Dynamic Pricing of Perishable Assets under Competition

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We study price competition in an oligopolistic market with a mix of substitutable and complementary products. Each firm has a fixed initial stock of items and competes in setting prices to sell them over a finite sales horizon. Customers sequentially arrive at the market, make a purchase choice and then leave immediately with some likelihood of no-purchase. The choice probability depends on the time of purchase, the product attributes and the current prices. Assuming deterministic customer arrival rates, we show that any equilibrium strategy has a simple structure, involving a finite set of *shadow prices* measuring capacity externalities that firms exert on each other: equilibrium prices are resolved from a one-shot competitive game with the *current-time* demand structure, taking into account the *time-invariant* capacity externalities. The former reflects the transient demand side at every moment and the latter captures the aggregate supply constraint over time. Such a structure sheds light on dynamic revenue management problems under competition, which helps capture the essence of the problems under demand uncertainty.

1. Introduction

Airlines face the problem of competing in setting prices to sell a fixed capacity of seats over a finite sales horizon before planes depart. Online travel sites, such as Expedia, gather information and list flight fares almost in real time among competitive airlines. Enabled by such price transparency, customers comparison-shop online among various airlines' differentiated products based on flights' attributes and prices. This trend poses tremendous challenges for airlines to profitably maintain pricing responsiveness to competitors' strategies. The real-time competitive pricing problem is further complicated by the fact that the aggregate market demands and their elasticities evolve over time. For example, over the sales horizon, leisure-class customers tend to arrive earlier hunting for bargain tickets, and business-class customers tend to arrive later, willing to pay the full price.

Revenue management (RM) techniques help firms set the right price at the right time to maximize revenue. It has been successfully applied to airline and many other industries. Nevertheless, traditional RM models typically assume a monopoly setting. The literature on competitive RM is relatively scant. This is partly due to the challenges imposed by this complex game of capacitated inter-temporal price competition, not to mention even more thorny problems of time-varying demand structures. A few stylized models have been successfully built to examine the game from various angles. However, no *structural* results for the general problem are known. Algorithmic approaches also have been implemented, with the hope to capture the whole dynamics and a focus on optimal policy computation. But without the guidance of structural results in algorithm design, computational approaches can suffer from the curse of dimensionality.

We aim at uncovering the structural nature of this competitive RM game. Specifically, we consider the setting where multiple capacity providers compete to sell their own fixed initial capacities of *differentiated*, substitutable or complementary, perishable items by varying prices over the same finite sales horizon. The customers' arrival rate and their price sensitivity vary over time. Customers sequentially arrive at the market, make a purchase choice and then leave immediately with some likelihood of no-purchase. The choice probability depends on the time of purchase, the product attributes and the current prices. The supply side of each firm is capacitated by the initial inventory level at the beginning of the sales horizon; there are no replenishment opportunities during the horizon. We formulate the game as a differential game in continuous time. In this formulation, we obtain structural results, capturing the nature of how transient market conditions and aggregate capacity constraints interact to determine inter-temporal pricing behavior in equilibrium. This structural nature would otherwise not be captured in a stylized one-shot model/a setting for a homogenous product, or would be lost in a discretize-time formulation. Such a structure arises in nature, because inter-temporal sales share aggregate capacity constraints over the entire sales horizon, and demands (e.g., arrival rates and choice probabilities) are independent of the inventory levels. Due to the structural results, the computation of infinite-dimensional equilibrium policies can be cast as finite-dimensional problems. These efficiently-computable solutions from the differential game can be used to derive heuristics for formulations that take into account demand uncertainty.

Our contribution is two fold. First, we focus on the first-order effect by assuming a deterministic arrival process. We show that the equilibrium strategy has a simple structure: There exists a finite set of shadow prices measuring aggregate capacity externalities that firms exert on each other; the equilibrium price at any point of time is simply the outcome from a one-shot price competition with the current-time demand structure, taking into account the time-invariant capacity externalities. A firm with ample capacity does not exert any externality on the price competition. A firm with limited capacity exerts *positive* externality by alleviating price competition among substitutable products, but exerts *negative* externality by undercutting the prices of other complementary products. Due to the structure, the computation of infinite-dimensional equilibrium pricing policies reduces to solving for finite-dimensional shadow prices.

Second, we show insights from the deterministic problem are valuable in capturing the essence of the stochastic problem under demand uncertainty. There is an active stream of literature on applying a computational approach, called approximate dynamic programming (ADP), to solve the stochastic problem in RM. We show that the shadow prices obtained from the deterministic problem coincide with the solution obtained from an *affine* ADP approach to the stochastic game. Moreover, applying the efficiently-computable solutions from the deterministic game, as pre-commitment or contingent pricing heuristics to the stochastic game, sustains as an asymptotic equilibrium, when demand and supply are large.

Literature Review. There is a growing body of literature on competitive RM. Depending on the chosen decision variables, RM is categorized as quantity-based or price-based, or a mix of the two. Netessine and Shumsky (2005) examine one-shot quantity-based games of booking limit control under both horizontal competition and vertical competition. Jiang and Pang (2010) study a one-shot oligopolistic competition in setting quantity decisions of booking limits in a network RM setting. These works ignore inter-temporal pricing behavior.

Oligopoly pricing, common in the economics and marketing literature, is gaining traction within the RM community. Unlike a standard oligopoly pricing setting, firms in an RM model are capacity constrained and pricing decisions need to be made over time. One line of research is to use *variational inequalities* to characterize inter-temporal price equilibrium with capacity constraints. Perakis and Sood (2006) address a discrete-time stochastic game of setting prices and protection levels by using variational inequalities and ideas from robust optimization. Mookherjee and Friesz (2008) consider a discrete-time combined pricing, resource allocation, overbooking RM problem under demand uncertainty over networks and under competition. Kwon et al. (2009) study a continuous-time deterministic differential game of price competition with demand learning. Adida and Perakis (2010) consider a continuous-time deterministic differential game of joint pricing and inventory control where each firm's multiple products share production capacity. The authors also study the robust fluid model of the corresponding stochastic game. See also Friesz (2010, Chapter 10) for applications of finite or infinite dimensional quasi-variational inequality to various RM settings. This research stream aims at efficient algorithms to compute equilibrium prices. We focus on structural properties on inter-temporal equilibrium pricing behavior.

Some works assume that the competition is to sell a homogenous product. Granot et al. (2007) analyze a multi-period duopoly pricing game where a homogeneous perishable good is sold to consumers who visit one of the retailers in each period. Talluri and Martínez de Albéniz (2011) study perfect competition of a homogenous product in an RM setting under demand uncertainty and derive a closed-form solution to the equilibrium price paths. The authors show a structural property on the equilibrium policy that the seller with the lower equilibrium reservation value sells a unit at a price equal to the competitor's equilibrium reservation value. This structural property is due to the nature of Bertrand competition of a homogenous product that the seller is willing to undercut the competitor down to its own reservation value. The authors also show that the equilibrium sales trajectory is such that firms alternatively serve as a monopoly and the firm with less capacity sells out before the firm with more capacity. We complement Talluri and Martínez de Albéniz (2011) by studying price competition of differentiated products and exploring its structural nature. To customers who shop around only for lowest fares, the products can be viewed more or less homogeneous. However, pricing transparency facilitated by third-party travel websites exposes the same price to various consumer segments with heterogenous price sensitivities, e.g., loyal customers and bargain hunters. The aggregate demand structure is closer to the case of differentiated products, and the resulting equilibrium behaviour is different from the case of a homogenous product.

Strategic consumer has been examined in the competitive RM setting with various assumptions on consumer behavior, such as what they know and how they behave. Levin et al. (2009) present a unified stochastic dynamic pricing game of multiple firms where differentiated goods are sold to finite segments of strategic customers who may *time* their purchases. The key insight is that firms may benefit from limiting the information available to consumers. Liu and Zhang (2012) study dynamic pricing competition between two firms offering vertically differentiated products to strategic consumers, where price skimming arises as a subgame perfect equilibrium. This model may be more applicable to the seasonal products, and less applicable to the airline industry where the average price trend is typically upward. We do not take consumers' strategic waiting behavior into account and admit this as a limitation. One may argue when the aggregate demand arrival process, as an input to our model, is calibrated from real data over repeated horizons, it should, to some extent, have captured the equilibrium waiting/purchase behavior of strategic consumers, e.g., strategic consumers may tend to arrive either at the beginning of the booking horizon or close to departure (Li et al. 2011). Our model may provide a more practical approach to address strategic consumer behavior. Firms can repeatedly, over horizons, solve the same problem with updated time-varying demand structures to address repeated interaction with strategic consumers.

Two papers closest to ours are Lin and Sibdari (2009) and Xu and Hopp (2006). The former proves the existence of a pure-strategy subgame perfect Nash equilibrium in a discrete-time stochastic game with a stationary MNL demand structure. The main difference, apart from the demand structure and the choice of how to model time, is that we focus on the structural nature of the game and its implications, beyond the existence and uniqueness results. Similar to our paper, the latter studies a dynamic pricing problem under oligopolistic competition in a continuous-review setting. The authors establish a weak perfect Bayesian equilibrium of the pricing game. There are several notable differences. Most significantly, the latter obtains a cooperative *fixed-pricing* equilibrium strategy in a perfect competition of a homogeneous product. We obtain *time-varying* pricing strategies for imperfect competition with differentiated products. Furthermore, the latter assumes a quasi-linear consumer utility function, and our demand structure allows for a more general consumer utility function.

In the extension, we study Markovian pricing equilibrium in a continuous-time dynamic stochastic game over a finite horizon. In the economics literature, Pakes and McGuire (1994) develop an algorithm for computing Markovian equilibrium strategies in a discrete-time infinite-horizon dynamic game of selling differentiated products. Fershtman and Pakes (2000) apply the algorithm to a collusive framework with heterogeneity among firms, investment, entry, and exit. Borkovsky et al. (2010) discuss an application of the homotopy method to solving these dynamic stochastic games. Farias et al. (2012) introduce a new method to compute Markovian equilibrium strategies in large-scale dynamic oligopoly models by approximating the best-response value function with a linear combination of basis functions. (See references therein for comprehensive review of this line of development.) We show that Markovian equilibrium of the continuous-time stochastic game is reduced to equilibrium of the differential game, if the value functions are approximated by affine functions. Moreover, instead of discretizing time to compute Markovian equilibrium of the stochastic game like Lin and Sibdari (2009), we show that heuristics suggested by the corresponding differential game are asymptotically optimal with large demand and supply.

2. The Model

We define some notation. $\mathbb{R}_+ \equiv [0, +\infty)$ and $\mathbb{R}_{++} \equiv (0, +\infty)$. x_i denotes the i^{th} component of vector $\vec{x}$, and $\vec{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)^{\mathrm{T}}$ is a sub-vector of $\vec{x}$ with components other than i. $\vec{e}_i$ denotes a vector with the i^{th} element 1 and all other elements 0's. For notation simplicity, 0 can denote a scalar or a vector of any dimension with all entries being zeros. LHS and RHS are shorthand for left-hand side and right-hand side, respectively. A function is said to be increasing (decreasing) when it is nondecreasing (nonincreasing).

We consider a market of m competing firms selling differentiated perishable assets over a finite horizon $[0, T]$. At time $t = 0$, each firm i has an initial inventory of C_i units of one differentiated product. We count the time forwards, and use t for the elapsed time and $s \equiv T - t$ for the remaining time.

2.1. Assumptions

Consumers sequentially arrive at the market and make a purchase choice based on attributes of the differentiated products and their current prices across the market. Both the arrival rate to the market and the choice probability can be time-dependent. We specify the aggregate demand rate function in a general way: at any time $t \in [0, T]$, the vector of demand rates $\vec{d}(t, \vec{p}(t))$ for all firms is time-dependent and influenced by the current market price vector $\vec{p}(t)$. The general form of the demand rate functions can allow for general consumer utility functions and general time-varying arrival processes. This demand rate function can be calibrated, over repeated sales horizon, from data of arrival rates to the market and inter-temporal price elasticities for the same origin-destination “local” market. We further assume that the demand rate function is public information. In the airline industry, firms typically have access to the same sources of pricing/sales data and have very similar or sometimes even identical forecasting systems.

We denote the revenue rate function for any firm i at time t by $r_i(t, \vec{p}) \equiv p_i d_i(t, \vec{p})$. We further make the following general assumptions on the demand rate functions.

Assumption 1 (Demand Rate). *The following assumptions hold for all i, t.*

(a) (Differentiability) *$d_i(t,\vec{p})$ is continuously differentiable in $\vec{p}$;*

(b) (Pseudo-Convexity) *$d_i(t,\vec{p})$ is pseudo-convex in p_i.*

As a technical note, the pseudo-convexity assumption is a slight relaxation of convexity: a function f is pseudo-convex on a non-empty open set X if for any $x, y \in X$, $(y-x)^{\mathrm{T}}\nabla_x f(x) \geqslant 0 \Rightarrow f(y) \geqslant f(x)$, where ∇_x is the gradient operator. f is pseudo-concave if and only if $-f$ is pseudo-convex. Pseudo-concavity is stronger than quasi-concavity but weaker than concavity. Unlike standard oligopoly pricing problems without capacity constraints, we resort to this weaker version of convexity on the demand rate function to account for general demand functions and deal with capacity constraints.

Assumption 2 (Revenue Rate). *The following assumptions hold for all i, t.*

(a) (Pseudo-Concavity) *$r_i(t,\vec{p})$ is pseudo-concave in p_i;*

(b) (Bounded Revenue) *There exists a function $\bar{R}_i(t)$ such that $r_i(t,\vec{p}) \leqslant \bar{R}_i(t)$ and $\int_0^T \bar{R}_i(t)\,dt < \infty$.*

The less-used pseudo-convexity/concavity assumptions on demand and revenue rates are resorted to accommodate MultiNomial Logit (MNL; see Anderson et al. 1992) demand functions:

$$d_i(t,\vec{p}) = \lambda(t)\frac{\beta_i(t)e^{-\alpha_i(t)p_i}}{a_0(t) + \beta_i(t)e^{-\alpha_i(t)p_i} + \sum_{j\neq i}\beta_j(t)e^{-\alpha_j(t)p_j}}, \tag{1}$$

where $\lambda(t), a_0(t), \alpha_i(t), \beta_i(t) > 0$ for all i. Moreover, Assumptions 1 and 2 are also satisfied by linear demand functions of substitutable products $d_i(t,\vec{p}) = a_i(t) - b_i(t)p_i + \sum_{j\neq i} c_{ij}(t)p_j$, where $a_i(t), b_i(t) > 0$ for all i and $c_{ij} \geqslant 0$ for all $j \neq i$. Lastly, we emphasize that all of our assumptions can accommodate the *co-existence* of differentiated substitutable and complementary products, e.g., $d_i(t,\vec{p}) = a_i(t) - b_i(t)p_i + \sum_{j\neq i} c_{ij}(t)p_j$, where $a_i(t), b_i(t) > 0$ for all i and $c_{ij} \in \mathbb{R}$ for all $j \neq i$.

It is easy to see that in a linear demand structure, each firm's feasible strategy set can possibly depend on competitors' strategies. This is so called *coupled strategy constraints* (coined by Rosen 1965). We make the following assumptions on the feasible strategy set of each firm.

Assumption 3 (Price Set). *The following assumptions hold for any competitors' prices $\vec{p}_{-i}$ for all i, t.*

(a) (Null Price) *There exists a unique null price $p_i^\infty(t,\vec{p}_{-i})$ such that $\lim_{p_i \to p_i^\infty(t,\vec{p}_{-i})} d_i(t,\vec{p}) = 0$ and $\lim_{p_i \to p_i^\infty(t,\vec{p}_{-i})} r_i(t,\vec{p}) = 0$, which is the* only *pricing option when a firm runs out of stock;*

(b) (Feasible Set) *Other than the null price that is available at any time and is the only option upon stockout, firm i chooses price from the set $\mathcal{P}_i(t,\vec{p}_{-i})$ that is a non-empty, compact and convex subset of $\{p_i \in \mathbb{R} \mid d_i(t,p_i,\vec{p}_{-i}) \geqslant 0\}$.*

(c) (Slater's Condition) *There exists $\bar{p}_i(t,\vec{p}_{-i}) \in \mathcal{P}_i(t,\vec{p}_{-i})$ such that $d_i(t,\bar{p}_i(t,\vec{p}_{-i}),\vec{p}_{-i}) < 1/T$.*

Assumption 3(a) makes sure that a firm immediately exits the market upon a stockout. In this case, consumers who originally prefer the firm that has run out of stock will *spill over* to the remaining firms that still have positive inventory. The spillover is endogenized from the demand structure according to consumers' preferences and product substitutability. For example, in any MNL demand function, ∞ is the null price, yielding the attraction value equal to zero. We further illustrate the spillover effect by the following example.

