Dynamic Programming Approach to Principal-Agent Problems

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Abstract

We consider a general formulation of the Principal-Agent problem from Contract Theory, on a finite horizon. We show how to reduce the problem to a stochastic control problem which may be analyzed by the standard tools of control theory. In particular, Agent's value function appears naturally as a controlled state variable for the Principal's problem. Our argument relies on the Backward Stochastic Differential Equations approach to non-Markovian stochastic control, and more specifically, on the most recent extensions to the second order case.

Key words. Stochastic control of non-Markov systems, Hamilton-Jacobi-Bellman equations, second order Backward SDEs, Principal-Agent problem, Contract Theory.

1 Introduction

Optimal contracting between two parties – Principal ("she") and Agent ("he"), when Agent's effort cannot be contracted upon, is a classical problem in Microeconomics, so-called Principal-Agent problem with moral hazard. It has applications in many areas of economics and finance, for example in corporate governance and portfolio management (see Bolton and Dewatripont (2005) for a book treatment). In this paper we develop a general approach to solving such problems in Brownian motion models, in the case in which Agent is paid only at the terminal time.

The first paper on continuous-time Principal-Agent problems is the seminal paper by Holmström and Milgrom (1987). They consider Principal and Agent with exponential utility functions and find that the optimal contract is linear. Their work was generalized by Schättler

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and Sung (1993, 1997), Sung (1995, 1997), Müller (1998, 2000), and Hellwig and Schmidt (2002). The papers by Williams (2009) and Cvitanić, Wan and Zhang (2009) use the stochastic maximum principle and Forward-Backward Stochastic Differential Equations (FBSDEs) to characterize the optimal compensation for more general utility functions, under moral hazard, also called the "hidden actions" case. Cvitanić and Zhang (2007) and Carlier, Ekeland and Touzi (2007) study the adverse selection case of "hidden type", in which Principal does not observe Agent's "intrinsic type". A more recent seminal paper in moral hazard setting is Sannikov (2008), who finds a tractable model for solving the problem with a random time of retiring the agent and with continuous payments to the agent, rather than a lump-sum payment at the terminal time. We leave for future research a study of Sannikov's model using our approach.

The main approach taken in the literature is to characterize Agent's value function and his optimal actions given an arbitrary contract payoff, and then to analyze the maximization problem of the principal over all possible payoffs. [1] This approach does not always work, because it may be hard to solve Agent's stochastic control problem given an arbitrary payoff, possibly non-Markovian, and it may also be hard for Principal to maximize over all such contracts. Furthermore, Agent's optimal control depends on the given contract in a highly nonlinear manner, rendering Principal's optimization problem even harder. For these reasons, in its most general form the problem was approached in the literature also by means of the calculus of variations, thus adapting the tools of the stochastic version of the Pontryagin maximal problem; see Cvitanić and Zhang (2012). Still, none of the standard approaches can solve the problem when Agent also controls the diffusion coefficient (if it has the dimension at least two), and not just the drift[2].

Our approach is different, and it works in great generality, including the latter problem. We restrict the family of admissible contracts to the contracts for which Agent would be able to solve his problem by dynamic programming. For such contracts, it is easy for Principal to identify what the optimal policy for Agent is - it is the one that maximizes the corresponding Hamiltonian. Moreover, the admissible family is such that Principal can apply standard methods of stochastic control. Finally, we show that under mild technical conditions, Agent's supremum over our admissible contracts is equal to the supremum over all possible contracts. We accomplish that by representing the value function of Agent's problem by means of the so-called second order BSDEs as introduced by Soner, Touzi and Zhang (2011), [30], (see also Cheridito, Soner, Touzi and Victoir (2007)). It turns out we also need to use the recent results of Possamaï, Tan and Zhou (2015), to bypass the regularity conditions in [30]. We successfully applied this approach to the above mentioned setup in which the agent controls the diffusion vector of the output process in Cvitanić, Possamai and Touzi (2015), a problem previously not solved in the literature. The approach developed here will also be used in Aïd, Possamaï and Touzi (2015) for a problem of optimal electricity tarification.

