} = (A - BP^{-1}Q_2^T)\ X + BP^{-1}B^T\ \bar{\lambda} \tag{28}$$

$$\dot{\bar{\lambda}} = (Q_1 - Q_2P^{-1}Q_2^T)\ X - (A^T - Q_2P^{-1}B^T)\ \bar{\lambda} \tag{29}$$

Because of the nature of equation (26), we assume that

$$\bar{\lambda}(t) = -\ [A_o(t)\ X(t) + R(t)\ \bar{\mu}] \tag{30}$$

where

$A_o(t)$ = nxn matrix of function of time to be determined.

$R(t)$ = nxp matrix of function of time to be determined.

The terminal conditions to be satisfied are

$$A_o(t_f) = \bar{g} \tag{31a}$$

$$R(t_f) = \bar{\Psi}^T \tag{31b}$$

Upon differentiating equation (30) and substituting $\dot{X}$ and $\dot{\bar{\lambda}}$ by their values in equations (28) and (29) and with the use of the value of $\bar{\lambda}$ given in equation (30), the following is obtained

$$[\dot{A}_O + A_O(A - BP^{-1}Q_2^T) + (Q_1 - Q_2P^{-1}Q_2^T) - A_O BP^{-1}B^TA_O$$

$$+ (A^T - Q_2P^{-1}B^T) A_O] \; X + [\dot{R} - A_OBP^{-1}B^TR$$

$$+ (A^T - Q_2P^{-1}B^T)R] \; \bar{\mu} \;\; = \;\; O \tag{32}$$

For this last equation to be satisfied it is sufficient that

$$\dot{A}_O = - A_OA - A^TA_O - Q_1 + (A_OB + Q_2)P^{-1}(Q_2^T + B^TA_O) \tag{33a}$$

$$R = (A_OBP^{-1}B^T - A^T + Q_2P^{-1}B^T) \; R \tag{33b}$$

Equation (33a) is known to be the matrix Ricatti equation. Its solution gives the nxn matrix $A_O(t)$. Once $A_O(t)$ is determined, $R(t)$ is found by integrating equation (33b). Still to be determined is the constant multiplier vector $(\bar{\mu})$. By substituting the value of $\bar{\lambda}$ given in equation (30) in equation (28), one may write

$$\dot{X} = C_1(t) \; X + C_2(t) \; \bar{\mu} \tag{34}$$

Consequently, it can be shown upon integrating equation (34) by using the method of variation of parameters, that the final state may be written as:

$$X(t_f) = D_1(t) \; X(t) + D_2(t) \; \bar{\mu} \tag{35}$$

Next, equation (35) is multiplied by $\overline{\Psi}$ to yield

$$\Psi_o = \overline{\Psi}\, D_1\, X + \overline{\Psi}\, D_2\, \overline{\mu} \tag{36}$$

If we let

$$S_1(t) = \overline{\Psi}\, D_1 \tag{37a}$$

$$S_2(t) = \overline{\Psi}\, D_2 \tag{37b}$$

the following results

$$\Psi_o = S_1\, X + S_2\, \overline{\mu} \tag{38}$$

with the boundary conditions

$$S_1\,(t_f) = \overline{\Psi} \tag{39a}$$

$$S_2\,(t_f) = 0 \tag{39b}$$

Equation (38) allows to solve for $\overline{\mu}$ as

$$\overline{\mu} = S_2^{-1}\, \Psi_o - S_2^{-1} S_1\, X \tag{40}$$

and equation (30) becomes

$$\overline{\lambda} = -\,[\, A_o - RS_2^{-1} S_1\,]\, X - RS_2^{-1} \Psi_o \tag{41}$$

Note that all variables in this equation are functions of time except Ψ_o which is a constant. Equation (27) is used to eliminate $\overline{\lambda}$ to yield an expression for the optimal control vector as

$$U^*(X, t) = -P^{-1}B^T R S_2^{-1} \Psi_o - P^{-1} \left\{ Q_2^T + B^T [A_0 - R S_2^{-1} S_1] \right\} X \tag{42}$$

$A_0(t)$ and $R(t)$ are already found as solutions of equations (33a,b), still remains to be determined are S_1 and S_2. Upon differentiating equation (38) one gets

$$\dot{S}_1 X + S_1 \dot{X} + \dot{S}_2 \overline{\mu} = 0 \tag{43}$$

Equation (28) is used to eliminate $\dot{X}(t)$ from equation (43). Then $\lambda(t)$ is replaced by its value in terms of X and $\overline{\mu}$ through use of equation (30). The result is

$$[\dot{S}_1 + S_1(A - BP^{-1}Q_2^T) - S_1 BP^{-1}B^T A_0] X$$

$$+ [\dot{S}_2 - S_1 BP^{-1}B^T R] \overline{\mu} = 0 \tag{44}$$

Again, this last equation is satisfied for the following set of equations

$$\dot{S}_1 = S_1 (BP^{-1}B^T A_0 - A + BP^{-1}Q_2^T) \tag{45a}$$

$$\dot{S}_2 = S_1 B P^{-1} B^T R \tag{45b}$$

By comparing equation (33b) to (45a) one finds that

$$S_1 = R^T \tag{46}$$

The optimal control vector may be written as

$$U^*(X,t) = - P^{-1}B^T R S_2^{-1} \Psi_0 - P^{-1} \left\{ Q_2^{\,T} + B^T [A_0 - R S_2^{\,-1} R^T] \right\} X \tag{47}$$

where the matrices $A_0(t)$, $R(t)$, and $S_2(t)$ are determined so as to satisfy the following

$$\dot{A}_0 = - A_0 A - A^T A_0 - Q_1 + (A_0 B + Q_2)\; P^{-1} \;(Q_2^{\,T} + B^T A_0) \tag{48}$$

$$\dot{R} = (A_0 B P^{-1} B^T - A^T + Q_2 P^{-1} B^T)\; R \tag{49}$$

$$S_2 = R^T (B P^{-1} B^T)\; R \tag{50}$$

Subject to the terminal conditions

$$A_0(t_f) = \bar{g} \tag{51}$$

$$R(t_f) = \bar{\Psi}^{\,T} \tag{52}$$

$$S_2(t_f) = 0 \tag{53}$$

Note that for the particular case where the final time is infinity and $Q_2 = 0$, the "reduced or degenerate Riccati equation results.

$$A^T A_O + A_O A + Q_1 - A_O B P^{-1} B^T A_O = 0 \tag{54}$$

One might also add that only the solution that yields a positive definite matrix A_O is acceptable in solving equation (54) It is obvious that equations (48) through (53) are in general not possible to solve analytically. In succeeding chapters some class of problems will be shown to yield an exact solution.

CHAPTER III

EXACT SOLUTION TO THE LINEAR OPTIMIZATION PROBLEM KNOWING THE POLES OF THE SYSTEM

In the dynamic optimization of linear systems especially when one is confronted with a two point boundary value problem, the difficulty in obtaining an exact solution lies in the success of matching boundary conditions. For a high order system this task becomes tedious and time consuming. For a given system and in most practical control problems, one is more interested in the optimal trajectories and control for a range of initial conditions. Thus, it is convenient and useful to have available a closed form solution for both the optimal control vector and optimal trajectories. A method of solution will be presented in this chapter, one which will assume full knowledge of the poles of the system.

1. Selection of State Variables and General Form for the Optimal Solution

Consider the system represented by its transfer function

$$G(s) = \frac{Y(s)}{U(s)} \tag{1}$$

where $Y(s)$ is the transform of the output $y(t)$

$U(s)$ is the transform of the input $u(t)$

If s_i $(i = 1, \ldots, n)$ are the set of distinct poles of the n^{th} order system, then the system may be written in the diagonal representation as

$$\dot{X} = \Lambda X + b\,u(t) \tag{2a}$$

$$y(t) = c^T X \tag{2b}$$

Equation (2a) is the state equation and equation (2b) is the output equation.

Λ is the diagonal matrix whose elements are the poles of the system, i.e.

$$\Lambda = \mathrm{DIAG}[s_i] \quad i = 1, \ldots, n$$

X is the state vector given as $X = [\chi_1 \ldots \chi_n]^T$

b is the residue vector given as $b = [\alpha_1 \ldots \alpha_n]^T$

$c = [1 \ldots 1]^T$

y(t) and u(t) are the scalar output and input respectively. $[\alpha_i]$ $i = 1, \ldots, n$ are the n residues and are given by

$$\alpha_i = \lim_{s \to s_i} (s-s_i)\, G(s) \quad i = 1, \ldots, n \tag{3}$$

Note that the poles are assumed to be distinct and single input single output is assumed.

Next, assume one wishes to minimize the cost function

$$V = \int_0^{t_f} \tfrac{1}{2} u^2 \, dt \tag{4}$$

subject to the constraint

$$\dot{X} = \Lambda X + b\,u(t) \tag{5a}$$

$$y(t) = c^T X \tag{5b}$$

The Hamiltonian may be constructed and the Pontryagin minimum principle used to yield

$$H = \tfrac{1}{2} u^2 + \bar{\lambda}^T (\Lambda X + bu) \tag{6}$$

where $\bar{\lambda}$ is the Lagrange multiplier vector. The minimum principle results in the following

$$u^* = -\sum_{i=1}^{n} \lambda_i \alpha_i \quad ; \quad i = 1, \ldots, n \tag{7}$$

where

$$\lambda_i(t) = \lambda_i(o) \exp(-s_i t); \; i = 1, \ldots, n \tag{8}$$

By using equations (1), (7) and (8) the transform of the optimal output $y^*(t)$ may be written as:

$$Y^*(s) = -\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\alpha_i \alpha_k \lambda_i(o)}{(s + s_i)(s - s_k)} \tag{9}$$

Equation (9) does not take into account any initial conditions, but one can solve the matrix differential equation given in equation (2) to yield a solution for the optimal trajectories as:

$$\chi_k^*(t) = \chi_k(o) \cdot \exp(s_k t) - \int_o^t \sum_{i=1}^{n} \alpha_i \alpha_k \lambda_i(o) \cdot \exp[s_k t - \tau(s_i + s_k)] \cdot d\tau \tag{10}$$

The general form for the optimal output solution is obtained by using equation (2b) and integration of equation (10), the following results

$$y^*(t) = \sum_{i=1}^{n} \chi_k(0)\cdot\exp(s_k t) + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\alpha_i\alpha_k}{s_i + s_k}\cdot\lambda_i(0)\left[\exp(-s_i t)-\exp(s_k t)\right] \quad (11)$$

2. <u>Determination of State Variables Initial Conditions</u>

In general, only the initial conditions for y, $\overset{(1)}{y}$, ..., $\overset{(n-1)}{y}$ are given, yet the optimal solution for the output is given in terms of $\chi_k(0)$, thus it is necessary to solve for $\chi_k(0)$ in terms of $y(0)$, $\overset{(1)}{y}(0)$, ..., $\overset{(n-1)}{y}(0)$.

Starting with equation (2) and differentiating (n-1) times and with the use of the state equations, the following set of equations is obtained

$$y(0) = \sum_{k=1}^{n} \chi_k(0) \quad (12)$$

$$\overset{(1)}{y}(0) = \sum_{k=1}^{n} s_k\chi_k(0) + \sum_{k=1}^{n} \alpha_k\, u^*(0) \quad (13)$$

$$\overset{(2)}{y}(0) = \sum_{k=1}^{n} s_k^2\, \chi_k(0) + \sum_{k=1}^{n} \alpha_k s_k\, u^*(0) + \sum_{k=1}^{n} \alpha_k\, \overset{(1)}{u}{}^*(0) \quad (14)$$

$$\vdots$$

$$\overset{(n-1)}{y}(0) = \sum_{k=1}^{n} s_k^{n-1}\cdot\chi_k(0) + \sum_{k=1}^{n} \alpha_k s_k^{n-2}\cdot u^*(0) + \cdots + \sum_{k=1}^{n} \alpha_k \overset{(n-2)}{u}{}^*(0) \quad (15)$$

Note that $\overset{(n)}{y} = \frac{d^n y}{dt^n}$ and $\overset{(n)}{u} = \frac{d^n u}{dt^n}$

Equations (12) through (15) may be represented by the following matrix equation

$$Y_O = V\, X(0) + \bar{\bar{H}}\, U_O \tag{16}$$

where

$$Y_O = \begin{bmatrix} y(0) & \overset{(1)}{y}(0) & \cdots & \overset{(n-1)}{y}(0) \end{bmatrix}^T \tag{17}$$

$$U_O = \begin{bmatrix} u^*(0) & \overset{(1)}{u^*}(0) & \cdots & \overset{(n-1)}{u^*}(0) \end{bmatrix}^T \tag{18}$$

$$X(o) = \begin{bmatrix} \chi_1(0) & \chi_2(0) \cdots & \chi_n(0) \end{bmatrix}^T \tag{19}$$

V is a "Vandermonde" matrix and is defined here in terms of the poles of the system in consideration as:

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ s_1^{n-2} & s_2^{n-2} & \cdots & s_n^{n-2} \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{bmatrix} \tag{20}$$

The matrix $\bar{\bar{H}}$ is given as:

$$\bar{\bar{H}} = \begin{bmatrix} \ddots & & & & \bigcirc \\ \Sigma\alpha_i & \ddots & & & \\ \Sigma\alpha_i s_i & \Sigma\alpha_i & \ddots & & \\ \Sigma\alpha_i s_i^2 & \Sigma\alpha_i s_i & \Sigma\alpha_i & \ddots & \\ \vdots & & & & \ddots \\ \Sigma\alpha_i s_i^{n-2} & & \cdots & & \Sigma\alpha_i \end{bmatrix} \tag{21}$$

Note that all summations are from (1 to n).

Upon solving for the vector X(0), one gets

$$X(0) = V^{-1}[Y_O - \bar{\bar{H}}\, U_O] \tag{22}$$

It should be noted that equation (22) is general, and applies to systems with or without zeros. The only stipulation again is that all the poles are assumed to be distinct, a condition without which no unique solution can be found for X(0).

Next, we take another approach to the problem and look at the original system without any zeros, i.e.

$$\frac{Y(s)}{U(s)} = G(s) = \left(\sum_{k=1}^{n+1} a_k\, s^{k-1}\right)^{-1}, \text{ with } a_{n+1} = 1 \tag{23}$$

Upon inversion back to the time domain, the following results:

$$\overset{(n)}{y} + a_n \cdot \overset{(n-1)}{y} + a_{n-1} \cdot \overset{(n-2)}{y} + \cdots a_1 y = u(t) \tag{24}$$

Keeping in mind the definition of the zero input response and the contributions of the initial conditions to the transform of derivatives, the following matrix relation is obtained

$$X(0) = F\,Y_0 \tag{25}$$

where F is given by

$$F = \begin{bmatrix} \beta_1 f_1(s_1) & \beta_1 f_2(s_1) & \cdots & \beta_1 \\ \beta_2 f_1(s_2) & \beta_2 f_2(s_2) & \cdots & \beta_2 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \beta_n f_1(s_n) & \beta_n f_2(s_n) & \cdots & \beta_n \end{bmatrix} \tag{26}$$

and β_i = residues of system without zeros; i = 1, ..., n. Note that $\beta_i = \alpha_i$ for no zeros present in the system. In general

$$\alpha_i = Z(s_i) \cdot \beta_i \tag{27}$$

where Z(s) is some function of the complex argument s for which $(Z(s) = 0)$ gives all possible zeros of the system. The elements $f_i(s_i)$ are defined as follows

$$f_1 = \sum_{k=2}^{n+1} a_k s^{k-2} \tag{28}$$

$$f_2 = \sum_{k=3}^{n+1} a_k s^{k-3} \tag{29}$$

$$\vdots$$

$$f_{n-1} = \sum_{k=n}^{n+1} a_k s^{k-n} \tag{30}$$

From equation (22) one may deduce that, for $\bar{\bar{H}} = 0$

$$X(o) = V^{-1} Y_o \tag{31}$$

But; $\bar{\bar{H}} = 0$ implies that

$$\sum_{i=1}^{n} \alpha_i = 0 \tag{32a}$$

$$\sum_{i=1}^{n} \alpha_i s_i = 0 \tag{32b}$$

$$\sum_{i=1}^{n} \alpha_i s_i^2 = 0 \tag{32c}$$

$$\vdots$$

$$\sum_{i=1}^{n} \alpha_i s_i^{n-2} = 0 \tag{32d}$$

Also, if the system does not possess any zeros, then $\alpha_i = \beta_i$ and

$$\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \beta_i s_i = \cdots = \sum_{i=1}^{n} \beta_i s_i^{n-2} = 0 \qquad (33)$$

Consequently, it can be shown that

$$V \cdot F = I_{nxn} \qquad (34)$$

where I_{nxn} is the n^{th} order identity matrix; thus

$$V^{-1} = F \qquad (35)$$

We can summarize the result the following way. If $(s_i,\ i = 1, \ldots, n)$ and $(\beta_i,\ i = 1, \ldots, n)$ are the poles and the residues respectively, and if no zeros are present then

$$\bar{\alpha} = \bar{\beta} = V^{-1} K \qquad (36)$$

where

$$k = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]^T \qquad (37)$$

$\bar{\alpha}$ and $\bar{\beta}$ are n-vectors

This represents an alternate form for the Heaviside expansion theorem. It also gives the inverse of the Vandermonde matrix.