Example 1 (Demand Structure with Spillover). For a duopoly with stationary linear demand rate functions $d_i(t,p_i,p_{-i}) = 1 - p_i + \gamma p_{-i},\ i = 1,2,\ \gamma < 1$, firm 1 can post a null price $p_1^\infty(t,p_2) = 1+\gamma p_2$, which is solved from $d_1(t,p_1,p_2) = 1-p_1+\gamma p_2 = 0$, to shut down its own demand. The resulting demand rate function for firm 2 with the spillover effect is $d_2(t,p_1^\infty(t,p_2),p_2) = 1-p_2+\gamma p_1^\infty(t,p_2) = (1+\gamma) - (1-\gamma^2)p_2$. The spillover-adjusted demand for firm 2 has a higher potential demand (higher intercept), and it is less price sensitive. ▫

Note that we do not exclude the possibility of shutting down demand by posting a null price in a firm's strategy before its stockout (see Example 2). The joint feasible price set at any time t is denoted by $\mathcal{P}(t) \equiv \big\{\vec{p} \mid p_i \in \mathcal{P}_i(t,\vec{p}_{-i}) \cup \{p_i^\infty(t,\vec{p}_{-i})\}, \forall\, i\big\}$. Assumption 3(c) guarantees that there exists a price control policy $\{\vec{p}^{\,o}(t) \in \mathcal{P}(t), 0 \leqslant t \leqslant T\}$ such that the capacity constraint is satisfied, i.e., $\int_0^T d_i(t,\vec{p}^{\,o}(t))\,dt < x_i$ for all $x_i \in \mathbb{N}$ and all i. This is the Slater constraint qualification, commonly used in nonlinear optimization to ensure that the Karush-Kuhn-Tucker(KKT)-type conditions are necessary for optimality. Assumption 3(c) is satisfied if the feasible price set contains sufficiently high prices.

We also assume that the salvage value of the asset at the end of the sales horizon is zero and that all other costs are sunk. We can always transform a problem with positive salvage cost c_i for firm i to a zero-salvage-cost case by changing variables from p_i to $p_i - c_i$ in the demand rate function.

2.2. The Model

We formulate a finite-horizon non-cooperative differential game, where demand is a deterministic fluid process (Dockner et al. 2000). In the extension, we will consider its stochastic counterpart where demand is a Poisson process. Firms compete in influencing demand rates by adjusting prices. At time $t \in [0,T]$, firm i sets its own price $p_i(t)$. We assume the following information structure throughout the paper.

Assumption 4 (Strong Information Structure). *All firms have perfect knowledge about each other's inventory levels at any time.*

This assumption is standard in game theory for seeking subgame perfect equilibrium. It used to be unrealistic, but nowadays inventory information in real time may be considered as being revealed in some way, since almost all online travel agencies and major airlines offer a feature of previewing seat availability from their websites.

Let us denote by $\vec{x}(t)$ the joint inventory level at time t, which is assumed to be a *continuous quantity* in the differential game. Let $\mathcal{X} \equiv \times_i [0, C_i]$ denote the state space of inventory in the market. Later, the inventory level will be discrete in the stochastic extension. We differentiate the following two types of strategies. In an open-loop strategy, firms make an irreversible *pre-commitment* to a future course of action at the beginning of the game. Alternatively, feedback strategies designate prices according to the current time and joint inventory level, which capture the *feedback* reaction of competitors to the firm's chosen course of action.

DEFINITION 1 (OPEN-LOOP STRATEGY). A joint open-loop strategy $\vec{p}(t)$ depends only on time t and the given initial joint inventory level $\vec{x}(0) = \vec{C}$. The set of all joint open-loop strategies such that $\vec{p}(t) \in \mathcal{P}(t)$, $\forall\, t$, is denoted by $\mathscr{P}_O$.

DEFINITION 2 (FEEDBACK STRATEGY). A joint feedback strategy $\vec{p}(t, \vec{x}(t))$ depends on time t and the current joint inventory level $\vec{x}(t)$. The set of all joint feedback strategies such that $\vec{p}(t, \vec{x}(t)) \in \mathcal{P}(t)$, $\forall\, t$, is denoted by $\mathscr{P}_F$.

Let $D[0, T]$ denote the space of all right-continuous real-valued functions with left limits defined on interval $[0, T]$, where the left discontinuities accommodate price jumps after a sale. Given joint pricing strategy $\vec{p} \in (D[0, T])^m$, we denote the total profit for any firm i by $J_i[\vec{p}] \equiv \int_0^T r_i(t, \vec{p}(t))\, dt$. Inventory drops at the demand rate, hence the vector of inventory evolves according to the following equation: for all i,

$$\dot{x}_i(t) = -d_i(t, \vec{p}(t)), \quad 0 \leqslant t \leqslant T, \quad x_i(0) = C_i.$$

Any firm i's objective is to maximize its own total revenue over the sales horizon subject to all capacity constraints at any time, i.e.,

$$\text{Problem } (\mathrm{P}_i) \qquad \max_{\{p_i(t), 0 \leqslant t \leqslant T\}} \int_0^T r_i(t, \vec{p}(t))\, dt$$

$$\text{s.t.} \qquad x_i(t) = C_i - \int_0^t d_i(v, \vec{p}(v))\, dv \geqslant 0, \quad 0 \leqslant t \leqslant T, \forall\, i. \tag{2}$$

Firms simultaneously solving its own revenue maximization problem subject to a joint set of constraints gives arise to a game with coupled strategy constraints (2) for all i, i.e., any firm's feasible strategy set depends on competitors' strategies through these capacity constraints. For this type of game, Rosen (1965) coined the term a *generalized* Nash game with coupled constraints; see also Topkis (1998) for a treatment of such generalized games. In the differential game, the pricing strategies are simultaneously presented by all firms before the game starts. If some pricing policy is not jointly feasible, then it will be eliminated from the joint feasible strategy space, in other words, all firms face a joint set of constraints in selecting feasible strategies such that their strategies remain credible. This explains why any firm i is also constrained by all firms' capacity constraints in its own revenue maximization problem (P_i).

The definitions of generalized Nash equilibrium for open-loop (OLNE) and feedback strategies (FNE) follow immediately. A generalized (omitted hereafter) OLNE (resp. FNE) is an m-tuple of open-loop (resp. feedback) strategies $\vec{p}^* \in (D[0,T])^m \cap \mathscr{P}_O$ (resp. $\mathscr{P}_F$) such that $\{p_i^*(t), 0 \leqslant t \leqslant T\}$ is the solution to problem (P_i) for all i. In a non-zero-sum differential game, open-loop and feedback strategies are generally different, either in form, or in terms of generated inventory trajectory and price path. However, we will demonstrate in §3.4 that re-solving OLNE continuously over time results in an FNE, which generates the same inventory trajectory and price path as those of the OLNE with the initial time and inventory level. Due to this relationship between OLNE and FNE, for convenience, we may loosely call an OLNE, an equilibrium strategy, in the following discussion of the differential game.

3. Equilibrium

In this section, we show equilibrium existence, and its uniqueness in an appropriate sense. We fully explore its structural properties, and illustrate them with examples.

3.1. Existence

To show the existence of an OLNE that is infinite-dimensional, we invoke an infinite-dimensional version of Kakutani's fixed point theorem (Bohnenblust and Karlin 1950).

Proposition 1 (Existence of OLNE). *We have the following equilibrium existence results.*

(i) If $p_i^\infty(t, \vec{p}_{-i}) \in \mathcal{P}_i(t, \vec{p}_{-i})$, there exists an OLNE.

(ii) For MNL demand structures, there exists an OLNE where firms do not use the null price ∞ at any time.

We list MNL demand structures separately. This is because any feasible price set containing MNL's null price ∞ will not be compact and convex, hence we need to treat them slightly differently.

3.2. Characterization

We follow the maximum principle of the differential game with constrained state space (Pontryagin et al. 1962) to derive the set of necessary conditions for OLNE, and then verify that they are also sufficient conditions under our assumptions on demand and revenue rate functions. Then we have the following full characterization of OLNE.

Proposition 2 (Characterization of OLNE). *The open-loop policy $\{\vec{p}^*(t) : 0 \leqslant t \leqslant T\}$, with its corresponding state trajectory $\{\vec{x}^*(t) : 0 \leqslant t \leqslant T\}$, is an OLNE if and only if there exists a matrix of nonnegative shadow prices $M \equiv [\mu_{ij}]_{m\times m} \geqslant 0$ such that the following conditions are satisfied for all i,*

(i) (Equilibrium Prices) *for t such that $x_i^*(t) > 0$,*

$$p_i^*(t) = \underset{p_i \in \mathcal{P}_i(t, \vec{p}^*_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}^*_{-i}(t))\}}{\arg\max} \left\{ r_i(t, p_i, \vec{p}^*_{-i}(t)) - \overbrace{\sum_j \mu_{ij} d_j(t, p_i, \vec{p}^*_{-i}(t))}^{capacity\ externality} \right\}; \tag{3}$$

(ii) (Market Exit) *if the set* $E_i = \{t \in [0,T] \mid x_i^*(t) = 0\}$ *is non-empty, then for* $t \in [\bar{t}_i, T]$ *where* $\bar{t}_i = \inf E_i$*, there exist decreasing shadow price processes* $\underline{\mu}_{ij}(t) \in [0, \mu_{ij}]$ *for all* j *that shut down firm* i*'s demand, i.e.,*

$$p_i^*(t) = p_i^\infty(t, \vec{p}_{-i}^*(t)) = \underset{p_i \in \mathcal{P}_i(t, \vec{p}_{-i}^*(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}^*(t))\}}{\arg\max} \left\{ r_i(t, p_i, \vec{p}_{-i}^*(t)) - \sum_j \underline{\mu}_{ij}(t) d_j(t, p_i, \vec{p}_{-i}^*(t)) \right\};$$

(iii) (Complementary Slackness) $\mu_{ij} x_j^*(T) = 0$ *for all* j.

The OLNE has a simple structure. First, there exists a finite set of shadow prices, independent of time, measuring capacity externalities that firms exert on each other. Second, the inter-temporal equilibrium prices at any time are solved from a one-shot price competition game with the current-time demand structure, taking into account time-invariant capacity externalities.

We now illustrate this structure in detail. First, let us focus on the shadow prices. The characterization says that in the differential game, in equilibrium the externality that one firm's capacity exerts on all firms is *constant* over time before the firm runs out of stock. Intuitively, the time-invariant capacity externalities are due to the fact that capacity constraints are imposed on the *total* sales over the entire sales horizon, and that the demand rate is independent of current inventory levels. The complementary slackness condition indicates when to, and when not to expect capacity externality. If at the end of the sales horizon, some firm, along the equilibrium inventory path, still has positive inventory, then this firm's capacity exerts no externalities at all. Otherwise, a non-zero externality will be expected. Whenever a firm's inventory level hits zero before the end of the horizon, the firm has to post an appropriate null price to exit the market, which is the only option to avoid taking orders but not being able to fulfill them. The demand system among the remaining firms with positive inventory will be adjusted to account for spillover, and the firm that has run out of stock no longer exerts any further externality on all other firms.

Second, let us see how the inter-temporal equilibrium prices emerge from the interaction between the current-time demand structure and aggregate supply constraints. We start with a one-shot monopoly problem to illustrate the self-inflicted capacity externality. In a monopoly market with a continuous downward-sloping demand curve $d(p)$, a revenue-maximizing firm with capacity C faces a one-shot pricing decision. The revenue maximization problem with capacity constraint can be written as $\max_p pd(p)$, s.t. $d(p) \leqslant C$. The optimal solution is the maximum between the market-clearing price $p^o = \inf\{p \mid d(p) \leqslant C\}$, and the revenue-maximizing price $p^* = \arg\max_p pd(p)$. The first-order condition of this problem taking into account capacity constraint is $\partial[pd(p) + \mu(C - d(p))]/\partial p = \partial[(p-\mu)d(p)]/\partial p = 0$, where $\mu \geqslant 0$ is the shadow price of capacity. If the firm has ample capacity such that $d(p^*) \leqslant C$, then the optimal price is the revenue-maximizing price p^* and the capacity constraint exerts no externality on setting the price (i.e., $\mu = 0$). If the firm has limited capacity such that $d(p^*) > C$, then the optimal price is the market-clearing price p^o and the capacity constraint exerts a positive externality on boosting the optimal price to be higher than p^* (i.e., $\mu > 0$).

Let us now get back to the inter-temporal price competition game. We have already explained that the externalities a firm's capacity exerts on all firms are time-invariant. Then the price equilibrium at any time is simply to solve a one-shot price competition game under the current-time demand structure that has been adjusted for spillover, taking into account time-invariant capacity externalities (see problem (3)). Now we illustrate how capacity externalities influence the equilibrium pricing inter-temporally. Let us fix an arbitrary time t and focus on the first order conditions of the maximization problem (3) of any firm i that has taken into account capacity externalities. On one hand, if firms i and j offer substitutable products, then $\partial d_j(t,\vec{p})/\partial p_i \geqslant 0$ and hence, $\mu_{ij}\partial d_j(t,\vec{p})/\partial p_i \geqslant 0$. Therefore firm j's scarce capacity exerts a *positive* externality on firm i: since firm j has limited capacity, it has a tendency to increase its own price as the self-inflicted capacity externality; due to the substitutability between products from firms i and j, the price competition between the two firms will be alleviated so that firm i can also post a higher price. On the other hand, if firms i and j offer complementary products, then $\partial d_j(t,\vec{p})/\partial p_i \leqslant 0$ and hence, $\mu_{ij}\partial d_j(t,\vec{p})/\partial p_i \leqslant 0$. Therefore firm j's scarce capacity exerts a *negative* externality on firm i: while firm j has a tendency to increase its own price, due to the complementarity between products from firms i and j, the price of firm i has to be undercut to compensate for the price increase of firm j.

Comparative Statics. Motivated by firms' self-interested behavior, we propose and investigate a special notion of equilibrium, *bounded rational equilibrium*, which facilitates analysis of comparative statics of equilibrium prices.

Definition 3 (Bounded Rational OLNE). A bounded rational OLNE has its matrix of constant (bounded rational) shadow prices satisfy $\mu_{ij} = 0$ for all $i \neq j$; namely, $M \equiv [\mu_{ij}]_{m\times m}$ is a diagonal matrix with the diagonal $[\mu_{ii}]_{m\times 1} \in \mathbb{R}^m_+$.

The bounded rational equilibrium may arise if in the best-response problem of each firm, only the firm's own capacity constraint is taken into account. The bounded rational equilibrium can be the only relevant equilibrium concept, if firms do not have competitors' inventory information and the equilibrium emerges from repeated best responses. In view of how capacity externalities influence price competition depending on the nature of product differentiability, we can obtain the following comparative statics of equilibrium prices in bounded rational OLNE with respect to capacity levels.

Proposition 3 (Comparative Statics of Bounded Rational OLNE in Capacity). *If all products are substitutable such that the price competition is (log-)supermodular, then a decrease in any firm's capacity leads to higher equilibrium prices in bounded rational OLNE for any firm. In a duopoly selling complementary products such that the price competition is (log-)submodular, a decrease in one firm's capacity leads to higher equilibrium prices for the firm itself and lower equilibrium prices for the other firm in bounded rational OLNE.*

Proposition 3 says that the lower any firm's capacity is, the higher the bounded rational OLNE equilibrium prices are at any time for all firms in the competition of selling substitutable products. This decreasing monotonicity of equilibrium prices in capacities is driven by the decreasing monotonicity of bounded rational shadow prices in capacities, as a natural extension of the monopoly case.

