

Dynamic Pricing of Limited Inventories When Customers Negotiate

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Abstract

Although take-it-or-leave-it pricing is the main mode of operation for many retailers, a number of retailers discreetly allow price negotiation when some haggle-prone customers ask for a bargain. At these retailers, the posted price, which itself is subject to dynamic adjustments in response to the pace of sales during the selling season, serves two important roles: (i) it is the take-it-or-leave-it price to many customers who do not bargain, and (ii) it is the price from which haggle-prone customers negotiate down.

In order to effectively measure the benefit of dynamic pricing and negotiation in such a retail environment, one must take into account the interactions among inventory, dynamic pricing, and negotiation. The outcome of the negotiation (and the final price a customer pays) depends on the inventory level, the remaining selling season, the retailer's bargaining power, and the posted price.

We model the retailer's dynamic pricing problem as a dynamic program, where the revenues from both negotiation and posted pricing are embedded in each period. We characterize the optimal posted price and the resulting negotiation outcome as a function of inventory and time. We also show that negotiation is an effective tool to achieve price discrimination, particularly when the inventory level is high and/or the remaining selling season is short even when implementing negotiation is costly.

Subject classification: inventory/production: uncertainty, stochastic; inventory/production: policies, marketing/pricing; dynamic programming: models, applications.

Area of review: Manufacturing, Service, and Supply Chain Operations

1. Introduction

We know all too well how common negotiation is when buying a car or furniture. Perhaps surprisingly, negotiation is also acceptable at a wider range of retailers who sell products primarily at posted prices. For example, a recent *New York Times* article reports that “a bargaining culture once confined largely to car showrooms and jewelry stores is taking root in major stores like Best-Buy, Circuit City and Home Depot, as well as mom-and-pop operations” (Richtel, 2008). Likewise, two reporters from *The Wall Street Journal* who negotiated at 50 retail stores during the holiday season of 2001, were successful in getting a discount from the posted price at 18 of these stores for products ranging from a personal digital assistant (\$20) to a fitness machine (\$50) (Agin and Collins, 2001). The stores at which they got discounts range from Eddie Bauer to Sunglass Hut, from Kenneth Cole to Ferragamo. Interestingly, many of the aforementioned retailers that allow some customers to negotiate, sell short life-cycle products: high-tech consumer electronics (as in the case of BestBuy or Circuit City) or style goods (as in the case of Kenneth Cole and Sunglass Hut). For such products, retailers frequently change the posted price during the course of the season, in an effort to improve their revenues from the limited inventory available over the short selling season. When making those pricing decisions, the retailer who wants to judiciously choose the posted price needs to account for not only the price-sensitivity of customers who do not negotiate but also the presence of bargainers who will negotiate the price down. In this paper, we focus on such retailers for whom take-it-or-leave-it pricing is the main mode of operation, but that nonetheless allow price negotiation when haggle-prone customers initiate it. We investigate how the presence of negotiation influences such a retailer’s dynamic adjustment of posted prices.

Both dynamic adjustment of posted prices (hereafter, dynamic pricing) and negotiation can improve revenues (compared to setting a fixed price throughout the entire selling season), particularly when the inventory is limited. Although both strategies may improve the retailer’s revenue, there are differences in how each one achieves this goal. Dynamic pricing adjusts the margin based on the inventory level relative to the remaining selling season: The posted price will increase when inventory is low, and will decrease when inventory is high. On the other hand, negotiation enables the retailer to extract more revenue from customers: The retailer can set a high posted price, and those customers with high willingness-to-pay may buy the product after little or no negotiation, while others with low willingness-to-pay will buy at discounted prices after negotiating with the retailer. Therefore, the retailer chooses a high posted price in order to improve the range of price discrimination enabled by negotiation.

Interestingly, negotiation and dynamic pricing can drive the posted price in the same or opposite direction depending on the inventory level. When there is little inventory of the product, the retailer's tendency to set high posted prices under negotiation will be reinforced by dynamic pricing. On the other hand, if the risk of excess inventory at the end of the season is significant, then dynamic pricing drives the posted price down to move the product faster, but such low posted prices reduce the retailer's ability to price discriminate via negotiation. These interactions among dynamic pricing, negotiation, and inventory motivate our paper, which fills a gap in the literature by considering the joint use of dynamic pricing and negotiation.

In this paper, we propose a model where negotiation and dynamic pricing take place together. We consider a retailer who has limited inventory at the beginning of a relatively short selling season. We divide the season into periods, each of them short enough so that at most one customer can arrive in a period. The customer population is comprised of two types of consumers — *price-takers* and *bargainers*. Price-takers either buy at the posted price or quit without purchasing. On the other hand, bargainers negotiate in the hope of getting a discount from the retailer. We model the outcome of the negotiation and integrate the revenue implication of negotiation outcome into a dynamic pricing model. The negotiation outcome depends on a number of factors, including: the retailer's posted price, which in turn depends on the inventory and time until the end of the selling season; the minimum price *at which the retailer is willing to sell to any customer*, which again depends on time and inventory; the bargainer's willingness-to-pay, and the relative bargaining powers of the retailer and the customer.

This model produces a number of interesting analytical and numerical results. We show that the optimal posted price of a retailer who allows negotiation includes a premium over that of a retailer who does not allow negotiation. This price premium helps the retailer extract more revenues from a bargainer with a high willingness-to-pay. Interestingly, even though the posted price itself decreases in the inventory level, the price premium becomes larger at high inventory levels. As the negotiating retailer adds a price premium, some customers will be worse off when buying from the negotiating retailer (compared to buying from the retailer that does not allow negotiation). At the same time, there will emerge a group of customers who will benefit by buying from the negotiating retailer.

We also consider a model where negotiation is costly and the retailer decides whether or not to allow negotiation in each period. We prove that, the higher the inventory and the shorter the time until the end of the season, the more the retailer favors negotiation. In addition, our numerical results illustrate that if a retailer had to choose between negotiation and dynamic pricing, it would

choose negotiation over dynamic pricing when the risk of excess inventory is high (if the inventory is high and/or the remaining time is low).

The remainder of this paper is organized as follows. Section 2 provides a survey of the relevant literature. Section 3 outlines our model where a negotiation model is embedded into a dynamic pricing problem. In Section 4, we derive analytical results regarding the effects of the negotiation. We then describe the results of our numerical study in Section 5. In Section 6, we provide an extension of our model to the case in which the posted price influences some bargainers' willingness to pay. The paper concludes with discussion and future research directions in Section 7. All proofs are provided in the appendix.

2. Literature Review

There has been a significant volume of research in dynamic pricing of limited inventories in the last decade. Starting with Gallego and van Ryzin (1994), and Bitran and Mondschein (1997), this research focuses on products whose inventory cannot be replenished during their relatively short selling season, and the key question is how the retailer should adjust the price of the product based on remaining time and inventory in order to maximize the total revenue over the selling season. For reviews of the literature, see Bitran and Caldentey (2003), and Elmaghraby and Keskinocak (2003). As the use of dynamic pricing has spread from airline and travel industries to the retail industry, researchers have studied many different dynamic pricing problems that correspond to specific business applications, such as dynamic pricing for multiple products (e.g., Zhang and Cooper, 2005, and Maglaras and Meissner, 2006), dynamic pricing in the presence of strategic consumers (e.g., Aviv and Pazgal, 2008, Elmaghraby, Gulcu and Keskinocak, 2008, Su, 2007, and Cho, Fan, and Zhou, 2009), the use of dynamic pricing and discounting when making product offers to customers (e.g., Netessine, Savin and Xiao, 2006, and Aydin and Ziya, 2008) and dynamic pricing when the demand in each period is affected by prices over multiple periods (e.g., Popescu and Wu, 2007, and Ahn, Gumus and Kaminsky, 2007). However, the existing work on dynamic pricing has not considered retail situations where some customers can initiate a negotiation on the price of the product. Our contribution is to investigate the interaction between dynamic pricing and negotiation, and to analyze the effect of negotiation on the retailer and the consumers in a setting where prices are adjusted dynamically over the selling season.

Another research topic closely related to our model is bargaining, which has been studied extensively in economics. For a detailed review of the theory and applications of bargaining, see Muthoo (1999). Two classic bargaining models in economics are the Nash bargaining solution

and the Rubinstein model. Under the Nash bargaining solution, two parties bargaining over a surplus split the difference between the total surplus and the sum of their reservation utilities (also known as disagreement payoffs). The classic Nash bargaining solution can be extended to the case where the two parties have different bargaining powers, in which case a larger portion of the total surplus is given to the more powerful party. This *generalized Nash bargaining solution* (also known as asymmetric Nash bargaining solution) is what we use to model the outcome of the negotiation between the retailer and the bargainer. The Rubinstein model, on the other hand, views bargaining as a series of alternating offers between two parties bargaining over a surplus. The Rubinstein model assumes that the two parties have full information regarding each other's utilities and they make alternating offers with a fixed time interval between two successive offers to maximize discounted utility. This bargaining process leads to a unique subgame perfect equilibrium where the parties immediately settle at the very beginning of the bargaining process. The equilibrium of the Rubinstein model yields the generalized Nash bargaining solution when the time interval between two offers approaches zero (or discount factor approaches one). Furthermore, a number of bargaining processes that can be modeled as variants of Rubinstein's, such as adding the possibility of negotiation breakdown or presence of inside or outside options, lead to outcomes that are slight variations of the Nash bargaining solution. (See Muthoo, 1999, for a detailed discussion.)

Another alternative for modeling the bargaining process and/or outcome would be a non-cooperative game-theoretic solution. However, there is sufficient experimental evidence, which suggests that such non-cooperative solutions are no better, if not worse, than GNBS in predicting the outcomes of actual bargaining situations (see, chapter 5 in Kagel and Roth, 1995, and chapter 4 in Davis and Holt, 1992). A particularly vivid illustration is the *ultimatum game*. In the ultimatum game, player A offers a share of a pot of money to player B. If player B accepts, the two divide the pot accordingly. If player B rejects, neither player gets anything. A rather obvious non-cooperative solution of this game would predict that player A would offer player B an infinitesimal amount of money and player B would accept. However, in almost all ultimatum games conducted around the world over several decades (for example, Guth, Schmittberger and Schwarze, 1982, Kahneman, Knetsch and Thaler, 1986, Henrich, Boyd, Bowles, Camerer, Fehr and Gintis, 2004), the outcome tends to veer toward much more equitable distributions of the money, suggesting the existence of forces that push the bargaining outcome further away from those predicted by the non-cooperative models, and closer to the outcomes predicted by a cooperative model such as the GNBS. Similar observations have been made for the *dictator game*, in which player A simply decides how much of her pot of money to share with player B, if any. In this setting, the non-cooperative equilibrium

would be for player A to offer no money at all to player B. Experimenting with this game under several different conditions, however, Hoffman, McCabe, Shachat, Smith (1994) found that at least as many as 30% of these players shared some of their money with the other player, suggesting once again that the players are not acting as single-minded income maximizers. In addition to these simple games, GNBS has received experimental support in the context of unstructured bargaining games where two parties negotiate freely until a settlement is reached (e.g., Nydegger and Owen, 1975) and in more complicated games where there are multiple terms to negotiate (e.g., negotiations in media purchasing, studied by Neslin and Greenhalgh, 1983).

Using the generalized Nash bargaining solution as a model of negotiation outcome is all the more attractive given the experimental support and its correspondence to the equilibria of a number of bargaining processes including Rubinstien's alternating offers game and its variants. In fact, GNBS has been a popular modeling tool for researchers who study take-it-or-leave-it pricing versus bargaining. For example, Bester (1993), Wang (1995), Arnold and Lippman (1998), Desai and Purohit (2004), and Roth, Woratschek and Pastowski (2006) all use GNBS to model the negotiation outcome. In particular, the last two papers use GNBS to model the outcome of negotiation between a retailer and an individual customer, as we do. In many of the aforementioned papers, GNBS is used to model the outcome of one buyer and one seller negotiating over one unit of an item. In the problem we study, however, the retailer will encounter a series of bargainers over time, and the outcome of negotiation with each bargainer will depend on the retailer's inventory and the remaining time until the end of the selling season. Thus, the seller's decision must factor in the inventory level and the time remaining until the end of the selling season.

There are several papers that compare take-it-or-leave-it pricing with bargaining. One such paper is by Riley and Zeckhauser (1983) who show that take-it-or-leave-it pricing is superior to haggling if the seller incurs a cost for bringing a new potential buyer. Wang (1995), on the other hand, uses the Nash bargaining solution to model the outcome of bargaining, and finds that bargaining is always preferable to take-it-or-leave-it pricing if the cost of implementing bargaining is not too high. Desai and Purohit (2004) analyze how two competing retailers choose whether to use take-it-or-leave-it pricing or negotiation and analyze equilibrium outcome. Terwiesch, Savin and Hann (2005) analyze an online retailer that uses a negotiation process where customers name their own prices, and derive the retailer's optimal threshold price above which the retailer accepts all offers. Our paper is different from these in one major aspect: We explicitly consider the effect of limited inventory and finite selling season, thus account for both the risk of excess inventory and the risk of shortage.

There is some recent work that incorporates negotiation among supply chain partners. For example, Wu (2004) uses the Rubinstein model to search for coordinating contracts in a supply chain with demand uncertainty. Gurnani and Shi (2006) use the Nash bargaining solution in order to model a contract design problem in a supply chain where the supplier’s reliability is private information. Iyer and Villas-Boas (2003) consider bargaining issues in distribution channels. Dukes and Gal-Or (2003) consider the relationship between advertisers and media outlets, using the Nash bargaining solution to model the negotiation outcome. Nagarajan and Sosic (2008) review and extend the research in cooperative bargaining in supply chains.

3. Model Description

We consider a retailer selling a limited inventory of a product over a predetermined selling season. We assume that the selling season is divided into T periods, each of which is short enough that at most one customer arrives in a given period, and we denote the probability that a customer arrives in a period by $\lambda \in [0, 1]$. A customer can be one of two types — a price-taker or a bargainer. Let $q \in (0, 1)$ be the proportion of bargainers in the customer population. Facing two types of customers, the retailer sets a posted price at the beginning of each period as well as a cut-off price below which it will not sell. A price-taker purchases the product at the posted price if her reservation price is high enough. On the other hand, a bargainer with a high enough reservation price negotiates with the retailer, and she purchases at a price between the cut-off price and the posted price.¹ The retailer’s goal is to choose the posted and cut-off prices in each period so as to maximize its total expected revenue over the selling season. Next we describe the model in detail.

3.1 Price-takers and Bargainers

Let r denote the reservation price of the customer (the maximum price that the customer is willing to pay for the product), unobservable to the retailer. From the retailer’s perspective, an arriving customer’s reservation price is a non-negative random variable R_c with a cumulative distribution function (cdf) $F(\cdot)$ and a probability density function (pdf) $f(\cdot)$. We assume that F is defined over the domain $[0, b]$ for some $0 < b < \infty$ and strictly increasing. We define $\bar{F}(\cdot) := 1 - F(\cdot)$. The choice of b is arbitrary: It simply implies that there exists a price above which no one is willing to pay for the product.

¹One can interpret the price-takers as consumers who incur prohibitively high-costs of negotiation. In that sense, our model captures two distinct classes of consumers that differ from one another in their costs of negotiation.

Upon arrival, all customers observe the posted price, but their subsequent behavior depends on their type. A price-taker buys the product if the posted price, p , is less than or equal to her reservation price, r , and quits otherwise. On the other hand, a bargainer observes the posted price and initiates a negotiation. Instead of focusing on a specific negotiation process, we model the outcome of the negotiation process between a bargainer and the retailer using the generalized Nash bargaining solution, under which the total surplus is split between the two parties based on their relative bargaining power (see, for example, Muthoo, 1999). As discussed in the literature review, the GNBS can be used to capture the bargaining outcomes of several bargaining processes. Let $\beta \in (0, 1)$ denote the retailer's bargaining power and $1 - \beta$ the customer's bargaining power. The outcome of the negotiation is determined by the retailer's cut-off price, c , the posted price, p , the customer's reservation price of the item, r , and each party's relative bargaining power. In particular, following the generalized Nash bargaining solution, the retailer with cut-off price c and a bargainer with valuation $r \geq c$ will agree on a final price $p_N^*(c, r)$ that maximizes the following objective function.

$$\begin{aligned} \max_{p_N} \quad & (r - p_N)^{1-\beta} (p_N - c)^\beta \\ \text{subject to} \quad & c \leq p_N \leq p, p_N \leq r \end{aligned} \tag{1}$$

Note from above that a bargainer with valuation r will obtain a surplus of $r - p_N^*(c, r)$, and the retailer's surplus will be $p_N^*(c, r) - c$. In that sense, the generalized Nash bargaining solution maximizes the product of the two parties' surpluses, where each surplus is weighted by the bargaining power of the respective party. Two observations from the above formulation are worth noting. First, only bargainers whose reservation prices are above the retailer's cut-off price, c , will be able to purchase the product. In other words, as one would expect, the negotiation between a retailer with cut-off price c and a bargainer with a reservation price $r < c$, is bound to fail. Second, note that the final price agreed by both parties can never exceed the posted price, p . That is, as one would expect, no bargainer will ever pay more than the posted price, although it is still possible that some bargainers may still buy at the posted price as an eventual outcome of the negotiation process.