3. Further Simplification of Problem

Rather than working with matrices whose elements belong to the complex field, it is convenient to decouple the complex matrix by breaking it into its real and imaginary parts. In this fashion, one is not faced with the tedious task of inverting complex non-Hermitian matrices. Thus one may write:

$$X(o) \quad = \quad X^R(o) \quad + \quad j\ X^I(o) \tag{38}$$

where the superscripts in capital letters imply real and imaginary. Similarly, the Vandermonde matrix V may be written as

$$V \quad = \quad V^R \quad + \quad j\ V^I \tag{39}$$

With any loss in generality the poles (s_k $k = 1, \ldots, n$) can be expressed in either polar form, i.e., $s_k = r_k \cdot \exp(j\ \theta_k)$, or in rectangular form $s_k = c_k + j\ d_k$. For the first case we have

$$V^R \quad = \quad \begin{bmatrix} 1 & \cdots & 1 \\ r_1 \cos\theta_1 & \cdots & r_n \cos\theta_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_1^{n-1} \cos(n-1)\theta_1 & \cdots & r_n^{n-1} \cos(n-1)\theta_n \end{bmatrix} \tag{40}$$

and

$$V^I = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ r_1 \text{Sin}\theta_1 & & \cdots & r_n \text{Sin}\theta_n \\ \vdots & & & \vdots \\ r_1^{n-1} \text{Sin}(n-1)\theta_1 & & \cdots & r_n^{n-1} \text{Sin}(n-1)\theta_n \end{bmatrix} \qquad (41)$$

For the second case

$$V^R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_n \\ \vdots & & & \vdots \\ \text{Real}[c_1 + jd_1]^{n-1} & & \cdots & \text{Real}[c_n + jd_n]^{n-1} \end{bmatrix} \qquad (42)$$

and

$$V^I = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ d_1 & d_2 & \cdots & d_n \\ \vdots & & & \vdots \\ \text{IM}(c_1 + jd_1)^{n-1} & & \cdots & \text{IM}(c_n + jd_n)^{n-1} \end{bmatrix} \qquad (43)$$

where (IM) implies imaginary part. Rewriting the equation

$$Y_0 = V\,X(0)$$

with the assumption that system does not possess any zeros, otherwise equation (16) is used, and because Y_O is always a real vector, one may write ($\tilde{0}$ is a n vector with elements = 0)

$$\begin{bmatrix} Y_O \\ \hline \tilde{0} \end{bmatrix} = \left[\begin{array}{c|c} V^R & -V^I \\ \hline V^I & V^R \end{array}\right] \begin{bmatrix} X^R(0) \\ \hline X^I(0) \end{bmatrix} \tag{44}$$

By letting:

$$X^{RI}(o) = \begin{bmatrix} X^R(0) \\ \hline X^I(0) \end{bmatrix}, \quad \text{2n vector} \tag{45}$$

$$Y_{OO} = \begin{bmatrix} Y_O \\ \hline \tilde{0} \end{bmatrix}, \quad \text{2n vector} \tag{46}$$

and

$$V^{RI} = \left[\begin{array}{c|c} V^R & -V^I \\ \hline V^I & V^R \end{array}\right], \quad \text{2n x 2n matrix} \tag{47}$$

the following is derived for the state vector initial condition.

$$X^{RI}(0) = \left(V^{RI}\right)^{-1} \cdot Y_{OO} \tag{48}$$

Note that a unique solution results provided none of the poles are equal.

4. Evaluation of Co-State Initial Conditions Fixed Initial and Final States

It will be assumed that the final time t_f is fixed and that we wish to minimize the performance function given in equation (4) such that the system is transferred from the given initial conditions represented by the known vector Y_O to the given final conditions represented by the vector Y_T. The state variables $\chi_k(t_f)$ evaluated at the final time may be found in this manner. Using the result of equation (10) the following holds

$$\chi_k(t_f) = \chi_k(0)\cdot\exp(s_k t_f) + \sum_{i=1}^{n} \frac{\alpha_i \alpha_k \lambda_i(0)}{s_i + s_k} \left[\exp(-s_i t_f) - \exp(s_k t_f)\right] \qquad (49)$$

In the same manner as equation (16) was derived one can deduce

$$Y_T = V\,X(t_f) + \bar{\bar{H}}\,U_T \qquad (50)$$

where

$$Y_T = \begin{bmatrix} y(t_f) & \overset{(1)}{y}(t_f) & \cdots & \overset{(n-1)}{y}(t_f) \end{bmatrix} \qquad (51)$$

$$U_T = \begin{bmatrix} u^*(t_f) & \overset{(1)}{u^*}(t_f) & \cdots & \overset{(n-1)}{u^*}(t_f) \end{bmatrix} \qquad (52)$$

$$X(t_f) = \begin{bmatrix} \chi_1(t_f) & \chi_2(t_f) & \cdots & \chi_n(t_f) \end{bmatrix} \qquad (53)$$

Note again that for no zeros present; $H = 0$ and

$$Y_T = V\,X(t_f) \qquad (54)$$

Consequently:

$$\overset{RI}{X}(t_f) = \left(V^{RI}\right)^{-1} Y_{TT} \tag{55}$$

where

$$Y_{TT} = \left[\begin{array}{c} Y_T \\ \hline \tilde{0} \end{array}\right], \quad \text{2n vector} \tag{56}$$

$$\overset{RI}{X}(t_f) = \left[\begin{array}{c} X^R(t_f) \\ \hline X^I(t_f) \end{array}\right], \quad \text{2n vector} \tag{57}$$

Next, we define:

$$R_T ik = \frac{\alpha_i \alpha_k}{s_i + s_k}\left[\exp(-s_i t_f) - \exp(s_k \cdot t_f)\right] \tag{58}$$

where

$$R_T = \text{matrix of elements } R_T ik = \left[R_T ik\right] \begin{array}{l} i = 1, \ldots, n \\ k = 1, \ldots, n \end{array}$$

Then, after some algeleraic manipulation and separation of real and imaginary parts, the following results

$$\bar{\lambda}^{RI}(0) = P_T^{-1} X^{RI}(t_f) - P_T^{-1} \Psi_T X^{RI}(0) \tag{59}$$

where

$$P_T = \left[\begin{array}{c|c} \text{REAL}(R_T^T) & -\text{IM}(R_T^T) \\ \hline \text{IM}(R_T^T) & \text{REAL}(R_T^T) \end{array}\right] \tag{60}$$

$$\Psi_T = \left[\begin{array}{c|c} \mathrm{REAL}[\exp(\Lambda t_f)] & -\mathrm{IM}[\exp(\Lambda t_f)] \\ \hline \mathrm{IM}\ [\exp(\Lambda t_f)] & \mathrm{REAL}[\exp(\Lambda t_f)] \end{array}\right] \tag{61}$$

and

$$\bar{\lambda}^{RI}(o) = \left[\begin{array}{c} \bar{\lambda}^{R}(o) \\ \hline \bar{\lambda}^{I}(o) \end{array}\right] \tag{62}$$

Keep in mind that

$$P_t(t = t_f) = P_T \tag{63a}$$

and

$$R_t(t = t_f) = R_T \tag{63b}$$

Note that the inverse of the matrix P_T exists if the poles are distinct.

The co-state initial conditions may now be written as:

$$\bar{\lambda}^{RI}(o) = P_T^{-1}\left(V^{RI}\right)^{-1}\cdot Y_{TT} - P_T^{-1}\ \Psi_T\left(V^{RI}\right)^{-1}\cdot Y_{OO} \tag{64}$$

For anytime t, the solution for the optimal state vector is given by:

$$X^{*RI}(t) = \left[\Psi_t - P_t P_T^{-1}\Psi_T\right]\left(V^{RI}\right)^{-1} Y_{OO} + P_t P_T^{-1}\left(V^{RI}\right)^{-1} Y_{TT} \tag{65}$$

In terms of the co-state vector initial condition

$$X^{*RI}(t) = \Psi_t\left(V^{RI}\right)^{-1} Y_{OO} + P_t\ \bar{\lambda}^{RI}(o) \tag{66}$$

The optimal output solution is

$$y^*(t) = \sum_{i=1}^{n} x_i^*(t) \text{ , or}$$

$$y^*(t) = \left[c^T \,\middle|\, \tilde{0}^T\right] \cdot X^{*RI}(t) \tag{67}$$

where

$$c^T = \left[1 \ 1 \ \cdots \ 1\right] \text{ , n vector}$$

Note that if the poles of the system are known, only two matrix inversions are required. Although in this derivation it was assumed that no zeros are present in the system, equation (65) can be adjusted to incorporate the matrix $\bar{\bar{H}}$, and $\bar{\alpha}$ will have to be used instead of $\bar{\beta}$. Also it is worthwhile to mention that, while this derivation applied to the case of fixed initial and final conditions, the case of fixed initial conditions and unspecified final ones yields a trivial solution for which the optimal control $u^* = 0$. In this particular case the optimal trajectory vector is given as:

$$X^{*RI}(t) = \Psi_t \left(V^{RI}\right)^{-1} Y_{OO} \tag{68}$$

An example problem will be next worked out to show the amount of computations required by the above derivation.

5. Example Problem

To illustrate the method of solution, let us consider the second order system represented by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{\pi^2}{s^2 + \pi^2} \tag{69}$$

We wish to minimize

$$V = \frac{1}{2}\int_0^1 u^2 \, dt$$

such that the system can be transferred from the initial conditions $y(0) = \dot{y}(0) = 1$, to the final conditions $y(1) = \dot{y}(1) = 0$.

First the Vandermonde matrix is found and inverted to yield:

$$V = \begin{bmatrix} 1 & 1 \\ j\pi & -j\pi \end{bmatrix} \quad \text{and} \quad V^{-1} = \begin{bmatrix} 0.5 & \frac{-j}{2\pi} \\ 0.5 & \frac{j}{2\pi} \end{bmatrix} \tag{70}$$

Note that here V is used instead of V^{RI} because of the simplicity of inverting a second order matrix with complex elements.

Next the state vectors initial conditions are computed by:

$$X(0) = V^{-1} Y_0 = V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus:

$$X(0) = \begin{bmatrix} 0.5 - \frac{j}{2\pi} \\ 0.5 + \frac{j}{2\pi} \end{bmatrix} \tag{71}$$

Then the residue vector $\overline{\alpha}$ is calculated by

$$\overline{\alpha} = Z(s_i)V^{-1} K = \pi^2 V^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-j\pi}{2} \\ \frac{j\pi}{2} \end{bmatrix} \tag{72}$$

Upon substitution of the values of α_1 and α_2 in equations (58) through (63), the following results.

$$P_T = \begin{bmatrix} 0 & \frac{-\pi^2}{4} \exp(j\pi) \\ \frac{-\pi^2}{4} \exp(-j\pi) & 0 \end{bmatrix} \quad ; \; P_T^{-1} = \begin{bmatrix} 0 & \frac{-4}{\pi^2} \exp(j\pi) \\ \frac{-4}{\pi^2} \exp(-j\pi) & 0 \end{bmatrix} \tag{73}$$

with

$$P_t = \frac{\pi}{4} \begin{bmatrix} \mathrm{Sin}(\pi t) & -\pi t \exp(j\pi t) \\ -\pi t \exp(-j\pi t) & \mathrm{Sin}(\pi t) \end{bmatrix} \tag{74}$$

Here it is worthwhile to note that for poles with no real parts, ℓ'Hospital rule is used in equation (58) to determine elements of P_t. For instance, writing

$$R_t ik = \frac{\alpha_i \alpha_k}{s_i + s_k} \left[\exp(-s_i t) - \exp(s_k t)\right] \tag{75}$$

and letting $s_i + s_k = \gamma$, then

$$R_t ik = \alpha_i \alpha_k \exp(-s_i t) \left[\frac{1 - \exp(t\gamma)}{\gamma}\right]$$

Noting that for no real parts present $\gamma = 0$, and

$$R_{t12} = \alpha_1 \alpha_2 \exp(-s_1 t) \cdot \lim_{\gamma \to 0} \frac{1 - \exp(\gamma t)}{\gamma} \tag{76}$$

or

$$R_{t12} = -t \, \alpha_1 \alpha_2 \exp(-s_1 t) \tag{77}$$

This situation may occur sometimes for higher order systems. Next the matrix Ψ_T is found as:

$$\Psi_t = \begin{bmatrix} \exp(j\pi t) & 0 \\ 0 & \exp(-j\pi t) \end{bmatrix} \text{ and } \Psi_T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{78}$$

The co-state initial conditions are found by using equation (59) realizing that $X(t_f) = 0$, the following is obtained:

$$\bar{\lambda}(0) = \begin{bmatrix} 0.2026 + j\,0.0646 \\ 0.2026 - j\,0.0646 \end{bmatrix} \tag{79}$$

The optimal open loop control may be found using

$$u^*(t) = -\sum_{i=1}^{n=2} \alpha_i \lambda_i(0) \exp(-s_i t) \tag{80}$$

Carrying out the summation and substituting one gets:

$$u^*(t) = 0.638 \operatorname{Sin}(\pi t) - 0.2026 \cos(\pi t) \tag{81}$$

The optimal output is found using equation (66) and (67). Upon substitution and matrix multiplication, the result is

$$y^*(t) = 0.318 \operatorname{Sin}(\pi t) + (1-t)\left[\cos(\pi t) + 0.318 \operatorname{Sin}(\pi t)\right] \tag{82}$$

This example illustrates very well the method of solution. It presents the solution to the optimal control problem for a quadratic cost in the control and given initial and final conditions. In the next chapter, an introductory study will be presented for a certain class of nonlinear systems, one that will be used later to establish an exact solution to the nonlinear class of problems in consideration.

CHAPTER IV

OPTIMAL CONTROL OF A CLASS OF NONLINEAR SYSTEMS

Although some numerical techniques are available to obtain an open loop control for nonlinear systems, a closed lo9p control is practically impossible. Even for the more simple cases, the Hamilton Jacobi equation is so hard to solve that everybody is satisfied with only approximate solutions and/or perhaps some numerical solution for the control $u^*(t)$ and the trajectory $X^*(t)$ as a function of time. In this chapter, a class of nonlinear system is considered, one that yields an exact solution for the closed loop control in some particular cases and some interesting results in all cases for the open loop control $u^*(t)$ and trajectory $X^*(t)$.

1. Problem Formulation and General Form of Solution

Consider the first order nonlinear system represented by

$$\dot{X} = aX + bu + cX^n \tag{1}$$

This represents a class of systems such that the nonlinearity exists in the power of the controlled variable X; a, b, c, n are given constants. We wish to find a closed loop control law $u^*(X,t)$ and/or an open loop control $u^*(t)$ coupled with an optimal trajectory $X^*(t)$ that will minimize the performance criteria

$$V = \tfrac{1}{2} \int_{t_o}^{t_f} (\alpha\chi^2 + \beta u^2)\ dt \tag{2}$$

subjected to the constraints given in equation (1). It should be noted that in most physical problems, the system depends on a prescribed range of initial conditions, thus they will be assumed to lie in a closed subset of a Eucledian Space, i.e.

$$\chi(t_o) \in R \quad \text{for } t \in [t_o, t_f]$$

For this reason, a closed loop solution is more desirable since it does not depend on any initial conditions. Of course, whenever an open loop control is required, a solution can be better useful for a given initial condition. Rather than specify now what type of problem we are dealing with, whether a fixed end problem or variable end problem, the problem will be formulated in general with the use of the Pontryagin Minimum Principle. So, if the Pontryagin control function is formed and the minimum principle used, the following is obtained

$$H[X(t),\lambda(t),u(t),t] = \tfrac{1}{2}[\alpha\chi^2+\beta u^2] + \lambda(a\chi+bu+c\chi^n) \tag{3}$$

With
$$\frac{\delta H}{\delta u} = 0 \tag{4a}$$

and
$$\frac{\delta H}{\delta \chi} = -\dot{\lambda} \tag{4b}$$

where H is the Pontryagin control function and λ is the Lagrange multiplier. Equations (4a) and (4b) yield

$$u^* = \frac{-b}{\beta}\lambda \tag{5}$$

and

$$\dot{\lambda} = -\alpha\chi - \lambda(a + nc\chi^{n-1}) \tag{6}$$

Upon differentiation of state equation (1) and substitution of u and λ, the following results:

$$\ddot{\chi} = nc^2\chi^{2n-1} + ac(n+1)\chi^n + \chi(a^2 + \frac{\alpha b^2}{\beta}) \tag{7}$$

Equation (7) represents a second order nonlinear differential equation whose solution gives the optimal trajectory $\chi^*(t)$. Now if equation (7) is multiplied by $2\dot{\chi}dt$ and integrated, the following is obtained

$$\dot{\chi}^2 = c^2\chi^{2n} + 2ac.\chi^n + \chi^2(a^2 + \frac{\alpha b^2}{\beta}) + \text{Constant} \tag{8}$$

Denote this constant by A_1. Next we solve for $\dot{\chi}$ and separate the variables to get the following integral equation.

$$\int_0^{\chi} \frac{\pm \quad d\chi}{[g^2 + \frac{\alpha b^2}{\beta}\chi^2 + A_1]^{\frac{1}{2}}} = t + A_2 \tag{9}$$

where A_2 is another constant and $g(\chi)$ is given by:

$$g(\chi) = a\chi + c\chi^{n} \tag{10}$$

The optimal control $u^{*}(\chi, t)$ may be found by substitution into the state equation as:

$$u^{*}(\chi, t) = \frac{1}{b}\left\{-g(\chi) \pm \left[g^{2} + \frac{\alpha b^{2}}{\beta}\chi^{2} + A_{1}\right]^{\frac{1}{2}}\right\} \tag{11}$$

At this point $u^{*}(\chi, t)$ as given in equation (11) does not qualify as an optimal closed loop control law because of the fact that the undetermined constants A_1 and A_2 depend on either the initial condition or the unspecified final condition.

2. Extension to Linear Case (C=0)

Case 1: t_f specified, $\chi(t_f)$ unspecified

The transversality condition as stated in equation (50) Chapter I may be reduced to the following:

$$\lambda(t_f) = 0 \tag{12}$$

In this case

$$u^{*}(\chi, t) = \frac{\chi}{b}\left\{-a \pm \left[a^{2} + \frac{\alpha b^{2}}{\beta}\left(1 - \frac{\chi^{2}(t_f)}{\chi^{2}(t)}\right)\right]^{\frac{1}{2}}\right\} \tag{13}$$

With a simple transformation of variable $z = \dfrac{\chi(t_f)}{\chi(t)}$ and with the use of the integral equation (9), the following is obtained after some algebraic manipulation and proper substitution.

$$u^{*}(\chi,t) = \frac{\chi}{b}\left\{-a-\zeta\frac{(\epsilon^2-\gamma^2\ \exp(2\theta))}{\epsilon^2+\gamma^2\ \exp(2\theta)}\right\} \tag{14}$$

for $a > 0$
$\alpha < 0$

and

$$u^{*}(\chi,t) = \frac{\chi}{b}\left\{-a+\zeta\frac{(\epsilon^2-\gamma^2\ \exp(-2\theta))}{\epsilon^2+\gamma^2\ \exp(-2\theta)}\right\} \tag{15}$$

for $a > 0$
$\alpha > 0$

where:

$$\zeta = \left(a^2 + \frac{\alpha b^2}{\beta}\right)^{\frac{1}{2}} \tag{16a}$$

$$\epsilon^2 = \left(1 + \frac{\beta a^2}{\alpha b^2}\right)^{-1} \tag{16b}$$

$$\theta = \zeta(t-t_f) \tag{17a}$$

$$\gamma = 1+(1-\epsilon^2)^{\frac{1}{2}} \tag{17b}$$

For the case where $\alpha < 0$, the following is obtained:

$$u^{*}(\chi,t) = \frac{\chi}{b}\left\{-a + \zeta\frac{[\epsilon^2+\gamma^2\ \exp(\pm 2\theta)]}{[-\epsilon^2+\gamma^2\exp(\pm 2\theta)]}\right\} \tag{18}$$

for $a > 0$
$a^2 + \frac{\alpha b^2}{\beta} > 0$

where
$$\zeta = \left(a^2 + \frac{\alpha b^2}{\beta}\right)^{\frac{1}{2}} \tag{19a}$$

$$\gamma = 1 + (1+\epsilon^2)^{\frac{1}{2}} \tag{19b}$$

$$\epsilon^2 = -\left\{1 + \frac{\beta a^2}{\alpha b^2}\right\}^{-1} \tag{20a}$$

$$\theta = \zeta(t-t_f) \tag{20b}$$

Note that for $a < 0$ and $a^2 + \frac{\alpha b^2}{\beta} < 0$, no minimum exists. For the special case when the final time is infinity ($t_f = \infty$), the following holds:

$$u^*(\chi) = \frac{\chi}{b}[-a-\zeta] \qquad a > 0,\ \alpha > 0 \tag{21}$$

$$u^*(\chi) = \frac{\chi}{b}[-a-\zeta] \qquad a < 0,\ \alpha > 0 \tag{22}$$

$$u^*(\chi) = \frac{\chi}{b}[-a\pm\zeta] \qquad a > 0,\ \alpha < 0 \tag{23}$$

It should be noted that the solutions for $u^*(\chi,t)$ and $u^*(\chi)$ are the solutions to the Ricatti equation for this problem. This different approach in solving this linear problem will play an important role in obtaining a solution for the non-linear problem.