3.3. Uniqueness

In our game, one firm's strategy set depends on competitors' strategies. Such a game is referred to as *generalized* Nash game in the literature. Rosen (1965) investigates the notion of *normalized* Nash equilibrium in the context of finite-dimensional generalized Nash games. In a series of papers (e.g., Carlson 2002), Carlson extends the idea to infinite-dimensional generalized Nash games. Similarly, for our differential game, we can define normalized Nash equilibrium that has the constant shadow prices related in a special way, and provide a sufficient condition to guarantee its uniqueness.

Definition 4 (Normalized OLNE). A normalized OLNE has its matrix of constant shadow prices specified by one vectors $\vec{\xi} \in \mathbb{R}_+^m$ as $\mu_{ij} = \xi_j$ for all i and j; namely, $M \equiv [\mu_{ij}]_{m\times m}$ is a matrix with all rows being equal. (See Hobbs and Pang 2007, Adida and Perakis 2010 for the same notion.)

Recall that in any firm i's revenue maximization, the shadow prices μ_{ij}, for all j, measure how much externality firm j's capacity exerts on firm i. The normalized Nash equilibrium can be interpreted qualitatively as that all firms use the *same* shadow prices in their best-response problems. It may be reasonable to argue that in the airline industry, each firm infer the same set of shadow prices from the commonly observed capacity levels across firms. This is because there are common business practices across the airline industry, e.g., most airlines use very similar or sometimes even identical RM systems, and they may use common external sources of data. If the normalized Nash equilibrium is concerned, its uniqueness can be obtained under the commonly used *strict diagonal dominance* (SDD) condition.

Proposition 4 (Unique Normalized OLNE). *If $d_i(t,\vec{p})$ is convex in p_j for all i,j,t and*

$$\frac{\partial^2 r_i(t,\vec{p})}{\partial {p_i}^2} + \sum_{j\neq i} \left| \frac{\partial^2 r_i(t,\vec{p})}{\partial p_i \partial p_j} \right| < 0 \tag{SDD}$$

for all i,t, then there exists a unique normalized OLNE.

To accommodate MNL demand structures, we provide a general sufficient condition to guarantee the uniqueness of bounded rational OLNE.

Proposition 5 (Unique Bounded Rational OLNE). *If $\partial d_i(t,\vec{p})/\partial p_i < 0$ for all i and t, and the Jacobian and Hessian matrix of the demand function $\vec{d}(t,\vec{p})$ with respect to $\vec{p}$ are negative semi-definite for all $\vec{p} \in \mathcal{P}(t)$ and all t, then there exists at most one bounded rational OLNE for any vector of diagonal shadow prices $[\mu_{ii}]_{m\times 1} \in \mathbb{R}_+^m$. Moreover, there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.*

We show that for any vector of nonnegative diagonal shadow prices, there exists a unique price equilibrium at any time, arising from the one-shot price competition game, with the diagonal shadow prices as the marginal costs. However, an arbitrary vector of diagonal shadow prices may not necessarily result in a bounded rational OLNE. Only if the entire price trajectory indeed satisfies the equilibrium characterization, then this vector of diagonal shadow prices does correspond to an equilibrium, with a unique joint equilibrium pricing policy. By the existence result of Proposition 1 which essentially shows the existence of a bounded rational OLNE, we know that there exists *at least one* vector of diagonal shadow prices such that its corresponding bounded rational OLNE is unique. As an immediate result of Propositions 4 and 5, we can provide the following sufficient conditions for linear demand structures to guarantee the uniqueness of bounded rational OLNE.

COROLLARY 1. *For any linear demand structure with $\vec{d}(t,\vec{p}) = \vec{a}_t - B_t\vec{p}$, where $\vec{a}_t \in \mathbb{R}^m_{++}$, $B_t \in \mathbb{R}^{m\times m}$ is a diagonally dominant matrix with diagonal entries positive and off-diagonal entries non-positive,*

(i) there exists a unique normalized OLNE;

(ii) there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.

Moreover, as an immediate result of Gallego et al. (2006), we have the following corollary.

COROLLARY 2. *For the MNL demand structure (1), there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.*

3.4. Feedback Nash Equilibrium

So far we have fully characterized OLNE. Next we will establish a connection between OLNE and FNE. Given any point of time t with a joint inventory level $\vec{x}(t)$, firms can solve a differential game, denoted by $\mathrm{P}(t,\vec{x}(t))$, with a remaining sales horizon $[t,T]$ and a current inventory level $\vec{x}(t)$, as the initial condition. We denote by $\vec{p}^f(t,\vec{x})$ the mapping from the initial condition $(t,\vec{x})$ of the differential game $\mathrm{P}(t,\vec{x}(t)=\vec{x})$, to the equilibrium prices $\vec{p}^*(t)$ of an OLNE at its initial time t. Intuitively, $\vec{p}^f(t,\vec{x})$ is re-solving OLNE for any initial condition $(t,\vec{x})$. If OLNE is unique as discussed in the previous subsection, the designation of the mapping $\vec{p}^f(t,\vec{x})$ is unambiguous. In some other scenarios, even OLNE may not be unique, a natural *focal* point can be the Pareto-dominant equilibrium. For example, for any diagonal shadow prices $[\mu_{ii}]_{m\times 1} \in \mathbb{R}^m_+$, if the revenue rate function $r_i(t,p_i,\vec{p}_{-i})$ for all i has increasing differences in $(p_i,\vec{p}_{-i})$ for any t, then multiple bounded rational OLNE may arise, but there is a largest one that is preferred by all firms (Bernstein and Federgruen 2005, Theorem 2).

Open-loop strategy is a static concept. For a given initial time and joint inventory level, it specifies a time-dependent control path. Feedback strategy is a dynamic concept and specifies reactions to all possibilities of current time and joint inventory levels. However, in our RM differential game, a joint feedback strategy, that solves an OLNE at every time instant with the current joint inventory level, is an FNE and can generate the same equilibrium price/inventory *trajectory* as an OLNE.

This is, again, due to the structural nature of our RM differential game. By the characterization of OLNE, the set of shadow prices to determine an OLNE for a differential game $\mathrm{P}(t, \vec{x}(t))$ only depends on the game's initial time and inventory, namely, time t and the joint inventory level $\vec{x}(t)$. Hence, the prices for the current time t in the re-solving feedback strategy are uniquely determined by the current time t and the current joint inventory level $\vec{x}(t)$, and are independent of future inventory levels. Successively re-solving the open-loop problem for the current time will keep updating prices solved from the current shadow prices that have fully captured the *time-invariant* capacity externality over the remaining horizon. In the extreme case when all firms have ample capacities and hence there is no capacity externality, both the re-solving feedback strategy and open-loop equilibrium reduce to a one-shot price competition with zero marginal costs at any time, independent of any inventory levels. Hence, the re-solving mapping $\vec{p}^f(t, \vec{x})$ is indeed an FNE by definition (Starr and Ho 1969), and the existence of OLNE also guarantees the existence of FNE.

Starting from any given initial time and joint inventory level, there exists an OLNE by Proposition 1. Since the shadow prices along the equilibrium inventory trajectory in *this* OLNE are *constant* by Proposition 2, the re-solving FNE's prices determined by those shadow prices evolve along the same price trajectory and result in the same inventory trajectory as predicted by the *very* OLNE. The same type of behavior has been observed in the monopoly RM problem (Maglaras and Meissner 2006).

Proposition 6 (Feedback Equilibrium). *The re-solving strategy $\vec{p}^f(t, \vec{x})$ is an FNE of the differential game. For any initial condition $(t_0, \vec{x}(t_0))$, the equilibrium price and inventory trajectories under the re-solving FNE are the same as those under its corresponding OLNE with the very initial condition $(t_0, \vec{x}(t_0))$.*

We have fully characterized OLNE and identified an FNE in a feedback form that results in coincidental price and inventory trajectories as the OLNE. We caution that this coincidence holds only for the deterministic problem. For problems with random demand, the price and inventory trajectories under open-loop and re-solving feedback strategies, in general, are different. However, one can surmise that since a deterministic problem provides the first-order approximation to the corresponding stochastic problem, the feedback strategies obtained from the deterministic problem should serve as a reasonably good heuristic for the stochastic problem. We will provide more rigorous arguments for this claim in §4.

3.5. Applications

From the characterization, we know that the inter-temporal equilibrium prices are jointly determined by the current-time market condition (on the inter-temporal demand side) and time-independent shadow prices reflecting capacity externality (on the aggregate supply side). Next we illustrate with a couple of examples how these two-sided influences interact to determine the *inter-temporal* equilibrium pricing behavior. Each example comes with its own theme, set to illustrate a set of managerial insights under the framework. These insights cannot be gained by analyzing a one-shot capacitated competition model. Though the analysis is conducted for OLNE, the same type of inter-temporal behavior can also be sustained at an FNE, by Proposition 6.

3.5.1. Alternating Monopoly In a dynamic Bertrand-Edgeworth competition of a homogenous product (Talluri and Martínez de Albéniz 2011), along the trajectory of a non-cooperative subgame perfect equilibrium, firms may take turns to be a monopoly, avoiding a head-to-head fierce competition. Is this phenomenon unique to the homogenous-product price competition? For an RM game of differentiated products, can such an outcome be sustained in equilibrium? The answer is *yes*, but it depends on the inter-temporal demand structure.

First, we show that for any MNL demand structure, it is impossible to have an alternating monopoly in equilibrium. We prove this result by contradiction. Suppose an alternating monopoly is in equilibrium. In an MNL demand structure, for any finite price of a product, no matter how high it is, there always exists a positive demand rate. Due to this nature of MNL, we can show that it is beneficial for any firm to deviate by evening out a sufficiently small amount of inventory from its own monopoly period to a competitor's monopoly period. Hence, we can reach the following conclusion (see the Appendix for a rigorous proof).

Proposition 7. *In a differential game with an MNL demand structure and the strategy space for any firm being the full price space $\mathbb{R}_+$, an alternating monopoly cannot be sustained in equilibrium.*

Next, in the following example we show that for a linear inter-temporal demand structure, it is indeed possible to have an alternating monopoly. This is because due to the nature of a linear structure, demand can be zero when price is sufficiently high. Hence, if a firm is exerted a sufficiently high positive externality by the competitor's capacity, then it is possible for the null price to be optimal even before stocking out.

Example 2 (Avoid Head-to-Head). Consider a duopoly with an inter-temporal linear demand rate function: $d_i(t,p_i,p_{-i}) = 1 - p_i + \gamma_H p_{-i}$, $d_{-i}(t,p_i,p_{-i}) = 1 - p_{-i} + \gamma_L p_i$ for $t \in [(i-1)T/2, iT/2)$, $i = 1,2$, $0 < \gamma_L < \gamma_H < 1$. The feasible price set $\mathcal{P}_i(t,p_{-i}) = \{p_i \geqslant 0 \mid d_i(t,p_i,p_{-i}) \geqslant 0\}$. In this demand structure, firm i, $i=1,2$, is more sensitive to the competitor's price reduction in the i^{th} period; in other words, in period i, firm i is less preferred and firm $-i$ is more preferred by consumers. Suppose both firms have limited capacity $C_1 = C_2 = 1$ relative to the sales horizon T that is assumed to be sufficiently large. We propose a joint policy that two firms alternately sell as a monopoly for one half of the sales horizon: $p_i^*(t) = p^* \equiv \frac{(1+\gamma_H)-2/T}{1-\gamma_H\gamma_L}$, $p_{-i}^*(t) = p_{-i}^{\infty}(t,p_i^*(t)) = 1+\gamma_L p^*$ for $t \in [(i-1)T/2, iT/2)$, $i=1,2$. The proposed joint policy is such that firm i, $i=1,2$ will be a monopoly to sell off its capacity $C_i = 1$ using the i^{th} half of the sales horizon $[0,T]$.

One intuitive way to verify the proposed joint policy as an OLNE is to examine if there is an incentive for any firm to unilaterally deviate from this policy (see the Appendix for such a verification). By Proposition 2, an alternative way of verifying OLNE is to check against the characterization. It is easy to see that the proposed joint price policy and the shadow price process $\mu_{ij}(t) = \frac{p^*-2/T}{1-\gamma_L}$ for all $t \in [0,T)$ and $i,j = 1,2$, indeed satisfy the characterization. ◻

From the angle of shadow prices, it is not only easy to verify OLNE, but also is intuitive to see how capacity externalities interact with the inter-temporal demand structure to determine equilibrium behavior. In Example 2, the externality exerted by the competitor's scarce capacity is significant to force the firm to shut down demand in the half horizon when its product is less-preferred by consumers.

3.5.2. Effective Sales Horizon We illustrate by an example that in equilibrium (all) firms may not fully utilize the nominal whole sales horizon $[0,T]$ due to competitors' limited capacity, even under a stationary demand structure. This poses a stark contrast to the monopoly case (Gallego and van Ryzin 1994) where the full sales horizon is always used in the optimal solution under a stationary demand structure.

Example 3 (Head-to-Head). Consider a duopoly with a stationary and symmetric MNL demand rate function: $d_i(t,p_i,p_{-i}) = \frac{e^{-p_i}}{a_0+e^{-p_i}+e^{-p_{-i}}}$, $t \in [0,T], a_0 > 0, i = 1,2$. Other than the null price ∞, firms choose price from the set $\mathcal{P}_i(t,p_{-i}) = [0, L-1]$ where L is sufficiently large. Suppose both firms have limited capacity $C_1 = C_2 = 1$ relative to the sales horizon T that is assumed to be sufficiently large. We will show that for *any* $\tau \in [a_0 \exp(p^*) + 2, T]$, where p^* is the price equilibrium without capacity constraints characterized by the equation $a_0(1-p) + 2\exp(-p) = 0$, the following joint open-loop policy

$$p_i(t) = p_{-i}(t) = \begin{cases} \ln\left(\frac{\tau-2}{a_0}\right) & t \in [0,\tau), \\ \infty & t \in [\tau, T], \end{cases}$$

is an OLNE. With the proposed joint policy, both firms price at $\ln\left(\frac{\tau-2}{a_0}\right) \geqslant p^*$ until the sellout at time $t = \tau \leqslant T$; both firms earn a total revenue $\ln\left(\frac{\tau-2}{a_0}\right)$ that is increasing in τ. Let us verify that the proposed policy is indeed an equilibrium by checking whether there is any incentive to deviate. First, given the competitor's strategy fixed, it is not beneficial for any firm to shorten its effective sales horizon within period $[0,\tau)$. Second, it may seem that a firm may improve its profit by evening out a small amount of capacity from period $[0,\tau)$ and selling it as a monopoly in period $[\tau,T]$. Such a deviation would be profitable if the amount is made sufficiently small and it is jointly feasible. However, it is clear that the firm's deviation of evening out some capacity by price increase in period $[0,\tau)$ will make its competitor sell more than its capacity. Thus such a deviation is not jointly feasible and will not sustain in equilibrium, though it is unilaterally feasible. Hence the proposed joint open-loop policy is an OLNE, where both firms do not fully utilize the whole sales horizon.

Again, we can verify OLNE by checking against the characterization. It is easy to see that the proposed joint price policy and the shadow price process for all i, $\mu_{ii}(t) = L$, $\mu_{i,-i}(t) = L[1 + a_0 \exp(p^*)]$

for $t \in [0,\tau)$ and $\mu_{ii}(t) = L$, $\mu_{i,-i}(t) = 0$ for $t \in [\tau, T]$, indeed satisfy the OLNE characterization. There are infinite number of such equilibria differing in the length of effective sales horizon. Among all such equilibria, the one utilizing the whole sales horizon Pareto-dominates all other, and is the unique bounded rational OLNE. □

The discrepancy from the monopoly case is due to that in the oligopoly, as firms pre-commit to OLNE, they also take into account competitors' capacity constraint to make the pre-commitment credible. Example 3 clearly illustrates that the externalities exerted by the competitor's scarce capacity may limit a firm's feasible option and force each other to engage in fierce head-to-head competition.