It follows from (1) that the final price $p_N^*(c, r)$ is either the posted price, p , or the convex

combination of r and c that optimally splits the surplus, whichever is smaller:

$$\begin{aligned}
p_N^*(c, r) &= \min \left\{ p, \arg \max_{p_N} \left[(r - p_N)^{1-\beta} (p_N - c)^\beta \right] \right\} \\
&= \min \{ p, \beta r + (1 - \beta)c \} \\
&= \begin{cases} p & \text{if } r \geq \frac{p - (1-\beta)c}{\beta}; \\ \beta r + (1 - \beta)c & \text{if } c \leq r < \frac{p - (1-\beta)c}{\beta}. \end{cases} \tag{2}
\end{aligned}$$

Observe that the retailer can increase the transaction price by picking a higher cut-off price, but only at the cost of not being able to sell to the bargainers whose reservation prices fall below the cut-off price.

Note also the significance of β , which reflects the fact that one of the two parties may be in a better position to influence the transaction price. For example, the retailer could extract a higher price if the customer has few outside options and is in a weak bargaining position. If $\beta \rightarrow 1$, the retailer has all the bargaining power and extracts the entire surplus by selling the item at customer's reservation price, r or the posted price p , whichever is smaller. On the other hand, if $\beta \rightarrow 0$, any bargainer with reservation price c and higher extracts the entire surplus by paying the cut-off price c for the item.

Also notice from (2) that only bargainers whose reservation price is considerably higher than the posted price will pay the posted price as the final outcome of the negotiation. For example, if $p = 100$, $c = 70$ and $\beta = 0.25$, only bargainers whose willingness-to-pay is greater than \$190 (i.e., \$90 above the posted price of \$100) will purchase the product at \$100. This feature of our model is consistent with what we observe from our daily shopping behavior.

Remark 1. In our model, we assume that a customer's reservation price is an intrinsic quality of an individual consumer, drawn from the reservation price distribution of the consumer population. While this interpretation of reservation price is natural, we also consider in Section 6 a model where some bargainers adjust their reservation prices after observing the posted price. All the results of the paper continue to hold even in this case (see Proposition 9). On the other hand, we allow the retailer to rationally decide the minimum price it is willing to accept from any customer (i.e., the cut-off price). This is in keeping with the fact that any profit-maximizing retailer will strategically choose the price to quote to the customer in models of dynamic pricing. In our setting, in addition to the usual posted price, the retailer has another pricing lever to control who buys, in the form of the cut-off price. Instead of selling to a bargainer whose reservation price is too low, the retailer is better off to keep the item for future periods. As we will soon observe, the retailer's benefit

from keeping a unit of the item for future periods, which we will refer to as the marginal value of inventory, evolves with the pace of sales (i.e., the item's inventory level relative to the remaining selling horizon). In that sense, the retailer does not attach a stationary and intrinsic value to a unit of the item, but it continues to adjust the value of the item throughout the selling season instead. Hence, it makes sense for the retailer to adjust the lowest price it is willing to accept for the item in each period. A number of papers that compare take-it-or-leave-it pricing and bargaining also use the endogenous cut-off price (e.g., Wang, 1995; Arnold and Lippman, 1998).

3.2 The Retailer's Problem

At the beginning of period t , $t = 1, \dots, T$, the retailer, given y units in inventory, sets the posted price, p and the cut-off price, c . In a given period, if a price-taker arrives, the revenue accrued is simply the posted price p , provided that the price-taker's reservation price exceeds the posted price, which happens with probability $\bar{F}(p)$. On the other hand, if a bargainer arrives, a revenue will be accrued as long as the bargainer's reservation price is above the cut-off price, c . As stated in (2), if the bargainer's reservation price is between c and $\frac{p-(1-\beta)c}{\beta}$, the revenue accrued from the bargainer is a weighted average of the bargainer's reservation price and the cut-off price, and if the bargainer's reservation price exceeds $\frac{p-(1-\beta)c}{\beta}$, the revenue accrued is the posted price p . Given y units in inventory in period t , the retailer's problem of setting the posted price and the cut-off price is given by the following optimality equations:

$$\begin{aligned} V_t(y) &= \max_{p, c: p \geq c} J_t(p, c, y) \text{ for } y > 0, t = 1, \dots, T, \text{ and} \\ V_0(y) &= 0 \text{ for } y \geq 0, \text{ and } V_t(0) = 0 \text{ for } t = 1, \dots, T, \end{aligned} \quad (3)$$

where

$$\begin{aligned} J_t(p, c, y) &= \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} [\beta x + (1-\beta)c] f(x) dx + \bar{F}(c) V_{t-1}(y-1) \right] \\ &\quad + \lambda(1-q) [p \bar{F}(p) + \bar{F}(p) V_{t-1}(y-1)] \\ &\quad + [1 - \lambda(q \bar{F}(c) + (1-q) \bar{F}(p))] V_{t-1}(y). \end{aligned}$$

Note that our formulation admits take-it-or-leave-it pricing: If $p = c$, the retailer is selling the item at the posted price to all customers.

Letting $\Delta_t(y) = V_t(y) - V_t(y-1)$, which represents the marginal value of inventory, we can

rewrite $J_t(p, c, y)$ as

$$J_t(p, c, y) = \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} [\beta x + (1-\beta)c] f(x) dx - \bar{F}(c) \Delta_{t-1}(y) \right] + \lambda(1-q) \bar{F}(p) [p - \Delta_{t-1}(y)] + V_{t-1}(y). \quad (4)$$

Let $p_t^*(y)$ and $c_t^*(y)$ denote, respectively, the optimal posted price and the optimal cut-off price in period t with y units of inventory.²

4. Analysis

In this section we first highlight important technical properties of the retailer's expected revenue-to-go function, $J_t(p, c, y)$. We then explore the effect of negotiation on the retailer and the customers (both price-takers and bargainers). For the time being, we assume that negotiation is costless. We will subsequently analyze the setting where negotiation is costly.

4.1 Properties of the Retailer's Objective Function

Observe from (4) that the retailer's expected revenue-to-go with t periods to go and y units in inventory, $J_t(p, c, y)$, is a weighted sum of the contribution from a bargainer, given by

$$\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} [\beta x + (1-\beta)c] f(x) dx - \bar{F}(c) \Delta_{t-1}(y) + V_{t-1}(y),$$

and the contribution from a price-taker, given by

$$\bar{F}(p) [p - \Delta_{t-1}(y)] + V_{t-1}(y),$$

where each contribution is weighted by the probability that the corresponding type of customer arrives: λq for a bargainer and $\lambda(1-q)$ for a price-taker.

If we only examine the revenue contribution from a bargainer in isolation, the effect of the posted price, p is straight-forward: Because the posted price forms a ceiling on the price to be paid by a bargainer, the larger the posted price, the larger the contribution from a bargainer. Likewise, the contribution from a price-taker in isolation follows a form that is well-studied in the revenue management literature.³ Despite such simple behavior on the part of these two separate contributions, the retailer's expected revenue-to-go function $J_t(p, c, y)$, which is the weighted sum of

²When solving for $p_t^*(y)$ it suffices to consider $c \leq b$ and $p \leq b$, where b is the upper bound of the customers' reservation price distribution. If c exceeded b , then no bargainer would purchase. Given c must be less than or equal to b , it can be checked that the retailer cannot do any better by setting $p > b$ than it could by setting $p = b$.

³See Ziya, Ayhan and Foley (2004) for conditions under which this contribution becomes unimodal or concave in the price, p .

the two contributions, is not necessarily well-behaved and may exhibit rather complicated behavior in the posted price, p . To see why, notice that the two contributions may add up to yield a function $J_t(p, c, y)$ that is bimodal in p , which would then be hard to analyze. Such technical challenges are exacerbated by the retailer's need to determine the cut-off price, c , jointly with the posted price, p .

To impose a structure on $J_t(p, c, y)$ so that it is amenable to analysis, some of our subsequent results will use the following two assumptions.

Assumption 1. *The cdf of R_c , $F(\cdot)$, has increasing failure rate. That is, $\frac{f(\cdot)}{F(\cdot)}$ is increasing.*

Assumption 2. *The fraction of bargainers in the population, q , is less than $1/2$.*

The first of these assumptions is innocuous as it restricts the reservation price distribution to have an increasing failure rate, a restriction that is satisfied by a large range of probability distributions, including uniform, Weibull, normal, and exponential, and their truncated versions.

The second assumption is very much in line with the practical domain of our model, because we are mainly interested in retailers who do not necessarily use negotiation as their main sales format, but allow it for a small proportion of customers who choose to negotiate. In fact, anecdotal evidence from Richtel (2008) supports this assumption: approximately one-quarter of customers try to bargain at a BestBuy store in Minnesota.

The following lemma states that, when Assumptions 1 and 2 hold, the retailer's objective function is well-behaved in the sense that there exists a unique stationary point that also yields the optimal solution.

Lemma 1. *Consider a retailer with y units of inventory and t periods to go until the end of the season. Suppose that Assumptions 1 and 2 hold. There exists a unique pair of the posted price, p , and the cut-off price, c , that satisfy the first order conditions for $J_t(p, c, y)$, and this pair maximizes $J_t(p, c, y)$.*

We should note that not all IFR distributions require Assumption 2. For any IFR distribution that has a non-decreasing density, Lemma 1 holds even in the absence of the restriction imposed by Assumption 2 (i.e., even when the fraction of bargainers is more than $1/2$). One interesting special case is obtained when the reservation price is uniformly distributed (i.e., the density $f(\cdot)$ is constant over $[0, b]$). The next lemma formally states the structural result for the uniform distribution. Furthermore, in a numerical study with q values greater than $1/2$, we observed that the result of Lemma 1 continues to hold if the reservation price distribution is exponential or Weibull. In

short, it appears that, for certain reservation price distributions (e.g., IFR distributions with non-decreasing densities as well as the exponential and Weibull distributions), Lemma 1 continues to hold even in the absence of Assumption 2.

Lemma 2. *If the reservation price distribution, F , is uniform over the interval $[0, b]$, then, for any $q \in (0, 1)$, there exists a unique pair of the posted price, p , and the cut-off price, c , that satisfy the first order conditions for $J_t(p, c, y)$, and this pair maximizes $J_t(p, c, y)$.*

Given the analytical tractability afforded by Assumptions 1 and 2 (as witnessed by Lemma 1), we will work with these assumptions in the remainder of the paper. We should note, however, that *not all of our results require both assumptions*. Whenever these assumptions can be relaxed (e.g., Propositions 1 and 6), we will remark so. In addition, if we assume that F is uniform (as in Lemma 2), not only do we not need Assumption 2 but we also obtain closed-form expressions for the optimal posted price and cut-off price. Some of our results use the assumption that F is uniform to take advantage of closed-form expressions. Whenever the uniform assumption is made to obtain a result, we will remark so in the statement of the result.

4.2 Effect of Negotiation

When negotiation is costless, it is not difficult to observe that a retailer that allows negotiation will obtain a higher revenue than a retailer that does not allow negotiation. After all, a negotiating retailer could always set both its posted price and its cut-off price equal, thereby eliminating any room for negotiation and, in effect, reverting to take-it-or-leave-it pricing. When a retailer allows negotiation, the retailer's ability to price discriminate among bargainers enables the retailer to improve its revenue. In fact, Proposition 1 below shows that the larger the fraction of bargainers in the population, the higher the retailer's revenue.⁴

Proposition 1. *Consider a retailer that has y units of inventory with t periods to go until the end of the season. If the fraction of bargainers, q , increases, then the retailer's optimal expected revenue, $V_t(y)$, increases.*

Unlike a price-taker, a bargainer does not walk away when presented with a high posted price. Therefore, the presence of bargainers allows the retailer a higher degree of pricing flexibility. Hence, the larger the fraction of bargainers, the larger the gain that the retailer can realize from the pricing flexibility enabled by negotiation. To further explore the effect of negotiation on the retailer's prices,

⁴As the proof reveals, the result holds under general conditions that do not require Assumptions 1 and 2 at all.

we define the following auxiliary optimization problem, which represents a benchmark where the retailer does not allow negotiation in period t (i.e., the retailer uses take-it-or-leave-it pricing in period t) while following the optimal policy under negotiation from next period onward (i.e., the revenue-to-go from period $t - 1$ onward is $V_{t-1}(\cdot)$ as defined by (3)):

$$V_t^{TL}(y) = \max_p \{ \lambda \bar{F}(p)p + \lambda \bar{F}(p)V_{t-1}(y-1) + (1 - \lambda \bar{F}(p))V_{t-1}(y) \}, y > 0, t = 1, \dots, T \quad (5)$$

Let $p_t^{TL}(y)$ denote the optimal solution to (5), that is, given t periods to go with y units in inventory, $p_t^{TL}(y)$ is the optimal *take-it-or-leave-it price* of a retailer that does not allow negotiation in period t . The next proposition highlights the effect of negotiation on the prices.

Proposition 2. *Consider a retailer that has y units of inventory with t periods to go until the end of the season. Then:*

- (a) *The optimal posted price under negotiation is greater than or equal to the optimal take-it-or-leave-it price, i.e., $p_t^*(y) \geq p_t^{TL}(y)$.*
- (b) *The optimal cut-off price under negotiation is less than or equal to the optimal take-it-or-leave-it price, i.e., $c_t^*(y) \leq p_t^{TL}(y)$.*
- (c) *Suppose the customer's reservation price distribution, $F(\cdot)$, is uniform over $[0, b]$. Let $p_t^{avg}(y)$ denote the average of all prices paid by bargainers who purchase the product. The average price paid by a bargainer is less than or equal to the optimal take-it-or-leave-it price, i.e., $p_t^{avg}(y) \leq p_t^{TL}(y)$.*

Proposition 2(a),(b) together show that the optimal posted price and the optimal cut-off price straddle the optimal take-it-or-leave-it price. If negotiation were not allowed in period t , the retailer would charge the price $p_t^{TL}(y)$. Under negotiation, however, the retailer chooses a posted price that includes a premium on top of $p_t^{TL}(y)$, while at the same time choosing a cut-off price below $p_t^{TL}(y)$, with the intention of selling to some bargainers at prices even below the take-it-or-leave-it price. Hence, an individual bargainer may end up purchasing at a price above or below the take-it-or-leave-it price, depending on her willingness to pay. This is a manifestation of the price discrimination enabled by negotiation.

Proposition 2(c) shows that the average price paid by a bargainer is less than the take-it-or-leave-it price, which is the price the bargainers would have to pay if negotiation were not allowed. In that sense, on average, bargainers are better off when negotiation is allowed. While the assumption of uniform reservation price distribution facilitates the analysis, it does not seem to drive the result stated in Proposition 2(c). We observed through a numerical study that the same result holds when the reservation price follows exponential and Weibull distributions.

Because the price paid by a bargainer increases in his or her reservation price, bargainers with lower reservation prices may benefit from negotiation while bargainers with higher reservation prices may be worse off in the presence of negotiation. As for price-takers, none of them is better off with a negotiating retailer, because the negotiating retailer is adding a premium to its posted price with the intention to negotiate with the bargainers. The following proposition formalizes the effect of negotiation on customers (see Figure 1 for a graphical representation).

Proposition 3. *Consider a retailer that has y units of inventory with t periods to go until the end of the season. No price-taker is better off if negotiation is allowed in period t . As for bargainers, there exists a threshold reservation price such that the bargainer is better off if her reservation price is below the threshold and worse off otherwise.*

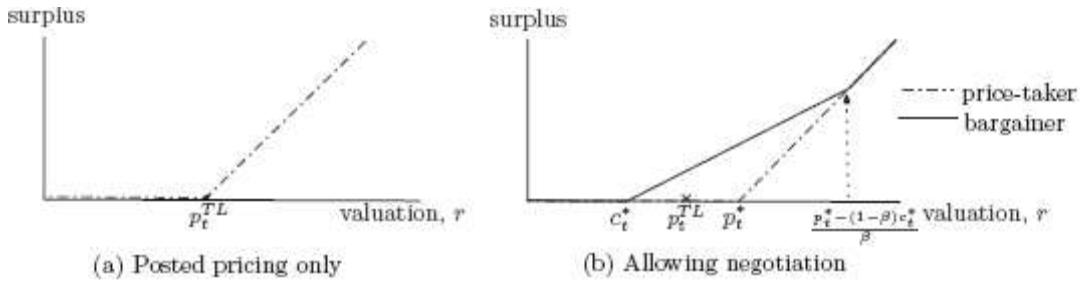


Figure 1: Consumer surplus as a function of the customer's reservation price (or, valuation). Panel (a) shows the surplus when negotiation is not allowed, in which case a customer buys only if her valuation is above the take-it-or-leave-it price, $p_t^{TL}(y)$. Panel (b) shows the surplus when negotiation is allowed, in which case the surplus is different for bargainers and price-takers.