Case 2. t_f specified, $\chi(t_f) = \chi_f$ specified

The problem now is much more complicated because one cannot use $\lambda(t_f) = 0$ any more, but instead one has to use the given condition χ_f. Upon using equations (47) through (53) in Chapter II and substituting the given quantities, the following is obtained.

$$u^*(\chi,t) = \frac{-1}{\beta}\, b\, \frac{r(t)}{S_2(t)}\, \chi_f - \frac{1}{\beta}\, b\, [a_0(t) - \frac{r^2(t)}{S_2(t)}] \qquad (24)$$

where $a_0(t)$, $r(t)$ and $S_2(t)$ satisfy the following:

$$\dot{a}_o = -2a_o(t)\, a - \alpha + a_o^2(t)\, \frac{b^2}{\beta} \qquad (25)$$

$$\dot{r} = (\frac{a_o b^2}{\beta} - a)\, r(t) \qquad (26)$$

$$\dot{S} = \frac{b^2 r^2}{\beta} \qquad (27)$$

with the boundary conditions:

$$a_0(t_f) = 0 \qquad (28a)$$

$$r(t_f) = 1 \qquad (28b)$$

$$S_2(t_f) = 0 \qquad (28c)$$

Note that here, $\overline{\Psi} = 1$ and $\Psi_o = \chi_f$. After some algebraic manipulation and separation of variables, the following is obtained

$$a_0(t) = \frac{\alpha}{\zeta} \cdot \frac{\text{Sinh } \varphi}{\text{Cosh}\varphi - \frac{a}{\zeta} \text{ Sinh}\varphi} \tag{29}$$

$$r(t) = (\text{Cosh}\varphi - \frac{a}{\zeta} \text{ Sinh}\varphi)^{-1} \tag{30}$$

$$S_2(t) = \frac{-b^2}{\beta\zeta} \left\{ \frac{\text{Sinh } \varphi}{\text{Cosh}\varphi - \frac{a}{\zeta} \text{ Sinh}\varphi} \right\} \tag{31}$$

where

$$\zeta = (a^2 + \frac{\alpha}{\beta} b^2)^{\frac{1}{2}} \tag{32}$$

$$\varphi = -\zeta(t - t_f) \tag{33}$$

After substitution the values of $a_0(t)$, $r(t)$ and $S_2(t)$ in equation (24), the following results

$$u^*(\chi, t) = \frac{\zeta \chi_f}{b \text{ Sinh}\varphi} - \chi(t) \cdot \frac{\frac{\alpha b}{\zeta\beta} \text{ Sin}^2\text{h}\varphi + \frac{\zeta}{b}}{\text{Sinh}\varphi(\text{Cosh}\varphi - \frac{a}{\zeta} \text{ Sin}\varphi)} \tag{34}$$

Note that there are conjugate points at $t^* = [t_f - \frac{1}{\zeta} \text{ arctanh}(\frac{\zeta}{a})]$

3. <u>Nonlinear Case C ‡ 0</u>

<u>Case 1 $\chi(t_f)$ Unspecified, t_f specified and finite</u>

Assume that we start at t_o and that we wish to drive the system from a given initial state χ_o to an unknown final state $\chi(t_f)$ while minimizing the performance function described in the beginning of the chapter. The integral equation (9) becomes

$$\pm(t-t_o) = \int_{\chi_o}^{\chi} \frac{d\chi}{[g^2 + \frac{\alpha b^2}{\beta}\chi^2 + A_1]^{\frac{1}{2}}} \tag{35}$$

Since the final state is not specified then, $\lambda(t_f) = 0$ and $u^*[x(t_f), t_f] = 0$. Thus equation (11) becomes:

$$u^*(\chi, t) = \frac{1}{b}\left[-g \pm \sqrt{g^2 + \frac{\alpha b^2}{\beta}(\chi^2 - \chi^2(t_f))}\right] \tag{36}$$

The final reachable state may be determined by

$$\pm(t_f - t_o) = \int_{\chi o}^{\chi(t_f)} \frac{d\chi}{\left\{g^2 + \frac{\alpha b^2}{\beta}[\chi^2 - \chi^2(t_f)]\right\}^{\frac{1}{2}}} \tag{37}$$

Note that in this case only the (+) is acceptable in equation (36) since $\lambda(t_f) = u(t_f) = 0$. The optimal control law in equation (36) is inadmissible because it depends on the initial condition χ_o through equation (37). One may conclude that, although a feed back control law is impossible to obtain for a finite final time, an open loop control may be found for a given initial state χ_o. The problem is tremendously simplified and at least for an open loop solution only a numerical integration is required.

<u>Case 2 $\chi(t_f)$ Specified, t_f Specified</u>

In this case we wish to drive the system from the given initial state χ_o to the final state χ_f. Equation (35) still holds. The undetermined constant A_1 is found using

$$\pm(t_f - t_o) = \int_{X_o}^{X_f} \frac{dX}{\left[g^2 + \frac{\alpha b^2}{\beta} X^2 + A_1\right]^{\frac{1}{2}}} \tag{38}$$

With X_o, X_f and t_f known A_1 may be found. Now the open loop solution is found by using both equations (35) and (38). The closed loop control law is not possible in this case since A_1 depends on the initial condition. In the next chapter, for infinite time optimization and some other particular cases an exact solution will be determined for the closed loop control law.

Trivial Case $\alpha = 0$

In this case $u^* = 0$ and by integration one finds that the optimal trajectory is given by:

$$\frac{X^*(t)}{X_o} = \left\{ \frac{a \exp[a(n-1)(t-t_o)]}{a + cX_o^{\,n-1}\,(1-\exp[a(n-1)(t-t_o)])} \right\}^{\frac{1}{n-1}} \tag{39}$$

with the final state given by:

$$\frac{X^*(t_f)}{X_o} = \left\{ \frac{a \exp[a(n-1)(t_f-t_o)]}{a + cX_o^{\,n-1}\,(1-\exp[a(n-1)(t_f-t_o)])} \right\}^{\frac{1}{n-1}} \tag{40}$$

In the next chapter an exact solution to the Hamilton Jacobi Equation will be developed one that will be more useful than the open loop solution established in this chapter.

CHAPTER V

EXACT SOLUTION VIA HAMILTON JACOBI EQUATION

1. Case of Infinite Time Optimization

An exact solution to the Hamilton Jacobi Equation is in most cases impractical if not impossible for nonlinear systems. In this chapter an exact solution will be found for the case of infinite time optimization and some other cases where the final time is free and final state specified. The open loop control and trajectories will be determined and a feedback control law will be established.

The general form for the Hamilton Jacobi Equation may be taken from equation (20) in Chapter II

$$\frac{\delta V}{\delta t}[X(t),t] + \overline{H}^{*}\left[\frac{\delta V}{\delta X}, X, t\right] = 0 \tag{1}$$

The Hamiltonian $\overline{H}^{*}$ is determined by using equation (3) in Chapter III and we write

$$H[X(t),\lambda(t),u(t),t] = \tfrac{1}{2}(\alpha X^2+\beta u^2) + \lambda(aX+bu+cX^n) \tag{2}$$

with $$u^{*}[\lambda(t)] = \frac{-b}{\beta}\lambda \tag{3}$$

and $$\lambda = +\frac{\partial V}{\partial X} \tag{4}$$

Upon substitution of λ and u^* in terms of $\frac{\delta V}{\delta \chi}$ the following results

$$\overline{H}^* = \tfrac{1}{2}\alpha\chi^2 + g(\chi) \cdot \frac{\delta V}{\delta \chi} - \tfrac{1}{2}\frac{b^2}{\beta}\left(\frac{\delta V}{\delta \chi}\right)^2 \tag{5}$$

Consequently, equation (1) becomes

$$\frac{\delta V}{\delta t} + \tfrac{1}{2}\alpha\chi^2 + g(\chi)\frac{\delta V}{\delta \chi} - \tfrac{1}{2}\frac{b^2}{\beta}\left(\frac{\delta V}{\delta \chi}\right)^2 = 0 \tag{6}$$

Equation (6) represents the Hamilton Jacobi Equation for the problem. For the case when $t_f \to \infty$, the dependency of V on time is eliminated $(\frac{\delta V}{\delta t} = 0)$ and equation (6) simplifies to:

$$-\left(\frac{dV}{d\chi}\right)^2 + \frac{2\beta}{b^2} g(\chi)\frac{dV}{d\chi} + \frac{\alpha\beta}{b^2}\chi^2 = 0 \tag{7}$$

with the boundary condition $V(o) = 0$. (8)

Upon solving for $\frac{dV}{d\chi}$ we get

$$\frac{dV}{d\chi} = + \frac{\beta}{b^2} g(\chi) \pm \left[\frac{\beta^2}{b^4} g^2(\chi) + \frac{\alpha\beta}{b^2}\chi^2\right]^{\frac{1}{2}} \tag{9}$$

where again

$$g(\chi) = a\chi + c\chi^n \tag{10}$$

The performance function $V(\chi)$ may be found by integration, and we write

$$V(\chi) = + \frac{\beta}{b^2} \int_0^{\chi} g(\chi)\ d\chi \pm \int_0^{\chi} [\frac{\beta^2}{b^4} g^2(\chi) + \frac{\alpha\beta}{b^2} \chi^2]^{\frac{1}{2}}\ d\chi + C \tag{11}$$

At this time we do not need to carry the integration. The optimal feed back law may be derived using

$$u^* = -\frac{b}{\beta} \frac{\delta V}{\delta \chi} \tag{12}$$

or

$$u^*[\chi(t)] = \frac{1}{b} \left\{ -g(\chi) \pm (g^2 + \frac{\alpha}{\beta} b^2 \chi^2)^{\frac{1}{2}} \right\} \tag{13}$$

As it can be seen, this is exactly the same result as equation (4.36) for $\chi(t_f) = 0$. This is obvious because when $t_f \rightarrow \infty$ the Hamiltonian $\rightarrow 0$, thus

$$\overline{H}^*[\chi^*, u^*, \lambda^*, t]|_{t_f \rightarrow \infty} = 0 \tag{14}$$

which implies that
$$\chi(t_f)|_{t_f \rightarrow \infty} = 0 \tag{15}$$

So $u^*(\chi)$ as given in equation (13) is an exact solution to the Hamilton-Jacobi Equation. It is also interesting to notice that for the most general case when t_f is finite for which $\frac{\delta V}{\delta t} \neq 0$, the exact solution is impossible to obtain because of the dependency of the control on $\chi(t_f)$. Nevertheless, a relation may be developed between $u^*(\chi, t)$ and $u^*(\chi)$.

Here
$$\mathbf{u}^*(\chi) = \mathbf{u}^*(\chi,t)\big|_{t_f \to \infty} \tag{16}$$

and
$$\mathbf{u}^*(\chi,t) = \frac{1}{b}\left\{-g \pm \left[-\frac{\alpha b^2}{\beta}\chi^2(t_f) + (g+bu^*(\chi))^2\right]^{\frac{1}{2}}\right\} \tag{17}$$

$\mathbf{u}^*(\chi)$ = optimal closed loop control for $t_f = \infty$

$\mathbf{u}^*(\chi,t)$ = optimal closed loop control for t_f finite

Now, once the optimal control law $u^*(\chi)$ is obtained the optimum trajectory $\chi^*(t)$ may be determined in the following manner. We start with equation (4.35) where $\chi(t_f) = 0$ and write

$$\pm(t-t_o) = \int_{\chi_o}^{\chi} \frac{d\chi}{(g^2 + \frac{\alpha b^2}{\beta}\chi^2)^{\frac{1}{2}}} \tag{18}$$

Where the initial condition $\chi(t_o) = \chi_o$ (given). Upon transformation of variables

$$\nu = c\chi^{n-1} \tag{19}$$

$$d\nu = (n-1)c\chi^{n-2}d\chi \tag{20}$$

Equation (18) becomes

$$\pm(t-t_o) = \int_{\nu_o}^{\nu} \frac{d\nu}{(n-1)\ \nu[\frac{\alpha b^2}{\beta} + (a+\nu)^2]^{\frac{1}{2}}} \tag{21}$$

Note that
$$\nu_o = c\,\chi_o^{n-1} \tag{22}$$

By defining
$$\gamma^2 = a^2 + \frac{\alpha b^2}{\beta} \tag{23}$$

$$\theta' = (n-1)(t-t_o) \tag{24}$$

The integral equation now becomes

$$\pm\theta' = \int_{\nu_o}^{\nu} \frac{d\nu}{\nu(\nu^2+2a\nu+\gamma^2)^{\frac{1}{2}}} \tag{25}$$

The integration of equation (25) may be performed, and after some algebraic manipulation, one gets:

$$\chi^*(t) = \left\{\frac{-\frac{2}{c}\gamma r_o \exp(\pm\theta)}{1-\left[r_o\exp(\pm\theta)-\frac{a}{\gamma}\right]^2}\right\}^{\frac{1}{n-1}} \tag{26}$$

valid for $\quad a^2 + \frac{\alpha b^2}{\beta} > 0$

where

$$r_o = \frac{\gamma + \left\{\frac{\alpha b^2}{\beta} + (a+c\chi_o^{n-1})^2\right\}^{\frac{1}{2}}}{c\chi_o^{n-1}} + \frac{a}{\gamma} \tag{27}$$

$$\theta = \gamma(n-1)(t-t_o) \tag{28}$$

For $\alpha < 0$, $\quad \alpha^2 + \frac{\alpha b^2}{\beta} < 0$

In this case, although a solution exists for the optional trajectory $\chi^*(t)$, it does not correspond to an admissible control $u^*(\chi)$. In this case χ^* is given by:

$$\chi^*(t) = \frac{-\frac{\gamma^2}{c}}{a-(-\epsilon^2)^{\frac{1}{2}}\sin(\theta+\theta_o)} \tag{29}$$

where θ_o is determined by

$$\mathrm{Sin}(\theta_o) = \frac{ac\,\chi_o^{n-1} + \gamma^2}{c\chi_o^{n-1}(-\epsilon^2)^{\frac{1}{2}}} \tag{30}$$

and

$$\theta = (n-1)(t-t_o)(-\gamma^2)^{\frac{1}{2}} \tag{31a}$$

$$\epsilon^2 = \frac{\alpha b^2}{\beta} \tag{31b}$$

For an optimum trajectory to exist it can be shown that the following should be satisfied

$$\epsilon^2 + (c\chi^{n-1} + a)^2 < 0 \tag{32}$$

Equation (17) clearly indicates that

$$u^*(\chi) = \frac{1}{b}\left\{-g \pm \chi\ [\epsilon^2+(a+c\chi^{n-1})^2]^{\frac{1}{2}}\right\} \tag{33}$$

which shows that $u^*(\chi)$ is complex and is not admissible. This is a logical result since $\chi^*(t_f)$ in equation (29) does not come out to be zero, a condition which should be satisfied in order for $\bar{H}\,\big|_{t_f \to \infty} = 0$. Consequently the only solution that yields an admissible control $u^*(\chi)$ is the one for which $\alpha > 0$, i.e

$$\chi^*(t) = \left\{\frac{-\frac{2}{c}\,\gamma\, r_0\, \exp(\pm\theta)}{1-[r_0\exp(\pm\theta) - \frac{a}{\gamma}]^2}\right\}^{\frac{1}{n-1}} \tag{34}$$

with

$$\dot{u}^*(\chi) = \frac{1}{b}\left\{-g \pm [g^2 + \frac{\alpha b^2}{\beta}\chi^2]^{\frac{1}{2}}\right\} \tag{35}$$

and both $\alpha,\beta > 0$.

Although equations (34) and (35) indicate that the optimal solution is not unique, it can be easily shown that only one of the solution is admissible. For instance, looking at the optimal trajectory given in equation (34), there exist some values of time for which $\chi^*(t)$ becomes unbounded. In order to avoid these singularities which make the closed loop control inadmissible, one rejects the solution that yields a singular point. Thus, if there exists a finite time $\bar{t}$ in

$[t_o, t_f]$ such that $\chi^*(t)$, hence $u^*(\chi)$ becomes unbounded, then the control $u^*(\chi)$ is inadmissible. Equation (34) clearly shows that the denominator is zero for

$$\exp(\pm\,\theta) = K_1, K_2 \tag{36}$$

where

$$K_1 = \frac{1 + \frac{a}{\gamma}}{r_o} \tag{37}$$

$$K_2 = \frac{-1 + \frac{a}{\gamma}}{r_o} \tag{38}$$

If either K_1 and K_2 are greater than 1 then there exists a $\bar{\theta}$ such that

$$\exp(\bar{\theta}) = K_1, K_2 \tag{39}$$

and the (+) in equation (34) constitutes an inadmissible trajectory. On the other hand, if either K_1 and K_2 are less than 1 then there exists a $\bar{\theta}$ such that

$$\exp(-\bar{\theta}) = K_1, K_2 \tag{40}$$

and the (-) becomes inadmissible. So:

$$\chi^*(t) = \left\{ \frac{\frac{2\gamma}{r_o c} \exp(-\theta)}{(\exp(-\theta) - K_1)(\exp(-\theta) - K_2)} \right\}^{\frac{1}{n-1}} \tag{41}$$

for $K_1, K_2 > 1$

and

$$\chi^*(t) = \left\{ \frac{\frac{2\gamma}{r_0 c} \exp(\theta)}{(\exp(\theta)-K_1)(\exp(\theta)-K_2)} \right\}^{\frac{1}{n-1}} \tag{42}$$

$$\text{for } K_1, K_2 < 1$$

Now, the admissible control $u^*(\chi)$ is unique and it can be shown that since for the case when $t_f = \infty$, $\chi(t_f) = 0$, $\dot{\chi}^*(0) < 0$. Thus the optimal feed back control law is

$$u^*(\chi) = \frac{1}{b} \left\{ -g(\chi) + \left(g^2 + \frac{\alpha b^2}{\beta}\chi^2\right)^{\frac{1}{2}} \right\} \tag{43}$$

$$\text{for } \chi(0) < 0$$

and

$$u^*(\chi) = \frac{1}{b} \left\{ -g(\chi) - \left(g^2 + \frac{\alpha b^2}{\beta}\chi^2\right)^{\frac{1}{2}} \right\} \tag{44}$$

$$\text{for } \chi(0) > 0$$

An example problem will be next worked out to illustrate the method.