4. Extension to the Stochastic Game

We extend the differential game to account for demand uncertainty by considering its stochastic-game counterpart in continuous time. We will show that the solution suggested by the differential game captures the essence and provides a good approximation for the stochastic game. The stochastic game formulation can be viewed as a game version of the optimal dynamic pricing problem considered in Gallego and van Ryzin (1994). Firms compete in influencing stochastic demand intensity by adjusting prices. More specifically, demand for a product is assumed to be a non-homogeneous Poisson process with Markovian intensities, instead of deterministic rates. At time $t \in [0,T]$, firm i applies its own non-anticipating price policy $p_i(t)$. Let $N_i^{\vec{u}}(t)$ denote the number of items sold up to time t for firm i under joint pricing policy $\vec{u}$. A demand for any firm i is realized at time t if $dN_i^{\vec{u}}(t) = 1$. We denote the joint Markovian allowable pricing policy space by $\mathscr{P}$, where any joint allowable pricing policy $\vec{u} = \{\vec{p}(t, \vec{n}(t)), 0 \leqslant t \leqslant T\}$ satisfies $\vec{p}(t, \vec{n}(t)) \in \mathcal{P}(t)$ for all t and $\int_0^T dN_i^{\vec{u}}(t) \leqslant C_i$ almost sure (a.s.) for all i. By the Markovian property of $\mathscr{P}$, we mean that the price policy offered by any firm is a function of the elapsed time and current joint inventory level; that is, $\vec{p}(t, \vec{n}(t)) = \vec{p}\left(t, C_1 - N_1^{\vec{u}}(t), C_2 - N_2^{\vec{u}}(t), \ldots, C_m - N_m^{\vec{u}}(t)\right), 0 \leqslant t \leqslant T$. We want to analyze strategies with Markovian properties, and again assume the strong information structure.

Given pricing policy $\vec{u} \in \mathscr{P}$, we denote the expected profit for any firm i by $G_i[\vec{u}] \equiv E[\int_0^T p_i(t, \vec{n}(t))\, dN_i^{\vec{u}}(t)]$. The goal of any firm i is to maximize its total expected profit over the sales horizon. A joint pricing policy $\vec{u}^* \in \mathscr{P}$ constitutes a Nash equilibrium if, whenever any firm modifies its policy away from the equilibrium, its own payoff will not increase. More precisely, $\vec{u}^*$ is called a Markovian equilibrium strategy if $G_i\left[u_i, \vec{u}^*_{-i}\right] \leqslant G_i\left[\vec{u}^*\right]$ for $[u_i, \vec{u}^*_{-i}] \in \mathscr{P}$ and all i. By applying Brémaud (1980, Theorem VII.T1) to the context of the RM stochastic game, we show that the following set of Hamilton-Jacobi-Bellman (HJB) equations is a sufficient condition for the Markovian equilibrium strategy.

Proposition 8 (Stochastic RM Game). *If functions $V_i(s, \vec{n}) : [0,T] \times \{\mathbb{Z}^m \cap \mathcal{X}\} \mapsto \mathbb{R}_+$ for all i are differentiable with respect to time $s \equiv T - t$ and satisfy the following set of HJB equations simultaneously*

$$-\frac{\partial V_i(s, \vec{n})}{\partial t} = \sup_{p_i \in \mathcal{P}_i(t, \vec{p}_{-i}) \cup \{p_i^{\infty}(t, \vec{p}_{-i})\}} \left\{ r_i(t, \vec{p}) - \nabla \vec{V}_i(s, \vec{n})^{\mathrm{T}} \vec{d}(t, \vec{p}) \right\}, \quad n_i > 0, \tag{4}$$

where $\nabla\vec{V}_i(s,\vec{n}) \equiv (\Delta V_{i,1}(s,\vec{n}), \Delta V_{i,2}(s,\vec{n}), \ldots, \Delta V_{i,m}(s,\vec{n}))$ *and* $\Delta V_{i,j}(s,\vec{n}) \equiv V_i(s,\vec{n}) - V_i(s,\vec{n}-\vec{e}_j)$, *with boundary conditions for all* i, *(i)* $V_i(0,\vec{n}) = 0$ *for all* $\vec{n}$ *and (ii)* $V_i(s,\vec{n}) = 0$ *if* $n_i \leqslant 0$ *for all* s, *and* $p_i^*(t,\vec{n}) : [0,T] \times \{\mathbb{Z}^m \cap \mathcal{X}\} \mapsto \mathbb{R}_+$ *achieves the supremum in the HJB equation (4) for any firm* i *at all* $(t,\vec{n})$, *then* $\vec{u}^* = \{\vec{p}^*(t,\vec{n})\} \in \mathscr{P}$ *is a Markovian equilibrium strategy.*

For a discrete-time version of the stochastic game under a stationary MNL demand structure, Lin and Sibdari (2009) demonstrate the existence of a Markovian equilibrium strategy by backward-inductively solving the set of HJB equations. However, other than the existence result, no further structural results are known. For the continuous-time stochastic game, we focus on exploring the natural link between the stochastic game and the differential game that has a simple and intuitive structural characterization.

4.1. Affine Functional Approximations

In the stochastic game, we adopt an *affine* functional approximation to the value functions: $V_i(s,\vec{n}) \approx W_i(s,\vec{n}) \equiv \int_t^T \theta_i(v)\,dv + \vec{w}_i(t)^{\mathrm{T}}\vec{n}$ for all i, t and $\vec{n}$, where $s \equiv T-t$, and $\vec{w}_i(t) \geqslant 0$ is a piece-wise continuously differentiable function. If we restrict $\vec{w}_i(t) = \vec{w}_i$ for all $t \in [0,T)$ and $\vec{w}_i(T) = 0$, the approximation is called a *quasi-static affine* functional approximation (Adelman 2007). The term $\theta_i(t)$ approximates the marginal value of time-to-go and $\vec{w}_i(t)$ approximates the marginal value of capacity at time t. We show that the first-order approximated capacity marginal value process $\vec{w}_i(t)$ is exactly equal to the shadow price process $\vec{\mu}_i(t)$ in the differential game. By Proposition 2, in any OLNE, the shadow price process is constant before the stockout, hence we do not lose generality by restricting the approximation to a quasi-static affine approximation.

Proposition 9 (Affine Approximation to Stochastic Game). *A joint strategy satisfies the conditions obtained from an affine or quasi-static affine functional approximation to the value functions in the set of HJB equations (4) with boundary conditions omitted if and only if it is an OLNE in the differential game.*

We approximate the value functions in the set of HJB equations (4), which is a sufficient condition for Markovian equilibrium in the stochastic game, with the boundary conditions omitted. We show this approximation yields a set of conditions that admits and only admits OLNE as a solution. The treatment of omitting the boundary conditions in the approximation is essentially due to that the affine approximation to the value functions calls for open-loop strategies which satisfy boundary conditions only along the optimal state trajectories, while the boundary conditions of the HJB equations are in a feedback form in nature.

Proposition 9 suggests that the shadow prices obtained from the differential game provide a good approximation to the marginal values of capacities in the stochastic game. Under the affine functional approximation, the stochastic game reduces to the differential game with nonnegative constraints imposed on the inventory levels at any point of time. This is equivalent to imposing nonnegative constraints on the inventory levels at the end of the sales horizon, which is the differential game obtained under the quasi-static affine functional approximation. In a discrete-time monopolistic network RM setting with *exogenous* prices of resources, Adelman (2007) demonstrates the following three problems yield *different* solutions: the deterministic problem of replacing the random demand by its mean, the resulting problems of approximating the value functions of the stochastic problem by affine and by quasi-static affine functions. We show that under the dynamic pricing setting in continuous time, the three problems are *equivalent*, even with time-varying demand structure and under competition.

4.2. Heuristics as Asymptotic Equilibria

The differential game can be analyzed to suggest tractable and efficiently-computable heuristics to its stochastic counterpart. Next we propose heuristics suggested by OLNE and FNE of the differential game, and show that they are equilibria in an asymptotic sense for the stochastic game, in the limiting regime where the potential demand and capacity are proportionally scaled up. Specifically, using k as an index, we consider a sequence of problems with demand rate function $\vec{d}^k(t,\vec{p}) = k\vec{d}(t,\vec{p})$ and capacity $\vec{C}^k = k\vec{C}$, and let k increase to infinity; hereafter, a superscript k will denote quantities that scale with k.

Definition 5 (Asymptotic NE). In the stochastic game, $\vec{u}^* \in \mathscr{P}$ is called an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games if for any $\epsilon > 0$, there exists l such that for all $k > l$, $\frac{1}{k} G_i^k[u_i, \vec{u}^*_{-i}] \leqslant \frac{1}{k} G_i^k[\vec{u}^*] + \epsilon$ for all $(u_i, \vec{u}^*_{-i}) \in \mathscr{P}$.

The quantity ϵ here refers to a small amount *relative* to the profit under asymptotic equilibrium, more than which a firm's profit cannot be improved by a unilateral deviation.

Next we assume each firm implements any heuristic suggested by the differential game in the stochastic game up to the stopping time when it runs out of stock. After stockout, firms simply post a null price to avoid taking orders that cannot be fulfilled.

Pre-commitment. Even under the strong information structure, firms may pre-commit to open-loop policies. This decision of pre-commitment can arise when firms trade off between price pre-commitment and pricing flexibility. As contingent pricing in response to demand uncertainty may intensify competition, price pre-commitment can result in higher revenues for firms than price flexibility under competition (see Xu and Hopp 2006 for a discussion on this in the context of a homogeneous product). We show that the pre-commitment to an OLNE is asymptotic optimal under competition.

Proposition 10 (OLNE as Asymptotic NE). *Any OLNE of the differential game is an asymptotic NE among pre-committed jointly allowable open-loop strategies in the limiting regime of the sequence of scaled stochastic games.*

Contingent Pricing. Under the strong information structure, the inventory level of any firm in real time is public information. The re-solving feedback strategy $\vec{p}^f(t,\vec{x})$ in the differential game provides a heuristic in feedback form for the stochastic game. Due to the differentiability of the demand structure in prices for any time (Assumption 1(a)), by Proposition 2 and implicit function theorem, we can show that the re-solving FNE $\vec{p}^f(t,\vec{x})$ is piecewise continuous in the current inventory level $\vec{x}$ for all t. By extending Maglaras and Meissner (2006) to the game context, we show that this heuristic is an asymptotic NE in the limiting regime as demand and supply grow proportionally large.

PROPOSITION 11 (FNE AS ASYMPTOTIC NE). *The re-solving FNE heuristic $\vec{p}^f(t,\vec{x})$ is an asymptotic NE in the limiting regime of the sequence of scaled stochastic games.*

5. Conclusion

Current RM practice of legacy airlines is carried out with a pricing team designing fares and another operations team allocating capacity to fare classes. This flaw is exacerbated by low cost carriers offering fares with few or no restrictions and by Internet-enabled price transparency. RM researchers and practitioners are trying to integrate pricing and capacity allocation into a single system that takes into account pricing and quality attributes of the products available to customers at the time of purchase. The challenge, of course, is the complexity of solving such systems.

We have shown that such inter-temporal pricing problems under competition, formulated as a differential game, has a simple structure in nature. The structure sheds light on how transient market conditions and aggregate supply constraints interact to determine inter-temporal equilibrium pricing behavior. It is encouraging that the existence, and uniqueness (in the sense of normalized or bounded rational equilibrium), of the equilibrium can be established for most of the commonly-used demand rate functions, including MNL and linear demand functions. Moreover, by the structural characterization of equilibrium, the infinite-dimensional time-varying equilibrium pricing policy can be determined by the finite set of shadow prices measuring capacity externalities. Due to this structure, the equilibrium computation can be significantly facilitated, and be cast as a finite-dimensional nonlinear complementarity problem. Lastly, we show that the equilibrium solutions from the differential game can provide pre-committed or contingent heuristic policies, capturing the first-order effect, for its stochastic counterpart. No doubt that the re-solving feedback heuristic that is dynamically easy-to-implement and provably asymptotic optimal, should be of practical interest to airline managers.

Appendix. Proofs.

Proof of Proposition 1. We will apply the infinite-dimensional fixed point theorem (Bohnenblust and Karlin 1950, Theorem 5; see Electronic Companion B). As a sketch, to apply the fixed point theorem, the regularity assumptions on feasible price sets, demand differentiability in prices, and pseudo-convexity of one's demand in its own price ensure the coupled joint strategy space of the game with relaxed constraints is convex, closed and compact. The demand differentiability and bounded revenue assumptions ensure that the total profit $J_i[\vec{p}]$ is a continuous functional in the joint policy $\vec{p}(t)$, hence the best-response correspondence as a set-valued map is non-empty and has a closed graph. The pseudo-concavity of one's revenue rate in its own price ensures the best-response correspondence is convex. To apply the theorem, we consider the following set-valued function $\mathcal{B}[\vec{p}] = \times_i \mathcal{B}_i[\vec{p}_{-i}]$, for $\vec{p} \in S = \times_i S_i$, where $\mathcal{B}_i[\vec{p}_{-i}] = \arg\max_{p_i \in S_i} J_i[p_i, \vec{p}_{-i}]$, for $\vec{p}_{-i} \in \times_{j \neq i} S_j$ and $S_i \equiv \left\{p_i(t), 0 \leqslant t \leqslant T \mid p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)), \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \leqslant C_i, \forall\, t, \vec{p}_{-i} \in (D[0,T])^{m-1}\right\}$. For $p_i^{\infty}(t, p_{-i}) \in \mathcal{P}_i(t, \vec{p}_{-i})$, $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t))$ is equivalent to $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^{\infty}(t, \vec{p}_{-i}(t))\}$. For MNL demand structures, in a best response, it is not beneficial for a firm to use a null price at any time if only constrained with its own capacity: suppose in the best response there is a period of time where a firm use a null price, then the firm can even out a small amount of capacity from other time to sell in this period; the capacity constraint for the firm is not violated; as long as the feasible set $\mathcal{P}_i(t, \vec{p}_{-i}(t)), \forall\, t$ is sufficiently large to accommodate the prices to sell the sufficiently small amount so that the total profit is improved, we reach a contradiction. Hence, we can use the unilateral feasible set as defined in S_i where $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t))$ also for MNL. For MNL demand structures, the OLNE, of which we show the existence in this proof as a solution to a fixed-point problem, does not use the null price (i.e., ∞) at any time. But there can be other OLNE that indeed uses a null price for positive measurable set of time (see Example 3).

Step 1. We show that S is convex. It suffices to show that S_i is convex for all i. Since $d_i(t, p_i, \vec{p}_{-i})$ is continuously differentiable (Assumption 1(a)) and pseudo-convex in p_i (Assumption 1(b)), then $\int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv$ is pseudo-convex, hence quasi-convex, in $\{p_i(v), 0 \leqslant v \leqslant t\}$. Then its lower level set $\left\{p_i(v), 0 \leqslant v \leqslant t \mid \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \leqslant C_i\right\}$ is convex. Hence, since $\mathcal{P}_i(t, \vec{p}_{-i}(t))$ is convex for all t (Assumption 3(b)), S_i is convex by the fact that the intersection of any collection of convex sets is convex.

Step 2. We show that S is weakly closed. It suffices to show that S_i is (strongly) closed for all i. Since $\mathcal{P}_i(t, \vec{p}_{-i})$ is compact in $\mathbb{R}$ for all t (Assumption 3(b)), hence $\mathcal{P}_i(t, \vec{p}_{-i})$ is closed for all t and $\vec{p}_{-i}$ by the fact that in an Euclidean space every compact set is closed. Then $\{p_i(t), 0 \leqslant t \leqslant T \mid p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)), \forall\, t\}$ is closed for all $\{\vec{p}_{-i}(t), 0 \leqslant t \leqslant T\}$ by the fact that the product of closed sets is closed. Since $d_i(v, \vec{p})$ is continuous in $\vec{p}$, by Cesari (1983, Theorem 10.8.i), $\int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv$ is a lower semi-continuous functional in $\{p_i(v), 0 \leqslant v \leqslant t\}$. By an equivalent definition of lower semi-continuity (Royden 1988, Problem 2.50(c)), the integral functional's lower level set $\{p_i(v), 0 \leqslant v \leqslant t \mid \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \leqslant C_i\}$ is closed. Therefore, S_i is closed by the fact that the intersection of any collection of closed sets is closed.