If the retailer allows negotiation, a bargainer whose reservation price is at the threshold (which, incidentally, can be shown to be equal to $\frac{p_t^{TL}(y) - (1-\beta)c_t^*(y)}{\beta}$), ends up paying exactly $p_t^{TL}(y)$ (see (2)), which is also the price that the retailer would charge and the bargainer would pay if negotiation were not allowed. Hence, the bargainer at the threshold is not affected by whether or not negotiation is allowed. However, negotiation benefits bargainers with lower reservation prices. In fact, there are two different ways in which such bargainers benefit from negotiation. First, consider bargainers whose reservation prices are higher than $p_t^{TL}(y)$, but less than the threshold, $\frac{p_t^{TL}(y) - (1-\beta)c_t^*(y)}{\beta}$. Such bargainers would be able to afford the product even if negotiation were not allowed, but they now manage to purchase the product at a lower price than $p_t^{TL}(y)$. Hence, they are better off because they pay less for the product. Second, consider the bargainers whose reservation prices are higher than the cut-off price $c_t^*(y)$, but less than $p_t^{TL}(y)$. Such bargainers would not be able to afford the product if negotiation were not allowed, but they do purchase the product under

negotiation. Hence, negotiation benefits such bargainers by enabling them to purchase the product.

Remark 2. In Propositions 2 and 3, we compare two policies that are different only in period t : one of which disallows negotiation in period t and the other allows negotiation in period t . Both policies allow negotiation from period $t - 1$ onward. Will the results of Propositions 2 and 3 continue to hold if one policy allows negotiation throughout the horizon and the other does not allow negotiation in any period of the horizon? To test this, we conducted a numerical study using three different reservation price distributions (uniform over $[0, 50]$, truncated exponential over $[0, 150]$ with mean 20, and truncated Weibull over $[0, 150]$ with shape parameter 2 and scale parameter 50), three different values for the arrival rate ($\lambda \in \{0.2, 0.5, 0.7\}$), three different values for the retailer's bargaining power, ($\beta \in \{0.2, 0.5, 0.7\}$), five different values for the proportion of bargainers in the population, ($q \in \{0.05, 0.2, 0.35, 0.5, 0.8\}$), with inventory levels ranging from 1 to 15 with $T = 15$ periods to go until the end of the season. All in all, 2,025 different problem instances are considered. In all instances, numerical results support Propositions 2 and 3 even when the comparison is between two retailers one of which allows negotiation throughout the horizon while the other retailer never does.

Proposition 3 shows that the existence of bargainers makes the price-takers worse off once the retailer allows negotiation. A follow-up question is: What happens to the posted price and cut-off price if more customers become bargainers? The next proposition answers this question.

Proposition 4. *Consider a retailer that has y units of inventory with t periods to go until the end of the season. Suppose the fraction of bargainers in the population, q , increases in period t . Then:*

- (a) *The optimal posted price, $p_t^*(y)$, increases.*
- (b) *The optimal cut-off price, $c_t^*(y)$, increases.*

The posted price forms a ceiling on the revenue that can be extracted from a bargainer. Hence, the retailer can do better price discrimination among bargainers by increasing the posted price. On the flip side, however, every bit of increase in the posted price translates into more price-takers being left out. If the fraction of bargainers, q , is small, then the retailer needs to mind the large revenue stream from price-takers and charges a low posted price so as not to turn away too many price-takers. As the fraction of bargainers increases, the retailer puts less emphasis on the price-takers and starts to increase the posted price to better price-discriminate among bargainers. Unlike the choice of the posted price, where the trade-off is between the revenue from bargainers and the

revenue from price-takers, the choice of the cut-off price concerns only the revenue from bargainers. After the fraction q increases, the retailer increases the cut-off price, because it would like to take advantage of the now-higher posted price: An increase in the posted price raises the ceiling on what bargainers can be charged, and the retailer can take advantage of this higher ceiling by raising the cut-off price, which in turn increases the transaction price for those bargainers who purchase. The manifestations of Proposition 4 are abundant in practice. For example, in consumer electronics, local stores that advertise their willingness “to deal” tend to attract more bargainers and their posted prices are higher than those of national chains that are more discreet about their willingness to haggle. A recent price comparison revealed that Big George, a consumer electronics store in Ann Arbor, posted a price of \$1,979 for a plasma HDTV (to be specific, a Panasonic 50” VIERA 1080p Plasma HDTV with AR Filter) while BestBuy’s posted price for the same TV on the same day was \$1499.99.

Another determinant of the prices under negotiation is the distribution of bargaining power. The following result shows how the optimal posted price and the optimal cut-off price are influenced by the retailer’s bargaining power, β , when the customer’s reservation price is uniformly distributed.

Proposition 5. *Suppose the customer’s reservation price distribution, $F(\cdot)$ is uniform between $[0, b]$. Consider a retailer that has y units of inventory with t periods to go until the end of the season. Then:*

- (a) *The optimal posted price $p_t^*(y)$ is increasing in the retailer’s bargaining power, β .*
- (b) *The optimal cut-off price $c_t^*(y)$ is decreasing in the retailer’s bargaining power, β .*

To see the intuition behind Proposition 5(a), first notice that an increase in the retailer’s bargaining power translates into an increase in the transaction price. Because the transaction price for any bargainer is bounded by the posted price, the retailer cannot fully take advantage of its increased bargaining power when the posted price is low. Hence, the retailer increases the posted price in order to obtain higher prices from bargainers (in particular, those with very high reservation price).

According to Proposition 5(b), the cut-off price decreases when the retailer’s bargaining power increases, because the cut-off price and bargaining power are economic substitutes (in the precise sense that the retailer’s revenue is submodular in the cut-off price and bargaining power as we show in the proof of the proposition). On a more intuitive level, when the retailer’s bargaining power increases, the retailer now finds that it can extract higher transaction prices from bargainers across-the-board. Thus, the retailer can now afford to sell to bargainers with slightly lower reservation

prices, safe in the knowledge that those bargainers will now pay higher prices than they would have when the retailer had less bargaining power. In order to serve such bargainers, the retailer reduces the cut-off price.

An interesting implication of Proposition 5 is that, when the retailer's bargaining power increases, the price-takers are now worse off (since the posted price is higher), but more bargainers are now able to purchase the product (since the cut-off price is lower). In a sense, this result suggests that the retailer's bargaining power benefits bargainers with low reservation prices at the expense of price-takers.

Extending the proposition to cover non-uniform reservation price distributions is formidable due to a highly non-linear effect of β onto the revenue-to-go function. However, in our numerical study, we observed that the same result holds for non-uniform distributions as well. Fixing β at one of three values (0.2, 0.5, 0.7), the numerical study uses two different reservation price distributions, namely exponential and Weibull, three different values for the arrival rate, λ , five different values for the proportion of bargainers in the population, q , with inventory levels ranging from 1 to 15 with $T = 15$ periods to go until the end of the season. Thus, for each value of $\beta \in \{0.2, 0.5, 0.7\}$, we consider 450 different combinations of parameter values, resulting in a total of 1,350 instances. In all instances, we find that the optimal prices decrease as β increases. This numerical observation suggests that the result of Proposition 5 holds even when the reservation price distribution is exponential or Weibull. Therefore, we believe that while the uniform assumption facilitates the analysis, it does not drive the result.

As for the effects of time and inventory level, Gallego and van Ryzin (1994) and Bitran and Mondschein (1997) show that, in the case of take-it-or-leave-it pricing, the price decreases if there is more inventory or less time until the end of the season. The following proposition shows that the posted price and the cut-off price behave similarly when the retailer allows negotiation.⁵

Proposition 6. *The optimal posted price and cut-off price, $p_t^*(y)$ and $c_t^*(y)$, are decreasing in the stock level y and increasing in the number of remaining periods, t .*

Proposition 6 has implications for the probability that an arriving bargainer will make a purchase. Note that the probability that a bargainer purchases is the probability that her reservation price exceeds the cut-off price, i.e., $\bar{F}(c_t^*(y))$. Given that the cut-off price decreases in the stock level and increases in the remaining time, it follows that the higher the stock level and the shorter the time until the end of the selling season, the higher the probability that a bargainer will end up

⁵As the proof reveals, this result does not require that Assumptions 1 and 2 hold.

making a purchase. Likewise, the probability that a price-taker purchases is the probability that her reservation price exceeds the posted price, i.e., $\bar{F}(p_t^*(y))$, and this probability is also higher when the stock level is higher and the remaining time is shorter. No matter whether the customer is a bargainer or price-taker, the larger the stock level relative to the remaining time, the higher the probability that an arriving customer will purchase.

As stated in Proposition 2(c), the average price paid by a bargainer who purchases the product, $p_t^{avg}(y)$, undercuts the optimal take-it-or-leave-it price, $p_t^{TL}(y)$. The following corollary to Proposition 2(c) further examines the gap between these two prices and illustrates the effect of time and inventory on a negotiating retailer's willingness to make a deal.

Corollary 1. *Suppose the customer's reservation price distribution, $F(\cdot)$, is uniform over $[0, b]$. Then, the difference between the take-it-or-leave-it price and the average price paid by a bargainer, $p_t^{TL}(y) - p_t^{avg}(y)$, increases in the stock level and decreases in the time remaining until the end of the season.*

The implication of the corollary is that the more overstocked a negotiating retailer is, the larger the discounts it is willing to give from the take-it-or-leave-it price. We should note that although Proposition 2(c) is robust to the assumption of uniformly distributed reservation prices, Corollary 1 is not.

4.3 When Negotiation is Costly

So far we have assumed that negotiation is costless, in which case the retailer always benefits from negotiation. In this section we extend our model to the case where the retailer incurs a cost K if it allows negotiation in the current period. In the presence of negotiation cost, the retailer does not necessarily wish to negotiate in each and every period. Hence, in this section we endogenize the decision to negotiate. This extension also allows us to highlight the interaction between dynamic pricing and negotiation to a greater degree. In particular, we will show the effect of time and inventory level on the decision to negotiate.

At the beginning of each period the retailer first decides whether or not to allow negotiation in that period. If the retailer allows negotiation, the sequence of events in that period is the same as before. If the retailer does not allow negotiation, then all customers are effectively rendered price-takers who will purchase at the posted price provided that their reservation price exceeds the posted price. Given y units in inventory in period t , the retailer's problem is given by the following

optimality equations:

$$\begin{aligned}\tilde{V}_t(y) &= \max\{\tilde{V}_t^{TL}(y), \tilde{V}_t^N(y) - K\} && \text{for } y > 0, t = 1, \dots, T, \\ \tilde{V}_0(y) &= 0 \text{ for } y \geq 0, \text{ and } \tilde{V}_t(0) = 0 \text{ for } t = 1, \dots, T,\end{aligned}\quad (6)$$

where

$$\begin{aligned}\tilde{V}_t^{TL}(y) &= \max_p \left\{ \lambda \bar{F}(p)p + \lambda \bar{F}(p)\tilde{V}_{t-1}(y-1) + (1 - \lambda \bar{F}(p))\tilde{V}_{t-1}(y) \right\}, \\ \tilde{V}_t^N(y) &= \max_{p \geq c} \left\{ \begin{aligned} &\lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} (\beta x + (1-\beta)c) f(x) dx + \bar{F}(c)\tilde{V}_{t-1}(y-1) \right] \\ &+ \lambda(1-q) \left[p \bar{F}(p) + \bar{F}(p)\tilde{V}_{t-1}(y-1) \right] + [1 - \lambda(q\bar{F}(c) + (1-q)\bar{F}(p))] \tilde{V}_{t-1}(y) \end{aligned} \right\}\end{aligned}\quad (7)$$

The retailer's decision to negotiate boils down to whether or not the revenue improvement due to negotiation, enabled by price discrimination, exceeds the cost of negotiation, K .

Suppose that the retailer has y units of inventory and t periods to go. Define $\tilde{p}_t^{TL}(y)$ to be the take-it-or-leave-it price the retailer will choose if the retailer does not allow negotiation in period t . Also, define $\tilde{p}_t^*(y)$ and $\tilde{c}_t^*(y)$ to be the posted price and cut-off price that the retailer will choose if the retailer allows negotiation in period t . Following a logic similar to that used in the proof of Proposition 2, it can be shown that the take-it-or-leave-it price, $\tilde{p}_t^{TL}(y)$ is between the cut-off price and posted price of the retailer who allows negotiation, $\tilde{c}_t^*(y)$ and $\tilde{p}_t^*(y)$. In other words, compared to a retailer that does not allow negotiation, a negotiating retailer charges a *price premium* $\tilde{p}_t^*(y) - \tilde{p}_t^{TL}(y)$ and plans to yield bargainers a *negotiation allowance* up to $\tilde{p}_t^{TL}(y) - \tilde{c}_t^*(y)$. The following proposition highlights the effect of the inventory and time on the price premium and negotiation allowance.

Proposition 7. *Suppose the customer's reservation price distribution, $F(\cdot)$ is uniform between $[0, b]$. Consider a retailer that has y units of inventory with t periods to go until the end of the season. Then:*

- (a) *The price premium, $\tilde{p}_t^*(y) - \tilde{p}_t^{TL}(y)$, is increasing in the stock level y and decreasing in the number of remaining periods, t .*
- (b) *The negotiation allowance, $\tilde{p}_t^{TL}(y) - \tilde{c}_t^*(y)$, is increasing in the stock level y and decreasing in the number of remaining periods, t .*

If there is more inventory or less time until the end of the season, the risk of excess inventory becomes more important to the retailer than the risk of stock-out. In such a case, both the non-negotiating retailer and negotiating retailer will decrease the posted prices in an effort to move

inventory quickly. At the same time, as Proposition 7 illustrates, the negotiating retailer increases the price premium and negotiation allowance. Consequently, the gap between the lowest and highest prices extracted from bargainers increases, even as the posted price and the cut-off price themselves decrease. The implication is that if there is more inventory or less time until the end of the season, the retailer engages in negotiation more aggressively and does a finer price discrimination.

Building on the observation that the retailer does a more aggressive price discrimination when the inventory is high and the remaining time is shorter and recalling that the benefits of negotiation accrue due to the ability to price discriminate, one would conjecture that if there is more inventory or less time until the end of the season, then the retailer benefits more from negotiation. This partly explains why the reporters from the Wall Street Journal were able to negotiate the price at a wide range of stores during the holiday season. Indeed, the following proposition formalizes the effect of inventory and remaining time on the retailer's decision to allow negotiation.

Proposition 8. *Suppose that the customers' reservation price distribution, $F(\cdot)$ is uniform between $[0, b]$. If it is optimal for the retailer to negotiate when there are $y > 0$ units of inventory with $t > 1$ periods to go, then:*

- (a) *it is also optimal to allow negotiation if it has $y + 1$ units of inventory, and*
- (b) *it is also optimal to allow negotiation if it has $t - 1$ periods to go until the end of the season.*

The proposition above shows that, as the inventory level increases or the remaining time decreases, the retailer becomes more willing to incur the cost of negotiation, so that it can enjoy the benefits of price discrimination. An example is depicted in Figure 2. To see the intuition behind this result, consider the effect of inventory at a given time. When the inventory level is low relative to the remaining selling horizon, the retailer is not particularly worried about the risk of leftover inventory and, therefore, it is reluctant to sell to customers with low reservation prices. In such a case, a negotiating retailer would set both the posted price and the cut-off price at high levels, which means the retailer would not have much of a bandwidth to price discriminate. Thus, the benefit from price discrimination may not be worth the cost of implementing negotiation. On the other hand, when the inventory level is high, the risk of leftover inventory looms large and the retailer is much more willing to part with its inventory even at relatively low prices. In such a case, a negotiating retailer can still charge a high posted price, but is willing to settle for a low cut-off price. Thus, fewer bargainers will quit and the price paid by a purchasing customer is one of a wide range of prices. In other words, the higher the inventory level, the finer the price discrimination enabled by negotiation, thus making it worthwhile to incur the cost of negotiation.

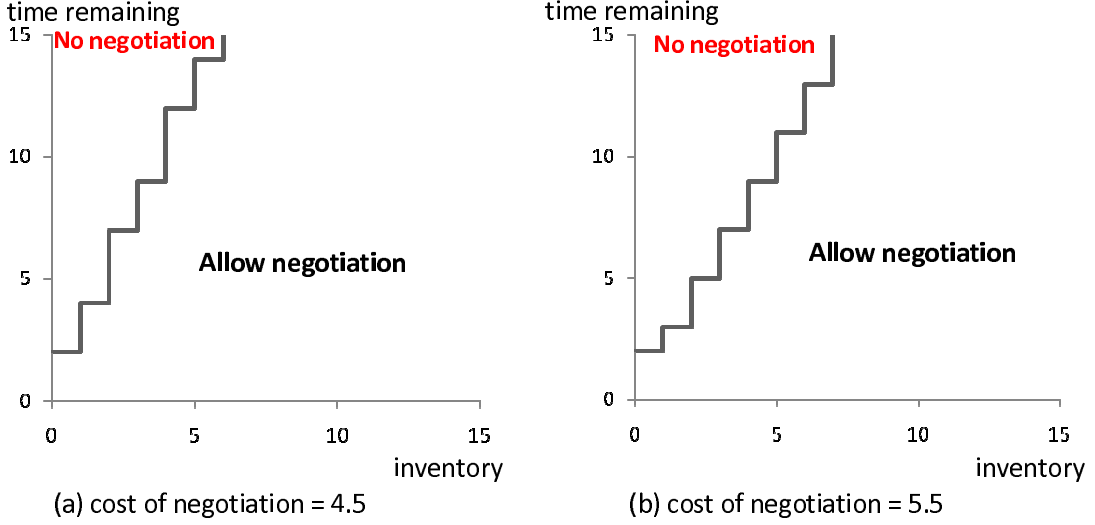


Figure 2: Pricing strategies – take-it-or-leave-it pricing only or negotiation – that the retailer chooses when it has y units of inventory with t periods to go until the end of the season at two different costs of negotiation (K): 4.5 (left) and 5.5 (right) when $\lambda = 0.7$, $\beta = 0.7$, $q = 0.7$, and $F(\cdot) \sim \text{Weibull}[0, 150]$ with shape parameter 2 and scale parameter 50.