2. Example I

To illustrate this method of solution, consider the problem of minimizing the performance function

$$V = \frac{1}{2} \int_0^\infty (\chi^2 + u^2)\, dt \tag{45}$$

Subjected to the nonlinear differential constraint

$$\dot{\chi} = -\chi^3 + \chi + u \tag{46}$$

The optimal closed loop control is determined using equations (43) and (44) with:

$$n = 3$$

$$a = \alpha = \beta = b = 1$$

$$c = -1$$

$$g(\chi) = \chi - \chi^3$$

Thus,

$$u^*(\chi) = \chi^3 - \chi \pm (\chi^6 - 2\chi^4 + 2\chi^2)^{\frac{1}{2}} \tag{47}$$

where again the (+) is the branch for negative initial conditions and the (-) for positive initial conditions. For this particular example consider the cases where $\chi(0) = \pm 1$ and $\chi(0) = \pm 3$. For $\chi(0) = \pm 1$, the solution for the optimal trajectory is

$$\chi^*(t) = \pm \left\{ \frac{4.828 \exp(2.828t)}{[0.707 + 1.707 \exp(2.828t)]^2 - 1} \right\}^{\frac{1}{2}} \tag{48}$$

For $\chi(0) = \pm 3$, the solution is:

$$\chi^*(t) = \pm \left\{ \frac{0.9785 \exp(2.828t)}{[0.707 + 0.346 \exp(2.828t)]^2 - 1} \right\}^{\frac{1}{2}} \tag{49}$$

The optimal return function or $V(\chi)$ is determined by direct integration as

$$V(\chi) = \frac{\chi^2}{2} - \frac{\chi^4}{4} + \frac{1}{4}\left\{(\chi^2-1)(\chi^4-2\chi^2+2)^{\frac{1}{2}} + \text{Log}[\chi^2-1+(\chi^4-2\chi^2+2)^{\frac{1}{2}}]\right\} + 0.575$$
$$\text{for } \chi(0) > 0 \tag{50}$$

$$V(\chi) = \frac{\chi^2}{2} - \frac{\chi^4}{4} - \frac{1}{4}\left\{(\chi^2-1)(\chi^4-2\chi^2+2)^{\frac{1}{2}} + \text{Log}[\chi^2-1+(\chi^4-2\chi^2+2)^{\frac{1}{2}}]\right\} - 0.575$$
$$\text{for } \chi(0) < 0 \tag{51}$$

This represents the exact solution to the Hamilton Jacobi Equation for this problem. Plots for $u^*(\chi)$ and a family of $\chi^*(t)$ are shown in figure I, II.

3. <u>Case When Final Time is Free, Final State Specified</u>

In the previous section we assumed that the final time is extended to infinity and an exact solution was obtained. In this section, the performance index will be altered slightly as to include variation in the undetermined final time. The final state $\chi(t_f) = \chi_f$ will be assumed known. The performance function is written as:

$$V = \theta(t_f) + \int_{t_o}^{t_f} (\tfrac{1}{2}\alpha\chi^2 + \tfrac{1}{2}\beta u^2)\, dt \tag{52}$$

Upon adjoining the extra equality constraints to the problem, the following transversality conditions result

$$\overline{H}^*[\chi(t_f), \lambda(t_f), t_f] = -\frac{\delta\theta}{\delta t_f} \tag{53}$$

Keeping in mind equation (4.11) we may write

$$u^*(\chi, t) = \frac{1}{b}\left\{-g(\chi) \pm [g^2 + \frac{\alpha b^2}{\beta}\chi^2 + A_1]^{\frac{1}{2}}\right\} \tag{54}$$

Using equation (53) it can be shown that

$$A_1 = \frac{2b^2}{\beta}\frac{\delta\theta}{\delta t_f} \tag{55}$$

and the optimal control law is given as

$$u^*(\chi, t) = \frac{1}{b}\left\{-g(\chi) \pm [g^2 + \frac{b^2}{\beta}(\alpha\chi^2 + 2\frac{d\theta}{dt_f})]^{\frac{1}{2}}\right\} \tag{56}$$

where $\frac{d\theta}{dt_f}$ replaced $\frac{\delta\theta}{\delta t_f}$ because $\theta = \theta(t_f)$. Of course equation (56) could have been derived by direct solution of the Hamilton Jacobi Equation. For instance, equation (1) yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}\alpha\chi^2 + g(\chi)\,\frac{\delta V}{\delta \chi} - \frac{1}{2}\,\frac{b^2}{\beta}\left(\frac{\delta V}{\delta \chi}\right)^2 = 0 \tag{57}$$

$$\text{with } V[\chi(t_f), t_f] = \theta(t_f)$$

Solving for $\frac{\delta V}{\delta \chi}$ one gets:

$$\frac{\delta V}{\delta \chi} = \frac{\beta}{b^2}\, g(\chi) \pm \left[\frac{\beta^2}{b^4}\, g^2(\chi) + \frac{\alpha\beta}{b^2}\left(\chi^2 + 2\frac{\delta V}{\delta t}\right)\right]^{\frac{1}{2}} \tag{58}$$

By comparing this equation to the optimal control equation we find that

$$\frac{\delta V}{\delta t} = \frac{d\theta}{dt_f}$$

Note that because of the complicated nature of the solution, we will assume that $\theta(t_f)$ is linear in t_f which makes the control in equation (56) admissible. In this manner, u^* is only a function of χ, thus $\frac{\delta V}{\delta t}$ = constant and a solution is possible for $V(\chi, t)$. The optimal trajectory $\chi^*(t)$ again satisfies the following integral equation

$$\pm(t - t_o) = \int_{\chi_o}^{\chi} \frac{d\chi}{\left[g^2 + \frac{b^2}{\beta}\left(\alpha\chi^2 + 2\frac{d\theta}{dt_f}\right)\right]^{\frac{1}{2}}} \tag{59}$$

For a given initial condition $\chi(t_o) = \chi_o$, the trajectory χ^* is completely determined. The final time t_f may be found by the following

$$t_f = t_o \pm \int_{\chi_o}^{\chi_f} \frac{d\chi}{[g^2+\frac{b^2}{\beta}(\alpha\chi^2+2\frac{d\theta}{dt}_f]^{\frac{1}{2}}} \tag{60}$$

An example problem will be next worked out to illustrate the method.

4. Example II

To illustrate the method of solution, consider the problem of minimizing the performance function:

$$V = t_f + \frac{1}{2}\int_0^{t_f} (u^2 + 3\chi^2)dt \tag{61}$$

subjected to the nonlinear differential constraint

$$\dot{\chi} = \chi^2 + u(t) \tag{62}$$

such that the system is driven from any arbitrary initial state $\chi(0)$ to the origin. Note that this represents a nonlinear minimum time problem. Equation (56) gives the optimal feed back law $u^*(\chi)$ as

$$u^*(\chi) = -\chi^2 \pm [\chi^4+3\chi^2+2]^{\frac{1}{2}} \tag{63}$$

where(+)is used for $\chi(0) < 0$ and(-) for $\chi(0) > 0$. Assume next that we wish to find a family of optimal trajectories for $\chi(0) = \pm 2$ and $\chi(0) = \pm 1$. The optimal trajectory satisfies equation (59), thus

$$t = \pm \int_{\chi_o}^{\chi} \frac{d\chi}{(\chi^4+3\chi^2+2)^{\frac{1}{2}}} \tag{64}$$

By making the complex transformation

$$\chi = -j \operatorname{Sin}(\varphi) \tag{65}$$

the integral in equation (64) becomes:

$$t = \pm 0.707 j \int_{\varphi_o}^{\varphi} \frac{d\varphi}{(1-K^2 \operatorname{Sin}^2\varphi)^{\frac{1}{2}}} \tag{66}$$

where in this case $K^2 = 0.50$. This integral may be recognized as an elliptic integral of the first kind which is written as

$$F(K,\varphi) = \int_{o}^{\varphi} \frac{d\varphi}{(1-K^2 \operatorname{Sin}^2\varphi)^{\frac{1}{2}}} \tag{67}$$

By breaking the integral in equation (66) into two parts the following results

$$t = \pm 0.707 j \, [F(K,\varphi) - F(K,\varphi_o)] \tag{68}$$

Here φ, φ_o are complex numbers. Next another transformation is introduced to get rid of the complex arguments,

$$\varphi = j\tilde{\varphi} \tag{69}$$

where $\tilde{\varphi}$ is real, and using an important property of the elliptic integrals that is written as:

$$F(K,\varphi) = jF(K',\tilde{\varphi}) \tag{70}$$

where $$K' = (1-K^2)^{\frac{1}{2}} \tag{71}$$

and $$\tilde{\varphi} = -j\varphi \tag{72}$$

Equation (68) becomes

$$t = \pm 0.707 \, [F(K',\tilde{\varphi})-F(K',\tilde{\varphi}_o)] \tag{73}$$

Note that because of equation (72) we have

$$\chi = \operatorname{Sinh}(\tilde{\varphi}) \tag{74}$$

For the case when $\chi_o = \pm 1$, $\tilde{\varphi}_o = 50.4^o$ and $F(K',\tilde{\varphi}_o) = 0.935$. When $\chi_o = \pm 2$, $\tilde{\varphi}_o = 83^o$ and $F(K',\tilde{\varphi}_o) = 1.68$. Also note that since $\chi_f = 0$, then $F(K',\tilde{\varphi}_f) = 0$. The final time t_f is determined by

$$t_f = \pm 0.707 \, [F(K',\tilde{\varphi}_f)-F(K',\tilde{\varphi}_o)] \tag{75}$$

Thus, for $\chi_o = \pm 1$, $t_f = 0.66$ second

and $\chi_o = \pm 2$, $t_f = 1.19$ second

Now the solution is completely determined. The optimal return function $V(X)$, a solution of the Hamilton Jacobi equation is given by

$$V(X,t) = \frac{X^3}{3} \pm \int_0^X (X^4+3X^2+2)^{\frac{1}{2}}\, dX + \theta(t)+C_1 \tag{76}$$

where $\theta(t) = t$ and C_1 is a constant of integration. Here (+) for $X(0) > 0$ and (-) for $X(0) < 0$

let
$$I(X) = \int_0^X (X^4+3X^2+2)^{\frac{1}{2}}\, dX \tag{77}$$

equation (76) becomes:

$$V(X,t) = \frac{X^3}{3} \pm I(X) + t \tag{78}$$

Note that because of the boundary condition $(V(0,t_f) = t_f)$, $C_1 = 0$. The solution for $I(X)$ may be shown to be a combination of elliptic integrals of the first and second kind. It is omitted here because it does not add anyting to the control problem. Plots for $u^*(X)$ and a family of $X^*(t)$ are shown in figure III, IV.

5. General Classification of Problem ($t_f = \infty$)

It was seen that an exact solution to the Hamilton Jacobi Equation was found for a one dimensional nonlinear class of problems. In most physical systems one is dealing with a multidimensional problem. In this section a class of m dimensional system will be derived such that the solution obtained previously apply for the m-dimensional case.

Consider the class of systems such that:

$$\overset{(m)}{y}(t) = g(\overset{(m-1)}{y}) + b\,u(t) \tag{79}$$

where $\overset{(m)}{y}(t) = \frac{d^m y}{dt^m}$ and g is a nonlinear function in the (m-1)th derivative of the variable y. Assume next, that we consider a class of problems in which we wish to minimize the cost function

$$V = \tfrac{1}{2}\int_{t_o}^{t_f}(\alpha x_m^{\,2} + \beta u^2)\,dt \tag{80}$$

where the state variables are chosen the following way

$$x_1 = y \tag{81}$$

$$\dot{x}_1 = x_2 \tag{82}$$

$$\vdots$$

$$\dot{x}_{m-1} = x_m \tag{83}$$

$$\dot{x}_m = g + bu \tag{84}$$

The Hamilton Jacobi Equation for this class of problem takes the following form:

$$\frac{\delta V}{\delta t} - \frac{1}{2}\frac{b^2}{\beta}\left(\frac{\delta V}{\delta X_m}\right)^2 - g(X_m) \cdot \frac{\delta V}{\delta X_m} + \tfrac{1}{2}\,\alpha X_m{}^2 - \sum_{i=1}^{m-1} \frac{\delta V}{\delta X_i}\cdot X_{i+1} = 0 \qquad (85)$$

If no constraints are applied to the final state and if the final time is extended to infinity, it is easy to show that:

$$\frac{\delta V}{\delta t} = \frac{\delta V}{\delta X_1} = \frac{\delta V}{\delta X_2} = \cdots = \frac{\delta V}{\delta X_{m-1}} = 0 \qquad (86)$$

Hence, one can solve for $\frac{dV}{dX_m}$ as:

$$\frac{dV}{dX_m} = -\frac{\beta}{b^2}\, g(X_m) \pm \left[\frac{\beta^2}{b^4}\, g^2 + \frac{\alpha\beta}{b^2}\, X_m{}^2\right]^{\frac{1}{2}} \qquad (87)$$

The optimal control is given by:

$$u^* = \frac{1}{b}\left\{-g(X_m) \pm \left[g^2 + \frac{\alpha b^2}{\beta} X_m{}^2\right]^{\frac{1}{2}}\right\} \qquad (88)$$

The optimal trajectory $X_m^*(t)$ is given as:

$$\pm(t-t_o) = \int_{X_m(0)}^{X_m} \frac{dX_m}{\left[g^2 + \frac{\alpha b^2}{\beta} X_m{}^2\right]^{\frac{1}{2}}} \qquad (89)$$

With the knowledge of the m-vector X(0), the desired optimal output $y^*(t)$ is found by successive integration

$$y^*(t) = \underbrace{\iint\cdots\int}_{m-1^{th}\ \text{integration}} X_m(t)\, dt \qquad (90)$$

with $$X(o) = [\chi_1(o)\ \chi_2(o)\cdots\chi_m(o)]^T \tag{91}$$

To illustrate this result, consider the problem of minimizing

$$V = \frac{1}{2}\int_0^\infty (\dot{y}^2 + u^2)\ dt \tag{92}$$

subjected to:

$$\ddot{y} = -\dot{y}^3 + \dot{y} + u \tag{93}$$

with $$y(o) = \dot{y}(o) = 1$$

By letting $\dot{y} = \chi$, the optimal control is given by:

$$u^* = \dot{y}^3 - \dot{y} - (\dot{y}^6 - 2\dot{y}^4 + 2\dot{y}^2)^{\frac{1}{2}} \tag{94}$$

and

$$\dot{y}^*(t) = \left\{\frac{4.828\ \exp\ (2.828t)}{[0.707 + 1.707\exp(2.828t)]^2 - 1}\right\}^{\frac{1}{2}} \tag{95}$$

It can be shown by transformation of variables that $y^*(t)$ is an elliptic integral of the first kind. An exact solution for $y^*(t)$ becomes more and more involved for higher order systems. At any rate the problem becomes just a mere integration. For this particular 2nd order system the integration could have been carried out to yield an elliptic integral of the first kind.

DATA

± χ	|u*|
0.20	0.47
0.30	0.68
0.40	0.86
0.50	1.00
0.60	1.096
0.70	1.143
0.80	1.138
0.90	1.087
1.0	1.000
1.5	0.526
2.0	0.320
2.5	0.236
3.0	0.186