Step 3. We show that S is compact. It suffices to show that S_i is compact for all i. Since $\mathcal{P}_i(t,\vec{p}_{-i}(t))$ is a compact set for any fixed $\vec{p}_{-i} \in \times_{j\neq i} S_j$ (Assumption 3(b)), the set $\{p_i(t), 0 \leqslant t \leqslant T \mid p_i(t) \in \mathcal{P}_i(t,\vec{p}_{-i}(t))\}$ is compact by Tychonoff's Theorem. Since S_i is closed (Step 2) and is a subset of the compact set $\{p_i(t), 0 \leqslant t \leqslant T \mid p_i(t) \in \mathcal{P}_i(t,\vec{p}_{-i}(t))\}$, then S_i is compact by the fact that a closed subset of a compact set is compact.

Step 4. We show that for any $\vec{p} \in S$, $\mathcal{B}[\vec{p}]$ is non-empty. It suffices to show that for any $\vec{p}_{-i} \in \times_{j\neq i} S_j$, $\mathcal{B}_i[\vec{p}_{-i}]$ is non-empty. Under Assumptions 1(a) and 2(b), $J_i[p_i,\vec{p}_{-i}]$ is a weakly continuous functional in p_i on S_i for any fixed $\vec{p}_{-i} \in \times_{j\neq i} S_j$ by Cesari (1983, Theorem 10.8.v). Hence, the continuous functional $J_i[p_i,\vec{p}_{-i}]$ that is bounded above (by Assumption 2(b)) can attain its maximum on the compact set S_i by an infinite-dimensional version of the extreme value theorem (Luenberger 1968, Theorem 2.13.1). Therefore, $\mathcal{B}_i[\vec{p}_{-i}]$ is non-empty.

Step 5. We show that for any $\vec{p} \in S$, $\mathcal{B}[\vec{p}]$ is convex. It suffices to show that for any $\vec{p}_{-i} \in \times_{j\neq i} S_j$, $\mathcal{B}_i[\vec{p}_{-i}] = \arg\max_{p_i \in S_i} J_i[p_i,\vec{p}_{-i}]$ is convex. Since $\mathcal{B}_i[\vec{p}_{-i}]$ is non-empty, let p_i^* denote an element of the set. The pseudo-concavity of the integral functional $J_i[p_i,\vec{p}_{-i}] = \int_0^T r_i(t,p_i(t),\vec{p}_{-i}(t))\,dt$ in $\{p_i(t), 0 \leqslant t \leqslant T\}$ is assured by Assumption 2(a) that $r_i(t,\vec{p}(t))$ is pseudo-concave in $p_i(t)$ at each instant of time t. Hence, $J_i[p_i,\vec{p}_{-i}]$ is quasi-concave in p_i. Then by one of equivalent definitions of quasi-concavity, $\{p_i \mid J_i[p_i,\vec{p}_{-i}] \geqslant J_i[p_i^*,\vec{p}_{-i}]\}$ is convex for any $\vec{p}_{-i} \in \times_{j\neq i} S_j$. Hence, $\mathcal{B}_i[\vec{p}_{-i}] = \{p_i \mid J_i[p_i,\vec{p}_{-i}] \geqslant J_i[p_i^*,\vec{p}_{-i}]\} \cap S_i$ is convex since S_i is convex (Step 1).

Step 6. We show that the graph $\mathcal{B}$ is weakly closed. Let $\{(\vec{x}^n,\vec{y}^n)\}_{n=1}^{\infty}$ be a sequence in $S \times S$ that converges weakly to $(\vec{x},\vec{y}) \in S \times S$ such that $\vec{x}^n \in \mathcal{B}[\vec{y}^n]$, i.e., $J_i[p_i,\vec{y}^n_{-i}] \leqslant J_i[x_i^n,\vec{y}^n_{-i}]$ for all $p_i \in S_i$ and all i. Under Assumptions 1(a) and 2(b), $J_i[\vec{p}]$ is weakly continuous in $\vec{p}$ by Cesari (1983, Theorem 10.8.v), hence $J_i[p_i,\vec{y}_{-i}] = \lim_{n\to\infty} J_i[p_i,\vec{y}^n_{-i}] \leqslant \lim_{n\to\infty} J_i[x_i^n,\vec{y}^n_{-i}] = J_i[x_i,\vec{y}_{-i}]$ for all $p_i \in S_i$ and all i. Then $\vec{x} \in \mathcal{B}[\vec{y}]$.

Step 7. Notice that $\bigcup_{\vec{p}\in S} \mathcal{B}(\vec{p})$ is a subset of S, which is compact by Step 3. This ensures that $\bigcup_{\vec{p}\in S} \mathcal{B}(\vec{p})$ is contained in a sequentially weakly compact set.

Step 8. Combining all of the above steps, we are ready to apply Theorem 1. Thus, $\mathcal{B}[\vec{p}]$ has a fixed point on S, namely, there exists an OLNE to the following differential game with relaxed constraints: given competitors' open-loop price policies $\{\vec{p}_{-i}(t), 0 \leqslant t \leqslant T\}$, each player i is to simultaneously $\max_{p_i \in D[0,T]} \int_0^T r_i(t,\vec{p}(t))\,dt$ such that $p_i(t) \in \mathcal{P}_i(t,\vec{p}_{-i}(t))$ for all t, $d_i(t,\vec{p}(t)) \geqslant 0$ for all t, $C_i - \int_0^t d_i(v,\vec{p}(v))\,dv \geqslant 0$ for all t. In contrast with the original differential game, the firms in the game with relaxed constraints have bounded rationality and ignore the nonnegative demand and capacity constraints of competitors in their best responses. From the perspective of dual variables, the game with relaxed constraints is to set $\mu_{ij}(t) = 0$ for all t and $j \neq i$ and all i.

Step 9. We argue that any OLNE of the game with relaxed constraints is one of the original game. Suppose $\{\vec{p}^*(t), 0 \leqslant t \leqslant T\}$ is an OLNE of the game with relaxed constraints, namely, given $\{\vec{p}^*_{-i}(t), 0 \leqslant t \leqslant T\}$, $\{p^*_i(t), 0 \leqslant t \leqslant T\}$ for all i maximizes $\int_0^T r_i(t, p_i(t), \vec{p}^*_{-i}(t))\, dt$ subject to $p_i(t) \in \mathcal{P}_i(t, \vec{p}^*_{-i}(t))$, $d_i(t, p_i(t), \vec{p}^*_{-i}(t)) \geqslant 0$ and $\int_0^t d_i(v, p_i(v), \vec{p}^*_{-i}(v))\, dv \leqslant C_i$ for all t. Thus we must have the OLNE satisfies the joint constraints, i.e., $p^*_j(t) \in \mathcal{P}_j(t, \vec{p}^*_{-j}(t))$, $d_j(t, \vec{p}^*(t)) \geqslant 0$ and $\int_0^t d_j(v, \vec{p}^*(v))\, dv \leqslant C_j$ for all t and all j. Therefore given $\{\vec{p}^*_{-i}(t), 0 \leqslant t \leqslant T\}$, $\{p^*_i(t), 0 \leqslant t \leqslant T\}$ for all i also maximizes $\int_0^T r_i(t, p_i(t), \vec{p}^*_{-i}(t))\, dt$ subject to $(p_i(t), \vec{p}^*_{-i}(t)) \in \mathcal{P}(t)$, $d_j(t, p_i(t), \vec{p}^*_{-i}(t)) \geqslant 0$ and $\int_0^t d_j(v, p_i(v), \vec{p}^*_{-i}(v))\, dv \leqslant C_j$ for all t and all j, namely, $\{\vec{p}^*(t), 0 \leqslant t \leqslant T\}$ is an OLNE of the original game. ▫

Proof of Proposition 2. Introducing piecewise continuously differentiable costate variable $\vec{\mu}_i(t) = (\mu_{ij}(t), \forall j)$ for all i and t, we define the Hamiltonians $H_i : [0,T] \times \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ by $H_i(t, \vec{x}, \vec{p}(t), \vec{\mu}_i(t)) \equiv r_i(t, \vec{p}(t)) - \sum_j \mu_{ij}(t) d_j(t, \vec{p}(t))$ for all i and t. Additionally, we have the state constraint $\vec{x}(t) \geqslant 0$. Hence we define the Lagrangians $L_i : [0,T] \times \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ by $L_i(t, \vec{x}, \vec{p}(t), \vec{\mu}_i(t), \vec{\eta}_i(t)) \equiv r_i(t, \vec{p}(t)) - \sum_j \mu_{ij}(t) d_j(t, \vec{p}(t)) + \sum_j \eta_{ij}(t) x_j(t)$ for all i and t, where Lagrangian multipliers $\eta_{ij}(t)$ for all i, j are piecewise continuous. Any OLNE $\{\vec{p}(t), 0 \leqslant t \leqslant T\}$, its corresponding costate trajectory $\{\vec{\mu}_i(t), 0 \leqslant t \leqslant T\}$ for all i, its corresponding Lagrange multiplier trajectory $\{\vec{\eta}_i(t), 0 \leqslant t \leqslant T\}$ for all i and its equilibrium state trajectory $\{\vec{x}(t), 0 \leqslant t \leqslant T\}$ need to satisfy the following set of *necessary* conditions (namely, the Pontryagin Maximum Principle; see Pontryagin et al. 1962, Sethi and Thompson 2005, Section 4.2): for all t and all i, j,

$$p_i(t) = \arg\max_{p_i \in \mathcal{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}} H_i(t), \tag{5}$$

$$-\frac{\partial \mu_{ij}(t)}{\partial t} = \frac{\partial L_i}{\partial x_j} = \eta_{ij}(t), \tag{6}$$

$$\mu_{ij}(T) x_j(T) = 0, \quad \mu_{ij}(T), x_j(T) \geqslant 0, \tag{7}$$

$$\eta_{ij}(t) x_j(t) = 0, \quad \eta_{ij}(t), x_j(t) \geqslant 0, \tag{8}$$

together with the jump conditions that $\mu_{ij}(t)$ for all i, j may jump down at the junction time when $x_j(t)$ hits zero, and the kinematic equation $\frac{\partial x_i(t)}{\partial t} = -d_i(t, \vec{p}(t))$, $x_i(0) = C_i$ for all i that is obvious from the context. Under pseudo-concavity of the revenue rate function and pseudo-convexity of the demand rate function, the Hamiltonian H_i is pseudo-concave in p_i for all t and i. By Arana-Jiménez et al. (2008), the maximum principle is also a *sufficient* condition for an OLNE.

Now we equivalently simplify the set of conditions (5)-(8) together with the jump conditions, by eliminating $\eta_{ij}(t)$. First, for all $t < \sup\{v \in [0,T] \mid x_j(v) > 0\}$, we have $x_j(t) > 0$, thus $\eta_{ij}(t) = 0$ for all i by the complementary slackness condition (8). Consider ODE (6): the piecewise continuously differentiable costate trajectory $\mu_{ij}(t)$ with derivative equal to zero everywhere (not almost everywhere) must be constant, which we can denote as μ_{ij}, before the inventory of firm j hits zero (if it happens). For t such that $x_i(t) > 0$ and $x_j(t) = 0$ for some j, in the Hamiltonian maximization (5) of firm i we can still set $\mu_{ij}(t) = \mu_{ij}$ even though $x_j(t)$ has hit zero, since at such a time firm j is simultaneously forced to post a null price and hence the term $\mu_{ij}(t) d_j(t, \vec{p}(t))$ is zero regardless of choices of $\mu_{ij}(t)$. Second, in Eq.(5), it is understood that for $t \in [\bar{t}_i, T]$ (if the set E_i is non-empty), an appropriate costate variable process, denoted by $\mu_{\bar{ij}}(t)$ to distinguish from μ_{ij}, can be chosen such that a null price is the optimal solution to the Hamiltonian H_i and hence the state $x_i(t)$ stays at zero. By Eq.(6), such a piecewise continuously differentiable process $\mu_{\bar{ij}}(t)$ is a decreasing process, which may have a jump-down discontinuity at the junction time $\bar{t}_i$. Lastly, consider the transversality condition (7). If $x_j(T) > 0$, then $\mu_{ij}(T) = \mu_{ij}$ for all i, hence condition (7) is equivalent to $\mu_{ij} x_j(T) = 0$. If $x_j(T) = 0$, condition (7) and $\mu_{ij} x_j(T) = 0$ always hold regardless of choices of shadow prices. Therefore, we can reach the equivalent set of simplified necessary and sufficient conditions as described in the proposition. ▫

Proof of Proposition 3. First, consider the competition of substitutable products. For an arbitrary firm, given its competitors' price paths fixed, we consider its best-response problem. In this firm's constrained optimization problem, let us tighten its capacity constraint, which is the only one, since we focus on the bounded rational OLNE. This tightened capacity constraint increases the constant shadow price corresponding to this constraint in the best-response problem and leads to a pointwise higher best-response price path. Because the best-response correspondences of all other firms remain unchanged and are increasing in competitors' price paths by (log-)supermodularity, the resulting equilibrium prices are higher (see Topkis 1998). Second, consider the competition of two complementary products. The result can be obtained just by playing the trick of reversing the order of the strategy set of one firm (see Vives 1999, Remark 2.20). ▫

Proof of Proposition 4. Let $\Gamma[\vec{p}, \vec{q}] \equiv \int_0^T \sum_i r_i(t, p_1(t), \ldots, p_{i-1}(t), q_i(t), p_{i+1}(t), \ldots, p_m(t))\, dt$. It is a commonly-used technique of applying the appropriate fixed point theorem to the set-valued mapping $\arg\max_{\vec{q}} \left\{ \Gamma[\vec{p}, \vec{q}] \mid C_i - \int_0^T d_i(t, \vec{q}(t))\, dt \geqslant 0, \forall\, i \right\}$ to show the existence of Nash equilibrium of the original problem. Following the same procedure as the proof of Proposition 1 and noticing that $d_i(t, \vec{q})$ is convex in q_j for all i, j, t, we can verify the existence of a fixed point $\vec{p}^*$ such that $\Gamma[\vec{p}^*, \vec{p}^*] = \max_{\vec{q}} \left\{ \Gamma[\vec{p}^*, \vec{q}] \mid C_i - \int_0^T d_i(t, \vec{q}(t))\, dt \geqslant 0, \forall\, i \right\}$. Such a fixed point $\vec{p}^*$ is a Nash equilibrium with shadow prices satisfying $\mu_{ij} = \xi_j$ for all i, j, where $\vec{\xi}$ is the Lagrangian multipliers in the maximization problem. Following the same procedure as the proof of Rosen (1965, Theorem 4) and noticing that condition (SDD) is a sufficient condition for pointwise strict diagonal concavity and hence integrally, the desired result can be obtained. ▫

Proof of Proposition 5. Let $\Pi_i(t, \vec{p}) \equiv r_i(t, \vec{p}) - \mu_{ii} d_i(t, \vec{p})$. The first order derivative is $\partial \Pi_i / \partial p_i = d_i(t, \vec{p}) + (p_i - \mu_{ii}) \partial d_i(t, \vec{p}) / \partial p_i$. The second order derivatives are $\partial^2 \Pi_i / \partial {p_i}^2 = 2 \partial d_i(t, \vec{p}) / \partial p_i + (p_i - \mu_{ii}) \partial^2 d_i(t, \vec{p}) / \partial {p_i}^2$ and $\partial^2 \Pi_i / \partial p_i \partial p_j = \partial d_i(t, \vec{p}) / \partial p_j + (p_i - \mu_{ii}) \partial^2 d_i(t, \vec{p}) / \partial p_i \partial p_j, \forall\, j \neq i$. Hence, $\partial^2 \Pi_i / \partial {p_i}^2 + \sum_{j \neq i} \partial^2 \Pi_i / \partial p_i \partial p_j = \partial d_i(t, \vec{p}) / \partial p_i + \sum_j \partial d_i(t, \vec{p}) / \partial p_j + (p_i - \mu_{ii}) \sum_j \partial^2 d_i(t, \vec{p}) / \partial p_i \partial p_j$. Under the stipulations of the proposition, the Hessian of $\vec{\Pi}(t, \vec{p})$ and its leading principal submatrices are negative definite at $\vec{p} = \vec{p}^*$ satisfying $\partial \Pi_i(t, \vec{p}) / \partial p_i = 0$ for all i such that $x_i(t) > 0$, since (i)