Remark 3. Both Propositions 7 and 8 use the assumption that the reservation price distribution is uniform. To test whether these results are robust, we conducted a numerical study using two non-uniform reservation price distributions (truncated exponential, and truncated Weibull). We vary the arrival rate, the retailer’s bargaining power, the proportion of bargainers, the initial inventory level, and the remaining selling season, leading to a total of 1,350 different problem instances. We observed that, in all of 1,350 instances, Proposition 8 (i.e., the effect of inventory and time on the gain from negotiation) holds. This observation leads us to conclude that the main result of this section, that the retailer benefits more from negotiation when there is more inventory or less remaining time, is robust to the distribution assumption. On the other hand, Proposition 7 (i.e., the effect of inventory and time on the price differences) does not necessarily hold under non-uniform distributions. When the reservation price distribution is not uniform, a change in price triggers a non-linear change in the portion of consumers who buy. Hence, the shape of the distribution becomes a very important input into the exact dependence of the optimal price on time and inventory. Consequently, when one focuses on the difference between two prices ($p_t^*(y) - p_t^{TL}(y)$ or $p_t^{TL}(y) - c_t^*(y)$) as Proposition 7 does, the shape of the distribution may interact with time and inventory in such complicated ways that the optimal price difference may end up being non-monotonic in time and/or inventory.

5. Numerical Study

We conduct a numerical study to gain further managerial insights into the use of negotiation along with dynamic pricing. We identify the scenarios under which negotiation benefits the retailer the most, and we investigate when the retailer prefers dynamic pricing over negotiation.

In our numerical study we consider several different combinations of parameter values. We use three different values for each of the probability that a customer arrives in a given period ($\lambda \in \{0.2, 0.5, 0.7\}$) and the retailer's bargaining power ($\beta \in \{0.2, 0.5, 0.7\}$), and we use five different values for the proportion of bargainers ($q \in \{0.05, 0.2, 0.35, 0.5, 0.8\}$). We also consider three different reservation price distributions:

- (i) $F(\cdot) \sim$ uniform over $[0, 50]$,
- (ii) $F(\cdot) \sim$ exponential over $[0, 150]$ with mean 20, and
- (iii) $F(\cdot) \sim$ Weibull $[0, 150]$ with shape parameter 2 and scale parameter 50.

Note that this parameter set results in 135 different combinations of λ , β , q and $F(\cdot)$ distribution. We first compare two retailers, one that allows negotiation and the other that does not. This comparison allows us to gain insights into the benefits of negotiation. We consider a 15-period selling season and we vary the starting inventory level from 1 to 15. For each starting inventory level and under all 135 combinations of λ , β , q , and $F(\cdot)$ distribution described above, we solve the dynamic program associated with each retailer and determine the retailer's optimal expected revenue, resulting in 2,025 different problem instances. For each problem instance, we measure the percentage revenue improvement from negotiation and summarize the results in Table 1.

A number of observations can be made from Table 1. For example, the larger the retailer's bargaining power, β , the larger the revenue improvement due to negotiation. Likewise, the larger the proportion of bargainers, q , the larger the revenue improvement due to negotiation. Furthermore, we observe that the retailer's benefit from negotiation tends to be larger under exponential and Weibull distributions for F , compared to uniform. Weibull and exponential reservation prices have heavier tails compared to uniform, indicating a larger portion of customers with high reservation prices. The retailer can extract more revenue out of those customers through negotiation, which results in the benefits from negotiation being higher under Weibull and exponential distributions. We also report that the percentage revenue improvement is larger when starting inventory level is higher. This is in line with the results of the previous section, where we discussed why higher inventory levels enable finer price discrimination, leading to higher benefits from negotiation.

Table 2 summarizes the magnitude of percentage improvement for all 2,025 problem instances

		F Uniform			F Exponential			F Weibull		
β	q	Mean (Std)	Max	Min	Mean (Std)	Max	Min	Mean (Std)	Max	Min
0.2	0.05	0.44 (0.10)	0.50	0.13	0.73 (0.10)	0.79	0.40	0.46 (0.09)	0.53	0.21
0.2	0.2	1.78 (0.42)	2.04	0.54	3.04 (0.43)	3.27	1.64	1.91 (0.38)	2.16	0.86
0.2	0.35	3.15 (0.76)	3.62	0.96	5.51 (0.80)	5.94	2.95	3.45 (0.70)	3.92	1.54
0.2	0.5	4.56 (1.11)	5.25	1.37	8.16 (1.20)	8.82	4.33	5.10 (1.05)	5.80	2.25
0.2	0.8	7.50 (1.87)	8.69	2.21	14.37 (2.19)	15.59	7.48	8.89 (1.90)	10.19	3.83
0.5	0.05	1.10 (0.26)	1.26	0.34	1.74 (0.25)	1.87	0.94	1.04 (0.19)	1.17	0.49
0.5	0.2	4.57 (1.11)	5.25	1.37	7.36 (1.07)	7.95	3.92	4.38 (0.84)	4.94	2.03
0.5	0.35	8.30 (2.67)	9.59	2.43	13.70 (2.08)	14.86	7.14	8.13 (1.63)	9.21	3.68
0.5	0.5	12.29 (3.14)	14.28	3.50	20.92 (3.31)	22.79	10.66	12.37 (2.57)	14.10	5.45
0.5	0.8	21.27 (5.72)	24.99	5.72	39.36 (6.84)	43.40	18.98	23.10 (5.25)	26.77	9.54
0.7	0.05	1.56 (0.37)	1.78	0.48	2.40 (0.34)	2.59	1.30	1.40 (0.25)	1.56	0.68
0.7	0.2	6.52 (1.60)	7.52	1.93	10.39 (1.54)	11.24	5.47	5.96 (1.11)	6.69	2.81
0.7	0.35	12.03 (3.07)	13.96	3.44	19.71 (3.10)	21.45	10.08	11.25 (2.22)	12.71	5.14
0.7	0.5	18.12 (4.78)	21.20	4.98	30.74 (5.12)	33.70	15.20	17.44 (3.62)	19.88	7.67
0.7	0.8	32.60 (9.31)	38.88	8.19	60.30 (11.47)	67.38	27.49	34.09 (8.11)	39.86	13.62

Table 1: Summary Statistics for Percentage Revenue Improvement from Negotiation

tested. In more than 72% of instances, the percentage revenue improvement is greater than 3%. In about 38% of instances, the revenue improvement is greater than 10%. These numbers suggest that, even when there is cost for implementing negotiation, the retailer can realize significant benefit from negotiation.

% Improvement	< 1%	1 – 3%	3 – 5%	5 – 10%	10 – 20%	20 – 30%	> 30%	Total
Number of cases	178	378	218	485	396	187	183	2025

Table 2: Frequency Table for Percentage Revenue Improvement

Note that both negotiation and dynamic pricing are tools that a retailer can use to improve its revenue. We next compare the benefits from each of these two strategies from the perspective of the retailer who is using neither dynamic pricing nor negotiation in the status quo. Consider a retailer who picks the the optimal static take-it-or-leave-it price at the beginning of the selling season (i.e., the retailer uses the same price throughout the selling season). Figure 3 illustrates, as a function of the retailer’s initial inventory, the revenue improvement the retailer would realize by switching to dynamic pricing (without negotiation) or negotiation (without dynamic pricing). When inventory level is low, the retailer would like to start with a very high price with the intention of reducing the price later in the season if the product is not selling well. Thus, the benefit from dynamic pricing exceeds the benefit from negotiation at low inventory levels. On the other hand, when inventory

level is high, the retailer’s primary concern is to move inventory before the end of the season, in which case negotiation proves to be an effective tool, since the retailer can still price discriminate without reducing the chances of making a sale. Hence, at high inventory levels, the benefit from negotiation exceeds the benefit from dynamic pricing.

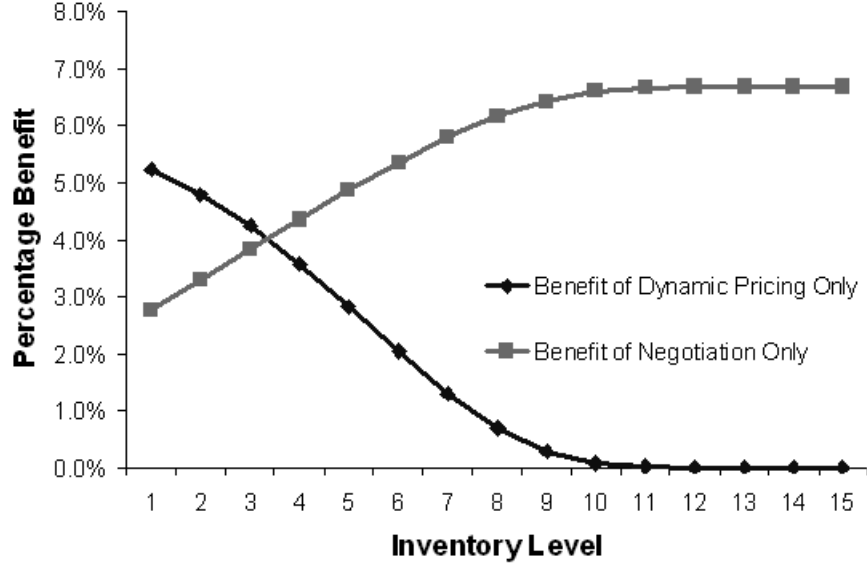


Figure 3: Consider a retailer who is currently using static pricing with no negotiation. The figure shows the percentage revenue improvement when switching to dynamic pricing only and switching to negotiation only. Here, $\lambda = 0.7$, $q = 0.2$, $\beta = 0.7$ and $F(\cdot)$ Weibull $[0, 150]$ with shape parameter 2 and scale parameter 50.

6. Reservation Prices Adjusted on the basis of the Posted Price

So far, following the precedence of much of the literature on dynamic pricing, we assumed that a customer’s reservation price for the product is intrinsic to the customer, and we assumed that each individual’s reservation price is drawn from an exogenous distribution. In a retail environment with bargaining, however, some bargainers may use the posted price as a benchmark to adjust (or calibrate) their valuation for the product. In this section, we relax the assumption that the customer’s reservation price is an intrinsic attribute and examine whether this alters any of the main insights derived from our original model.

In this extension model, we allow bargainers to adjust down their reservation price if their initial reservation price exceeds the posted price; hence, we refer to it as the adjusted reservation price (ARP) model. Specifically, if a bargainer’s initial reservation price, r , exceeds the posted price, p ,

the bargainer will adjust her reservation price down from r to $r - \eta(r - p)$, where $\eta \in [0, 1)$. For each individual consumer, the magnitude of the adjustment, $\eta(r - p)$, depends on the difference between their reservation price and the posted price: The higher the reservation price is, the bigger the adjustment. As for bargainers whose reservation prices are below the retailer's posted price, we assume that they will not adjust their reservation price. In other words, we make an implicit assumption that when the posted price already exceeds an individual's reservation price, the posted price does not provide any signal to the individual about the item's value.

Note that the ARP model captures several different scenarios that could arise. At one extreme, if $\eta = 0$, the ARP model is equivalent to our original model: the reservation price is an intrinsic attribute and is not affected by the posted price. At the other extreme, if $\eta \rightarrow 1$, then the bargainers' reservation prices will be censored at the posted price; all customers whose reservation prices used to be above the posted price will now have a reservation price equal to the posted price. Between these two extreme values, the reservation prices of high-valuation customers are gradually adjusted downward by the posted price.

The outcome of negotiation depends on a bargainer's initial reservation price, r , and the retailer's two prices, p and c . As in the original model, a bargainer with $r < c$ will not purchase. If the bargainer's initial reservation price, r , exceeds the cut-off price, c , her final transaction price is as follows.

$$p_N^*(c, r) = \begin{cases} \beta r + (1 - \beta)c & \text{if } c \leq r \leq p; \\ \beta(r - \eta(r - p)) + (1 - \beta)c & \text{if } p < r \leq \frac{(1 - \beta\eta)p - (1 - \beta)c}{\beta(1 - \eta)}; \\ p & \text{if } r \geq \frac{(1 - \beta\eta)p - (1 - \beta)c}{\beta(1 - \eta)}. \end{cases} \quad (9)$$

As equation (9) shows, if the initial reservation price is sufficiently low, the posted price does not affect the price that a bargainer eventually pays. Likewise, if the initial reservation price is very high, even with the downward adjustment, a bargainer will end up paying the posted price as a result of negotiation. For remaining bargainers, the ARP model leads to a lower transaction price compared to the original model, as a result of the reservation price adjustment that occurs in the ARP model. With these transaction prices, the retailer's expected revenue function, $J_t(p, c, y)$, now becomes:

$$J_t(p, c, y) = \lambda q \left[\int_{\frac{(1 - \beta\eta)p - (1 - \beta)c}{\beta(1 - \eta)}}^b p f(x) dx + \int_p^{\frac{(1 - \beta\eta)p - (1 - \beta)c}{\beta(1 - \eta)}} [\beta(x - \eta(x - p)) + (1 - \beta)c] f(x) dx \right. \\ \left. + \int_c^p [\beta x + (1 - \beta)c] f(x) dx - \bar{F}(c) \Delta_{t-1}(y) \right] + \lambda(1 - q) \bar{F}(p) [p - \Delta_{t-1}(y)] + V_{t-1}(y). \quad (10)$$

Once the original version of the expected revenue function is replaced with the revised version given above, it can be shown that all our results continue to hold under this model as well.

Proposition 9. *All previous results hold under the adjusted reservation price model.*

Proposition 9 attests that all of our main insights from the original model remain valid even when the posted price shapes the reservation prices of bargainers with high willingness to pay.

Remark 4. We note that an implicit assumption of the ARP model is that the posted price can only have a downward influence on the bargainer’s reservation price. In such a setting, if the posted price exceeds the customer’s reservation price, then the posted price does not hold any informational value for the bargainer. However, if one considers an environment where the posted price carries sophisticated information about the product quality, then it is conceivable that a customer may adjust her reservation price upward after discovering that the posted price exceeds her reservation price. Such situations where the posted price serves as a sophisticated signal of the product’s value is beyond the scope of this paper. For a treatment of a retail environment where the customers’ reservation prices are shaped by the posted price (and a retailer willfully takes advantage of this by overpricing), see Wathieu and Bertini (2007) and references therein.

7. Conclusion

In this paper we investigate the effect of negotiation on the dynamic pricing of a retailer with limited inventory. Specifically, our focus is on retailers who primarily sell products at posted prices, but allow negotiation when haggle-prone customers initiate it. For instance, it was reported that approximately 25% of customers negotiate at a BestBuy store in Minnesota and that major retailers like Home Depot give salespeople more latitude to negotiate the price.

We have modeled the negotiation between the retailer and a bargainer using the generalized Nash bargaining solution (GNBS), and embedded its outcome into the dynamic pricing problem. Our negotiation model allows us to capture interactions among key drivers of the retailer and the bargainer’s decisions: the retailer’s marginal value of inventory, the bargainer’s reservation price, and their relative bargaining power. Our model captures a spectrum of outcomes that may arise in practice: negotiation fails and no purchase occurs, the bargainer obtains a discount or the bargainer fails to get a discount and buys at the posted price.

To the retailer in our model, the posted price serves a dual purpose. This is the price that any price-taker will pay if she decides to purchase and, at the same time, it works as the price ceiling

for a bargainer. While the revenue from a bargainer will increase in the posted price (giving more room to bargain), a higher posted price will price out a substantial portion of price-takers. Thus, when choosing the posted price, the retailer must balance the revenue from bargainers with revenue from price-takers. This balancing act must be carried out in an environment where the retailer must choose a *cut-off price*, which is the minimum price the retailer is willing to accept from a bargainer and which determines the bargainers that will be priced out of the market. Furthermore, our retailer operates in a business environment where the posted price (as well as the cut-off price) will be adjusted in response to inventory level and remaining time until the end of the selling season.

We show that a retailer who allows negotiation chooses a posted price and a cut-off price that straddle the posted price of a retailer who does not allow negotiation. The bargainers then pay prices that range from the cut-off price to the posted price. This is a manifestation of the price discrimination enabled by negotiation. The retailer's benefit from allowing negotiation increases in the fraction of bargainers q and the retailer's bargaining power, β . In addition, the larger the fraction of bargainers in the population, the higher the retailer's posted price. This observation is borne in practice as local stores that promote their willingness to deal (e.g., Big George, a consumer electronics store in Ann Arbor) charge higher sticker prices than national stores who are more discreet about their willingness to negotiate (e.g., BestBuy). Despite posted prices being driven up by negotiation, we show that bargainers whose reservation prices are below a threshold are better off when negotiation is allowed.