The rest of the paper is structured as follows: We describe the model and the Principal-Agent problem in Section 2. We introduce the restricted family of admissible contracts in

[1] For a recent different approach, see Evans, Miller and Yang (2015). For each possible Agent's control process, they characterize contracts that are incentive compatible for it. However, their setup is less general, and it does not allow for volatility control, for example.

[2] An exception is Mastrolia and Possamaï (2015) (see also the related and independent work Sung (2015)), which studies a particular case of the Principal-Agent problem in the presence of ambiguity regarding the volatility component. Though close to our formulation, the problem in these two papers is not covered by our framework.

Section 3. Finally, we show that the restriction is without loss of generality in Section 4.

2 The Principal-Agent problem

2.1 The canonical space of continuous paths

Let $T > 0$ be a given terminal time, and $\Omega := C^0([0,T], \mathbb{R}^d)$ the set of all continuous maps from $[0,T]$ to $\mathbb{R}^d$, for a given integer $d > 0$. The canonical process on Ω is denoted by X, *i.e.*

$$X_t(x) = x(t) = x_t \quad \text{for all} \quad x \in \Omega,\ t \in [0,T],$$

and the corresponding (raw) canonical filtration by $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$, where

$$\mathcal{F}_t \quad := \quad \sigma(X_s,\ s \leq t),\ t \in [0,T].$$

We denote by $\mathbb{P}_0$ the Wiener measure on $(\Omega, \mathcal{F}_T)$, and for any $\mathbb{F}-$stopping time τ, by $\mathbb{P}_\tau$ the regular conditional probability distribution of $\mathbb{P}_0$ w.r.t. $\mathcal{F}_\tau$ (see Stroock and Varadhan (1979)), which is actually independent of $x \in \Omega$ by independence and stationarity of the Brownian increments.

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a semi-martingale measure if X is a semi-martingale under $\mathbb{P}$. Then, on the canonical space Ω, there is a $\mathbb{F}-$progressively measurable process (see e.g. Karandikar (1995)), denoted by $\langle X \rangle = (\langle X \rangle_t)_{0 \leq t \leq T}$, which coincides with the quadratic variation of X, $\mathbb{P}-$a.s. for all semi-martingale measure $\mathbb{P}$. We next introduce the $d \times d$ non-negative symmetric matrix $\widehat{\sigma}_t$ such that

$$\widehat{\sigma}_t^2 := \limsup_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{t-\varepsilon}}{\varepsilon},\ t \in [0,T].$$

A map $\Psi : [0,T] \times \Omega \longrightarrow E$, taking values in any Polish space E will be called $\mathbb{F}-$progressive if $\Psi(t,x) = \Psi(t, x_{\cdot \wedge t})$, for all $t \in [0,T]$ and $x \in \Omega$.

2.2 Controlled state equation

A control process $\nu = (\alpha, \beta)$ is an $\mathbb{F}-$adapted process with values in $A \times B$ for some subsets A and B of finite dimensional spaces. The controlled process takes values in $\mathbb{R}^d$, and is defined by means of the controlled coefficients:

$$\lambda : \mathbb{R}_+ \times \Omega \times A \longrightarrow \mathbb{R}^n,\ \text{bounded, with } \lambda(\cdot, \alpha)\ \mathbb{F} - \text{progressive for any } \alpha \in A,$$
$$\sigma : \mathbb{R}_+ \times \Omega \times B \longrightarrow \mathcal{M}_{d,n}(\mathbb{R}),\ \ \text{bounded, with } \sigma(\cdot, \beta)\ \mathbb{F} - \text{progressive for any } \beta \in B,$$

for a given integer n, and where $\mathcal{M}_{d,n}(\mathbb{R})$ denotes the set of $d \times n$ matrices with real entries. For all control process ν, and all $(t,x) \in [0,T] \times \Omega$, the controlled state equation is defined by the stochastic differential equation driven by an $n-$dimensional Brownian motion W,

$$X_s^{t,x,\nu} = x(t) + \int_t^s \sigma_r(X^{t,x,\nu}, \beta_r)\big[\lambda_r(X^{t,x,\nu}, \alpha_r)dr + dW_r\big],\ s \in [t,T], \tag{2.1}$$

and such that $X_s^{t,x,\nu} = x(s),\ s \in [0,t]$.