$\partial d_i(t,\vec{p})/\partial p_i < 0$, (ii) $p_i - \mu_{ii}\,|_{\vec{p}=\vec{p}^*} = -d_i(t,\vec{p})/(\partial d_i(t,\vec{p})/\partial p_i)\,|_{\vec{p}=\vec{p}^*} \geqslant 0$ for all i such that $x_i(t) > 0$, and (iii) that the negative semi-definity is preserved under additivity. By the Poincaré-Hopf index theorem, for any set of shadow prices $\{\mu_{ii}, \forall\, i\}$ with $\mu_{ij} = 0$ for all $i \neq j$ at any time t, there exists a unique price vector satisfying $\partial \Pi_i(t,\vec{p})/\partial p_i = 0$ for any set of firms $\{i \mid x_i(t) > 0\}$ together with the rest of firms posting the null prices. Moreover, by the proof of Proposition 1, there exists a bounded rational OLNE, hence for its corresponding diagonal shadow prices, the bounded rational OLNE is unique. □

Proof of Proposition 7. We prove by contradiction. Suppose at the joint open-loop policies $\vec{p}^*$, there exist two intervals $(b, b+\delta)$ and $(\tilde{b}, \tilde{b}+\delta)$ such that for some firm i, $p_i^*(t)$ is infinite on $(b, b+\delta)$ and $p_i^*(t)$ is finite on $(\tilde{b}, \tilde{b}+\delta)$, for any firm $j \neq i$, $p_j^*(t)$ is infinite on $(\tilde{b}, \tilde{b}+\delta)$, namely, firm i is the monopoly on $(\tilde{b}, \tilde{b}+\delta)$ and some firm other than firm i is the monopoly on $(b, b+\delta)$. Let $\bar{p}_i(t,\epsilon) \equiv \inf\{p_i \geqslant 0 \mid d_i(t, p_i, \vec{p}^*_{-i}(t)) = \epsilon/\delta\}$, which is finite for MNL demand rate functions. Let $\tilde{p}_i(t,\epsilon) \equiv \inf\{p_i \geqslant 0 \mid d_i(t, p_i, \vec{p}^*_{-i}(t)) = d_i(t, \vec{p}^*(t)) - \epsilon\}$. Since $\bar{p}_i(t,\epsilon)$ is decreasing in ϵ and $\lim_{\epsilon\to 0} \bar{p}_i(t,\epsilon) = \infty$, there exists sufficiently small $\tilde{\epsilon}(t) > 0$ for $t \in (\tilde{b}, \tilde{b}+\delta)$ such that $\bar{p}_i(t + b - \tilde{b}, \tilde{\epsilon}(t)) > p_i^*(t)$. We construct a price policy for firm i,

$$p_i(t) = \begin{cases} \bar{p}_i(t, \tilde{\epsilon}(t + \tilde{b} - b)) & \text{if } t \in (b, b+\delta), \\ \tilde{p}_i(t, \tilde{\epsilon}(t)) & \text{if } t \in (\tilde{b}, \tilde{b}+\delta), \\ p_i^*(t) & \text{otherwise.} \end{cases}$$

Under the above policy, firm i evens out a small quantity $\tilde{\epsilon}(t)$ at any time $t \in (\tilde{b}, \tilde{b}+\delta)$ to the corresponding time $t + b - \tilde{b}$ in $(b, b+\delta)$.

First, we check if the constructed policy is jointly feasible. It is obviously feasible for firm i as its total sales is unchanged. To see the feasibility for other firms, we check the derivative $(\partial d_j(t, a_i, \vec{a}_{-i})/\partial a_i) \cdot (\partial a_i^{-1}(t, d_i, \vec{a}_{-i})/\partial d_i)$, where $a_i = \beta_i(t)\exp(-\alpha_i(t) p_i)$ is the attraction value of firm i, $a_i^{-1}(t, d_i, \vec{a}_{-i}) = [\sum_{k\neq i} a_k(t)] d_i/[\lambda(t) - d_i]$ is the inverse function of $d_i(t, a_i, \vec{a}_{-i}) = \lambda(t) a_i(t)/[a_i(t) + \sum_{k\neq i} a_k(t)]$. This derivative captures the impact, on firm j's sales, of firm i's maintaining a small change in its sales by changing its price p_i while the competitor's price $\vec{p}_{-i}$ is fixed. It is easy to verify that

$$\frac{\partial d_j(t, a_i, \vec{a}_{-i})}{\partial a_i} \frac{\partial a_i^{-1}(t, d_i, \vec{a}_{-i})}{\partial d_i} = -\frac{a_j(t)}{\sum_{k\neq i} a_k(t)}.$$

The deviation of firm i will cause any firm j's sales to stay the same for $t \in (\tilde{b}, \tilde{b}+\delta)$ since $a_j(t) = 0$ for $t \in (\tilde{b}, \tilde{b}+\delta)$, and decrease by $a_j(t)\epsilon(t)/[\sum_{k\neq i} a_k(t)]$ for $t \in (b, b+\delta)$. As long as $\epsilon(t)$ is sufficiently small, firm i's deviation does not violate competitors' capacity constraints and state positivity, thus the deviation is feasible in the generalized Nash game with coupled constraints.

Next, we compare the profit before and after the deviation. Under the original policy p_i^*, firm i earns $p_i^*(t) d_i(t, \vec{p}_i^*(t))$ for any time $t \in (\tilde{b}, \tilde{b}+\delta)$ and 0 for any time in $(b, b+\delta)$. Under the constructed policy, firm i earns $\tilde{p}_i(t,\tilde{\epsilon}(t))d_i(t,\tilde{p}_i(t,\tilde{\epsilon}(t)),\vec{p}^*_{-i}(t)) = \tilde{p}_i(t,\tilde{\epsilon}(t))(d_i(t,\vec{p}^*_i(t)) - \tilde{\epsilon}(t)) > p^*_i(t)(d_i(t,\vec{p}^*_i(t)) - \tilde{\epsilon}(t))$ for any time $t \in (\tilde{b}, \tilde{b}+\delta)$ and $\bar{p}_i(t+b-\tilde{b},\tilde{\epsilon}(t))\tilde{\epsilon}(t)$ in the corresponding time in $(b, b+\delta)$. The constructed policy has a positive improvement in profit over p^*_i. We see a contradiction. ▫

Example 2. Verification of OLNE. Since firms have limited capacity relative to the sales horizon, their revenues depend on how high prices can be set to sell the capacity. It is definitely worse off for any firm i to sell faster in its monopoly period by setting a price lower than the market-clearing price p^* that sells off capacity over the half horizon. This rules out the possibility that firms want to have a monopoly sales horizon shorter than $T/2$. What about setting a price higher than p^*? Suppose firm i deviates by evening out ϵ amount of inventory from its monopoly period and competing in selling the ϵ amount with the competitor in firm $-i$'s originally monopoly period. First, we check if such a deviation is jointly feasible. It is obviously feasible for firm i as its total sales is unchanged. To see the feasibility for firm $-i$, we check the derivative $(\partial d_{-i}(t,p_i,p_{-i})/\partial p_i)(\partial p_i^{-1}(t,d_i,p_{-i})/\partial d_i)$, where $p_i^{-1}(t,d_i,p_{-i})$ is the inverse function of $d_i(t,p_i,p_{-i})$. This derivative captures the impact, on firm $-i$'s sales, of firm i's small change in its sales by varying its price p_i while the competitor's price p_{-i} is fixed.

$$\frac{\partial d_{-i}(t,p_i,p_{-i})}{\partial p_i}\frac{\partial p_i^{-1}(t,d_i,p_{-i})}{\partial d_i} = \begin{cases} -\gamma_L & \text{if } t \in [(i-1)T/2, iT/2)\,, \\ -\gamma_H & \text{otherwise.} \end{cases}$$

The deviation will cause the sales of firm $-i$ to increase by $\gamma_L\epsilon$ amount in firm i's monopoly period and to decrease by $\gamma_H\epsilon$ amount in firm $-i$'s originally monopoly period. The total sales of firm $-i$ will decrease by $(\gamma_H - \gamma_L)\epsilon$ amount under firm i's deviation, which remains feasible for firm $-i$ for all $\epsilon \in [0,1)$. Next, we fix firm $-i$'s policy at $\{p^*_{-i}(t), 0 \leqslant t \leqslant T\}$ to see firm i's payoff under the deviation of evening out the ϵ amount. The highest price $\bar{p}$ firm i can sell the ϵ amount is $\bar{p}$ such that $(1-\bar{p}+\gamma_L p^*)T/2 = \epsilon$. We solve $\bar{p} = 1 + \gamma_L p^* - 2\epsilon/T$. The highest price firm i can sell the $1-\epsilon$ amount in its monopoly period is $\tilde{p}$ such that $[1-\tilde{p}+\gamma_H(1+\gamma_L p^*)]T/2 = 1-\epsilon$. We solve $\tilde{p} = p^* + 2\epsilon/T$. The profit firm i can earn under the deviation is

$$\tilde{p}(1-\epsilon) + \bar{p}\epsilon = p^* + \epsilon\left[\frac{\gamma_L - \gamma_H + \gamma_H\gamma_L}{1-\gamma_H\gamma_L} + \frac{2(2-\gamma_L-\gamma_H\gamma_L)}{T(1-\gamma_H\gamma_L)} - \frac{4\epsilon}{T}\right] < p^*$$

for all $\epsilon \in [0,1)$ provided that $\gamma_L - \gamma_H + \gamma_H\gamma_L < 0$ and $T > \frac{2(2-\gamma_L-\gamma_H\gamma_L)}{-(\gamma_L-\gamma_H+\gamma_H\gamma_L)}$. Hence if γ_L is sufficiently small relative to γ_H and T is sufficiently large, the proposed joint policy is indeed an OLNE where firms are alternating monopolies. ▫

Proof of Proposition 8. First, to generalize the HJB as a sufficient condition of the optimal control strategy for a single firm to the game context, we require the Markovian equilibrium strategy satisfies the set of HJB equations simultaneously. Second, to apply Brémaud (1980, Theorem VII.T1), the only condition that needs to be verified is the boundedness of the value functions which are guaranteed by Assumption 2(b). Lastly, the boundary conditions that $V_i(s,\vec{n})=0$ if $n_i=0$ for all s enforce that upon a stockout a null price is the only option. Hence, the joint strategy satisfying the set of HJB equations must be in $\mathscr{P}$. ▫

Proof of Proposition 9. Under the affine functional approximation with boundary conditions omitted, the set of HJB equations (4) becomes:

$$\theta_i(t)-\left(\frac{\partial w_{i1}(t)}{\partial t},\ldots,\frac{\partial w_{im}(t)}{\partial t}\right)^{\mathrm{T}}\vec{n}=\sup_{p_i\in\mathcal{P}_i(t,\vec{p}_{-i})\cup\{p_i^{\infty}(t,\vec{p}_{-i})\}}\left\{r_i(t,\vec{p})-\vec{w}_i(t)^{\mathrm{T}}\vec{d}(t,\vec{p})\right\},\ \vec{n}\in\mathbb{Z}^m\cap\mathcal{X}, \tag{9}$$

for all i and t. Taking difference between Eq.(9) evaluated at $(t,\vec{n})$ and at $(t,\vec{n}-\vec{e}_j)$ for all j, we obtain $\frac{\partial w_{ij}(t)}{\partial t}=0$ for all i, j and t. Since $w_{ij}(t)$ is piecewise continuously differentiable, $w_{ij}(t)$ must be a constant, which we denote by w_{ij}. Hence, we do not lose generality by restricting the functional approximation to a quasi-static affine functional approximation with boundary conditions omitted.

It has been shown that the HJB equation for a discrete-time monopolistic RM problem can be equivalently stated as an optimization problem (Adelman 2007). In Electronic Companion D, we show that it is also true for continuous-time problems. Specifically, we show that if $\vec{V}^*(s,\vec{n})$ solves the set of HJB equations (4) for the continuous-time stochastic game, and a differentiable function $\vec{V}(s,\vec{n})$ is a feasible solution to a game where any firm i simultaneously solves the following functional optimization problem given competitors' strategy $\vec{p}_{-i}(t,\vec{n})$:

$$\begin{aligned}&\min_{\{V_i(\cdot,\cdot)\}} V_i(T,\vec{C})\\ &\text{s.t.} -\frac{\partial V_i(s,\vec{n})}{\partial t}\geqslant\left\{r_i(t,\vec{p}(t,\vec{n}))-\nabla\vec{V}_i(s,\vec{n})^{\mathrm{T}}\vec{d}(t,\vec{p}(t,\vec{n}))\right\},\ \forall\,\vec{p}(t,\vec{n})\in\mathcal{P}(t),\forall\,(t,\vec{n}).\end{aligned}$$

Hence the equilibrium value function $\vec{V}(T,\vec{C})$ at the initial time $t=0$ and state $\vec{n}=\vec{C}$ can be obtained by solving the functional optimization game. Under the affine functional approximation, we can approximate the functional minimization problem for any firm i as follows:

$$\begin{aligned}(\mathrm{D}_i)\quad &\min_{\vec{w}_i\geqslant 0}\quad \int_0^T\theta_i(t)\,dt+\vec{w}_i^{\mathrm{T}}\vec{C}\\ &\text{s.t.}\quad \theta_i(t)\geqslant r_i(t,\vec{p}(t))-\vec{w}_i^{\mathrm{T}}\vec{d}(t,\vec{p}(t)),\quad\forall\,\vec{p}(t)\in\mathcal{P}(t),\quad\forall\,t.\end{aligned}$$

Since (D_i) is a minimization problem, it is optimal to set

$$\theta_i(t)=\max_{p_i(t)\in\mathcal{P}_i(t,\vec{p}_{-i}(t))\cup\{p_i^{\infty}(t,\vec{p}_{-i}(t))\}}\left\{r_i(t,\vec{p}(t))-\vec{w}_i^{\mathrm{T}}\vec{d}(t,\vec{p}(t))\right\},\quad\forall\,t$$

in the objective function. Then the objective of any firm i becomes

$$\min_{\vec{w}_i\geqslant 0}\left[\max_{p_i(t)\in\mathcal{P}_i(t,\vec{p}_{-i}(t))\cup\{p_i^{\infty}(t,\vec{p}_{-i}(t))\},\forall\,t}\left\{\int_0^T r_i(t,\vec{p}(t))\,dt+\vec{w}_i^{\mathrm{T}}\left(\vec{C}-\int_0^T\vec{d}(t,\vec{p}(t))\,dt\right)\right\}\right].$$

This is equivalent to the maximization problem $\max_{p_i(t)\in\mathcal{P}_i(t,\vec{p}_{-i}(t))\cup\{p_i^{\infty}(t,\vec{p}_{-i}(t))\},\forall\,t}\int_0^T r_i(t,\vec{p}(t))\,dt$ with capacity constraints $\vec{C}-\int_0^T\vec{d}(t,\vec{p}(t))\,dt\geqslant 0$ dualized by the vector $\vec{w}_i\geqslant 0$. Strong duality holds here since this continuous-time maximization primal problem has pseudo-concave objective function and quasi-convex constraints (the LHS's of "$\leqslant$" constraints are quasi-convex), and both primal and dual are feasible (Zalmai 1985). For each firm to simultaneously solve the maximization problem with open-loop strategies subject to joint capacity constraints is exactly to solve the differential game with an initial condition $(t, \vec{C})$ for OLNE. ▫