Our results suggest that negotiation is an effective tool that eases the tension between revenue per sale and the risk of excess inventory at the end of the selling season, which is a key trade-off in dynamic pricing. The high posted price increases the revenue per sale without compromising the chance of making a sale as some customers can purchase at discounted prices. This is particularly helpful when the inventory level is high relative to the remaining selling season. In such cases, as negotiation mitigates the risk of excess inventory, the retailer can raise the posted price substantially higher than the retailer who does not negotiate, and the benefit of using negotiation increases in the inventory level. On the other hand, when inventory level is low, the risk of excess inventory is already low, thus the retailer is primarily interested in selling at high prices. As a result, the additional benefit from negotiation is not as significant. Consequently, we prove that when negotiation is costly and the retailer can choose whether or not to allow negotiation, higher inventory level and/or less time till the end of selling horizon makes the retailer more willing to negotiate. This explains why so many retailers that typically do not allow negotiation were willing to haggle with the WSJ reporters during the holiday season.

While our model has some limiting assumptions, we could relax several of them to extend the results without any major changes in insights. For example, most of our results would extend to the case where the problem parameters are time-dependent (one exception is monotonicity of prices in time). In addition, we could allow for multiple bargainer types, each with a different value of bargaining power relative to the retailer, but the insights that would result from such an extension are already captured through our two customer types, bargainers and price-takers. One of our assumptions is that the retailer commits to a cut-off price below which it will not sell. If the retailer cannot commit to such a price, then one could fix the cut-off price to be zero. All our structural results would continue to hold, but negotiation would be less attractive than our numerical results suggest.

Throughout the paper, we use GNBS to model the bargaining outcome. An implicit assumption of GNBS is that the bargaining outcome is not affected by how much information the two parties have about one another. Therefore, GNBS does not recognize the potential effects of information asymmetry. In particular, under GNBS, the retailer and the bargainer will strike a deal as long as both parties have something to gain from a trade, i.e., as long as the retailer's cut-off price is below the bargainer's reservation price. In contrast, when there is information asymmetry about valuations, there would be instances when the two parties would not be able to trade even though the bargainer's reservation price exceeds the seller's cut-off price, and both have something to gain from trade. Therefore, by using the GNBS to model the outcome of the negotiation, we might be overestimating the frequency with which the two parties reach a deal. However, it should be noted that, even in the case of two-sided information asymmetry, it would be reasonable to assume that, if a deal is struck, the transaction price is still determined by GNBS as if there existed no information asymmetry. For example, in a bargaining game where parties make sequential offers after a strategically-chosen delay, Cramton (1992) constructs an equilibrium such that the trade occurs at the price that would prevail if there were no information asymmetry. Of course, one could try to embed a bargaining model, which explicitly recognizes the information asymmetry about valuations, in the dynamic pricing problem. However, bargaining models with information asymmetry tend to present many analytical challenges (e.g., multiple equilibria, mixed equilibrium) even when the buyer's and the retailer's valuations for the item are exogenously drawn. In our setting, where the retailer has a supply of items to sell over a finite horizon, the retailer's valuation of the item is a function of time, inventory and future actions of the retailer. This complexity inherent in the retailer's valuation makes it a formidable challenge to build a model that will yield analytical insights. Incorporating information asymmetry is a challenging yet worthwhile future

research direction.

There are several further research questions that one can ask about the interaction between dynamic pricing and negotiation. In many bargaining situations, price is not the only dimension along which the buyer and the retailer can negotiate. One could enrich our model to analyze negotiation over product attributes or terms of financing. For example, it would be interesting to analyze how a retailer can use negotiation to shift demand among substitutable products (e.g., buy-up or buy-down) or how the terms of sales, e.g., buying in cash, leasing, buying through financing, affect the outcome of negotiation. In addition, one could investigate the effect of negotiation on the retailer's initial inventory choice, which might highlight the implications of negotiation further up in the supply chain. Another interesting line of research would be to analyze the effects of negotiation when customers can learn the prices paid by previous customers.

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Online Supplement

Dynamic Pricing of Limited Inventories When Customers Negotiate by Chia-wei Kuo, Hyun-soo Ahn and Göker Aydın

A. Proofs of Results in Section 4

Proof of Lemma 1

We first prove that, for a given $p \in [\Delta_{t-1}(y), b)$, $J_t(p, c, y)$ is strictly unimodal in c for $c \in [\Delta_{t-1}(y), p]$. To do so, we prove the following claims: (i) $\left. \frac{\partial J_t(p, c, y)}{\partial c} \right|_{c=\Delta_{t-1}(y)} \geq 0$, (ii) $\frac{\partial^2 J_t(p, c, y)}{\partial c^2} < 0$ whenever $\frac{\partial J_t(p, c, y)}{\partial c} = 0$, and (iii) $\left. \frac{\partial J_t(p, c, y)}{\partial c} \right|_{c=p} \leq 0$.

The first and second derivatives of $J_t(p, c, y)$ (given by (4)) with respect to c are

$$\frac{\partial J_t(p, c, y)}{\partial c} = \lambda q \left[f(c)(\Delta_{t-1}(y) - c) + (1 - \beta) \left(F \left(\frac{p - (1 - \beta)c}{\beta} \right) - F(c) \right) \right] \text{ and (A-1)}$$

$$\frac{\partial^2 J_t(p, c, y)}{\partial c^2} = \lambda q \left[f'(c)(\Delta_{t-1}(y) - c) - (2 - \beta)f(c) - \frac{(1 - \beta)^2}{\beta} f \left(\frac{p - (1 - \beta)c}{\beta} \right) \right]. \text{ (A-2)}$$

Noting that $p \geq \Delta_{t-1}(y)$, claims (i) and (iii) follow from (A-1). To show claim (ii), note from (A-1) and (A-2)

$$\left. \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \right|_{\frac{\partial J_t(p, c, y)}{\partial c} = 0} = \lambda q \left[-f'(c) \frac{(1 - \beta) \left(F \left(\frac{p - (1 - \beta)c}{\beta} \right) - F(c) \right)}{f(c)} - (2 - \beta)f(c) - \frac{(1 - \beta)^2}{\beta} f \left(\frac{p - (1 - \beta)c}{\beta} \right) \right] \text{ (A-3)}$$

If $f'(c) \geq 0$, then all three terms in the brackets of (A-3) are negative. To see why the first term is negative, note that $p \geq c$, thus $\frac{p - (1 - \beta)c}{\beta} \geq c$. Hence, claim (ii) holds if $f'(c) \geq 0$. Now, consider the case that $f'(c) < 0$. First, note that

$$(1 - \beta) \left(F \left(\frac{p - (1 - \beta)c}{\beta} \right) - F(c) \right) = (1 - \beta) \left(\bar{F}(c) - \bar{F} \left(\frac{p - (1 - \beta)c}{\beta} \right) \right) < \bar{F}(c).$$

Also note that

$$-(2 - \beta)f(c) \leq -f(c) \text{ for all } c \in [\Delta_{t-1}(y), p] \text{ and } \beta \in (0, 1).$$

The last two inequalities imply that, given $f'(c) < 0$,

$$-f'(c) \frac{(1 - \beta) \left(F \left(\frac{p - (1 - \beta)c}{\beta} \right) - F(c) \right)}{f(c)} - (2 - \beta)f(c) < -f'(c) \frac{\bar{F}(c)}{f(c)} - f(c). \text{ (A-4)}$$

Now, observe that $-f'(c) \frac{\bar{F}(c)}{f(c)} - f(c) \leq 0$ (because F is IFR and, thus, $f^2(\cdot) + f'(\cdot)\bar{F}(\cdot) \geq 0$). Hence, the first two terms in the brackets of (A-3) add up to a negative number, and the third term in the brackets of (A-3) is also negative, concluding the proof of claim (ii).

Exploiting the strict unimodality of $J_t(p, c, y)$ in $c \in [\Delta_{t-1}(y), p]$ for given $p \in [\Delta_{t-1}(y), b)$, let $c^*(p)$ be the unique optimal value of c at a given p . Define the induced function $J_t^*(p, y) := J_t(p, c^*(p), y)$. We now show that $J_t^*(p, y)$ is strictly unimodal in p for $p \in [\Delta_{t-1}(y), b)$.

We prove the unimodality of $J_t^*(p, y)$ by showing (iv) $\left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=\Delta_{t-1}(y)} \geq 0$, (v) $\left. \frac{d^2 J_t^*(p, y)}{dp^2} \right|_{p=b} < 0$ whenever $\left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=b} = 0$, and (vi) $\left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=b} \leq 0$.

To prove claims (iv) and (vi), we utilize the envelope theorem:

$$\left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=\Delta_{t-1}(y)} = \left. \frac{\partial J_t(p, c, y)}{\partial p} \right|_{p=\Delta_{t-1}(y), c=c^*(p)} \quad \text{and} \quad \left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=b} = \left. \frac{\partial J_t(p, c, y)}{\partial p} \right|_{p=b, c=c^*(p)}.$$

The partial derivative of $J_t(p, c, y)$ with respect to p is

$$\frac{\partial J_t(p, c, y)}{\partial p} = \lambda q \bar{F} \left(\frac{p - (1 - \beta)c}{\beta} \right) + \lambda(1 - q) [\bar{F}(p) + f(p)(\Delta_{t-1}(y) - p)] \quad (\text{A-5})$$

Now, claim (iv) follows directly from (A-5). Claim (vi) also follows from (A-5) by noting that $\bar{F}(x) = 0$ for $x \geq b$ and if $p = b$, then $\frac{p - (1 - \beta)c}{\beta} - b = \frac{1 - \beta}{\beta}(b - c) \geq 0$.

In what follows, we write c^* as a shorthand notation for $c^*(p)$. To conclude that $J_t^*(p, y)$ is strictly unimodal in p for $p \in [\Delta_{t-1}(y), b)$, it remains to prove claim (v). To that end, first note that:

$$\frac{d^2 J_t^*(p, y)}{dp^2} = \left. \frac{\partial^2 J_t(p, c, y)}{\partial p^2} \right|_{c=c^*} + \frac{dc^*}{dp} \left. \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \right|_{c=c^*} \quad (\text{A-6})$$

Using the implicit function theorem, we have

$$\frac{dc^*}{dp} = - \frac{\left. \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \right|_{c=c^*}}{\left. \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \right|_{c=c^*}} \quad (\text{A-7})$$

Substituting (A-7) in (A-6), we obtain

$$\frac{d^2 J_t^*(p, y)}{dp^2} = \frac{\left. \frac{\partial^2 J_t(p, c, y)}{\partial p^2} \right|_{c=c^*} \left. \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \right|_{c=c^*} - \left(\left. \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \right|_{c=c^*} \right)^2}{\left. \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \right|_{c=c^*}} \quad (\text{A-8})$$

Since $J_t(p, c, y)$ is strictly unimodal in c for given p , as we proved in the first part of this lemma, the denominator is always negative. Therefore, it suffices to show that the numerator in (A-8) is strictly positive. Note that

$$\frac{\partial^2 J_t(p, c, y)}{\partial p^2} = - \frac{\lambda q}{\beta} f \left(\frac{p - (1 - \beta)c}{\beta} \right) + \lambda(1 - q) [-2f(p) + f'(p)(\Delta_{t-1}(y) - p)], \quad (\text{A-9})$$

$$\frac{\partial^2 J_t(p, c, y)}{\partial c^2} = \lambda q \left[f'(c)(\Delta_{t-1}(y) - c) - (2 - \beta)f(c) - \frac{(1 - \beta)^2}{\beta} f \left(\frac{p - (1 - \beta)c}{\beta} \right) \right] \quad (\text{A-10})$$

$$\frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} = \lambda q \frac{1 - \beta}{\beta} f \left(\frac{p - (1 - \beta)c}{\beta} \right) \quad (\text{A-11})$$

Using the expressions above, one can check that the numerator in (A-8) can be written as

$$\lambda(1-q) [-2f(p) + f'(p)(\Delta_{t-1}(y) - p)] \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{c=c^*} - \frac{\lambda^2 q^2}{\beta} f\left(\frac{p - (1-\beta)c^*}{\beta}\right) (f'(c^*)(\Delta_{t-1}(y) - c^*) - (2-\beta)f(c^*))$$

Notice that, when $c = c^*$, $\frac{\partial J_t(p, c, y)}{\partial c} = 0$. Using this fact, we can utilize (A-1) to substitute for $\Delta_{t-1}(y) - c^*$ in the above expression:

$$\lambda(1-q) [-2f(p) + f'(p)(\Delta_{t-1}(y) - p)] \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{c=c^*} - \frac{\lambda^2 q^2}{\beta} f\left(\frac{p - (1-\beta)c^*}{\beta}\right) \left(-f'(c^*) \frac{(1-\beta) \left(F\left(\frac{p-(1-\beta)c^*}{\beta}\right) - F(c^*) \right)}{f(c^*)} - (2-\beta)f(c^*) \right)$$

Now, the term $-f'(c^*) \frac{(1-\beta) \left(F\left(\frac{p-(1-\beta)c^*}{\beta}\right) - F(c^*) \right)}{f(c^*)} - (2-\beta)f(c^*)$ is negative as shown earlier in this proof (see the earlier arguments where we prove (A-3) is negative). Furthermore, $\frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{c=c^*}$ is negative (since $J_t(p, c, y)$ is strictly unimodal in c for given p). Thus, to conclude the proof of claim (v), it suffices to show that $-2f(p) + f'(p)(\Delta_{t-1}(y) - p)$ is also negative when $\frac{dJ_t^*(p, y)}{dp} = \frac{\partial J_t(p, c, y)}{\partial p} \Big|_{c=c^*} = 0$. Given $\frac{\partial J_t(p, c, y)}{\partial p} \Big|_{c=c^*} = 0$, we can utilize (A-5) to substitute for $\Delta_{t-1}(y) - p$ in $-2f(p) + f'(p)(\Delta_{t-1}(y) - p)$ and we obtain:

$$-2f(p) + f'(p)(\Delta_{t-1}(y) - p) = -2f(p) - f'(p) \frac{q\bar{F}\left(\frac{p-(1-\beta)c^*}{\beta}\right) + (1-q)\bar{F}(p)}{(1-q)f(p)}. \quad (\text{A-12})$$

If $f'(p) \geq 0$, then (A-12) is clearly negative and the proof is done. Consider now the case that $f'(p) < 0$. Therefore:

$$\begin{aligned} -2f(p) + f'(p)(\Delta_{t-1}(y) - p) &= -2f(p) - f'(p) \frac{q\bar{F}\left(\frac{p-(1-\beta)c^*}{\beta}\right) + (1-q)\bar{F}(p)}{(1-q)f(p)} \\ &\leq -2f(p) - f'(p) \frac{\bar{F}(p)}{(1-q)f(p)} \\ &\leq -2f(p) - 2f'(p) \frac{\bar{F}(p)}{f(p)} \\ &< 0 \end{aligned}$$

where the first inequality holds because $c^* \leq p$ and, thus, $\bar{F}\left(\frac{p-(1-\beta)c^*}{\beta}\right) \leq \bar{F}(p)$; the second inequality holds because $q \leq \frac{1}{2}$; and the last inequality holds because F is IFR and, thus, $f^2(\cdot) + f'(\cdot)\bar{F}(\cdot) \geq 0$.

Thus, we have shown that (A-12) is negative, which concludes the proof of claim (v), which in turn concludes the proof of the lemma.

Proof of Lemma 2

In the proof of Lemma 1, observe that the restriction $q \leq \frac{1}{2}$ is used only when proving (A-12) is negative. When, F is uniform over $[0, b]$, $f' = 0$ and, hence, (A-12) is negative no matter what q is. This observation yields the lemma.