A weak solution of (2.1) is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{P}[X_{\cdot\wedge t} = x_{\cdot\wedge t}] = 1$, and

$$X_\cdot - \int_t^\cdot \sigma_r(X, \beta_r)\lambda_r(X, \alpha_r)dr, \quad \text{and} \quad X_\cdot X_\cdot^\top - \int_t^\cdot (\sigma_r\sigma_r^\top)(X, \beta_r)dr,$$

are $(\mathbb{P}, \mathbb{F})-$martingales on $[t, T]$.

For such a weak solution $\mathbb{P}$, there is an $n-$dimensional $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$, and $\mathbb{F}-$adapted, $A \times B-$valued processes $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ such that[3]

$$X_s = x_t + \int_t^s \sigma_r(X, \beta_r^{\mathbb{P}}) \left[\lambda_r(X, \alpha_r^{\mathbb{P}})dr + dW_r^{\mathbb{P}}\right], \; s \in [t, T], \; \mathbb{P} - \text{a.s.} \tag{2.2}$$

In particular, we have

$$\widehat{\sigma}_t^2 \quad = \quad (\sigma_t\sigma_t^\top)(X, \beta_t^{\mathbb{P}}), \quad dt \otimes d\mathbb{P} - \text{a.s.}$$

The next definition involves an additional map

$$c : \mathbb{R}_+ \times \Omega \times A \times B \longrightarrow \mathbb{R}_+, \text{ measurable, with } c(\cdot, u) \; \mathbb{F} - \text{progressive for all } u \in A \times B,$$

which represents Agent's cost of effort.

Throughout the paper we fix a real number $p > 1$.

Definition 2.1. *A control process ν is said to be admissible if SDE* (2.1) *has a weak solution, and for any such weak solution $\mathbb{P}$ we have*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \sup_{a\in A} |c_s(X, a, \beta_s^{\mathbb{P}})|^p ds\right] \quad < \quad \infty. \tag{2.3}$$

We denote by $\mathcal{U}(t, x)$ the collection of all admissible controls, $\mathcal{P}(t, x)$ the collection of all corresponding weak solutions of (2.1)*, and $\mathcal{P}_t := \cup_{x\in\Omega}\mathcal{P}(t, x)$.*

Notice that we do not restrict the controls to those for which weak uniqueness holds. Moreover, by Girsanov theorem, two weak solutions of (2.1) associated with (α, β) and (α', β) are equivalent. However, different diffusion coefficients induce mutually singular weak solutions of the corresponding stochastic differential equations.

For later use, we introduce an alternative representation of sets $\mathcal{P}(t, x)$. We first denote for all $(t, x) \in [0, T] \times \Omega$:

$$\Sigma_t(x, b) := \sigma_t\sigma_t^\top(x, b), \quad b \in B, \quad \text{and} \quad B_t(x, \Sigma) := \left\{b \in B : \sigma_t\sigma_t^\top(x, b) = \Sigma\right\}, \quad \Sigma \in \mathcal{S}_d^+.$$

For an $\mathbb{F}-$progressively measurable process β with values in B, consider then the SDE driven by a $d-$dimensional Brownian motion W

$$X_s^{t,x,\beta} = x_t + \int_t^s \Sigma_r^{1/2}(X, \beta_r)dW_r, \; s \in [t, T], \tag{2.4}$$

[3]Brownian motion $W^{\mathbb{P}}$ is defined on a possibly enlarged space, if $\widehat{\sigma}$ is not invertible $\mathbb{P}-$a.s. We refer to [24] for the precise statements.