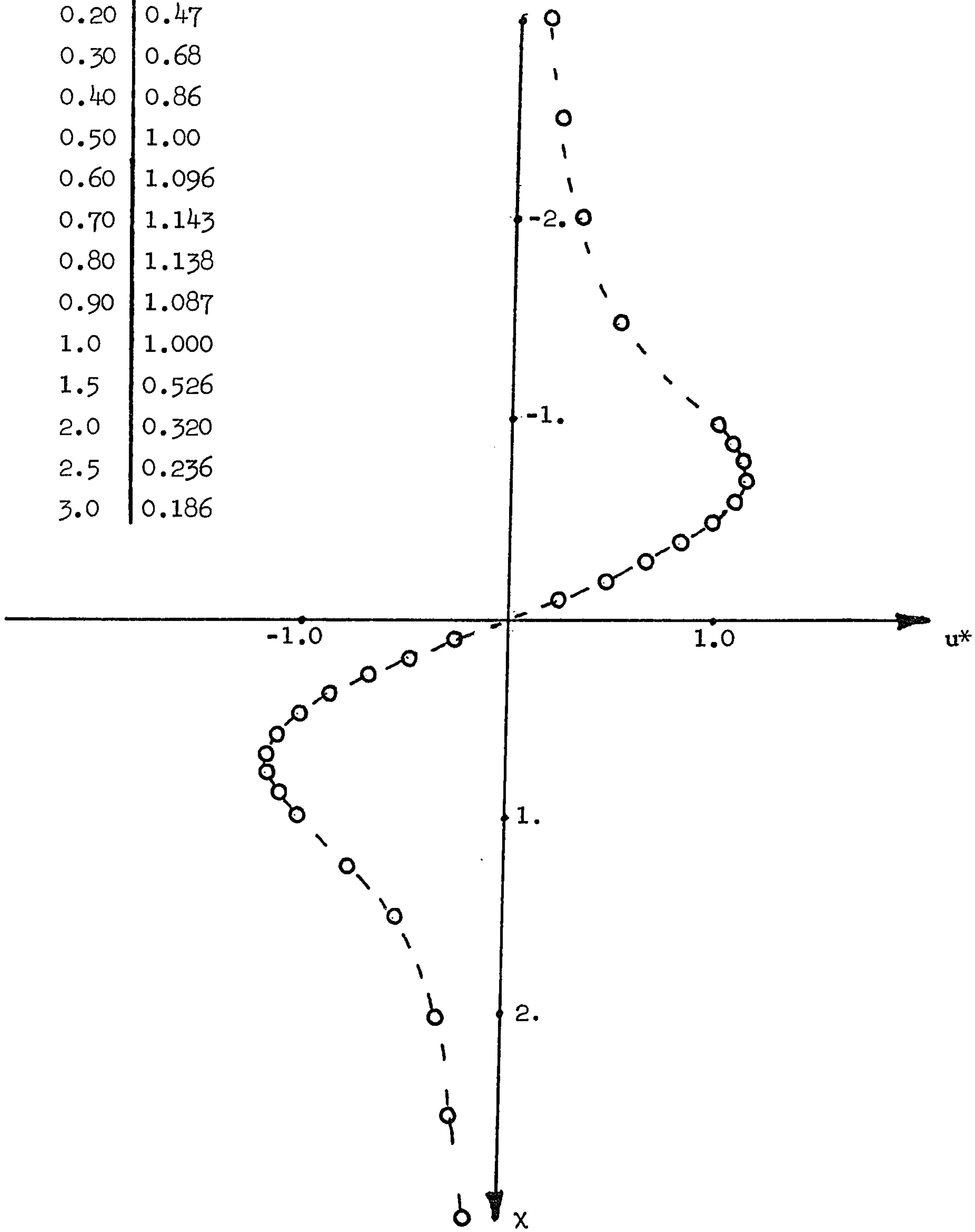
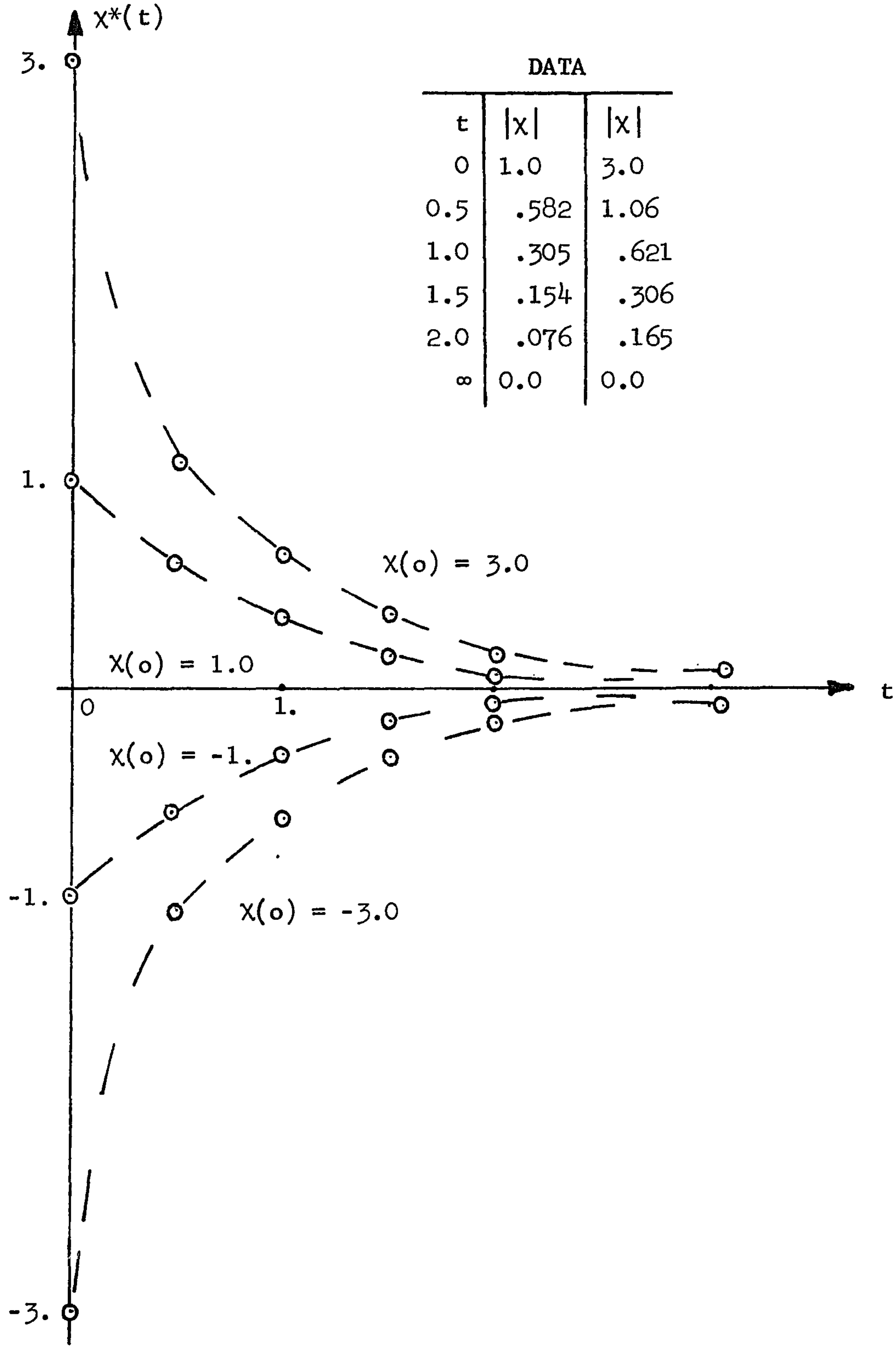


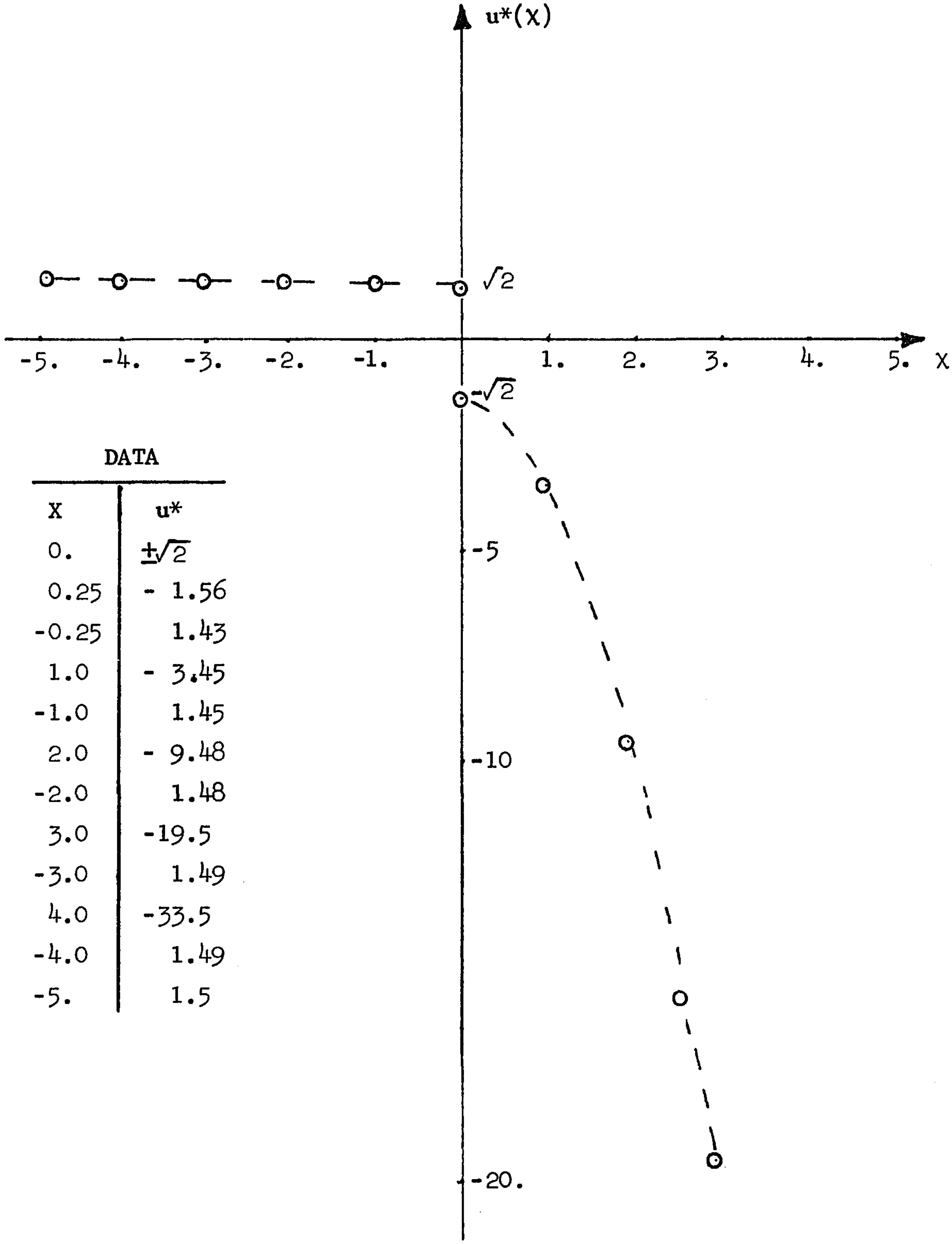
Figure 2. Optimal Control Law u*(χ) for Example I.



DATA

t	|x|	|x|
0	1.0	3.0
0.5	.582	1.06
1.0	.305	.621
1.5	.154	.306
2.0	.076	.165
∞	0.0	0.0

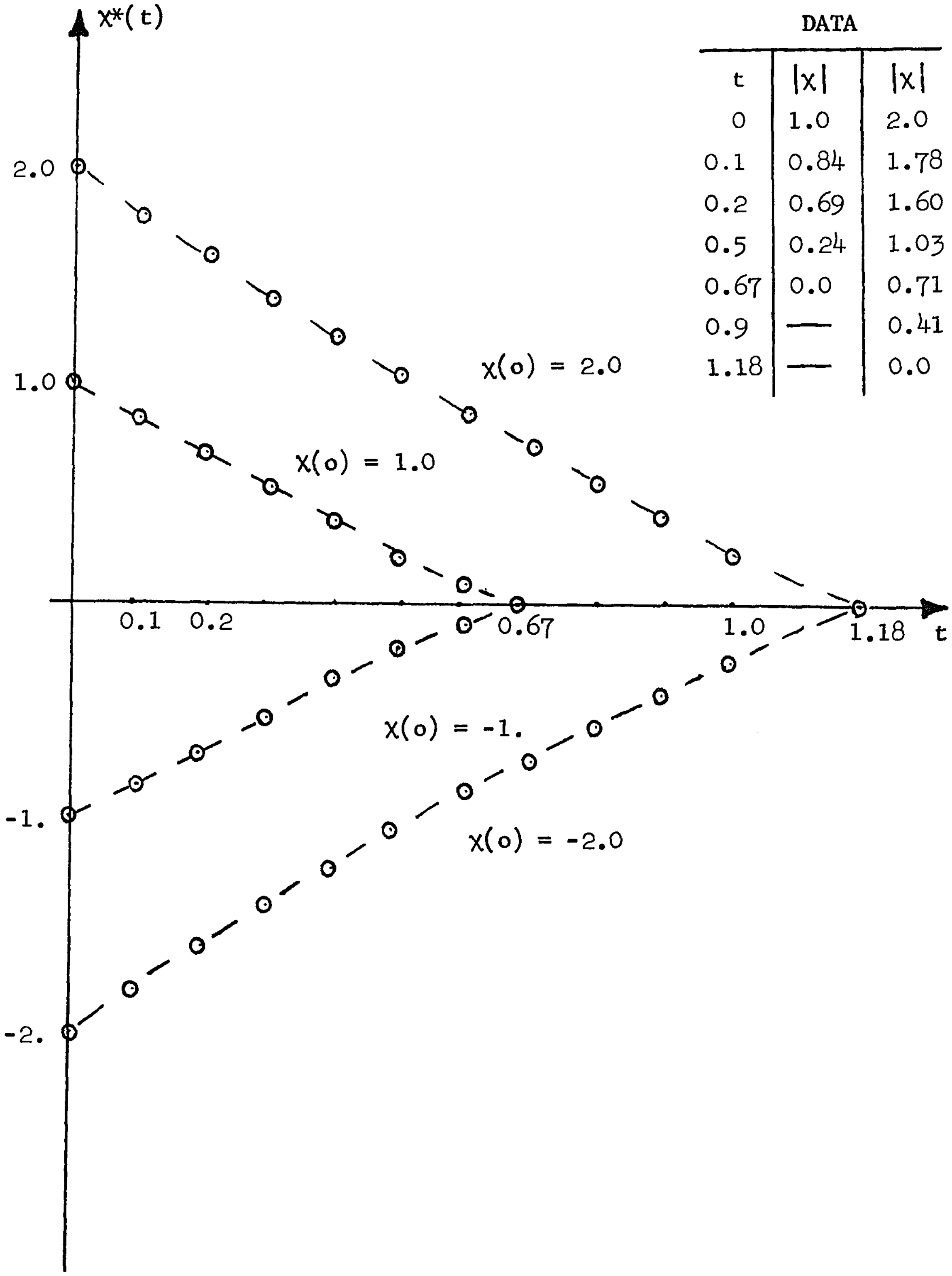
Figure 3. Family of Optimal Trajectories for Example I.



DATA

X	u*
0.	$\pm\sqrt{2}$
0.25	- 1.56
-0.25	1.43
1.0	- 3.45
-1.0	1.45
2.0	- 9.48
-2.0	1.48
3.0	-19.5
-3.0	1.49
4.0	-33.5
-4.0	1.49
-5.	1.5

Figure 4. Optimal Control Law $u^*(\chi)$ for Example II.



DATA

t	|x|	|x|
0	1.0	2.0
0.1	0.84	1.78
0.2	0.69	1.60
0.5	0.24	1.03
0.67	0.0	0.71
0.9	—	0.41
1.18	—	0.0

Figure 5. Family of Optimal Trajectories for Example II.

CHAPTER VI

NONLINEAR CONTROL OF FEEDBACK SYSTEMS

The theory of nonlinear control systems is stated as follows from reference [10].

"The most important thing about feedback systems is that they can be controlled either linearly or nonlinearly. If the system has inherent nonlinear characteristics, it is possible to introduce linear compensating networks to improve the system performance. If optimum system performance desired in a control system cannot be obtained by combination of linear transfer functions, it is most desirable to introduce nonlinearities into the control system. If the system has inherent nonlinear characteristics, linear compensation and nonlinear control can both be introduced."

In short, the newest aspect of feedback system control is the development of the theory of nonlinear control system.

In the discussion and analysis of feedback systems only systems with nonlinear control in the feedback branch will be studied. Although, present efforts are mostly concentrated in introducing nonlinear control in the forward branch, nonlinear control in the feedback branch is considered mainly because the analysis is simpler and the equations describing the system are much easier to handle.

The important thing that one has to forsee in the study of nonlinear systems is that almost all the methods that have been used for linear systems cannot be applied to nonlinear systems. The principle of superposition is not applicable to nonlinear systems. Heaviside, Fourrier, and Laplace transforms, which have aided in transient solutions and supplied transfer functions in feedback systems, are no longer applicable. Nevertheless, by proper selection of the state variables and for a certain class of nonlinear systems and performance functions exact solutions will be obtained for the dynamic optimization problem.

1. Formulation of the Problem Representing Nonlinear Control In Feedback Branch, and Selection of State Variables

Consider the feedback control system shown in the block diagram of Figure I, where $G_1(s)$ and $G_2(s)$ are the linear transfer functions respectively in the forward and the feedback branch, G_{n2} being the nonlinear function in the feedback branch. Thus, if the controller is in the feedforward branch as shown, the activating signal (e) is equal to the difference between the input function (u) and the feedback signal (b).

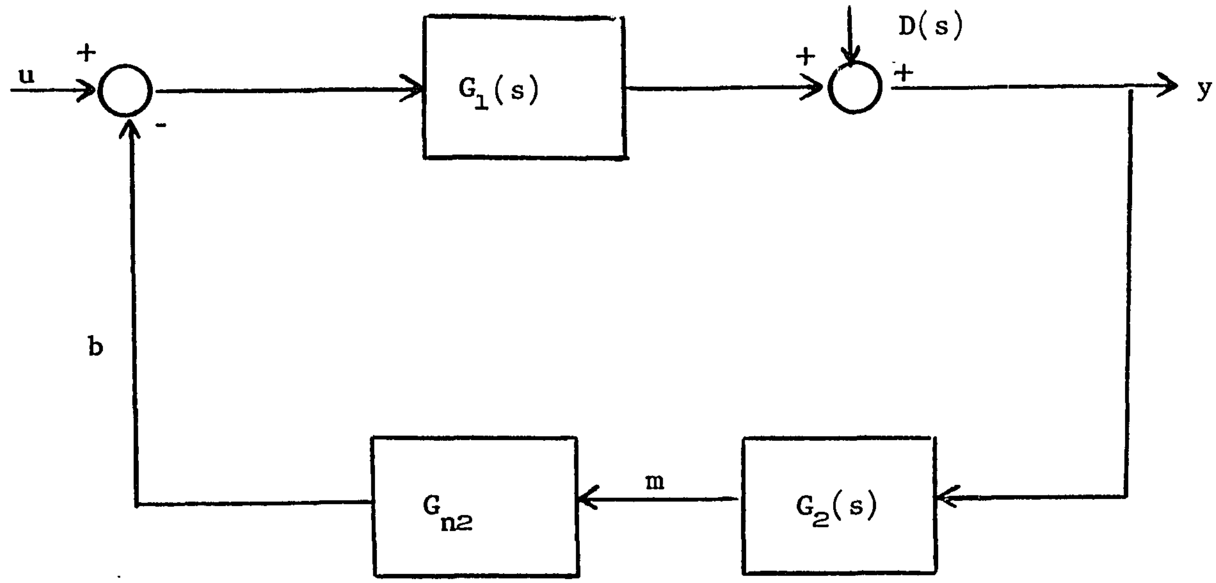


Figure 6: Block Diagram of the Nonlinear System

All the required notations in this chapter are as follows:

$U(t)$, $U(s)$	=	Input function, its Laplace transformation
$b(t)$, $B(s)$	=	Feedback singnal, its Laplace transformation
$e(t)$, $E(s)$	=	Actuating signal, its Laplace transformation
$m(t)$, $M(s)$	=	Control effort, its Laplace transformation
$Y(t)$, $Y(s)$	=	Output variable, its Laplace transformation
$d(t)$, $D(s)$	=	Disturbance function, its Laplace transformation
s	=	complex number
t_o, t_f	=	initial and final time

It will be assumed that the system represented in Fig. I belongs to a class C_o such that the following is satisfied

a) $G_1(s)$ is a linear transfer function in feedforward branch. It does not possess any zeros and its poles are distinct. We write

$$G_1(s) = \left[\sum_{i=0}^{n} a_i \, s^i \right]^{-1} \tag{1}$$

where all roots of $\sum_{i=0}^{n} a_i s_i = 0$ are distinct. If S_1 is the set of all of these poles, then:

$$s_i \in S_1 \; ; \quad i=1,\cdots,n$$

b) $G_2(s)$ is a linear transfer function in the feedback branch of the form

$$G_2(s) = \left[\sum_{i=0}^{k} b_i \, s^i \right]^{-1} \tag{2}$$

when $k < n$ and all roots of $G_2^{-1} = 0$ are distinct. If S_2 is the set containing all of these poles, then

$$s_i \in S_2 \; ; \quad i=1,\cdots,k$$

It will be also assumed that

$$s_i \left\{ i=1,\cdots,n \right\} \notin S_2$$

and $$s_i \left\{ i=1,\cdots,k \right\} \notin S_1$$

i.e. $$S_1 \cap S_2 = \emptyset \text{ (empty set)}$$

c) d(t) belongs to a class of disturbance function such that:

d(t) is bounded and continuous with continuous derivatives in $[t_0, t_f]$

With all these assumptions in mind, one may now write the equations relating the different variables of the system

$$E(s) = U(s) - B(s) \tag{3}$$

$$Y(s) = D(s) + E(s)\, G_1(s) \tag{4}$$

$$M(s) = Y(s)\, G_2(s) \tag{5}$$

Let X be the nth dimensional state vector such that

$$p(t) = c^T X \tag{6}$$

with $$c^T = [1\cdots1]^T, \qquad X = [x_1 \cdots x_n]^T \tag{7}$$

It is obvious that p(t) is the output to the system whose transfer function is $G_1(s)$ and input e(t).