Proof of Proposition 1

First, notice that by rearranging the terms, we can rewrite (3) as follows:

$$V_t(y) = \max_{p, c: p \geq c} J_t(p, c, y)$$

where

$$\begin{aligned} J_t(p, c, y) = & \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} (\beta x + (1-\beta)c) f(x) dx + \bar{F}(c) V_{t-1}(y-1) + F(c) V_{t-1}(y) \right] \\ & + \lambda(1-q) [p \bar{F}(p) + \bar{F}(p) V_{t-1}(y-1) + F(p) V_{t-1}(y)] \\ & + (1-\lambda) V_{t-1}(y). \end{aligned}$$

Define

$$B(p, c) = \int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} (\beta x + (1-\beta)c) f(x) dx.$$

Notice that $B(p, c)$ is the expected revenue from a bargainer who arrives in period t . Then,

$$\begin{aligned} J_t(p, c, y) = & \lambda q [B(p, c) + \bar{F}(c) V_{t-1}(y-1) + F(c) V_{t-1}(y)] \\ & + \lambda(1-q) [p \bar{F}(p) + \bar{F}(p) V_{t-1}(y-1) + F(p) V_{t-1}(y)] + (1-\lambda) V_{t-1}(y). \end{aligned} \quad (\text{A-13})$$

Let $p_t^*(y)$ and $c_t^*(y)$ be the retailer's optimal posted and cut-off prices that maximize $J_t(p, c, y)$. In preparation for the proof, we first prove the following claim.

$$\begin{aligned} & B(p_t^*(y), c_t^*(y)) + \bar{F}(c_t^*(y)) V_{t-1}(y-1) + F(c_t^*(y)) V_{t-1}(y) \\ & \geq p_t^*(y) \bar{F}(p_t^*(y)) + \bar{F}(p_t^*(y)) V_{t-1}(y-1) + F(p_t^*(y)) V_{t-1}(y) \text{ for all } t \geq 1, y \geq 1. \end{aligned} \quad (\text{A-14})$$

Suppose the inequality (A-14) does not hold. Then:

$$\begin{aligned} J_t(p_t^*(y), p_t^*(y), y) - V_t(y) &= J_t(p_t^*(y), p_t^*(y), y) - J_t(p_t^*(y), c_t^*(y), y) \\ &= \lambda q [B(p_t^*(y), p_t^*(y)) + \bar{F}(p_t^*(y)) V_{t-1}(y-1) + F(p_t^*(y)) V_{t-1}(y)] \\ &\quad - \lambda q [B(p_t^*(y), c_t^*(y)) + \bar{F}(c_t^*(y)) V_{t-1}(y-1) + F(c_t^*(y)) V_{t-1}(y)] \end{aligned}$$

Since setting the cut-off price equal to the posted price is equivalent to using take-it-or-leave-it pricing: $B(p, p) = p\bar{F}(p)$, we have

$$\begin{aligned} J_t(p_t^*(y), p_t^*(y), y) - V_t(y) &= \lambda q [p_t^*(y)\bar{F}(p_t^*(y)) + \bar{F}(p_t^*(y))V_{t-1}(y-1) + F(p_t^*(y))V_{t-1}(y)] \\ &\quad - \lambda q [B(p_t^*(y), c_t^*(y)) + \bar{F}(c_t^*(y))V_{t-1}(y-1) + F(c_t^*(y))V_{t-1}(y)] \\ &> 0. \end{aligned}$$

where the inequality follows from the supposition that inequality (A-14) does not hold. However, this inequality contradicts the fact that $V_t(y)$ is the optimal expected revenue-to-go. Thus, (A-14) must hold.

In the remainder of the proof, we write $V_t(y)$ and $J_t(p, c, y)$ as $V_t(y|q)$ and $J_t(p, c, y|q)$ to make explicit the dependence on q . Also, let $p_t^*(y|q)$ and $c_t^*(y|q)$ be the retailer's optimal posted and cut-off prices for given q . We prove that $V_t(y|q)$ increases in q through induction on t . First, we prove $V_1(y|q)$ increases in q .

Pick a pair of q values, q^o and q' such that $q^o > q'$. Then, applying the fact that $V_0(\cdot) = 0$:

$$\begin{aligned} V_1(y|q^o) &= \max_{p, c: p \geq c} J_1(p, c, y|q^o) \\ &\geq J_1(p_1^*(y|q'), c_1^*(y|q'), y|q^o) \\ &= \lambda q^o B(p_1^*(y|q'), c_1^*(y|q')) + \lambda(1 - q^o)\bar{F}(p_1^*(y|q'))p_1^*(y|q') \\ &\geq \lambda q' B(p_1^*(y|q'), c_1^*(y|q')) + \lambda(1 - q')\bar{F}(p_1^*(y|q'))p_1^*(y|q') \\ &= V_1(y|q'). \end{aligned}$$

where the first inequality comes from the fact that $p_1^*(y|q')$ and $c_1^*(y|q')$ are a suboptimal pair of posted price and cut-off price when $q = q^o$, the second inequality comes from equation (A-14) and the fact that $q^o > q'$. Therefore, $V_1(y|q)$ increases in q .

Suppose now that $V_k(y|q)$ is increasing in q for $k = 1, \dots, t-1$ and $y \geq 0$. Then:

$$\begin{aligned} V_t(y|q^o) &= \max_{p, c: p \geq c} J_t(p, c, y|q^o) \\ &\geq J_t(p_t^*(y|q'), c_t^*(y|q'), y|q^o) \\ &= \lambda q^o [B(p_t^*(y|q'), c_t^*(y|q')) + \bar{F}(c_t^*(y|q'))V_{t-1}(y-1|q^o) + F(c_t^*(y|q'))V_{t-1}(y|q^o)] \\ &\quad + \lambda(1 - q^o) [\bar{F}(p_t^*(y|q'))p_t^*(y|q') + \bar{F}(p_t^*(y|q'))V_{t-1}(y-1|q^o) + F(p_t^*(y|q^o))V_{t-1}(y|q^o)] \\ &\quad + (1 - \lambda)V_{t-1}(y|q^o) \end{aligned}$$

Applying the induction hypothesis that $V_{t-1}(y|q)$ is increasing in q , we note $V_{t-1}(\cdot|q^o) \geq V_{t-1}(\cdot|q')$

and we obtain:

$$\begin{aligned} V_t(y|q^o) &\geq \lambda q^o [B(p_t^*(y|q'), c_t^*(y|q')) + \bar{F}(c_t^*(y|q'))V_{t-1}(y-1|q') + F(c_t^*(y|q'))V_{t-1}(y|q')] \\ &\quad + \lambda(1-q^o) [\bar{F}(p_t^*(y|q'))p_t^*(y|q') + \bar{F}(p_t^*(y|q'))V_{t-1}(y-1|q') + F(p_t^*(y|q'))V_{t-1}(y|q')] \\ &\quad + (1-\lambda)V_{t-1}(y|q') \end{aligned}$$

Finally, using $q^o \geq q'$ and inequality (A-14), we get

$$\begin{aligned} V_t(y|q^o) &\geq \lambda q' [B(p_t^*(y|q'), c_t^*(y|q')) + \bar{F}(c_t^*(y|q'))V_{t-1}(y-1|q') + F(c_t^*(y|q'))V_{t-1}(y|q')] \\ &\quad + \lambda(1-q') [\bar{F}(p_t^*(y|q'))p_t^*(y|q') + \bar{F}(p_t^*(y|q'))V_{t-1}(y-1|q') + F(p_t^*(y|q'))V_{t-1}(y|q')] \\ &\quad + (1-\lambda)V_{t-1}(y|q') \\ &= V_t(y|q') \end{aligned}$$

Therefore, $V_t(y|q)$ increases in q , concluding the induction.

Proof of Proposition 2

Throughout this proof, recall from Lemma 1 the definitions $c^*(p) = \arg \max_c \{J_t(p, c, y)\}$ and $J_t^*(p, y) := J_t(p, c^*(p), y)$. Recall also from Lemma 1 that $J_t^*(p, y)$ is strictly unimodal in p and $J_t(p, c, y)$ is strictly unimodal in c for a given p .

Proof of (a): Note that, using the envelope theorem, we can write:

$$\frac{dJ_t^*(p, y)}{dp} = \left. \frac{\partial J_t(p, c, y)}{\partial p} \right|_{c=c^*(p)}.$$

Substituting for $\frac{\partial J_t(p, c, y)}{\partial p}$ from equation (A-5), it follows that

$$\frac{dJ_t^*(p, y)}{dp} = \lambda q \bar{F} \left(\frac{p - (1-\beta)c^*(p)}{\beta} \right) + \lambda(1-q) (\bar{F}(p) + f(p)(\Delta_{t-1}(y) - p)). \quad (\text{A-15})$$

Observe from Lemma B.1(b) that $p_t^{TL}(y)$ (abbreviated as p^{TL} in the remainder of the proof) satisfies the following identity:

$$\bar{F}(p^{TL}) + f(p^{TL})(\Delta_{t-1}(y) - p^{TL}) = 0. \quad (\text{A-16})$$

Now, using (A-15) and (A-16), we can write:

$$\left. \frac{dJ_t^*(p, y)}{dp} \right|_{p=p^{TL}} = \lambda q \bar{F} \left(\frac{p^{TL} - (1-\beta)c^*(p^{TL})}{\beta} \right) \geq 0. \quad (\text{A-17})$$

Hence, $J_t^*(p, y)$ is non-decreasing at $p = p^{TL}$. Given that $J_t^*(p, y)$ is strictly unimodal in p (by Lemma 1), it follows that the optimizer of $J_t^*(p, y)$ is at least as large as p^{TL} . The result now

follows, after noticing that the price p that maximizes $J_t^*(p, y)$ is in fact the optimal list price with t periods to go and y units of inventory, that is, $p_t^*(y)$.

Proof of (b): We first show that, for any given p , $J_t(p, c, y)$ is decreasing in c at $c = p^{TL}$. We use (A-1) to write

$$\left. \frac{\partial J_t(p, c, y)}{\partial c} \right|_{c=p^{TL}} = \lambda q \left[f(p^{TL})(\Delta_{t-1}(y) - p^{TL}) + (1 - \beta) \left(F \left(\frac{p - (1 - \beta)p^{TL}}{\beta} \right) - F(p^{TL}) \right) \right]. \quad (\text{A-18})$$

Note from (A-16) that $f(p^{TL})(\Delta_{t-1}(y) - p^{TL}) = -\bar{F}(p^{TL})$. Substituting for $f(p^{TL})(\Delta_{t-1}(y) - p^{TL})$ in (A-18), we obtain

$$\left. \frac{\partial J_t(p, c, y)}{\partial c} \right|_{c=p^{TL}} = \lambda q \left[-\bar{F}(p^{TL}) + (1 - \beta) \left(F \left(\frac{p - (1 - \beta)p^{TL}}{\beta} \right) - F(p^{TL}) \right) \right].$$

Noticing that $F \left(\frac{p - (1 - \beta)p^{TL}}{\beta} \right) - F(p^{TL}) = \bar{F}(p^{TL}) - \bar{F} \left(\frac{p - (1 - \beta)p^{TL}}{\beta} \right)$ and subsequent algebraic simplification of the above equality yields

$$\left. \frac{\partial J_t(p, c, y)}{\partial c} \right|_{c=p^{TL}} = \lambda q \left[-\beta \bar{F}(p^{TL}) - (1 - \beta) \bar{F} \left(\frac{p - (1 - \beta)p^{TL}}{\beta} \right) \right] \leq 0.$$

Hence, we proved that, at any given p , $J_t(p, c, y)$ is decreasing in c at $c = p^{TL}$. Now, given that $J_t(p, c, y)$ is strictly unimodal in c (as was shown in Lemma 1), it follows that p^{TL} is greater than or equal to $c^*(p)$ at any given p . In particular, if p is fixed to be the optimal list price $p_t^*(y)$, then $p^{TL} \geq c^*(p_t^*(y))$. Notice that $c^*(p_t^*(y))$ is in fact the optimal cut-off price of the retailer with y units of inventory and t periods to go, that is $c_t^*(y)$. This concludes the proof of part (b).

Proof of (c): From (2), it follows that the average price paid by a bargainer, denoted by $p_t^{avg}(y)$, is

$$\frac{1}{\bar{F}(c_t^*(y))} \left[\int_{\frac{p_t^*(y) - (1 - \beta)c_t^*(y)}{\beta}}^b p_t^*(y) f(x) dx + \int_{c_t^*(y)}^{\frac{p_t^*(y) - (1 - \beta)c_t^*(y)}{\beta}} [\beta x + (1 - \beta)c_t^*(y)] f(x) dx \right]. \quad (\text{A-19})$$

When F is uniform over $[0, b]$, it follows from Lemma 2 that $p_t^*(y)$ and $c_t^*(y)$ are given by the unique pair of p and c that satisfy the first-order conditions for $J_t(p, c, y)$. Substituting $F(x) = \frac{x}{b}$, $f(x) = \frac{1}{b}$ and $f'(x) = 0$ in (A-1) and (A-5), we obtain the following expressions for the first-order conditions when F is uniform over $[0, b]$:

$$\begin{aligned} \frac{\partial J_t(p, c, y)}{\partial c} &= \frac{\lambda q}{b} \left[\Delta_{t-1}(y) - c + (1 - \beta) \left(\frac{p - (1 - \beta)c}{\beta} - c \right) \right] = 0 \\ \frac{\partial J_t(p, c, y)}{\partial p} &= \lambda q \left(1 - \frac{p - (1 - \beta)c}{\beta b} \right) + \lambda(1 - q) \left(1 + \frac{\Delta_{t-1}(y) - 2p}{b} \right) = 0 \end{aligned}$$

Solving the above system of equations for p and c , we find that $p_t^*(y)$ and $c_t^*(y)$ are given by⁶

$$p_t^*(y) = \frac{b + \Delta_{t-1}(y)(1 - \beta q)}{2 - \beta q}, \quad (\text{A-20})$$

$$c_t^*(y) = \frac{(1 - \beta)b + \Delta_{t-1}(y)(1 + \beta - \beta q)}{2 - \beta q}. \quad (\text{A-21})$$

Substituting these expressions in (A-19), we find that the average price paid by a bargainer is

$$p_t^{avg}(y) = \frac{-(2b\beta q - 2b - b\beta - 3\Delta_{t-1}(y)\beta - 2\Delta_{t-1}(y)\beta^2 q^2 + 4\Delta_{t-1}(y)\beta q + 2\Delta_{t-1}(y)\beta^2 q - 2\Delta_{t-1}(y))}{2(\beta q - 2)(\beta q - \beta - 1)}.$$

As for the optimal take-it-or-leave-it price, when F is uniform over $[0, b]$, we substitute $F(x) = \frac{x}{b}$ and $f(x) = \frac{1}{b}$ in Lemma B.1(b) to find that

$$p_t^{TL}(y) = \frac{b + \Delta_{t-1}(y)}{2}.$$

Now, we can write

$$p_t^{TL}(y) - p_t^{avg}(y) = \frac{\beta(1 - q)(1 - \beta q)(b - \Delta_{t-1}(y))}{2(\beta q - 2)(\beta q - \beta - 1)}$$

where $\beta \in (0, 1)$, $q \in (0, 1)$ and $b \geq \Delta_{t-1}(y)$. Therefore, $p_t^{TL}(y) - p_t^{avg}(y) \geq 0$, which concludes the proof.

Proof of Proposition 3

Note from Proposition 2 that $c_t^*(y) \leq p_t^{TL}(y) \leq p_t^*(y)$. When the retailer allows negotiation, price-takers have to pay $p_t^*(y)$, whereas they have to pay $p_t^{TL}(y)$ when the retailer uses take-it-or-leave-it pricing. Hence, price-takers are not better off under negotiation.

As for bargainers, consider first a specific bargainer whose reservation price $r = \frac{p_t^{TL}(y) - (1 - \beta)c_t^*(y)}{\beta}$. Notice that $\frac{p_t^{TL}(y) - (1 - \beta)c_t^*(y)}{\beta} \in [c_t^*(y), \frac{p_t^*(y) - (1 - \beta)c_t^*(y)}{\beta}]$ (since $c_t^*(y) \leq p_t^{TL}(y) \leq p_t^*(y)$). It now follows from (3) that this specific bargainer will pay a price $\beta r + (1 - \beta)c_t^*(y) = p_t^{TL}(y)$. Therefore, all bargainers with reservation prices in the interval $[c_t^*(y), \frac{p_t^{TL}(y) - (1 - \beta)c_t^*(y)}{\beta}]$ will pay prices strictly below $p_t^{TL}(y)$ and they will be better off compared to take-it-or-leave-it pricing. On the other hand, all bargainers with reservation prices in the interval $(\frac{p_t^{TL}(y) - (1 - \beta)c_t^*(y)}{\beta}, b]$ will pay prices strictly above $p_t^{TL}(y)$ and they will be worse off compared to take-it-or-leave-it pricing. This concludes the proof.

Proof of Proposition 4

⁶Notice that when writing the first order conditions, one may think that it is necessary to consider possibility that $\frac{p - (1 - \beta)c}{\beta} \geq b$, in which case $F(\frac{p - (1 - \beta)c}{\beta}) = 1$ and $\bar{F}(\frac{p - (1 - \beta)c}{\beta}) = 0$. However, it can be shown that, with uniform distribution, it is never optimal to set p so that $\frac{p - (1 - \beta)c}{\beta} \geq b$.