Proof of Proposition 10. Suppose $\vec{p} = \{\vec{p}(t), 0 \leqslant t \leqslant T\} \in (D[0,T])^m \cap \mathscr{P}_O$ is any arbitrary joint open-loop policy subject to coupled constraints (2). Under policy $\vec{p}$, we denote by τ_i^k the minimum time of T and the random stopping time when the total sales process of firm i reach its original capacity in the k^{th} system. In the deterministic differential game, we denote by $\bar{t}_i$ the minimum time of T and the time when the total sales of firm i reach its original capacity in the unscaled system, which is also such a time in the scaled regimes without demand uncertainty. As dictated by $\vec{p}$, any firm i implements the open-loop policy $p_i(t)$ up to the time either $\bar{t}_i$ or τ_i^k whichever comes earlier, and posts only a null price afterwards. Without loss of generality, we index firms such that their deterministic stock-out times are ordered as $0 \leqslant \bar{t}_1 \leqslant \bar{t}_2 \leqslant \cdots \leqslant \bar{t}_n \leqslant T$. Let $N_i(\cdot)$ for all i denote independent unit rate Poisson processes. The functional strong law of large numbers for the Poisson process and composition convergence theorem assert that as $k \to \infty$, for any $\vec{p}(t)$,

$$\frac{N_i(kd_i(t, \vec{p}(t)))}{k} \to d_i(t, \vec{p}(t)) \quad \text{a.s., uniformly in } t \in [0,T]. \tag{10}$$

This suggests that in the stochastic system, the random stopping time τ_i^k should be close to its deterministic counterparts $\bar{t}_i$, at least the relative order, as k goes to infinity. For the time being, we suppose these stopping times are ordered almost sure as $\tau_1^k \leqslant \tau_2^k \leqslant \cdots \leqslant \tau_n^k$, as k becomes sufficiently large for the simplicity of exposure, which we will confirm shortly. Up to time τ_1^k, all firms implement $\vec{p}(t)$. Arguing by contradition and applying (10) to firm 1, one can easily conclude that $\tau_1^k \to \bar{t}_1$ a.s., as $k \to \infty$. The revenues of firm 1 extracted under the open-loop policy is $R_1^k[\vec{p}] \equiv \int_0^{\min\{\tau_1^k, \bar{t}_1\}} p_1(t)\, d\left(N_1(kd_1(t, \vec{p}(t)))\right)$ as k is sufficiently large, and $\frac{1}{k} R_1^k[\vec{p}] \to \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$ as $k \to \infty$. Recall the fact established in Gallego and van Ryzin (1994) that the solution of the deterministic pricing problem serves as an upper bound for the revenues extracted in the stochastic system and by Assumption 2(b), we have $G_1^k[\vec{p}] = E(R_1^k[\vec{p}]) \leqslant k \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$. By the bounded convergence theorem, $\frac{1}{k} G_1^k[\vec{p}] \to \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$.

The sales of firm 2 is $D_2^k[\vec{p}] \equiv$

$$\begin{cases} \int_0^{\tau_1^k} dN_2(kd_2(t, \vec{p}(t))) + \int_{\tau_1^k}^{\bar{t}_1} dN_2(kd_2(t, p_1^\infty(\vec{p}_{-1}(t)), \vec{p}_{-1}(t))) + \int_{\bar{t}_1}^{\min\{\tau_2^k, \bar{t}_2\}} dN_2(kd_2(t, \vec{p}(t))) & \text{if } \tau_1^k < \bar{t}_1, \\ \int_0^{\min\{\tau_2^k, \bar{t}_2\}} dN_2(kd_2(t, \vec{p}(t))) & \text{otherwise,} \end{cases}$$

as k is sufficiently large. Since $\tau_1^k \to \bar{t}_1$ a.s. as $k \to \infty$, the term $\int_{\tau_1^k}^{\bar{t}_1} dN_2(kd_2(t, p_1^\infty(\vec{p}_{-1}(t)), \vec{p}_{-1}(t)))$ is asymptotically negligible. By applying (10) to firm 2 and arguing by contradiction, one can conclude that $\tau_2^k \to \bar{t}_2$ a.s. as $k \to \infty$. The revenues of firm 2 extracted under the open-loop policy is

$$R_2^k[\vec{p}] \equiv \begin{cases} \int_0^{\tau_1^k} p_2(t)\, dN_2(kd_2(t,\vec{p}(t))) + \int_{\tau_1^k}^{\bar{t}_1} p_2(t)\, dN_2(kd_2(t,p_1^\infty(\vec{p}_{-1}(t)),\vec{p}_{-1}(t))) & \\ \quad + \int_{\bar{t}_1}^{\min\{\tau_2^k,\bar{t}_2\}} p_2(t)\, dN_2(kd_2(t,\vec{p}(t))) & \text{if } \tau_1^k < \bar{t}_1, \\ \int_0^{\min\{\tau_2^k,\bar{t}_2\}} p_2(t)\, dN_2(kd_2(t,\vec{p}(t))) & \text{otherwise,} \end{cases}$$

as k is sufficiently large, and $\frac{1}{k}R_2^k[\vec{p}] \to \int_0^{\bar{t}_2} r_2(t,\vec{p}(t))\, dt$ as $k \to \infty$. Moreover, $E(R_2^k[\vec{p}]) \leqslant k\int_0^{\bar{t}_2} r_2(t,\vec{p}(t))\, dt + \delta$, where a random variable δ bounds the revenue over $[\tau_1^k, \bar{t}_1]$ from the spillover sales from firm 1 and δ/k is asymptotically negligible. By the bounded convergence theorem, $\frac{1}{k}G_2^k[\vec{p}] = \frac{1}{k}E(R_2^k[\vec{p}]) \to \int_0^{\bar{t}_2} r_2(t,\vec{p}(t))\, dt$. Repeating the same argument, we conclude that $\frac{1}{k}G_i^k[\vec{p}] \to \int_0^{\bar{t}_i} r_i(t,\vec{p}(t))\, dt$ for all i, as $k \to \infty$. Applying this convergence result to an OLNE $\vec{p}^*$ and any of its unilateral deviation $(p_i, \vec{p}^*_{-i})$, we have $\frac{1}{k}G_i^k[\vec{p}^*] \to \int_0^{\bar{t}_i} r_i(t,\vec{p}^*(t))\, dt$ and $\frac{1}{k}G_i^k[p_i,\vec{p}^*_{-i}] \to \int_0^{\bar{t}_i} r_i(t,p_i(t),\vec{p}^*_{-i}(t))\, dt$ for all i, as $k \to \infty$. In other words, for any $\epsilon > 0$, there exist l such that for all $k > l$, $\left|\frac{1}{k}G_i^k[\vec{p}^*] - \int_0^{\bar{t}_i} r_i(t,\vec{p}^*(t))\, dt\right| < \frac{\epsilon}{2}$ and $\left|\frac{1}{k}G_i^k[p_i,\vec{p}^*_{-i}] - \int_0^{\bar{t}_i} r_i(t,p_i(t),\vec{p}^*_{-i}(t))\, dt\right| < \frac{\epsilon}{2}$. Since $\int_0^{\bar{t}_i} r_i(t,p_i(t),\vec{p}^*_{-i}(t))\, dt \leqslant \int_0^{\bar{t}_i} r_i(t,\vec{p}_i^*(t))\, dt$, then $\frac{1}{k}G_i^k[p_i,\vec{p}^*_{-i}] \leqslant \frac{1}{k}G_i^k[\vec{p}^*] + \epsilon$. $\square$

Proof of Proposition 11. Suppose $\vec{p}^c = \{\vec{p}^c(t,\vec{x}), 0 \leqslant t \leqslant T\} \in (D[0,T])^m \cap \mathscr{P}_F$ is any arbitrary joint feedback policy in the differential game subject to coupled constraints (2) and is piecewise continuous in $\vec{x}$. The cumulative demand for firm i up to time t driven by such a policy in the k^{th} system is denoted by $N_i(A_i^{c,k}(t))$, where $A_i^{c,k}(t) = \int_0^t kd_i(v,\vec{p}^c(v,\frac{1}{k}\vec{X}^{c,k}(v)))\, dv$, and $X_i^{c,k}(t) = \max(0, kC_i - N_i(A_i^{c,k}(t)))$ for all i denotes the remaining inventory for firm i at time t. Note that $A_i^{c,k}(0) = 0$, $A_i^{c,k}(t)$ is nondecreasing and $A_i^{c,k}(t_2) - A_i^{c,k}(t_1) \leqslant k\int_{t_1}^{t_2} d_{i,\max}(v)\, dv$, where $d_{i,\max}(v) \equiv \max_{\vec{p}(v)\in\mathcal{P}(v)} d_i(v,\vec{p}(v))$. This implies that the family of process $\{\frac{1}{k}A_i^{c,k}(t)\}$ for all i is equicontinuous, and therefore relatively compact. By the Ascoli-Arzelá Theorem, we can obtain a converging subsequence $\{k_m\}$ of the sequence $\{\frac{1}{k}A_i^{c,k}(t)\}$ such that $\frac{1}{k_m}A_i^{c,k_m}(t) \to \bar{A}_i^c(t)$ for all i in the following way: for $i = 1$, there exists a converging subsequence $\{k_1\}$, such that $\frac{1}{k_1}A_1^{c,k_1}(t) \to \bar{A}_1^c(t)$; for $i = 2$, along sequence $\{k_1\}$, there exists a converging subsequence of $\{k_2\}$, such that $\frac{1}{k_2}A_1^{c,k_2}(t) \to \bar{A}_1^c(t)$ and $\frac{1}{k_2}A_2^{c,k_2}(t) \to \bar{A}_2^c(t)$; we can repeat the process until we have a subsequence $\{k_m\}$ satisfying the desired property. Let $N_i(\cdot)$ for all i denote independent unit rate Poisson processes. Recall that the functional strong law of large numbers for the Poisson process asserts that $\frac{1}{k}N_i(kt) \to t$, a.s. uniformly in $t \in [0,T]$ as $k \to \infty$. By composition convergence theorem, along the subsequence $\{k_m\}$ we get that $\frac{1}{k_m}N_i(A_i^{c,k_m}(t)) \to \bar{A}_i^c(t)$ for all i, and therefore that $\bar{X}_i^{c,k_m}(t) \equiv \frac{1}{k_m}X_i^{c,k_m}(t)$ converges to a limit $\bar{x}_i^c(t)$ for all i; the two converging results hold a.s. uniformly in $t \in [0,T]$. Using the continuity of $\vec{d}(t,\vec{p})$ in $\vec{p}$ and the piecewise continuity of $\vec{p}^f(t,\vec{x})$ in $\vec{x}$, by Dai and Williams (1995, Lemma 2.4), we get that as $k_m \to \infty$, for all i, $\frac{1}{k_m}A_i^{c,k_m}(t) = \int_0^t d_i(v,\vec{p}^c(v,\frac{1}{k_m}\vec{X}^{c,k_m}(v)))\, dv \to \int_0^t d_i(v,\vec{p}^c(v,\vec{\bar{x}}^c(v)))\, dv$, a.s. uniformly in $t \in [0,T]$. Thus we get that as $k_m \to \infty$, for all i,

$$\bar{X}_i^{c,k_m}(t) = C_i - \frac{1}{k_m}N_i(A_i^{c,k_m}(t)) \to C_i - \int_0^t d_i(v,\vec{p}^c(v,\vec{\bar{x}}^c(v)))\, dv = \bar{x}_i^c(t), \tag{11}$$

a.s. uniformly in $t \in [0,T]$. This shows that the limiting state trajectories do not depend on the selection of the converging subsequence itself. Hence in the sequel we denote the converging sequence by k to simplify notation. The last equality in (11) shows that $\{\vec{\bar{x}}^c(t), 0 \leqslant t \leqslant T\}$ is the state trajectory generated by the feedback policy $\vec{p}^c$ in the differential game. By the piecewise continuity of $\vec{p}^c(t,\vec{x})$ in $\vec{x}$, we have as $k \to \infty$, $\vec{p}^c(t, \vec{\bar{X}}^{c,k}(t)) \to \vec{p}^c(t, \vec{\bar{x}}^c(t))$, a.s. uniformly in $t \in [0,T]$. Again by Dai and Williams (1995, Lemma 2.4), the revenue extracted under the feedback strategy $\vec{p}^c$ after normalization is, for all i, as $k \to \infty$, $\frac{1}{k}\int_0^T p_i^c(t, \vec{\bar{X}}^{c,k}(t)) dN_i(A_i^{c,k}(t)) \to \int_0^T r_i(t, \vec{p}^c(t, \vec{\bar{x}}^c(t)))\, dt$, a.s. By Assumption 2(b) and the bounded convergence theorem, we have

$$\frac{1}{k} G_i^k[\{\vec{p}^c(t, \vec{\bar{X}}^{c,k}(t)), 0 \leqslant t \leqslant T\}] = \frac{1}{k} E\left(\int_0^T p_i^c(t, \vec{\bar{X}}^{c,k}(t)) dN_i(A_i^{c,k}(t))\right) \to \int_0^T r_i(t, \vec{p}^c(t, \vec{\bar{x}}^c(t)))\, dt. \tag{12}$$

We apply the convergence result (12) to the FNE $\vec{p}^f$: for any $\epsilon > 0$, there exists l_1 such that for all $k > l_1$, $\left|\frac{1}{k} G_i^k[\{\vec{p}^f(t, \vec{\bar{X}}^{f,k}(t)), 0 \leqslant t \leqslant T\}] - \int_0^T r_i(t, \vec{p}^f(t, \vec{\bar{x}}^f(t)))\, dt\right| < \frac{\epsilon}{2}$. We apply the convergence result (12) to any unilateral deviation $\vec{p}^c = (p_i^c, \vec{p}_{-i}^f) \in (D[0,T])^m \cap \mathscr{P}_F$ as a feedback policy in the differential game subject to coupled constraints (2): for the same $\epsilon > 0$, there exists l_2 such that for all $k > l_2$, $\left|\frac{1}{k} G_i^k[\{\vec{p}^c(t, \vec{\bar{X}}^{c,k}(t)), 0 \leqslant t \leqslant T\}] - \int_0^T r_i(t, \vec{p}^c(t, \vec{\bar{x}}^c(t)))\, dt\right| < \frac{\epsilon}{2}$. Since $\vec{p}^c$ is a unilateral deviation from the FNE $\vec{p}^f$ in the differential game, we have $\int_0^T r_i(t, \vec{p}^c(t, \vec{\bar{x}}^c(t)))\, dt \leqslant \int_0^T r_i(t, \vec{p}^f(t, \vec{\bar{x}}^f(t)))\, dt$. Then for all $k > \max(l_1, l_2)$, $\frac{1}{k} G_i^k[\{\vec{p}^c(t, \vec{\bar{X}}^{c,k}(t)), 0 \leqslant t \leqslant T\}] \leqslant \frac{1}{k} G_i^k[\{\vec{p}^f(t, \vec{\bar{X}}^{f,k}(t)), 0 \leqslant t \leqslant T\}] + \epsilon$. ◻

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Electronic Companion
Dynamic Pricing of Perishable Assets under Competition

A. Demand Structures

We consider several of the most frequently-used classes of demand functions and verify that they indeed satisfy pseudo-convexity of the demand rate function and pseudo-concavity of the revenue rate function.

General Time-Varying Attraction Models

In the attraction models, customers choose each firm with probability proportional to its attraction value. Specifically, we have the following demand rate functions: for all i,

$$d_i(t,\vec{p}) = \lambda(t)\frac{a_i(t,p_i)}{\sum_{j=0}^{m} a_j(t,p_j)},$$

where $\lambda(t) > 0$, $a_i(t,p_i) \geqslant 0$ is the attraction value for firm i at time t, and

$$a_0(t) \equiv a_0(t,p_0) > 0$$

is interpreted as the value of the no-purchase option at time t. We emphasize that in order to have pseudo-convexity of the demand rate function holds with respect to one's own price (Proposition 1(i)), we need the no-purchase value to be positive. Since $\lambda(t)$ is always positive, it does not have impact on the signs of derivatives we will consider, hence we drop it in the following discussion.