One may write

$$P(s) = \frac{E(s)}{a_n(s-s_1)(s-s_2)\cdots(s-s_n)} \tag{8}$$

where P(s) is the transform of p(t). Upon applying the result of Chapter III, the following results:

$$P(s) = \frac{1}{a_n}\left\{\sum_{i=1}^{n} \frac{\alpha_i\, E(s)}{s-s_i}\right\} \tag{9}$$

Note that α_i $(i=1,\cdots,n)$ are the residues. Upon assigning a state whose transform is given by:

$$X_i(s) = \frac{\alpha_i E(s)}{a_n(s-s_i)} \qquad i=1,\cdots,n \tag{10}$$

the following state equation results

$$\dot{X} = \Lambda_1 X + \frac{\bar{\alpha}}{a_n} \cdot e(t) \tag{11}$$

with output

$$p(t) = c^T X \tag{12}$$

and

$$y(t) = d(t) + p(t) \tag{13}$$

Here
$$\bar{\alpha} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T \tag{14}$$

$$\Lambda_1 = \begin{bmatrix} s_1 & & & \\ & s_2 & & 0 \\ & & \ddots & \\ 0 & & & s_n \end{bmatrix} \tag{15}$$

The control effort is related to y(t) by the following relation

$$y(t) = \sum_{i=0}^{k} b_i \, m^{(i)}(t) \tag{16}$$

Now, using the same procedure for the transfer function $G_2(s)$ as the one used for $G_1(s)$ and with the help of equation (5), one can write

$$M(s) = \frac{Y(s)}{b_k(s-s_{1+n})(s-s_{2+n})\cdots(s-s_{k+n})} \tag{17}$$

where $s_{n+1}, \cdots, s_{n+k}$ are the k-distinct poles of $G_2(s)$. By assigning the state whose transform is given by:

$$Z_{i+n}(s) = \frac{\gamma_i \, Y(s)}{b_k(s-s_{i+n})} \tag{18}$$

where γ_i $(i=1,\cdots,k)$ are the residues of the system described in equation (17), the following state equation results

$$\dot{Z} = \Lambda_2 Z + \frac{\bar{\gamma}}{b_k} \cdot y(t) \tag{19}$$

with output
$$m(t) = c^T \cdot Z \tag{20}$$

Here,
$$\bar{\gamma} = \begin{bmatrix} \gamma_1 & \cdots & \gamma_k \end{bmatrix}^T \tag{21}$$

and
$$\Lambda_2 = \begin{bmatrix} s_{1+n} & & \\ & \ddots & \\ & & s_{k+n} \end{bmatrix} \tag{22}$$

The next step is to combine the two transfer functions $G_1(s)$ and $G_2(s)$ by recognizing that

$$M(s) = D(s) \cdot G_2(s) + E(s) \cdot G_1(s) \cdot G_2(s) \tag{23}$$

or:

$$M(s) = \frac{E(s)}{(a_n b_k)(s-s_1)\cdots(s-s_n)(s-s_{1+n})\cdots(s-s_{k+n})} \tag{24}$$

$$+ \frac{D(s)}{b_k(s-s_{1+n})\cdots(s-s_{k+n})}$$

Let $\varphi(t)$ be a function of time such that its transform is given by

$$\varphi(s) = \frac{D(s)}{b_k(s-s_{1+n})\cdots(s-s_{k+n})} \tag{25}$$

Consequently:
$$\varphi(s) = \sum_{i=1}^{k} \frac{\gamma_i \cdot D(s)}{b_k(s-s_{i+n})} \tag{26}$$

Upon inversion back to time domain and with the use of the Convolution theorem the following is obtained.

$$\varphi(t) = \sum_{i=1}^{k} \frac{\gamma_i}{b_k} \int_0^t d(\tau) \cdot \exp\left[s_{i+n}(t-\tau)\right] d\tau \tag{27}$$

Since we are dealing with a finite summation, we can write

$$\varphi(t) = \int_0^t \sum_{i=1}^{k} \frac{\gamma_i}{b_k} d(\tau) \cdot \exp\left[s_{i+n}(t-\tau)\right] d\tau \tag{28}$$

The first term of equation (24) may be written as:

$$\text{1st term} = \sum_{i=1}^{n+k} \frac{E(s)\ \epsilon_i}{b_k a_n(s-s_i)} \tag{29}$$

where ϵ_i $(i=1,\cdots,k+n)$ are the residues of the product $G_1(s)\cdot(G_2(s)$. Note that equation (29) is only valid for distinct poles, which was the assumption at the beginning of this chapter. By selecting the state variables whose transforms are:

$$R_i(s) = \frac{\epsilon_i \, E(s)}{b_k a_n (s-s_i)} \; ; \quad i=1,\cdots,n+k \tag{30}$$

the following state equation results

$$\dot{\bar{r}} = \Lambda \, \bar{r} + \frac{\bar{\epsilon}}{b_k a_n} \cdot e(t) \tag{31}$$

with output equation as

$$m(t) = \varphi(t) + \sum_{i=1}^{n+k} r_i(t) \tag{32}$$

where, $$\bar{r} = \left[r_1 \cdots r_{n+k} \right]^T, \quad n+k \quad \text{vector} \tag{33}$$

$$\bar{\epsilon} = \left[\epsilon_1 \cdots \epsilon_{n+k} \right]^T, \quad n+k \quad \text{vector} \tag{34}$$

and

$$\Lambda = \begin{bmatrix} s_1 & & & \bigcirc \\ & s_2 & & \\ & & \ddots & \\ \bigcirc & & & s_{n+k} \end{bmatrix} \tag{35}$$

$\varphi(t)$ is given in equation (28).

It is important to note that, although we are dealing with a nonlinear system with input function $u(t)$, it is easy to see how the actuating signal $e(t)$ plays the role of an input function to the linear system represented in equations (11) through (15). The nonlinear network G_{n2} in the feedback branch is assumed to give the following functional relation between the feedback system signal $b(t)$ and the control effort $m(t)$ and its derivatives. Thus,

$$b(t) \;=\; g_{n2}\left[m, \overset{(1)}{m}, \ldots ,\right] \tag{36}$$

In the next chapter the optimization problem will be formulated and a solution will be otained for a specified initial condition. The differential system described in equation (31) through (35) will be used as the constraint to the minimization problem.

CHAPTER VII

OPTIMIZATION SOLUTION OF A FEEDBACK SYSTEM

1. Problem Formulation

In most feedback systems linear or nonlinear it is desirable to achieve a best performance. This is obtained by minimizing the cost function of the form

$$V = \frac{1}{2} \int_{t_0}^{t_f} e^2 \, dt \tag{1}$$

subjected to the constraints, as stated in Chapter VI, equations 31, 32 and 36.

$$\dot{\overline{r}} = \Lambda \overline{r} + \underset{b_k \overline{a}_n}{\overline{\varepsilon}} \cdot e(t) \tag{2}$$

$$m(t) = \varphi(t) + \sum_{i=1}^{n+k} \tag{3}$$

and

$$b(t) = g_{n_2} \tag{4}$$

Such that the system represented in these equations is driven from a specified initial condition to a specified final one. We denote these given conditions by

$$Y(t_0) = Y_0 = [y(0) \;\; \overset{(1)}{y}(0) \;\; \cdots \;\; \overset{(n-1)}{y}(0)]^T \tag{5}$$

and

$$Y(t_f) = Y_T = [y(t_f) \overset{(1)}{y}(t_f) \cdots \overset{(n-1)}{y}(t_f)]^T \tag{6}$$

Next, it is worthwhile to show that the dynamic optimization of the nonlinear system with respect to either the input u or the function g_{n2}, reduces to the optimization of the linear system with respect to the actuating signal e. For instance, consider the system given in equations (2) through (4) and eliminate the third constraint $b(t) = g_{n2}$ by substituting $e(t)$ by $u - g_{n2}$. The following is obtained:

$$\dot{\bar{r}} = \Lambda \bar{r} + \frac{\bar{\varepsilon}}{b_k a_n} [u(t) - g_{n2}] \tag{7}$$

with output

$$m(t) = \varphi(t) + \sum_{i=1}^{n+k} r_i(t) \tag{8}$$

The cost function to be minimized becomes:

$$V = \frac{1}{2} \int_{t_0}^{t_f} \frac{1}{2} (u - g_{n2})^2 \, dt \tag{9}$$

Next, the Hamiltonian is constructed to yield:

$$H[\bar{r}, u(t), \lambda(t), t] = \frac{1}{2} (u - g_{n2})^2 + \bar{\lambda}^T \left\{ \Lambda \bar{r} + \frac{\bar{\varepsilon}}{b_k a_n} u - \frac{\bar{\varepsilon}}{b_k a_n} g_{n2} \right\} \tag{10}$$

where $\bar{\lambda}$ is the (n+k) co-state vector or Lagrange multiplier.

Now, with the use of the "Pontryagin Minimum Principle"

$$\frac{\delta H}{\delta u} = o \tag{11}$$

and

$$\frac{\delta H}{\delta \bar{r}} = -\dot{\bar{\lambda}} \tag{12}$$

We get:

$$u^*(t) - g^*_{n2} + \frac{\bar{\lambda}^T \bar{\epsilon}}{b_k a_n} = o \tag{13}$$

and

$$-\frac{\delta g^*_{n2}}{\delta \bar{r}} (u^* - g^*_{n2}) + \Lambda \bar{\lambda} - \frac{\delta}{\delta \bar{r}} \left\{ \frac{\bar{\lambda}^T \bar{\epsilon}}{b_k a_n} \quad g^*_{n2} \right\} = -\dot{\bar{\lambda}} \tag{14}$$

Note that the * indicates optimal conditions.

Equations (13) and (14) are simplified to yield the following:

$$\dot{\bar{\lambda}} = -\Lambda \bar{\lambda} \tag{15}$$

and

$$e^*(t) = -\frac{\bar{\lambda}^T \bar{\epsilon}}{b_k a_n} \tag{16}$$

This represents the same result if one minimizes the cost

$V = \frac{1}{2} \int_{t_o}^{t_f} e^2 \, dt$ subjected to the constraint

$$\dot{\bar{r}} = \Lambda \bar{r} + \frac{\bar{\epsilon}}{b_k a_n} \cdot e(t) \tag{17}$$

2. Optimal Trajectories and Output Determination

Assume that the system is to be driven from the given initial state Y_O to the given final state Y_T both given in equations (5) and (6). The optimal control effort $m^*(t)$ is found in a similar manner as equation (3.11), except now one takes into account the contribution of the disturbance function through the function $\varphi(t)$. For instance,

$$m^*(t) = \varphi(t) + \sum_{i=1}^{n+k} r_i(0).\exp(s_i t) + \sum_{i=1}^{n+k} \sum_{j=1}^{n+k} \frac{\epsilon_i \epsilon_j . \lambda_i(0)}{b_k{}^2 a_n{}^2 (s_i + s_j)} \left[\exp(-s_i t) - \exp(s_j t) \right] \tag{18}$$

where ϵ_i, $r_i(0)$, $\lambda_i(0)$ $(i = 1, \cdots, n + k)$ are still to be determined. With the use of equations (3.7) and (3.8), the optimal actuating signal is given by:

$$e^*(t) = - \sum_{i=1}^{n+k} \frac{\lambda_{i(0)} \ \epsilon_i \ \exp(-s_i t)}{a_n \ b_k} \tag{19}$$

The residue vector and the initial state vector are determined in the same manner as in Chapter III by decoupling the real and imaginary parts of vectors and matrices. Thus,

$$\bar{\epsilon} = V^{-1} K_O \tag{20}$$

or

$$\bar{\epsilon}^{RI} = \left(V^{RI}\right)^{-1} K_{OO} \tag{21}$$

where:

$$K_{OO} = \left[K_O{}^T \;\middle|\; 0_{1 \times k+n} \right]^T \qquad 2n + 2k \text{ vector} \tag{22}$$

$$K_O = [0 \quad 0 \quad \cdots \quad 1]^T \qquad n + k \text{ vector} \tag{23}$$

and

$$V^{RI} = \left[\begin{array}{c|c} V^R & -V^I \\ \hline V^I & V^R \end{array} \right] \qquad 2n + 2k \times 2n + 2k \text{ matrix} \tag{24}$$

Note that V is the Vandermonde matrix whose elements are defined in terms of the poles s_i $(1, \cdots, n+k)$.

Because of the presence of the transfer function $G_2(s)$ in the feedback branch, the problem requires also the knowledge of $(K-1)^{th}$ initial conditions for $m(t)$. For example, we will assume that in addition to equations (5) and (6) the following is known:

$$M(t_O) = \left[m(0) \quad \overset{(1)}{m}(0) \quad \cdots \quad \overset{(k-1)}{m}(0) \right]^T \tag{25}$$

and

$$M(t_f) = \left[m(t_f) \quad \overset{(1)}{m}(t_f) \quad \cdots \quad \overset{(k-1)}{m}(t_f) \right]^T \tag{26}$$

This is easily understood if the final n + k order differential equation is written with $m(t)$ as the variable, thus the requirement of the knowledge of $(k + n - 1)$ initial and final conditions.

Next, the initial condition vector $\bar{r}(0)$ is determined by the equation

$$\bar{r}(0) = V^{-1}\left\{\begin{bmatrix} M(0) \\ \hline \tilde{M}(0) \end{bmatrix} - \tilde{\varphi}_0\right\} \tag{27}$$

where $M(0) = M(t_0)$ is given in equation (25) and is a k vector. $\tilde{M}(0)$ is a n vector given by the following:

$$\tilde{M}(0) = \begin{bmatrix} \overset{(k)}{m}(0) & \overset{(k+1)}{m}(0) & \cdots & \overset{(k+n-1)}{m}(0) \end{bmatrix}^T \tag{28}$$

and

$$\tilde{\varphi}(0) = \tilde{\varphi}_0 = \begin{bmatrix} \varphi(0) & \overset{(1)}{\varphi}(0) & \cdots & \overset{(k+n-1)}{\varphi}(0) \end{bmatrix}^T \tag{29}$$

Still undetermined is the vector $\tilde{M}(0)$, but by using

$$y(t) = \sum_{i=0}^{k} b_i \overset{(i)}{m}(t) \tag{30}$$

and by expanding and solving for the K^{th} derivative of m(t) and evaluating at the initial and final times, one gets:

$$\overset{(k)}{m}(0) = \frac{y(0)}{b_k} - \sum_{i=0}^{k-1} b_i \overset{(i)}{m}(0) \tag{31}$$

$$\overset{(k+1)}{m}(0) = \frac{\overset{(1)}{y}(0)}{b_k} - \sum_{i=0}^{k-1} b_i \overset{(i+1)}{m}(0) \tag{32}$$

By successive substitution the vector $\tilde{M}(0)$ can be obtained. The same procedure may be applied to get $\tilde{M}(t_f)$. For instance,

$$\overset{(k)}{m}(t_f) = \frac{y(t_f)}{b_k} - \sum_{i=0}^{k-1} bi \overset{(i)}{m}(t_f) \tag{33}$$

and

$$\overset{(k+1)}{m}(t_f) = \frac{\overset{(1)}{y}(t_f)}{b_k} - \sum_{i=0}^{k-1} bi \overset{(i+1)}{m}(t_f) \tag{34}$$

Equation (27) is now completely determined. At the final time a similar procedure yields

$$\bar{r}(t_f) = V^{-1}\left\{\begin{bmatrix} M(t_f) \\ \hline \tilde{M}(t_f) \end{bmatrix} - \tilde{\varphi}_f\right\} \tag{35}$$

where

$$\tilde{\varphi}_f = \begin{bmatrix} \varphi(t_f) & \overset{(1)}{\varphi}(t_f) & \cdots & \overset{(k+n-1)}{\varphi}(t_f) \end{bmatrix}^T \tag{36}$$

The co-state initial conditions are obtained in the same manner as in Chapter III. For instance, we define the following terms:

$$R_t i\ell = \frac{\epsilon_i \epsilon_\ell}{b_k{}^2 a_n{}^2 (s_i + s_\ell)}\left\{\exp(-s_i t) - \exp(s_\ell t)\right\} \tag{37}$$

$$i, \ell = 1, \cdots, k+n$$

$$\Psi_t = \left[\begin{array}{c|c} \text{Real}[\exp(\Lambda t)] & -\text{Im}[\exp(\Lambda t)] \\ \hline \text{Im}[\exp(\Lambda t)] & \text{Real}[\exp(\Lambda t)] \end{array}\right] \tag{38}$$

Where again Real indicates real part and Im imaginary parts.

$$P_t = \left[\begin{array}{c|c} \mathrm{Real}(R_t{}^T) & -\mathrm{Im}(R_t{}^T) \\ \hline \mathrm{Im}(R_t{}^T) & \mathrm{Real}(R_t{}^T) \end{array}\right] \tag{39}$$

Note that:

$$P_T = P_t\ (t = t_f) \tag{40a}$$

and

$$\Psi_T = \Psi_t\ (t = t_f) \tag{40b}$$

With the help of equations (3-59) through (3.67), the co-state initial condition vector is determined by

$$\bar{\lambda}^{RI}(0) = P_T{}^{-1} \cdot \bar{r}^{RI}(t_f) - P_T{}^{-1}\ \Psi_T\ \bar{r}^{RI}(0) \tag{41}$$

with:

$$\bar{\lambda}^{RI}(0) = \left[\begin{array}{c} \lambda^R \\ \hline \lambda^I \end{array}\right] \quad \text{and} \quad \bar{r}^{RI}(0) = \left[\begin{array}{c} \bar{r}^R(0) \\ \hline \bar{r}^I(0) \end{array}\right] \tag{42}$$

The optimal trajectory vector is given by

$$\bar{r}^{*RI}(t) = \left[\Psi_t - P_t\ P_T{}^{-1}\ \Psi_T\right] \cdot \bar{r}^{RI}(0) + P_t\ P_T{}^{-1}\ \bar{r}^{RI}(t_f) \tag{43}$$

or even better as:

$$\bar{r}^{*RI}(t) = P_t\ \bar{\lambda}^{RI}(0) + \Psi_t\ \bar{r}^{RI}(0) \tag{44}$$

The optimal control effort as:

$$m^*(t) = \varphi(t) + \left[\begin{array}{c|c} C^T & 0_{1 \times k+n} \end{array}\right]^T \cdot \bar{r}^{*RI}(t) \tag{45}$$

where

$$C = [\,1 \;\; 1 \;\; \cdots \;\; 1\,]^T \text{, } n + k \text{ vector} \tag{46}$$

The optimal value of the actuating signal is:

$$e^*(t) = \frac{-1}{b_k a_n} \sum_{i=1}^{n+k} \lambda_i(t) \cdot \epsilon_i \tag{47}$$

Finally, for a given value of g_{n2} (nonlinear function of m), the required optimal input $u^*(t)$ that will minimize the given cost function is determined by:

$$u^*(t) = g^*_{n2} + e^*(t) \tag{48}$$

Here

$$g^*_{n2} = g_{n2} \left\{ m^*(t),\ t \right\} \tag{49}$$

The optimal output to the system $y^*(t)$ is obtained by using:

$$y^*(t) = \sum_{i=0}^{k} b_i \overset{(i)*}{m}(t) \tag{50}$$

Next, an example problem will be worked out to illustrate this method of solution.