Proofs of (a) and (b): It follows from Lemma 1 that the optimal posted price, $p_t^*(y)$ and the optimal cut-off price, $c_t^*(y)$, are given by the unique pair of p and c that satisfy the first order conditions of $J_t(p, c, y)$, that is,

$$\frac{\partial J_t(p_t^*(y), c_t^*(y), y)}{\partial p} = 0 \text{ and } \frac{\partial J_t(p_t^*(y), c_t^*(y), y)}{\partial c} = 0.$$

In the remainder of this proof, we use the shorthand notation p^* and c^* for $p_t^*(y)$ and $c_t^*(y)$. By implicit differentiation of the above identities with respect to q , we obtain:

$$\frac{dp^*}{dq} \frac{\partial^2 J_t(p, c, y)}{\partial p^2} \Big|_{p=p^*, c=c^*} + \frac{dc^*}{dq} \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \Big|_{p=p^*, c=c^*} + \frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*} = 0, \quad (\text{A-22})$$

$$\frac{dp^*}{dq} \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \Big|_{p=p^*, c=c^*} + \frac{dc^*}{dq} \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{p=p^*, c=c^*} + \frac{\partial^2 J_t(p, c, y)}{\partial c \partial q} \Big|_{p=p^*, c=c^*} = 0. \quad (\text{A-23})$$

We first prove that $\frac{dp^*}{dq} \geq 0$. Using (A-22) and (A-23), we can solve for $\frac{dp^*}{dq}$ to obtain the following:

$$\frac{dp^*}{dq} = \frac{-\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*} \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{p=p^*, c=c^*} + \frac{\partial^2 J_t(p, c, y)}{\partial c \partial q} \Big|_{p=p^*, c=c^*} \frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \Big|_{p=p^*, c=c^*}}{\frac{\partial^2 J_t(p, c, y)}{\partial p^2} \Big|_{p=p^*, c=c^*} \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{p=p^*, c=c^*} - \left(\frac{\partial^2 J_t(p, c, y)}{\partial p \partial c} \Big|_{p=p^*, c=c^*} \right)^2} \quad (\text{A-24})$$

To conclude that $\frac{dp^*}{dq} \geq 0$, we first note that the denominator of (A-24) is strictly positive as shown in the proof of Lemma 1. Therefore, it suffices to show that the numerator of (A-24) is non-negative. To that end, by taking the partial derivative of (A-1) with respect to q , notice that $\frac{\partial^2 J_t(p, c, y)}{\partial c \partial q} = \frac{1}{q} \frac{\partial J_t(p, c, y)}{\partial c}$. This observation, along with the fact that $\frac{\partial J_t(p, c, y)}{\partial c} \Big|_{p=p^*, c=c^*} = 0$, yields $\frac{\partial^2 J_t(p, c, y)}{\partial c \partial q} \Big|_{p=p^*, c=c^*} = 0$. Therefore, it suffices to show that $\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*} \frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{p=p^*, c=c^*} \leq 0$. Now, notice that $\frac{\partial^2 J_t(p, c, y)}{\partial c^2} \Big|_{p=p^*, c=c^*} < 0$, since $J_t(p, c, y)$ is strictly unimodal in c at a given p , as shown in Lemma 1. Furthermore, the cross-partial $\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*}$ is non-negative, as we show next. Taking the partial derivative of (A-5) with respect to q , we obtain:

$$\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} = \lambda \bar{F} \left(\frac{p - (1 - \beta)c}{\beta} \right) - \lambda [\bar{F}(p) + f(p)(\Delta_t(y) - p)] \quad (\text{A-25})$$

Because $\frac{\partial J_t(p, c, y)}{\partial p} \Big|_{p=p^*, c=c^*} = 0$, it follows from (A-5) that $\Delta_t(y) - p^* = -\frac{q\bar{F}\left(\frac{p^* - (1-\beta)c^*}{\beta}\right) + (1-q)\bar{F}(p^*)}{f(p^*)(1-q)}$. Substituting this expression for $\Delta_t(y) - p^*$ in (A-25) and subsequent algebraic simplification yields:

$$\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*} = \frac{\lambda}{1-q} \bar{F} \left(\frac{p^* - (1-\beta)c^*}{\beta} \right). \quad (\text{A-26})$$

Hence, $\frac{\partial^2 J_t(p, c, y)}{\partial p \partial q} \Big|_{p=p^*, c=c^*} \geq 0$, which concludes the proof of $\frac{dp^*}{dq} \geq 0$.

It remains to show that $\frac{dc^*}{dq} \geq 0$. The proof is similar. Using (A-22) and (A-23), we can solve for $\frac{dc^*}{dq}$ to obtain the following:

$$\frac{dc^*}{dq} = \frac{-\frac{\partial^2 J_t(p,c,y)}{\partial c \partial q} \Big|_{p=p^*,c=c^*} \frac{\partial^2 J_t(p,c,y)}{\partial p^2} \Big|_{p=p^*,c=c^*} + \frac{\partial^2 J_t(p,c,y)}{\partial p \partial q} \Big|_{p=p^*,c=c^*} \frac{\partial^2 J_t(p,c,y)}{\partial p \partial c} \Big|_{p=p^*,c=c^*}}{\frac{\partial^2 J_t(p,c,y)}{\partial p^2} \Big|_{p=p^*,c=c^*} \frac{\partial^2 J_t(p,c,y)}{\partial c^2} \Big|_{p=p^*,c=c^*} - \left(\frac{\partial^2 J_t(p,c,y)}{\partial p \partial c} \Big|_{p=p^*,c=c^*} \right)^2} \quad (\text{A-27})$$

Once again, we note that the denominator of (A-27) is strictly positive (as shown in Lemma 1). Also note that the cross-partial $\frac{\partial^2 J_t(p,c,y)}{\partial p \partial q} \Big|_{p=p^*,c=c^*}$ is non-negative and the cross-partial $\frac{\partial^2 J_t(p,c,y)}{\partial c \partial q} \Big|_{p=p^*,c=c^*} = 0$ as shown earlier. Hence, it suffices to show that $\frac{\partial^2 J_t(p,c,y)}{\partial p \partial c} \Big|_{p=p^*,c=c^*}$ is also non-negative. Note from (A-11) that $\frac{\partial^2 J_t(p,c,y)}{\partial p \partial c} = \frac{\lambda q(1-\beta)}{\beta} f\left(\frac{p-(1-\beta)c}{\beta}\right) \geq 0$. Hence, $\frac{\partial^2 J_t(p,c,y)}{\partial p \partial c} \Big|_{p=p^*,c=c^*}$ is non-negative.

Proof of Proposition 5

Taking the derivative of (A-20) with respect to β , we obtain:

$$\frac{dp_t^*(y)}{d\beta} = \frac{q(b - \Delta_{t-1}(y))}{(2 - \beta q)^2} \geq 0,$$

where the inequality follows from $b \geq \Delta_{t-1}(y)$. This proves part (a) of the proposition. Taking the derivative of (A-21) with respect to β , we obtain:

$$\frac{dc_t^*(y)}{d\beta} = \frac{(q - 2)(b - \Delta_{t-1}(y))}{(2 - \beta q)^2} \leq 0,$$

where the inequality comes from $q \in (0, 1)$ and $b \geq \Delta_{t-1}(y)$. This proves part (b) of the proposition.

Proof of Proposition 6

Let $A_t^*(y)$ denote the set of (p, c) pairs such that $p \geq c$ and that maximize $J_t(p, c, y)$, defined by (4). Notice that $A_t^*(y)$ is also the set of (p, c) pairs such that $p \geq c$ and that maximize $J(p, c, \Delta_{t-1}(y))$, where $J(p, c, \Delta)$ is defined in Lemma B.4. Lemma B.4 shows that $J(p, c, \Delta)$ is supermodular in (p, c, Δ) over the set $\{(p, c, \Delta) : p \geq c, \Delta \geq 0\}$. Therefore, $A_t^*(y)$ is an ascending set in $\Delta_{t-1}(y)$, which implies that there exist an optimal (p, c) pair that is increasing in $\Delta_{t-1}(y)$. Let $p_t^*(y)$ and $c_t^*(y)$ denote this optimal pair of list price and cut-off price.

Now, note that inequality $H3(t, y)$ in Lemma B.3 implies that $\Delta_t(y+1) \geq \Delta_t(y+2), y \geq 0, t = 1, \dots, T$. That is, $\Delta_{t-1}(y)$ is decreasing in y . Therefore, $p_t^*(y)$ and $c_t^*(y)$, which are increasing in $\Delta_{t-1}(y)$, are decreasing in the stock level, y . Likewise, the inequality $H1(t, y)$ in Lemma B.3 implies that $\Delta_{t+1}(y+1) \geq \Delta_t(y+1), y \geq 0, t = 1, \dots, T$. That is, $\Delta_{t-1}(y)$ is increasing in t . Therefore, $p_t^*(y)$ and $c_t^*(y)$, which are increasing in $\Delta_{t-1}(y)$, are increasing in the remaining time, t .

Proof of Corollary 1

As we already showed in the proof of Proposition 2(c):

$$p_t^{TL}(y) - p_t^{avg}(y) = \frac{\beta(1-q)(1-\beta q)(b - \Delta_{t-1}(y))}{2(\beta q - 2)(\beta q - \beta - 1)}$$

To show that $p_t^{TL}(y) - p_t^{avg}(y)$ increases in the stock level and decreases in the time remaining until the end of the season, we have

$$\frac{d(p_t^{TL}(y) - p_t^{avg}(y))}{d\Delta_{t-1}(y)} = \frac{-\beta(1-q)(1-\beta q)}{2(\beta q - 2)(\beta q - \beta - 1)} \leq 0,$$

where the inequality is from $\beta \in (0, 1)$ and $q \in (0, 1)$. Now, note that inequality $H3(t, y)$ in Lemma B.3 implies that $\Delta_t(y+1) \geq \Delta_t(y+2)$, $y \geq 0$, $t = 1, \dots, T$. That is, $\Delta_{t-1}(y)$ is decreasing in y . Therefore, $p_t^{TL}(y) - p_t^{avg}(y)$, which is decreasing in $\Delta_{t-1}(y)$, is increasing in the stock level, y . Likewise, the inequality $H1(t, y)$ in Lemma B.3 implies that $\Delta_{t+1}(y+1) \geq \Delta_t(y+1)$, $y \geq 0$, $t = 1, \dots, T$. That is, $\Delta_{t-1}(y)$ is increasing in t . Therefore, $p_t^{TL}(y) - p_t^{avg}(y)$, which is decreasing in $\Delta_{t-1}(y)$, is decreasing in the remaining time, t .

Proof of Proposition 7

We first obtain closed-form expressions for $\tilde{p}_t^*(y)$ and $\tilde{c}_t^*(y)$. Let $\tilde{\Delta}_t(y) = \tilde{V}_t(y) - \tilde{V}_t(y-1)$ for $y \geq 1$ and $t \geq 0$, and note that $\tilde{V}_t^N(y)$, given by (8), can be rewritten as:

$$\tilde{V}_t^N(y) = \max_{p \geq c} \tilde{J}_t^N(p, c, y),$$

where

$$\begin{aligned} \tilde{J}_t^N(p, c, y) = & \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} [\beta x + (1-\beta)c] f(x) dx - \bar{F}(c) \tilde{\Delta}_{t-1}(y) \right] \\ & + \lambda(1-q) \bar{F}(p) [p - \tilde{\Delta}_{t-1}(y)] + \tilde{V}_{t-1}(y). \end{aligned}$$

Notice that $\tilde{p}_t^*(y)$ and $\tilde{c}_t^*(y)$ are the pair of p and c values that maximize $\tilde{J}_t^N(p, c, y)$. Now, observe that $\tilde{J}_t^N(p, c, y)$ differs from $J_t(p, c, y)$, given by (4), only in that $\Delta_{t-1}(y)$ is now replaced by $\tilde{\Delta}_{t-1}(y)$. Therefore, by replacing $\Delta_{t-1}(y)$ in (A-20) and (A-21) with $\tilde{\Delta}_{t-1}(y)$, we obtain the following expressions for $\tilde{p}_t^*(y)$ and $\tilde{c}_t^*(y)$:

$$\begin{aligned} \tilde{p}_t^*(y) &= \frac{b + \tilde{\Delta}_{t-1}(y)(1 - \beta q)}{2 - \beta q}, \\ \tilde{c}_t^*(y) &= \frac{(1 - \beta)b + \tilde{\Delta}_{t-1}(y)(1 + \beta - \beta q)}{2 - \beta q}. \end{aligned}$$

Next, we obtain a closed-form expression for $\tilde{p}_t^{TL}(y)$. Note that $\tilde{V}_t^{TL}(y)$, given by (7), can be rewritten as:

$$\tilde{V}_t^{TL}(y) = \max_p \tilde{J}_t^{TL}(p, y),$$

where

$$\tilde{J}_t^{TL}(p, y) = \lambda \bar{F}(p) \left(p + \tilde{V}_{t-1}(y-1) \right) + (1 - \lambda \bar{F}(p)) \tilde{V}_{t-1}(y).$$

Notice that $\tilde{p}_t^{TL}(y)$ is the value of p that maximizes $\tilde{J}_t^{TL}(p, y)$. Observe also that $\tilde{J}_t^{TL}(p, y)$ differs from $J_t^{TL}(p, y)$, defined in Lemma B.1, only in that V is now replaced by \tilde{V} . Therefore, in Lemma B.1(b), we replace $\Delta_{t-1}(y)$ with $\tilde{\Delta}_{t-1}(y)$ and, using the uniform assumption, we substitute $F(x) = \frac{x}{b}$ and $f(x) = \frac{1}{b}$ to obtain the following expression for $\tilde{p}_t^{TL}(y)$:

$$\tilde{p}_t^{TL}(y) = \frac{b + \tilde{\Delta}_{t-1}(y)}{2}.$$

We are now ready to show the main result of this proposition. Putting together the expressions for $\tilde{p}_t^*(y)$, $\tilde{c}_t^*(y)$, and $\tilde{p}_t^{TL}(y)$, the price premium is given by

$$\tilde{p}_t^*(y) - \tilde{p}_t^{TL}(y) = \frac{\beta q (b - \tilde{\Delta}_{t-1}(y))}{2(2 - \beta q)},$$

and the negotiation allowance is given by

$$\tilde{p}_t^{TL}(y) - \tilde{c}_t^*(y) = \frac{\beta(2 - q)(b - \tilde{\Delta}_{t-1}(y))}{2(2 - \beta q)}.$$

Notice that both the price premium and the negotiation allowance decrease in $\tilde{\Delta}_{t-1}(y)$. The result now follows from Lemma B.2 $H3(t, y)$ and $H1(t, y)$, which state that $\tilde{\Delta}_t(y)$ decreases in y and increases in t .

Proof of Proposition 8

Proof of (a): We would like to show that if $\tilde{V}_t^N(y) - \tilde{V}_t^{TL}(y) \geq K$, then $\tilde{V}_t^N(y+1) - \tilde{V}_t^{TL}(y+1) \geq K$. In order to show this result, it suffices to show that

$$\tilde{V}_t^N(y+1) - \tilde{V}_t^N(y) \geq \tilde{V}_t^{TL}(y+1) - \tilde{V}_t^{TL}(y). \quad (\text{A-28})$$

$\tilde{V}_t^N(y)$ and $\tilde{V}_t^{TL}(y)$ are given by, respectively, (8) and (7). Notice that $\tilde{V}_t^{TL}(y)$ is in fact equivalent to $\tilde{V}_t^N(y)$ when $q = 0$. Thus, in order to prove (A-28), it suffices to show that $\tilde{V}_t^N(y+1) - \tilde{V}_t^N(y)$ increases in q . To that end, we first derive an expression for $\frac{d\tilde{V}_t^N(y)}{dq}$. To help with the proof, let

$\tilde{\Delta}_t(y) = \tilde{V}_t(y) - \tilde{V}_t(y-1)$ for $y \geq 1$ and $t \geq 0$, and note that $\tilde{V}_t^N(y)$, given by (8), can be rewritten as:

$$\tilde{V}_t^N(y) = \max_{p \geq c} \tilde{J}_t(p, c, y),$$

where

$$\begin{aligned} \tilde{J}_t(p, c, y) = & \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} [\beta x + (1-\beta)c] f(x) dx - \bar{F}(c) \tilde{\Delta}_{t-1}(y) \right] \\ & + \lambda(1-q) \bar{F}(p) [p - \tilde{\Delta}_{t-1}(y)] + \tilde{V}_{t-1}(y). \end{aligned}$$

Notice that $\tilde{J}_t(p, c, y)$ differs from $J_t(p, c, y)$, given by (4), only in that $\Delta_{t-1}(y)$ is now replaced by $\tilde{\Delta}_{t-1}(y)$. Therefore, the structural properties of $J_t(p, c, y)$ with respect to p and c continue to hold for $\tilde{J}_t(p, c, y)$. In particular, there exists a unique pair of p and c that satisfy the FOC for $\tilde{J}_t(p, c, y)$ and this pair optimizes $\tilde{J}_t(p, c, y)$. In what follows, we use the short-hand notation p^* and c^* for the posted price and cut-off price that optimize $\tilde{J}_t(p, c, y)$. First, note that

$$\begin{aligned} \frac{d\tilde{V}_t^N(y)}{dq} &= \frac{d\tilde{J}_t(p^*, c^*, y)}{dq} \\ &= \left. \frac{\partial \tilde{J}_t(p, c, y)}{\partial q} \right|_{p=p^*, c=c^*} + \frac{dp^*}{dq} \left. \frac{\partial \tilde{J}_t(p, c, y)}{\partial p} \right|_{p=p^*, c=c^*} + \frac{dc^*}{dq} \left. \frac{\partial \tilde{J}_t(p, c, y)}{\partial c} \right|_{p=p^*, c=c^*} \\ &= \left. \frac{\partial \tilde{J}_t(p, c, y)}{\partial q} \right|_{p=p^*, c=c^*} \quad \text{by the envelope theorem} \\ &= \lambda \left[\int_{\frac{p^*-(1-\beta)c^*}{\beta}}^b p^* f(x) dx + \int_{c^*}^{\frac{p^*-(1-\beta)c^*}{\beta}} (\beta x + (1-\beta)c^*) f(x) dx \right. \\ & \quad \left. - \bar{F}(c^*) \tilde{\Delta}_{t-1}(y) - \bar{F}(p^*) (p^* - \tilde{\Delta}_{t-1}(y)) \right]. \end{aligned} \quad (\text{A-29})$$

Now, when F is uniform over $[0, b]$, we have $F(x) = \frac{x}{b}$, $f(x) = \frac{1}{b}$. Furthermore, p^* and c^* are given by (see (A-20) and (A-21)):

$$\begin{aligned} p^* &= \frac{b + \tilde{\Delta}_{t-1}(y)(1-\beta q)}{2-\beta q}, \\ c^* &= \frac{(1-\beta)b + \tilde{\Delta}_{t-1}(y)(1+\beta-\beta q)}{2-\beta q}. \end{aligned}$$

Substituting for $F(x)$, $f(x)$, p^* and c^* in (A-29), we obtain the following expression for $\frac{d\tilde{V}_t^N(y)}{dq}$:⁷

$$\frac{d\tilde{V}_t^N(y)}{dq} = \frac{\lambda\beta(b - \tilde{\Delta}_{t-1}(y))^2}{2b(2-\beta q)^2}$$

Therefore:

$$\frac{d\left(\tilde{V}_t^N(y+1) - \tilde{V}_t^N(y)\right)}{dq} = \frac{\lambda\beta}{2b(2-\beta q)^2} \left(\tilde{\Delta}_{t-1}(y) - \tilde{\Delta}_{t-1}(y+1)\right) \left(2b - \tilde{\Delta}_{t-1}(y+1) - \tilde{\Delta}_{t-1}(y)\right)$$

⁷The algebra leading to this expression is very messy, so we do not attempt to reproduce the steps here.