Lemma 1 (Sufficient Condition of Pseudo-Convexity). *If a twice continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f'(x)=0 \Rightarrow f''(x)>0$, then f is pseudo-convex, i.e., for any x_1 and x_2, $(x_1-x_2)f'(x_2)\geqslant 0 \Rightarrow f(x_1)\geqslant f(x_2)$.*

Proof of Lemma 1. For each x_0 with $f'(x_0)=0$, we have $f''(x_0)>0$. This means that whenever the function f' reaches the value 0, it is strictly increasing. Therefore it can reach the value 0 at most once. If f' does not reach the value 0 at all, then f is either strictly decreasing or strictly increasing, and therefore pseudo-convex: if f is strictly decreasing, then $(x_1-x_2)f'(x_2)\geqslant 0 \Rightarrow x_1 \leqslant x_2 \Rightarrow f(x_1)\geqslant f(x_2)$; if f is strictly increasing, then $(x_1-x_2)f'(x_2)\geqslant 0 \Rightarrow x_1\geqslant x_2 \Rightarrow f(x_1)\geqslant f(x_2)$. Otherwise f' must reach the value 0 exactly once, say at x_0. Since $f''(x_0)>0$, it follows that $f'(x)<0$ for $x<x_0$, and $f'(x)>0$ for $x>x_0$. Therefore f is pseudo-convex: if $x_2=x_0$, we always have $f(x_1)\geqslant f(x_2)=f(x_0)$ for any x_1; if $x_2<x_0$, then $(x_1-x_2)f'(x_2)\geqslant 0\Rightarrow x_1\leqslant x_2\Rightarrow f(x_1)\geqslant f(x_2)$; and if $x_2>x_0$, then $(x_1-x_2)f'(x_2)\geqslant 0\Rightarrow x_1\geqslant x_2\Rightarrow f(x_1)\geqslant f(x_2)$. □

We have the following structural results on the general attraction demand functions. We assume $a_i(t)$ is twice continuously differentiable. For notation simplicity, we drop arguments and let $a_0\equiv a_0(t)>0$, $a_i\equiv a_i(t,p_i), a_i'\equiv \partial a_i(t,p_i)/\partial p_i$ and $a_i''\equiv \partial^2 a_i(t,p_i)/\partial {p_i}^2$.

Proposition 1 (Pseudo Properties of Attraction Models). *The following pseudo-properties of general attraction models hold:*

(i) if $a_i'' > (resp. <) 0$ for all i, $d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i for all i;

(ii) if $a_i > 0$, $a_i'' < (resp. >) 0$ for all i, $d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_j for all $j \neq i$;

(iii) if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$ for all i, $r_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i for all i.

Proof of Proposition 1. (i) Taking the first order derivative of $d_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial d_i}{\partial p_i} = \frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $d_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial^2 d_i}{\partial {p_i}^2} = \frac{a_i'' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \frac{2(a_i')^2 \sum_{j\neq i} a_j}{(\sum_j a_j)^3} = \frac{a_i'' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \left(\frac{\partial d_i}{\partial p_i}\right) \frac{2a_i'}{\sum_j a_j}.$$

Since $a_0 > 0$, then $\sum_{j\neq i} a_j > 0$. Hence whenever $\partial d_i/\partial p_i = 0$, $\partial^2 d_i/\partial {p_i}^2 > (resp. <) 0$ if $a_i'' > (resp. <) 0$. By Lemma 1, $d_i(t,\vec{p})$ is pseudo-convex (pseudo-concave) in p_i if $a_i'' > (resp. <) 0$.

(ii) Taking the first order derivative of $d_i(t,\vec{p})$ with respect to p_j,

$$\frac{\partial d_i}{\partial p_j} = -\frac{a_i a_j'}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $d_i(t,\vec{p})$ with respect to p_j,

$$\frac{\partial^2 d_i}{\partial {p_j}^2} = -\frac{a_i a_j''}{(\sum_j a_j)^2} + \frac{2a_i(a_j')^2}{(\sum_j a_j)^3} = -\frac{a_i a_j''}{(\sum_j a_j)^2} - \left(\frac{\partial d_i}{\partial p_j}\right) \frac{2a_j'}{\sum_j a_j}.$$

Whenever $\partial d_i/\partial p_j = 0$, $\partial^2 d_i/\partial {p_j}^2 > (resp. <) 0$ if $a_i > 0$, $a_j'' < (resp. >) 0$. By Lemma 1, $d_i(t,\vec{p})$ is pseudo-convex (pseudo-concave) in p_j for all $j \neq i$ if $a_i > 0$, $a_i'' < (resp. >) 0$ for all i.

(iii) Taking the first order derivative of $r_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial r_i}{\partial p_i} = d_i + p_i \frac{\partial d_i}{\partial p_i} = \frac{a_i}{\sum_j a_j} + p_i \frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $r_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{\partial d_i}{\partial p_i} + p_i \frac{\partial^2 d_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} + p_i \frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} \left(\frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j}\right).$$

Whenever $\partial r_i/\partial p_i = 0$, $p_i a_i' \sum_{j\neq i} a_j/(\sum_j a_j)^2 = -a_i/\sum_j a_j$, thus

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \frac{a_i}{\sum_j a_j} \left(\frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j}\right) = \frac{2a_i' - a_i a_i''/a_i'}{\sum_j a_j} > (resp. <) 0,$$

if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$. By Lemma 1, $r_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$. □

Combining Proposition 1 parts (i) and (ii), we immediately have the following corollary.

Corollary 1. *There exists no general attraction models such that $d_i(t,\vec{p})$ is pseudo-convex in p_j for all j.*

Corollary 1 shows the need of adopting the notion of bounded rational equilibrium in showing the existence of OLNE for the differential game.

The MNL demand structure assumes $a_i(t,p_i) = \beta_i(t)\exp(-\alpha_i(t)p_i)$, $\alpha_i(t), \beta_i(t) > 0$ for all i. Since $a_i'' = \alpha_i(t)^2\beta_i(t)\exp(-\alpha_i(t)p_i) > 0$ and $2a_i' - a_i a_i''/a_i' = -\alpha_i(t)\beta_i(t)\exp(-\alpha_i(t)p_i) < 0$, we have the following corollary as an immediate result of Proposition 1.

Corollary 2 (MNL). *For the MNL demand structure, $d_i(t,\vec{p})$ for all i is pseudo-convex in p_i, is pseudo-concave in p_j for all $j \neq i$ and $r_i(t,\vec{p})$ for all i is pseudo-concave in p_i.*

Linear Models

The demand rate function has the form of

$$d_i(t,\vec{p}) = a_i(t) - b_i(t)p_i + \sum_{j\neq i} c_{ij}(t)p_j,$$

where $a_i(t), b_i(t) > 0$ for all i and $c_{ij}(t) \in \mathbb{R}$ for all $j \neq i$. Since $\partial d_i(t,\vec{p})/\partial p_i = -b_i(t) < 0$, then for any p_i^1, p_i^2, if $(p_i^2 - p_i^1)\partial d_i(t,\vec{p})/\partial p_i = -(p_i^2 - p_i^1)b_i(t) \geqslant 0$, then $p_i^2 \leqslant p_i^1$ and hence $d_i(t,p_i^2,p_{-i}) \geqslant d_i(t,p_i^1,p_{-i})$. We have verified that $d_i(t,\vec{p})$ for all i is pseudo-convex in p_i. Since $r_i(t,\vec{p})$ for all i is strictly concave in p_i, it is pseudo-concave in p_i.

Proposition 2 (Linear Model). *For the linear demand structure, $d_i(t,\vec{p})$ for all i is pseudo-convex in p_i and $r_i(t,\vec{p})$ for all i is pseudo-concave in p_i.*

Note that these pseudo-properties do not use the signs of cross-price elasticity term c_{ij}'s, hence a linear demand structure of complementary products also satisfies Assumptions 1(b) and 2(a).

B. The Fixed Point Theorem

Theorem 1 (Bohnenblust and Karlin (1950, Theorem 5)). *Let X be a weakly separable Banach space with S a convex, weakly closed set in X. Let $\mathcal{B} : S \to 2^S\backslash\{\varnothing\}$ be a set-valued mapping satisfying the following:*

(a) $\mathcal{B}(x)$ is convex for each $x \in S$;

(b) The graph of $\mathcal{B}, \{(x,y) \in S \times S : y \in \mathcal{B}(x)\}$, is weakly closed in $X \times X$. That is, if $\{x_n\}$ and $\{y_n\}$ are two sequences in S such that $x_n \to x$, $y_n \to y$, weakly in X with $x_n \in \mathcal{B}(y_n)$, then necessarily we have $x \in \mathcal{B}(y)$;

(c) $\bigcup_{x\in S} \mathcal{B}(x)$ is contained in a sequentially weakly compact set;

Then there exists $x^* \in S$ such that $x^* \in \mathcal{B}(x^*)$.

C. HJB Equivalence

We establish the equivalency between the HJB equation (4) and the optimization problem by showing that any feasible solution $\vec{V}(t,\vec{n})$ to the optimization problem is an upper bound of the value function $\vec{V}^*(t,\vec{n})$ satisfying the HJB equation (4). We prove it by induction on the value of $\vec{e}^{\mathrm{T}}\vec{n}$, where $\vec{e}$ denotes a vector with all entries being ones. As an initial step, for $\vec{n}=0$ such that $\vec{e}^{\mathrm{T}}\vec{n}=0$, by the boundary conditions, $\vec{V}(t,\vec{n})=\vec{V}^*(t,\vec{n})=0$ for all t. Now suppose for all $\vec{n}$ such that $\vec{e}^{\mathrm{T}}\vec{n}=l_o$, we have $\vec{V}(t,\vec{n})\geqslant\vec{V}^*(t,\vec{n})$ for all t. Let us consider any $\vec{n}_o$ such that $\vec{e}^{\mathrm{T}}\vec{n}_o=l_o+1$. We further show by induction on time. As an initial step, for $s=0$, again by the boundary conditions, we have $\vec{V}(0,\vec{n}_o)=\vec{V}^*(0,\vec{n}_o)=0$. Suppose for some $s_o\geqslant 0$, we have $\vec{V}(s,\vec{n}_o)\geqslant\vec{V}^*(s,\vec{n}_o)$ for all $s\in[0,s_o]$. For any i, there exists $h>0$ small enough such that

$$\begin{aligned}
& V_i(s_o+h,\vec{n}_o)\\
={}& V_i(s_o,\vec{n}_o)+\frac{\partial V_i(s_o,\vec{n}_o)}{\partial s}h+o_1(h)\\
\geqslant{}& V_i(s_o,\vec{n}_o)+\{r_i(\vec{p}(T-s_o,\vec{n}_o))-\vec{d}(\vec{p}(T-s_o,\vec{n}_o))^{\mathrm{T}}\nabla\vec{V}_i(s_o,\vec{n}_o)\}h+o_1(h)\\
\geqslant{}& V_i(s_o,\vec{n}_o)+\{r_i(\vec{p}^*(T-s_o,\vec{n}_o))-\vec{d}(\vec{p}^*(T-s_o,\vec{n}_o))^{\mathrm{T}}\nabla\vec{V}_i(s_o,\vec{n}_o)\}h+o_1(h)\\
={}& [1-\vec{e}^{\mathrm{T}}\vec{d}(\vec{p}^*(T-s_o,\vec{n}_o))h]V_i(s_o,\vec{n}_o)\\
&\quad+\vec{d}(\vec{p}^*(T-s_o,\vec{n}_o))^{\mathrm{T}}(V_i(s_o,\vec{n}_o-\vec{e}_1),V_i(s_o,\vec{n}_o-\vec{e}_2),\ldots,V_i(s_o,\vec{n}_o-\vec{e}_m))h+o_1(h)\\
\geqslant{}& [1-\vec{e}^{\mathrm{T}}\vec{d}(\vec{p}^*(T-s_o,\vec{n}_o))h]V_i^*(s_o,\vec{n}_o)\\
&\quad+\vec{d}(\vec{p}^*(T-s_o,\vec{n}_o))^{\mathrm{T}}(V_i^*(s_o,\vec{n}_o-\vec{e}_1),V_i^*(s_o,\vec{n}_o-\vec{e}_2),\ldots,V_i^*(s_o,\vec{n}_o-\vec{e}_m))h+o_1(h)\\
={}& V_i^*(s_o+h,\vec{n}_o)+o_2(h),
\end{aligned}$$

where the first inequality is due to the feasibility of $\vec{V}(s,\vec{n})$ to the optimization problem, the second inequality is due to the inequality constraints in the optimization problem hold for all pricing strategies, and the third inequality is due to the induction hypothesis. Therefore, there exists a neighborhood $[s_o,s_o+h_o]$ with $h_o>0$ such that $V_i(s,\vec{n}_o)\geqslant V_i^*(s,\vec{n}_o)$ for all $s\in[s_o,s_o+h_o]$. □

D. Computation of OLNE

Friesz (2010, Chapter 10) formulates the equilibrium problem as an infinite-dimensional differential quasi-variational inequality and computes the generalized differential Nash equilibrium by a gap function algorithm. Adida and Perakis (2010) discretize the time horizon and solve for the finite-dimensional generalized Nash equilibrium by a relaxation algorithm. Instead, we explore the structural property of our differential game and cast the computation of OLNE as a much smaller size of finite-dimensional nonlinear complementarity problem (NCP).

By Proposition 2, the OLNE is equivalently characterized by the following m^2-dimensional NCP:

$$\mu_{ij}\left(C_j-\int_0^T d_j(t,\vec{p}^*(t;[\mu_{ij}]_{m\times m}))\,dt\right)=0,\ \text{for all } i,j,$$

$$C_j - \int_0^T d_j(t, \vec{p}^*(t; [\mu_{ij}]_{m\times m}))\, dt \geqslant 0, \text{ for all } i, j, \quad \mu_{ij} \geqslant 0, \text{ for all } i, j,$$

with appropriate ancillary decreasing shadow price processes $\mu_{ij}^-(t) \in [0, \mu_{ij}]$ for all i, j that can shut down demand upon a stockout, where $\vec{p}^*(t; [\mu_{ij}]_{m\times m})$ is the solution of (3) for any given matrix of shadow prices $[\mu_{ij}]_{m\times m} \geqslant 0$ at any time t that may have closed-form solutions in some cases, e.g., under linear demand structures. The process of computing the equilibrium candidate $\{\vec{p}^*(t; [\mu_{ij}]_{m\times m}), 0 \leqslant t \leqslant T\}$ involves solving a one-shot price competition at any time on an on-going basis from $t = 0$ while keeping checking whether firms have run out of inventory; whenever a firm's inventory process hits zero, we can check if there exists decreasing shadow price processes of shutting down demand: if so, the firm exits the market and the price competition afterwards only involves remaining firms of positive inventory with an updated demand function taking consideration of spillover; otherwise, the matrix of shadow prices does not sustain as equilibrium shadow prices. If a bounded rational OLNE is sought after, we can restrict $\mu_{ij}(t) = 0$ for all t and all $i \neq j$ and further reduce the NCP to an m-dimension problem. Upon a stockout, the checking of whether there exist appropriate decreasing shadow price processes to shut down demand is also much simplified for computation of bounded rational OLNE. For many commonly used demand structures, e.g., MNL and linear, there exists a unique equilibrium candidate $\{\vec{p}^*(t; [\mu_{ij}]_{m\times m}), 0 \leqslant t \leqslant T\}$ for any set of nonnegative shadow prices $[\mu_{ii}]_{m\times 1}$ with $\mu_{ij} = 0$ for all $i \neq j$. Mature computation algorithms for NCP with (i) a sub-loop of computing the equilibrium candidate and (ii) upon a stockout a sub-loop of checking whether null prices can be generated by decreasing shadow price processes, can be applied to identify OLNE that indeed satisfies the complementarity condition.