3. Example Problem

To illustrate the optimization scheme applied to a practical feedback system, let us consider the system represented in the block diagram of Fig. I. We will assume that the disturbance function d(t)

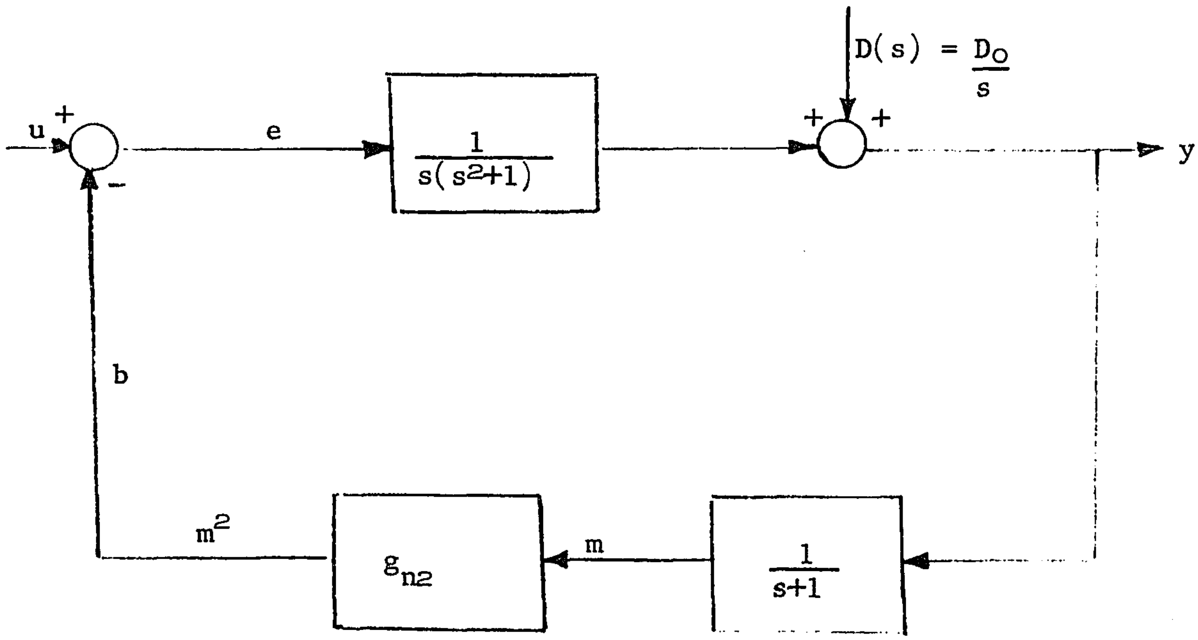


Figure 7: Block Diagram for the Nonlinear System

is a step function of magnitude D_O. Without any loss in generality D_O will be assumed equal to unity. Also consider the case where the nonlinear network g_{n2} in the feedback branch is specified and is

$$g_{n2}\,(m) = m^2 \tag{51}$$

We propose to transfer the system represented in the block diagram of (Fig. 7) from the given initial output vector Y_0 to the final output vector Y_T.

$$Y_0 = [y(0)\ \dot{y}(0)\ \ddot{y}(0)]^T \tag{52}$$

and

$$Y_T = [y(t_f)\ \dot{y}(t_f)\ \ddot{y}(t_f)]^T \tag{53}$$

such that the cost function $V = \frac{1}{2}\int_0^{2\pi} e^2\, dt$ is minimized. Assume that:

$$m(0) = y(0) = \dot{y}(0) = \ddot{y}(0) = 1 \tag{54}$$

$$m(2\pi) = y(2\pi) = \dot{y}(2\pi) = \ddot{y}(2\pi) = 3 \tag{55}$$

First, the poles of the system are ordered as:

$$s_1 = 0$$

$$s_2 = -j$$

$$s_3 = j$$

$$s_4 = -1$$

and the Vandermonde matrix V is:

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -j & j & -1 \\ 0 & -1 & -1 & 1 \\ 0 & j & -j & -1 \end{bmatrix} \tag{56}$$

The Vandermonde matrix is then inverted to yield:

$$V^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & .25(j-1) & -0.50 & -.25(j+1) \\ 0 & -.25(j+1) & -0.50 & .25(j-1) \\ 0 & -0.50 & 0 & -0.50 \end{bmatrix} \tag{57}$$

The residue vector $\bar{\epsilon}$ is found by using equation (20) $\bar{\epsilon} = V^{-1} K_O$ where

$$K_O = [0\ 0\ 0\ 1]^T \tag{58}$$

Thus:

$$\bar{\epsilon} = [1\ \ -0.25(j+1)\ \ 0.25(j-1)\ \ -0.50]^T \tag{59}$$

The initial and final state vectors may be determined by using equations (27) through (36) with

$$M(0) = 1 \quad \text{and} \quad M(t_f) = 3$$

$$\tilde{M}(o) = \begin{bmatrix} y(0)-m(0) \\ \dot{y}(0)-y(0)+m(0) \\ \ddot{y}(0)-\dot{y}(0)+y(0)-m(0) \end{bmatrix} \text{ and } \tilde{M}(t_f) = \begin{bmatrix} y(t_f)-m(t_f) \\ \dot{y}(t_f)-y(t_f)+m(t_f) \\ \ddot{y}(t_f)-\dot{y}(t_f)+y(t_f)-m(t_f) \end{bmatrix} \tag{60}$$

Upon substitution of the given values the following is obtained.

$$\tilde{M}(o) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{M}(t_f) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \tag{61}$$

Integration of equation (6.28) yields:

$$\varphi(t) = 1-e^{-t} \tag{62}$$

and by taking respective derivatives and evaluating at the initial and final time equations (29) and (36) become:

$$\tilde{\varphi}_0 = [0 \quad 1 \quad -1 \quad 1]^T \tag{63}$$

and

$$\tilde{\varphi}_f = [0.998 \; 0.0019 \; -0.0019 \; 0.0019]^T \tag{64}$$

Next by using

$$\bar{r}(0) = V^{-1}\left\{\begin{bmatrix} M(0) \\ - \; - \\ \tilde{M}(0) \end{bmatrix} - \tilde{\varphi}_0\right\} \tag{65}$$

and

$$\bar{r}(t_f) = V^{-1}\left\{\begin{bmatrix} M(t_f) \\ - \; - \\ \tilde{M}(t_f) \end{bmatrix} - \tilde{\varphi}_f\right\} \tag{66}$$

the state vector initial and final conditions are:

$$\bar{r}(0) = [1 \quad -0.50 \quad -0.50 \quad 1]^T \tag{67}$$

$$\bar{r}(t_f) = [5. \; -1.50 \quad -1.50 \quad 0.0019]^T \tag{68}$$

We then determine the matrices Ψ_t and Ψ_T by using equation (38).

$$\Psi_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \exp(-jt) & 0 & 0 \\ 0 & 0 & \exp(jt) & 0 \\ 0 & 0 & 0 & \exp(-t) \end{bmatrix} \tag{69}$$

$$\Psi_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.0019 \end{bmatrix} \tag{70}$$

Equation (37) allows the evaluation of all elements of $R_t i \ell$. After substitution of residues and poles in the equation, one gets

$$R_{11} = -t \tag{71}$$

$$R_{12} = \frac{1-j}{4} (1 - \exp(-jt)) \tag{72}$$

$$R_{13} = \frac{1+j}{4} (1 - \exp(jt)) \tag{73}$$

$$R_{14} = 0.50 (1 - \exp(-t)) \tag{74}$$

$$R_{21} = \frac{1-j}{4} (\exp(jt)-1) \tag{75}$$

$$R_{22} = \frac{-j}{8} \sin t \tag{76}$$

$$R_{23} = \frac{-t}{8} \exp(jt) \tag{77}$$

$$R_{24} = -\tfrac{1}{8} [\exp(jt) - \exp(-t)] \tag{78}$$

$$R_{31} = \frac{1+j}{4} (\exp(-jt)-1) \tag{79}$$

$$R_{32} = \frac{-t}{8} \exp(-jt) \tag{80}$$

$$R_{33} = \frac{j}{8} \sin t \tag{81}$$

$$R_{34} = -\tfrac{1}{8} [\exp(-jt) - \exp(-t)] \tag{82}$$

$$R_{41} = 0.5\,(\exp(t)-1) \tag{83}$$

$$R_{42} = -\tfrac{1}{8}\,[\exp(t) - \exp(-jt)] \tag{84}$$

$$R_{43} = -\tfrac{1}{8}\,[\exp(t) - \exp(jt)] \tag{85}$$

$$R_{44} = -\tfrac{1}{8}\,[\exp(t) - \exp(-t)] \tag{86}$$

It should be noted that whenever ($s_i + s_\ell = o$) which is the case for all conjugate poles, equation (37) becomes undeterminate. In this case, L'Hospital rule is used. For example:

$$R_t(23) = \frac{\epsilon_2\epsilon_3}{b_k^{\,2}a_n^{\,2}} \quad \exp(-s_2t) \quad \frac{\left\{1 - \exp[(s_2 + s_3)t]\right\}}{s_2 + s_3} \tag{87}$$

Thus by letting $\tau_o = s_2 + s_3$

$$R_t(23) = \frac{\epsilon_2\epsilon_3}{b_k^{\,2}a_n^{\,2}} \quad \exp(-s_2t). \quad \underset{\tau_o \to o}{\text{Limit}} \left[\frac{1 - \exp(\tau_o t)}{\tau_o}\right] \tag{88}$$

which yields

$$R_t(23) = \frac{-t\ \epsilon_2\epsilon_3}{b_k^{\,2}a_n^{\,2}} \quad \exp(-s_2t) \tag{89}$$

The matrix P_T evaluated at the final time is given as

$$P_T = R_T^{\;T}i\ell \tag{90}$$

Upon substitution, this matrix becomes:

$$P_T = \begin{bmatrix} -6.2832 & 0 & 0 & 266.4 \\ 0 & 0 & -0.7854 & -66.6 \\ 0 & -.7854 & 0 & -66.6 \\ 0.499 & -.12475 & -.12475 & -66.72 \end{bmatrix} \tag{91}$$

Its inverse is found to be:

$$P_T^{-1} \begin{bmatrix} -0.2971 & 0.276 & 0.276 & -1.7372 \\ 0.276 & -0.552 & -1.825 & 3.4745 \\ 0.276 & -1.825 & -0.552 & 3.4745 \\ -.00325 & 0.0065 & 0.0065 & -0.041 \end{bmatrix} \tag{92}$$

The co-state initial condition vector is next determined by using equation (41). After matrix multiplication, the result is:

$$\bar{\lambda}(0) = [-1.7404 \quad 3.481 \quad 3.481 \quad -0.026]^T \tag{93}$$

The optimal trajectory $\bar{r}^*(t)$ is obtained by using equation (44). The result is given as:

$$r_1^*(t) = 1.7405t + 1.7405\,(\cos t + \sin t) - 0.013\exp(t) - 0.7276 \tag{94}$$

$$r^{*R}_{2,3} = -0.43513t\cos t - 0.43513\sin t - 0.43513 - .06812\cos t + .00325\exp(t) \tag{95}$$

$$r_4^*(t) = 2.7373 \exp(-t) + 0.00325 \exp(t) - 0.8703 \cos t - 0.8703 \qquad (96)$$

Equation (45) yields the optimal control effort m* as:

$$m^*(t) = 1.7405t + 0.7341 \cos t + 0.8703 \sin t - 0.8703t \cdot \cos t$$

$$+ 1.7373 \exp(-t) - .00325 \exp(t) - 1.4681 \qquad (97)$$

and the optimal output y* is found as:

$$y^*(t) = 1.7405t + 0.7341 \cos t + 0.1362 \sin t - 0.8703t \cos t$$

$$+ 0.8703t \sin t - .0065 \exp(t) + 0.2724 \qquad (98)$$

The optimal actuating signal e* is:

$$e^*(t) = 1.7405 + 1.7405 (\cos t - \sin t) - .013 \exp(t) \qquad (99)$$

The value of the optimal input u* that will minimize the cost function in question is:

$$u^*(t) = [m^*(t)]^2 + e^*(t) \qquad (100)$$

where e* and m* are given in equations (97) and (99).

In this chapter an optimal solution was obtained valid for a specified initial condition. In the next chapter a solution will be obtained, one that is valid for a family of initial conditions.

CHAPTER VIII

SOLUTION TO THE NONLINEAR SERVOMECHANISM PROBLEM

1. Form of the Solution

In the preceeding chapter an optimal solution was determined for a given initial condition, in this one, an attempt will be made to obtain a feedback law valid for any initial condition. We will propose then, to minimize the performance function

$$V = \frac{1}{2} \int_{t_o}^{t_f} [\alpha\, m^2 + \beta e^2]\, dt \tag{1}$$

subjected to the constraint

$$\dot{\bar{r}} = \Lambda\, \bar{r} + \frac{\bar{\varepsilon}}{b_k a_n} \left\{ u - g_{n2} \right\} \tag{2}$$

with

$$m = \varphi(t) + c^T \bar{r} \tag{3}$$

$$c = [1 \cdots 1] \tag{4a}$$

and

$$e = u - g_{n2} \tag{4b}$$

These state equations represent the feedback system described in the block diagram of (Fig I). The function $\varphi(t)$ is a disturbance function and is given as in Chapter VII as:

$$\varphi(t) = \int_0^t \sum_{i=1}^{k} \frac{\gamma_i}{b_k} d(\tau) \cdot \exp[s_{i+n}(t-\tau)] \, d\tau \qquad (5)$$

First the final time will be assumed to be finite, and the final state unspecified.

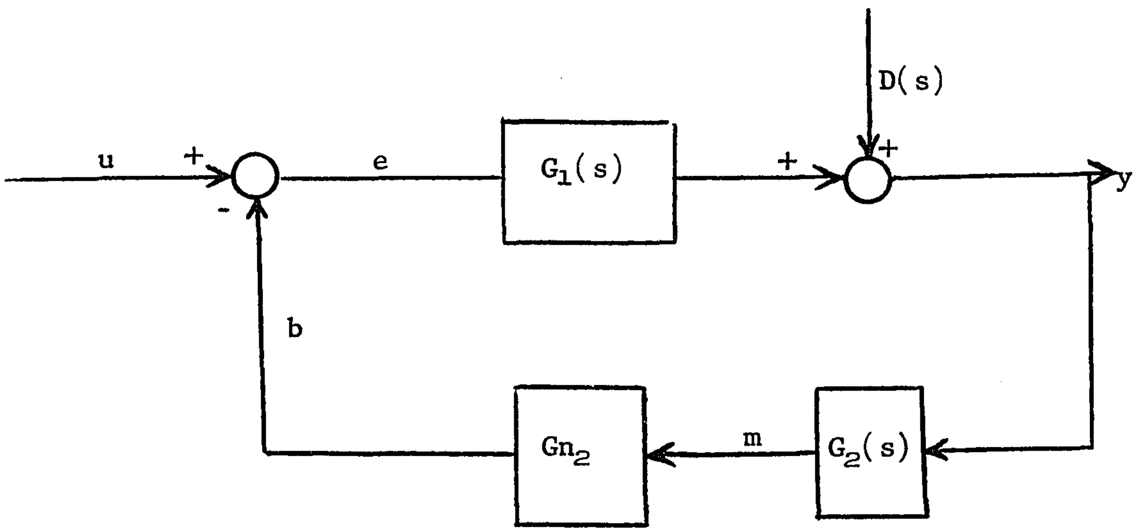


Figure 8. Block Diagram for the Nonlinear System

The cost function may be written as:

$$V = \frac{1}{2} \int_0^{t_f} \left\{ \alpha[\varphi(t) + c^T \bar{r}]^2 + \beta[u - g_{n2}(\bar{r})]^2 \right\} dt \qquad (6)$$

and the constraint

$$\dot{\bar{r}} = \Lambda \bar{r} + \frac{\bar{\epsilon}}{b_k a_n} \left\{ u - g_{n2}(\bar{r}) \right\} \qquad (7)$$

Next the minimum principle yields:

$$u^* = g_{n2}(\overline{r}^*) - \frac{1}{\beta b_k a_n} \overline{\varepsilon}^T \overline{\lambda} \tag{8}$$

and

$$\dot{\overline{\lambda}} = -\Lambda\overline{\lambda} - \alpha cc^T \overline{r} - \alpha c\varphi(t) \tag{9}$$

Because of the presence of the disturbance function $\varphi(t)$, one may assume that:

$$\overline{\lambda} = A_O(t)\cdot\overline{r} - T(t) \tag{10}$$

where $T(t)$ is a n+k vector function of time to be determined. After performing all the required substitutions in a similar manner as Chapter II, the following matrix equations are obtained.

$$\dot{A}_O = -A_O\Lambda - \Lambda A_O - \alpha cc^T + \frac{1}{\beta b_k^2 a_n^2} \cdot A_O \overline{\varepsilon}\,\overline{\varepsilon}^T A_O \tag{11}$$

and

$$\dot{T} = -\left[\Lambda - \frac{A_O \overline{\varepsilon}\,\overline{\varepsilon}^T}{\beta b_k^2 a_n^2}\right] \cdot T(t) + \alpha c\varphi(t) \tag{12}$$

Note that $A_O(t)$ is a symmetric matrix function of time. Equation (11) represents the Ricatti Equation for this problem. The boundary conditions to be satisfied are

$$[T(t_f)]_j = [A_O(t_f)]_{i,j} = 0 \tag{13}$$

$$i,j = 1, \ldots, k+n$$

The optimal value of u^* is determined by substitution of equation (10) into equation (8), thus

$$u^*(\bar{r},t) = g_{n2}(\bar{r}^*) - \frac{\bar{\epsilon}^T}{\beta b_k a_n} \left\{ A_O(t)\cdot\bar{r} - T(t) \right\} \qquad (14)$$

2. Further Simplification of the Problem

In this section some matrix operations will be performed to simplify the Ricatti Equation. For instance writing $A_O(t)$ as:

$$A_O(t) = \begin{bmatrix} a_{11}(t) & a_{12} & \cdots & a_{1n+k} \\ a_{21} & a_{22} & \cdots & a_{2n+k} \\ \vdots & & & \vdots \\ a_{n+k1} & a_{n+k2} & \cdots & a_{n+kn+k} \end{bmatrix} \qquad (15)$$

$$\text{with} \qquad a_{ij} = a_{ji} \qquad (16)$$

Let C be the matrix sum of $(A_O\Lambda + \Lambda A_O)$. Upon matrix multiplication one finds that C is given as

$$C = \begin{bmatrix} 2s_1 a_{11} & a_{12}(s_1+s_2) & \cdots & a_{1\,n+k}(s_1+s_{n+k}) \\ \vdots & & & \vdots \\ a_{1n+k}(s_1+s_{n+k}) & \cdots & & 2a_{n+kn+k} \cdot s_{n+k} \end{bmatrix} \tag{17}$$

Here $\quad C = [c_{i\ell}] = [a_{i\ell}\,(s_i + s_\ell)]\ i, \ell = 1, \cdots, n+k$

Finally, after considerable algebraic manipulation, the matrix Ricatti Equation reduces to

$$\dot{a}_{q\ell} = \left\{ \frac{1}{\beta b_k^{\,2} a_n^{\,2}} \cdot \sum_{i+1}^{n+k} \sum_{j=1}^{n+k} \epsilon_i \epsilon_j\ a_{qi} a_{j\ell} \right\} - \alpha - a_{q\ell}(s_q + s_\ell) \tag{18}$$

for $\quad q = 1,\ n+k$
$\quad \ell = 1,\ n+k$

The elements of the matrix $A_0(t)$ are determined as solution to equation (18). Once $A_0(t)$ is found equation (12) is integrated to yield $T(t)$.