with $X_s^{t,x,\beta} = x_s$ for all $s \in [0,t]$. A weak solution of (2.4) is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{P}[X_{\cdot\wedge t} = x_{\cdot\wedge t}] = 1$, and

$$X_\cdot \ \text{ and } \ X_\cdot X_\cdot^\top - \int_t^\cdot \Sigma_r(X,\beta_r)dr,$$

are $(\mathbb{P},\mathbb{F})-$martingales on $[t,T]$. Then, there is an $\mathbb{F}-$adapted process $\bar{\beta}^{\mathbb{P}}$ and some $d-$dimensional $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$ such that

$$X_s = x_t + \int_t^s \Sigma_r^{1/2}(X,\bar{\beta}_r^{\mathbb{P}})dW_r^{\mathbb{P}}, \ s \in [t,T], \ \mathbb{P}-\text{a.s.} \tag{2.5}$$

Definition 2.2. *A diffusion control process β is said to be admissible if the SDE* (2.4) *has a weak solution, and for all such solution $\mathbb{P}$, we have*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \sup_{a\in A} |c_s(X,a,\beta_s^{\mathbb{P}})|^p ds\right] \quad < \quad \infty.$$

We denote by $\mathcal{B}(t,x)$ the collection of all diffusion control processes, $\overline{\mathcal{P}}(t,x)$ the collection of all corresponding weak solutions of (2.4)*, and $\overline{\mathcal{P}}_t := \cup_{x\in\Omega}\overline{\mathcal{P}}(t,x)$.*

We emphasize that sets $\overline{\mathcal{P}}(t,x)$ are equivalent to sets $\mathcal{P}(t,x)$, in the sense that $\mathcal{P}(t,x)$ consists of probability measures which are equivalent to corresponding probability measures in $\overline{\mathcal{P}}(t,x)$, and vice versa. Indeed, for $(\alpha,\beta) \in \mathcal{U}(t,x)$ we claim that $\beta \in \mathcal{B}(t,x)$. To see this, denote by $\mathbb{P}^{\alpha,\beta}$ any of the associated weak solutions to (2.1). Then, there always is a $d\times n$ rotation matrix R such that, for any $(s,x,b) \in [0,T]\times\Omega\times B$,

$$\sigma_s(x,b) \quad = \quad \Sigma_s^{1/2}(x,b)R_s(x,b). \tag{2.6}$$

Since $d \le n$, and in addition Σ may be degenerate, notice that there may be many (and even infinitely many) choices of R, and in this case we may choose any measurable one. We next define $\mathbb{P}^\beta$ by

$$\frac{d\mathbb{P}^\beta}{d\mathbb{P}^{\alpha,\beta}} := \mathcal{E}\left(-\int_t^T \lambda_s(X,\alpha_s^{\mathbb{P}^{\alpha,\beta}})\cdot dW_s^{\mathbb{P}^{\alpha,\beta}}\right).$$

By Girsanov theorem, X is then a $(\mathbb{P}^\beta,\mathbb{F})-$martingale, which ensures that $\beta \in \mathcal{B}(t,x)$. In particular, the polar sets of $\mathcal{P}_0$ and $\overline{\mathcal{P}}_0$ are the same. Conversely, let us fix $\beta \in \mathcal{B}(t,x)$ and denote by $\mathcal{A}$ the set of $A-$valued and $\mathbb{F}-$progressively measurable processes. Then, we claim that for any $\alpha \in \mathcal{A}$, we have $(\alpha,\beta) \in \mathcal{U}(t,x)$. Indeed, let us denote by $\overline{\mathbb{P}}^\beta$ any weak solution to (2.4) and define

$$\frac{d\overline{\mathbb{P}}^{\alpha,\beta}}{d\overline{\mathbb{P}}^\beta} := \mathcal{E}\left(\int_t^T R_s(X,\bar{\beta}_s^{\overline{\mathbb{P}}^\beta})\lambda_s(X,\alpha_s)\cdot dW_s^{\overline{\mathbb{P}}^\beta}\right).$$