We can now conclude that $\tilde{V}_t^N(y+1) - \tilde{V}_t^N(y)$ is increasing in q , because:

- (i) $\frac{\lambda\beta}{2b(2-\beta q)^2} \geq 0$ since $b > 0$ and $\beta < 1$ and $q < 1$;
- (ii) $\tilde{\Delta}_{t-1}(y) - \tilde{\Delta}_{t-1}(y+1) \geq 0$ by Lemma B.2 $H3(t, y)$;
- (iii) $2b - \tilde{\Delta}_{t-1}(y) - \tilde{\Delta}_{t-1}(y+1) \geq 0$ because both $\tilde{\Delta}_{t-1}(y)$ and $\tilde{\Delta}_{t-1}(y+1)$ are smaller than b , the upper bound of the customer's reservation price.

Proof of (b): Using a line of arguments similar to those in part (a), it suffices to show that $\tilde{V}_t^N(y) - \tilde{V}_{t-1}^N(y)$ is decreasing in the fraction of the bargainers, q . Using the expression for $\frac{d\tilde{V}_t^N(y)}{dq}$ derived in part (a), we obtain:

$$\frac{d\left(\tilde{V}_t^N(y) - \tilde{V}_{t-1}^N(y)\right)}{dq} = \frac{\lambda\beta}{2b(2-\beta q)^2} \left(\tilde{\Delta}_{t-2}(y) - \tilde{\Delta}_{t-1}(y)\right) \left(2b - \tilde{\Delta}_{t-1}(y) - \tilde{\Delta}_{t-2}(y)\right).$$

Once again, $\frac{\lambda\beta}{2b(2-\beta q)^2} \geq 0$ and $2b - \tilde{\Delta}_{t-1}(y) - \tilde{\Delta}_{t-2}(y) \geq 0$. In addition, $\tilde{\Delta}_{t-2}(y) - \tilde{\Delta}_{t-1}(y) \leq 0$ by Lemma B.2 $H1(t, y)$. Hence, the derivative above is non-positive, and $\tilde{V}_t^N(y) - \tilde{V}_{t-1}^N(y)$ is decreasing in q .

Proof of Proposition 9

The proofs of all results can be reproduced by replacing $J_t(\cdot)$ with equation (10) and following analogous logic used in the proofs of the original model. The complete proof is available from the authors upon request.

B. Supporting Lemmas

Lemma B.1. *Consider a retailer that has y units of inventory with t periods to go. Let $J_t^{TL}(p, y)$ denote this retailer's expected profit if the retailer uses take-it-or-leave-it pricing with posted price p in period t and switches to negotiation thereafter. That is,*

$$J_t^{TL}(p, y) := \lambda\bar{F}(p)(p + V_{t-1}(y-1)) + (1 - \lambda\bar{F}(p))V_{t-1}(y).$$

Then:

- (a) $J_t^{TL}(p, y)$ is strictly unimodal in p .
- (b) The optimizer of $J_t^{TL}(p, y)$, denoted by $p_t^{TL}(y)$, is the unique solution to the first-order condition for $J_t^{TL}(p, y)$. That is, $p_t^{TL}(y)$ satisfies the following identity:

$$\left. \frac{\partial J_t^{TL}(p, y)}{\partial p} \right|_{p=p_t^{TL}(y)} = \bar{F}(p_t^{TL}(y)) + f(p_t^{TL}(y)) (\Delta_{t-1}(y) - p_t^{TL}(y)) = 0,$$

where $\Delta_{t-1}(y) = V_{t-1}(y) - V_{t-1}(y-1)$.

Proof of Lemma B.1

We prove the unimodality of $J_t^{TL}(p, y)$ in p by showing (i) $\left. \frac{\partial J_t^{TL}(p, y)}{\partial p} \right|_{p=0} \geq 0$, (ii) $\frac{\partial^2 J_t^{TL}(p, y)}{\partial p^2} < 0$ whenever $\frac{\partial J_t^{TL}(p, y)}{\partial p} = 0$, and (iii) $\left. \frac{\partial J_t^{TL}(p, y)}{\partial p} \right|_{p=b} \leq 0$.

Using $\Delta_{t-1}(y) = V_{t-1}(y) - V_{t-1}(y-1)$, we write the first and second derivatives of $J_t^{TL}(p, y)$ with respect to p as follows:

$$\frac{\partial J_t^{TL}(p, y)}{\partial p} = \lambda [\bar{F}(p) + f(p)(\Delta_{t-1}(y) - p)], \quad (\text{A-30})$$

$$\frac{\partial^2 J_t^{TL}(p, y)}{\partial p^2} = \lambda [-2f(p) + f'(p)(\Delta_{t-1}(y) - p)]. \quad (\text{A-31})$$

Claims (i) and (iii) follow directly (A-30). To show claim (ii), note from (A-30) and (A-31)

$$\left. \frac{\partial^2 J_t^{TL}(p, y)}{\partial p^2} \right|_{\left. \frac{\partial J_t^{TL}(p, y)}{\partial p} \right|_{p=0} = 0} = -\lambda \left(\frac{2f^2(p) + f'(p)\bar{F}(p)}{f(p)} \right) < 0. \quad (\text{A-32})$$

where the last inequality follows from the fact that F is IFR and, thus, $f^2(\cdot) + f'(\cdot)\bar{F}(\cdot) \geq 0$. Now that we have shown that $J_t^{TL}(p, y)$ is strictly unimodal in p , part (b) follows directly.

Lemma B.2. *The function $\tilde{V}_t(y)$, defined by (6), has the following three properties:*

$$H1(t, y) : \tilde{V}_{t+1}(y+1) - \tilde{V}_{t+1}(y) \geq \tilde{V}_t(y+1) - \tilde{V}_t(y), y \geq 0, t = 1, \dots, T.$$

$$H2(t, y) : \tilde{V}_{t+1}(y) - \tilde{V}_t(y) \geq \tilde{V}_{t+2}(y) - \tilde{V}_{t+1}(y), y \geq 0, t = 1, \dots, T.$$

$$H3(t, y) : \tilde{V}_t(y+1) - \tilde{V}_t(y) \geq \tilde{V}_t(y+2) - \tilde{V}_t(y+1), y \geq 0, t = 1, \dots, T.$$

Proof of Lemma B.2

Following Bitran and Mondschein (1993), we prove the result by induction on $t + y$. The three inequalities hold when $t + y = 0$. Suppose they hold for $t + y = m - 1$. We prove they hold when $t + y = m$ to complete the induction. To simplify the notation, we temporarily define:

$$B(p, c, y) = \int_{\frac{p-(1-\beta)c}{\beta}}^b pf(x)dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} (\beta x + (1-\beta)c)f(x)dx.$$

In addition, throughout the proof, let $p_t^*(y)$ and $c_t^*(y)$ denote the optimal posted price and cut-off price if the retailer chooses to negotiate in period t with y units in inventory (that is, the optimal solution to (8)) and let $p_t^{TL}(y)$ denote the optimal posted price if the retailer chooses not to negotiate in period t with y units in inventory (that is, the optimal solution to (7)).

$$(i) H1(t, y) : \tilde{V}_{t+1}(y+1) - \tilde{V}_{t+1}(y) \geq \tilde{V}_t(y+1) - \tilde{V}_t(y)$$

It is easy to show that the result holds at $y = 0$ for all $t = 1, \dots, T$. Suppose $y > 0$. Note that

$$\begin{aligned}\tilde{V}_{t+1}(y) &= \max\{\tilde{V}_{t+1}^{TL}(y), \tilde{V}_{t+1}^N(y) - K\} \\ &= \max\left\{\lambda\bar{F}(p_{t+1}^{TL}(y))(p_{t+1}^{TL}(y) + \tilde{V}_t(y-1)) + [1 - \lambda\bar{F}(p_{t+1}^{TL}(y))] \tilde{V}_t(y), \right. \\ &\quad \lambda q \left[B(p_{t+1}^*(y), c_{t+1}^*(y), y) + \bar{F}(c_{t+1}^*(y))\tilde{V}_t(y-1) \right] \\ &\quad + \lambda(1-q) \left[p_{t+1}^*(y)\bar{F}(p_{t+1}^*(y)) + \bar{F}(p_{t+1}^*(y))\tilde{V}_t(y-1) \right] \\ &\quad \left. + [1 - \lambda q\bar{F}(c_{t+1}^*(y)) - \lambda(1-q)\bar{F}(p_{t+1}^*(y))] \tilde{V}_t(y) - K \right\}.\end{aligned}$$

Subtracting $\tilde{V}_t(y)$ from both sides, we get

$$\begin{aligned}\tilde{V}_{t+1}(y) - \tilde{V}_t(y) &= \max\left\{\lambda\bar{F}(p_{t+1}^{TL}(y)) \left(p_{t+1}^{TL}(y) + \tilde{V}_t(y-1) - \tilde{V}_t(y) \right), \right. \\ &\quad \lambda q B(p_{t+1}^*(y), c_{t+1}^*(y), y) + \lambda(1-q)p_{t+1}^*(y)\bar{F}(p_{t+1}^*(y)) \\ &\quad \left. + [\lambda q\bar{F}(c_{t+1}^*(y)) + \lambda(1-q)\bar{F}(p_{t+1}^*(y))] \left(\tilde{V}_t(y-1) - \tilde{V}_t(y) \right) - K \right\}.\end{aligned}\tag{A-33}$$

Similarly,

$$\begin{aligned}\tilde{V}_{t+1}(y+1) &= \max\{\tilde{V}_{t+1}^{TL}(y+1), \tilde{V}_{t+1}^N(y+1) - K\} \\ &= \max\left\{\lambda\bar{F}(p_{t+1}^{TL}(y+1))(p_{t+1}^{TL}(y+1) + \tilde{V}_t(y)) + [1 - \lambda\bar{F}(p_{t+1}^{TL}(y+1))] \tilde{V}_t(y+1), \right. \\ &\quad \lambda q \left[B(p_{t+1}^*(y+1), c_{t+1}^*(y+1), y+1) + \bar{F}(c_{t+1}^*(y+1))\tilde{V}_t(y) \right] \\ &\quad + \lambda(1-q) \left[p_{t+1}^*(y+1)\bar{F}(p_{t+1}^*(y+1)) + \bar{F}(p_{t+1}^*(y+1))\tilde{V}_t(y) \right] \\ &\quad \left. + [1 - \lambda q\bar{F}(c_{t+1}^*(y+1)) - \lambda(1-q)\bar{F}(p_{t+1}^*(y+1))] \tilde{V}_t(y+1) - K \right\}.\end{aligned}$$

Now, we note that a retailer that has $y+1$ units of inventory with $t+1$ periods to go until the end of the season could use the prices that are optimal for a retailer that has y units of inventory with $t+1$ periods to go until the end of the season, i.e., $p_{t+1}^*(y)$, $c_{t+1}^*(y)$ and $p_{t+1}^{TL}(y)$, which are suboptimal. Therefore:

$$\begin{aligned}\tilde{V}_{t+1}(y+1) &\geq \max\left\{\lambda\bar{F}(p_{t+1}^{TL}(y))(p_{t+1}^{TL}(y) + \tilde{V}_t(y)) + [1 - \lambda\bar{F}(p_{t+1}^{TL}(y))] \tilde{V}_t(y+1), \right. \\ &\quad \lambda q \left[B(p_{t+1}^*(y), c_{t+1}^*(y), y+1) + \bar{F}(c_{t+1}^*(y))\tilde{V}_t(y) \right] \\ &\quad + \lambda(1-q) \left[p_{t+1}^*(y)\bar{F}(p_{t+1}^*(y)) + \bar{F}(p_{t+1}^*(y))\tilde{V}_t(y) \right] \\ &\quad \left. + [1 - \lambda q\bar{F}(c_{t+1}^*(y)) - \lambda(1-q)\bar{F}(p_{t+1}^*(y))] \tilde{V}_t(y+1) - K \right\}.\end{aligned}$$

Subtracting $\tilde{V}_t(y+1)$ from both sides of the above inequality, we obtain

$$\begin{aligned} \tilde{V}_{t+1}(y+1) - \tilde{V}_t(y+1) \geq \max \left\{ \lambda \bar{F}(p_{t+1}^{TL}(y)) \left(p_{t+1}^{TL}(y) + \tilde{V}_t(y) - \tilde{V}_t(y+1) \right), \right. \\ \lambda q B(p_{t+1}^*(y), c_{t+1}^*(y), y+1) + \lambda(1-q)p_{t+1}^*(y)\bar{F}(p_{t+1}^*(y)) \\ \left. + [\lambda q \bar{F}(c_{t+1}^*(y)) + \lambda(1-q)\bar{F}(p_{t+1}^*(y))] \left(\tilde{V}_t(y) - \tilde{V}_t(y+1) \right) - K \right\}. \end{aligned} \quad (\text{A-34})$$

From induction hypothesis $H3(t, y-1)$, we have

$$\tilde{V}_t(y) - \tilde{V}_t(y+1) \geq \tilde{V}_t(y-1) - \tilde{V}_t(y). \quad (\text{A-35})$$

Thus, from equations (A-33) – (A-35), we have

$$H1(t, y) : \tilde{V}_{t+1}(y+1) - \tilde{V}_t(y+1) \geq \tilde{V}_{t+1}(y) - \tilde{V}_t(y).$$

We have now completed the induction for $H1(t, y)$. The proof of Bitran and Mondschein (1993) can be adapted similarly to complete the induction for $H2(t, y)$ and $H3(t, y)$ as well.

Lemma B.3. *The function $V_t(y)$ has the following three properties:*

$$H1(t, y) : V_{t+1}(y+1) - V_{t+1}(y) \geq V_t(y+1) - V_t(y), y \geq 0, t = 1, \dots, T.$$

$$H2(t, y) : V_{t+1}(y) - V_t(y) \geq V_{t+2}(y) - V_{t+1}(y), y \geq 0, t = 1, \dots, T.$$

$$H3(t, y) : V_t(y+1) - V_t(y) \geq V_t(y+2) - V_t(y+1), y \geq 0, t = 1, \dots, T.$$

Proof of Lemma B.3

Omitted. Similar to the proof of the analogous result in Bitran and Mondschein (1993).

Lemma B.4. *Define $J(p, c, \Delta)$ as follows:*

$$J(p, c, \Delta) := \lambda q \left[\int_{\frac{p-(1-\beta)c}{\beta}}^b p f(x) dx + \int_c^{\frac{p-(1-\beta)c}{\beta}} (\beta x + (1-\beta)c) f(x) dx - \bar{F}(c)\Delta \right] + \lambda(1-q)\bar{F}(p)(p-\Delta).$$

Then, $J(p, c, \Delta)$ is supermodular in (p, c, Δ) over the set $\{(p, c, \Delta) : p \geq c, \Delta \geq 0\}$.

Proof of Lemma B.4

First, we note that the set $\{(p, c, \Delta) : p \geq c, \Delta \geq 0\}$ forms a lattice. To prove the result, it suffices

to show that the cross-partials $\frac{\partial^2 J(p,c,\Delta)}{\partial p \partial c}$, $\frac{\partial^2 J(p,c,\Delta)}{\partial p \partial \Delta}$ and $\frac{\partial^2 J(p,c,\Delta)}{\partial c \partial \Delta}$ are all non-negative (see Lemma 8.2 in Porteus, 2002, p.122). It is easy to check that these cross-partials are given by

$$\begin{aligned}\frac{\partial^2 J(p,c,\Delta)}{\partial p \partial c} &= \lambda q \frac{1-\beta}{\beta} f\left(\frac{p-(1-\beta)c}{\beta}\right) \\ \frac{\partial^2 J(p,c,\Delta)}{\partial p \partial \Delta} &= \lambda(1-q)f(p) \\ \frac{\partial^2 J(p,c,\Delta)}{\partial c \partial \Delta} &= \lambda q f(c)\end{aligned}$$

Observe now that all three cross-partials are non-negative, which concludes the proof.