3. Case of Infinite Time Optimization

For this case $\dot{a}_{q\ell}(t) = 0,\ q, \ell = 1, \cdots, k+n$ (19)

Equation (18) reduces to:

$$\alpha + a_{q\ell}(s_q + s_\ell) - \frac{1}{\beta b_k{}^2 a_n{}^2} \sum_{i=1}^{n+k} \sum_{j=1}^{n+k} \epsilon_i \epsilon_j \; a_{qi} a_{j\ell} = 0 \tag{20}$$

The vector $T(t)$ is now a solution to the following equation

$$\dot{T} = A(\Lambda, A_O) \cdot T + \alpha c \varphi(t) \tag{21}$$

where

$$A(\Lambda, A_O) = \frac{A_O \bar{\epsilon} \bar{\epsilon}^T}{\beta b_k{}^2 a_n{}^2} - \Lambda \tag{22}$$

Let ζ_i $(i = 1, n{+}k)$ be the eigen values of A, then by a simple transformation of variables:

$$T(t) = L\, V(t) \tag{23}$$

where

$$V(t) = [v_1 \; \ldots \; v_{n+k}]^T \tag{24}$$

$$T(t) = [T_1 \; \ldots \; T_{n+k}]^T \tag{25}$$

$$L = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{n+k} \\ \vdots & & & \vdots \\ \zeta_1^{n+k-1} & & & \zeta_{N+k}^{n+k-1} \end{bmatrix} \tag{26}$$

Equation (21) becomes:

$$\dot{V} = \tilde{\Lambda} V + L^{-1} \alpha c \varphi(t) \tag{27}$$

where

$$\tilde{\Lambda} = \begin{bmatrix} \zeta_1 & & & \\ & \zeta_2 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \zeta_{n+k} \end{bmatrix} \tag{28}$$

The boundary conditions to be satisfied are:

$$V(t_f) = L^{-1} T(t_f) \tag{29}$$

but since $T(t_f) = 0$, then $V(t_f) = 0 \,]_{t_f \to \infty}$ (30)

Upon integration of equation (27), the following results:

$$V(t) = \exp(\tilde{\Lambda} t) V(o) + \int_o^t \exp[\tilde{\Lambda}(t-\tau)] L^{-1} \alpha c \varphi(\tau) d\tau \tag{31}$$

which may be written as:

$$V(t) = \exp(\tilde{\Lambda} t) \left\{ V(0) + \int_0^t \exp(-\tilde{\Lambda}\tau) L^{-1} \alpha c\varphi(\tau) d\tau \right\} \tag{32}$$

Next, we use the boundary condition $V(t_f) = 0$ to solve for $V(0)$. Thus:

$$V(0) = -\int_{0}^{t_f} \exp(-\tilde{\Lambda}\tau)\; L^{-1}\; \alpha\; c\; \varphi(\tau)\; d\tau \tag{33}$$

Substituting $V(0)$ back into equation (32) yields:

$$V(t) = -\alpha \exp(\tilde{\Lambda}t) \left\{ \int_{t}^{t_f} \exp(-\tilde{\Lambda}\tau)\; L^{-1} c\; \varphi(\tau)\; d\tau \right\}_{t_f \to \infty} \tag{34}$$

Finally the optimal value of the actuating signal is

$$e^{*}(\bar{r}, t) = \frac{-\bar{\epsilon}^{\,T}}{\beta b_k a_n} \left\{ A_O \cdot \bar{r} - LV \right\} \tag{35}$$

with A_O given as solution to equation (20) and L given in equation (26). The optimal input is:

$$u^{*} = g_{n2}(\bar{r}^{*}) + e^{*}(\bar{r}, t) \tag{36}$$

An example problem will be worked out to illustrate this method of solution.

4. Example Problem

Consider the system represented in the block diagram of Fig. II.

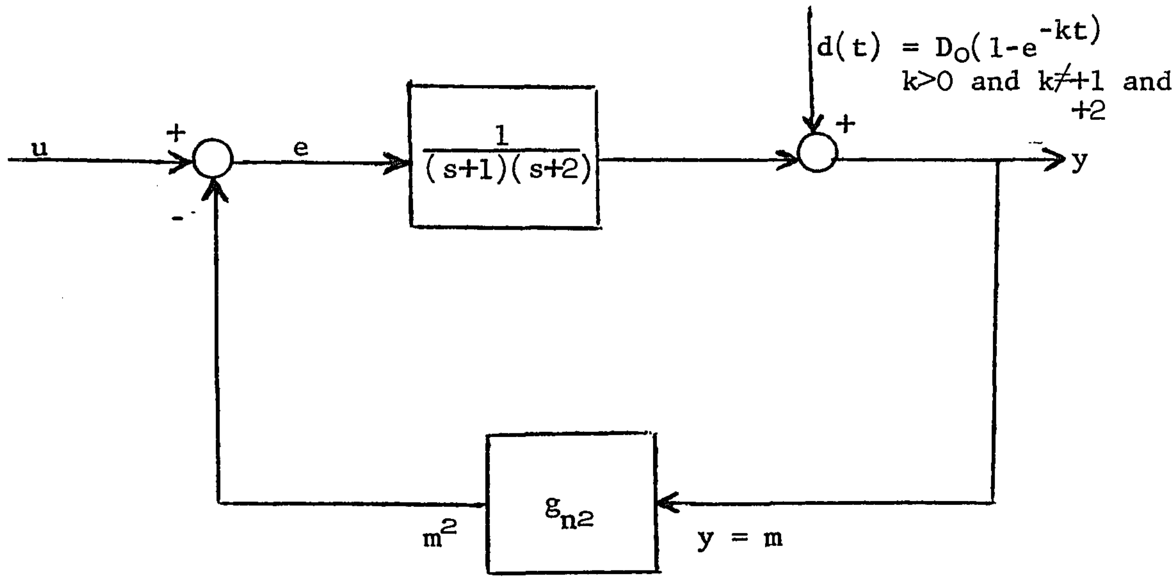


Figure 9. Block Diagram Representing the System.

We wish to minimize the performance function

$$V = \frac{1}{2}\int_0^\infty (m^2 + e^2)\,dt \tag{37}$$

subjected to the constraint given in the previous sections as:

$$\dot{\bar{r}} = \Lambda\bar{r} + \bar{\varepsilon}^T e(t) \tag{38}$$

$$m(t) = y(t) = c^T \bar{r} + \varphi(t) \tag{39}$$

The problem is reformulated as:

$$V = \frac{1}{2}\int_0^\infty \left\{ [c^T\bar{r} + \varphi(t)]^2 + \left[u-(\varphi(t) + c^T\bar{r})^2\right]^2 \right\} dt \tag{40}$$

with

$$\dot{\bar{r}} = \Lambda\bar{r} + \bar{\varepsilon}^T [u-g_{n2}] \tag{41}$$

The first step is to derive equation (20) noting here that:

$$\bar{\varepsilon} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \tag{42}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{43}$$

$$\varphi(t) = d(t) = D_0(1-e^{-kt}) \tag{44}$$

$$\alpha = \beta = a_n = b_k = 1 \tag{45}$$

and

$$c = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \tag{46}$$

When expanded equation (20) takes the following form:

$$1-2a_{11} - (a_{11}-a_{12})^2 = 0 \tag{47}$$

$$1-3a_{12} - (a_{11}- a_{12})(a_{12}- a_{22}) = 0 \tag{48}$$

$$1-4a_{22} - (a_{22}- a_{12})^2 = 0 \tag{49}$$

Noting that only the values of a_{ij} that will yield a positive definite matrix for A_O are acceptable, the solution for this set of nonlinear equations is:

$$a_{12} = 0.329$$

$$a_{11} = 0.487$$

$$a_{22} = 0.248$$

Thus,

$$A_O = \begin{bmatrix} 0.487 & 0.329 \\ 0.329 & 0.248 \end{bmatrix} \qquad (50)$$

Equation (22) yields $A(\Lambda, A_O)$ as:

$$A = \begin{bmatrix} 1.158 & -0.158 \\ 0.081 & 1.919 \end{bmatrix} \qquad (51)$$

and the Eigen values of A are:

$$\zeta_1 = 1.9018$$

$$\zeta_2 = 1.1752$$

The matrix L is given by

$$L = \begin{bmatrix} 1 & 1 \\ 1.9018 & 1.1752 \end{bmatrix} \qquad (52)$$

and

$$L^{-1} = \begin{bmatrix} -1.621 & 1.38 \\ 2.621 & -1.38 \end{bmatrix} \tag{53}$$

Also

$$\tilde{\Lambda} \quad \begin{bmatrix} 1.9018 & 0 \\ 0 & 1.1752 \end{bmatrix} \tag{54}$$

Upon integration of equation (34) the vector V(t) is determined as:

$$V_1(t) = D_O \left\{ 0.126 - \frac{0.241 \exp(-kt)}{K + 1.9018} \right\} \tag{55}$$

$$V_2(t) = D_O \left\{ -1.056 + \frac{1.241 \exp(-kt)}{K + 1.1752} \right\} \tag{56}$$

Equation (35) allows the determination of e^* after matrix and vector multiplication. The result is

$$e^*(\overline{r},t) = -0.158\, r_1 - 0.081\, r_2 + D_O \left\{ 0.07 - \frac{0.158 \exp(-kt)}{(k+1.1752)(k+1.9018)} \right\} \tag{57}$$

and the optimal input u^* is

$$u^* = [r_1 + r_2 + D_O(1-e^{-kt})]^2 - 0.158 r_1 - 0.081 r_2 \tag{58}$$

$$+ \; D_O \left\{ 0.07 - \frac{0.158 \exp(-kt)}{(k+1.1752)(k+1.9018)} \right\}$$

This clearly indicates that an exact solution is possible if the performance function is as the one defined in the beginning of this Chapter. An optimal solution for the output $y^*(t)$ is straightforward and is omitted here since it does not add anything to the study.

CHAPTER IX

DISCUSSION OF RESULTS, CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDIES

In this investigation of nonlinear systems, only nonlinearities in terms of powers of the output variable were considered. However, in some cases, by altering the form of the performance function nonlinearities in the powers of derivatives are amenable for solution by the method described. The dynamic optimization of linear systems was presented in the preliminary chapters to establish a solution to nonlinear systems.

The Hamilton Jacobi Equation was derived in a different manner than commonly encountered and is discussed in Chapters IV and V. Exact solutions to the Hamilton Jacobi Equation were shown possible for the one dimensional nonlinear system in particular, and for the n^{th} dimensional system that belongs to the class of problems defined at the end of Chapters IV and V in general. Prior to this study, even for the one dimensional nonlinear system, a series solution was assumed for the performance function V thus yielding an approximate solution for the feedback control law or optimal closed loop control. Thus, this study has a major contribution in that, one is able to start with a fairly general nonlinear system and minimize a certain cost function and obtain an exact solution for both the control and the optimal return.

Another interesting aspect was the fact that solutions to the open loop control and trajectories were obtained by transformation of variables that yielded elliptic integrals of the first kind. Example problems to illustrate the method of optimization proved very interesting. One for which the final time was extended to infinity and another for an unknown but finite time and a specified final state.

Although, this was by no means the most general type of problems that could be investigated, it represented indeed a closed form solution for the dynamic optimization of an important class of nonlinear systems. In practice, the optimal control law is found through the dynamic programming approach by computational solutions of some recursion relations. This method of solution would be very useful to get a quick and exact result for any first order system with any degree of nonlinearity in the output variable, for both the infinite time optimization and the nonlinear minimum time problem.

Other advantages such as having the solution to the linear control problem in terms of the poles of the system are noteworthy. In most feedback problems the systems are described in terms of transfer functions, thus it is convenient to start with the given poles and generate a closed form solution for both optimal output and control. In this manner, any backward integration or any attempt of matching boundary conditions are eliminated.

In the chapters involving automatic feedback systems, for the case of a specified initial condition the results of the earlier chapters were used. The nonlinear network was assumed to be in the feedback branch. The most interesting aspect of the optimization scheme was that for the particular performance function, the minimization of the nonlinear system was reduced to a linear minimization problem with the actuating signal as the control.

Finally, for the case of infinite time optimization and for the nonlinear servomechanism problem, the optimization yielded an interesting result for the optimal actuating signal. For both cases, example problems proved worthwhile as an illustration to the method of solution.

CONCLUSION

Exact solutions to the Hamilton Jacobi Equation are very rare cases for the most simple systems. This method of solution would not yield the exact result for all types of nonlinear systems, it would although, at least for the class of systems described in this study, give an exact solution and an easy formulation for the optimal control. The limitations might be listed as follows:

1. The disturbance function $d(t)$ was assumed to be bounded for any t in $[t_0, t_f]$, otherwise, at least for the infinite time optimization problem, the control law is not admissible. In addition to being bounded, it was assumed that both $d(t)$ and its derivatives were continuous.

2. Nonlinearities existed in the feedback branch, thus any feed forward analysis was omitted.

3. The performance function was assumed to be quadratic in terms of both the control and output variable.

4. For the case of the solution to the linear problem, the poles were assumed to be distinct, thus dispensing from any undersirable "Jordan Form" and allowing the existence of the inverse of the Vandermonde matrix.

5. For the optimization of nonlinear systems, nonlinearities existed in terms of the powers of the output variables for first order systems with quadratic performance in the output and the control.

Because of these limitations, some recommendations for further studies are indispensable. For instance, because of the nature of the solution obtained for infinite time optimization, some work needs to be done in the case of specified finite final time, a condition that introduces the term $\frac{\delta V}{\delta t}$ to the Hamilton Jacobi Equation. Also, other forms of performance functions need to be considered coupled with some other types of nonlinearities in terms of other variables, possibly the control. Finally, because of the Elliptic form of the integrals resulting from the solution to the optimal trajectory, it is conceivable that for the class of problems described in this study, through certain transformations the solution is always an elliptic integral of the first kind or second kind or even a combination of both.

BIBLIOGRAPHY

1. P. Franklin, Treatise on Advanced Calculus, John Wiley and Sons, Inc., New York. 1940.
2. Symposium on Nonlinear Circuit Analysis, Polytechnique Inst. of Brooklyn, New York, 1956.
3. E.T. Copson, Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, New York, 1935.
4. Wilbur R. LePage, Complex Variables and the Laplace Transform for Engineers, McGraw-Hill, Inc., New York, 1961.
5. H. Poincaré, Les Methodes Nouvelles de la Mécanique Céleste, Vols. 1-3, Paris, 1899.
6. S. Lefschetz, Existence of Periodic Solutions for Certain Differential Equations, Proc. Nat. Acad. Sci., U.S.A., 1943.
7. Alfred A. Wolf, Recurrence Relations in the Solution of a Certain Class of Nonlinear Systems, Trans. A.I.E.E., Vol. 78, Jan., 1960.
8. Izvestiya Vysshikh Uchebykh Zavedeniy, Elecktromekhanika, No. 8, 1965. (News of higher educational institutions, electromechanics.)
9. A.B. Farnell, C.E. Langenhop, and N. Levinson, Forced Periodic Solutions of a Stable Nonlinear System of Differential Equations, J. Maths, 1951.
10. Y.H. Ku, Analysis and Control of Nonlinear Systems, The Ronald Press Company, New York.

11. R.G. Malek, Application of the Convolution Techniques in the Solution and Determination of Stability of Nonlinear Systems, M.S. Thesis, 1968.

12. Andrew P. Sage, Optimum Systems Control, Prentice Hall, Inc., New Jersey, 1968.

13. S.J. Citron, Elements of Optimal Control, Holt, Rinehart and Winston, Inc., 1969.

14. Schultz and Melsa, State Functions and Linear Control Systems, McGraw-Hill, Inc., 1967.

15. M.A. Alam, A Constrained Optimal Control of Nonlinear Systems, I.E.E.E. Trans., April 1970.

16. Timothy and Bona, State Space Analysis, McGraw-Hill, Inc., New York, 1968.

17. Peter Dransfield, Engineering Systems and Automatic Control, Prentice-Hall, Inc., New Jersey, 1968.

18. Julius T. Tou, Modern Control Theory, McGraw-Hill, Electrical and Electronic Engineering Series.

19. A.S. Abbott and J.E. McIntyre, Guidance, Flight Mechanics and Trajectory Optimization, North American Aviation, Inc., Volume 10 - Dynamic Programming, April, 1968.

20. Abramowitz and Stegun, Handbook of Mathematical Functions, National Bureau of Standards.

21. Hsu and Meyer, Modern Control Principles and Applications, McGraw-Hill.

22. W.T. Cunningham, Nonlinear Analysis, McGraw-Hill.

23. J.N. Franklin, Matrix Theory, Prentice-Hall.

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