Then, by Girsanov Theorem, we have

$$\begin{aligned} X_s &= \int_t^s \Sigma_r^{1/2}(X,\bar{\beta}_r^{\overline{\mathbb{P}}^\beta})R_r(X,\bar{\beta}_r^{\overline{\mathbb{P}}^\beta})\lambda_r(X,\alpha_r)dr + \int_t^s \Sigma_r^{1/2}(X,\bar{\beta}_r^{\overline{\mathbb{P}}^\beta})d\overline{W}_r^{\overline{\mathbb{P}}^\beta} \\ &= \int_t^s \sigma_r(X,\bar{\beta}_r^{\overline{\mathbb{P}}^\beta})\lambda_r(X,\alpha_r)dr + \int_t^s \Sigma_r^{1/2}(X,\bar{\beta}_r^{\overline{\mathbb{P}}^\beta})d\overline{W}_r^{\overline{\mathbb{P}}^\beta}, \end{aligned}$$

where $\overline{W}^{\overline{\mathbb{P}}^\beta}$ is a $d-$dimensional $(\overline{\mathbb{P}}^{\alpha,\beta},\mathbb{F})-$Brownian motion. Hence, $(\alpha,\beta)\in\mathcal{U}(t,x)$. Moreover, setting

$$W^{\overline{\mathbb{P}}^\beta}_\cdot = R_\cdot(X,\overline{\beta}^{\overline{\mathbb{P}}^\beta}_\cdot)W^{\overline{\mathbb{P}}^{\alpha,\beta}}_\cdot + \int_t^\cdot R_s(X,\overline{\beta}^{\overline{\mathbb{P}}^\beta}_s)\lambda_s(X,\alpha_s)ds,$$

defines a Brownian motion under $\mathbb{P}^{\alpha,\beta}$. Since $\overline{\mathbb{P}}^\beta$ and $\overline{\mathbb{P}}^{\alpha,\beta}$ are equivalent, we have

$$\Sigma(X,\beta^{\overline{\mathbb{P}}^{\alpha,\beta}}) = \Sigma(X,\overline{\beta}^{\overline{\mathbb{P}}^\beta}),\ dt\otimes\overline{\mathbb{P}}^\beta - a.e.\ (\text{or } dt\otimes\overline{\mathbb{P}}^{\alpha,\beta} - a.e.),$$

that is $\beta_s^{\overline{\mathbb{P}}^{\alpha,\beta}}$ and $\overline{\beta}_s^{\overline{\mathbb{P}}^\beta}$ both belong to $B_s(X,\widehat{\sigma}_s^2(X))$, $dt\otimes\overline{\mathbb{P}}^\beta-$a.e. We can summarize everything by the following equality

$$\mathcal{P}(t,x) = \bigcup_{\alpha\in\mathcal{A}}\Big\{\mathcal{E}\Big(\int_t^T R_s(X,\overline{\beta}_s^{\mathbb{P}})\lambda_s(X,\alpha_s)\cdot dW_s^{\mathbb{P}}\Big)\cdot\mathbb{P} : \mathbb{P}\in\overline{\mathcal{P}}(t,x),\ R \text{ satisfying (2.6)}\Big\}. \quad (2.7)$$

2.3 Agent's problem

In the following discussion, we fix $(t,x,\mathbb{P})\in[0,T]\times\Omega\times\mathcal{P}(t,x)$, together with the associated control $\nu^{\mathbb{P}} := (\alpha^{\mathbb{P}},\beta^{\mathbb{P}})$. In our Principal-Agent problem, the canonical process X is called the *output* process, and the control $\nu^{\mathbb{P}}$ is usually referred to as Agent's *effort* or *action*. Agent is in charge of controlling the (distribution of the) output process by optimally choosing the effort process $\nu^{\mathbb{P}}$ in the state equation (2.1), while subject to cost of effort at rate $c(X,\alpha^{\mathbb{P}},\beta^{\mathbb{P}})$. Furthermore, Agent has a fixed reservation utility denoted by $R\in\mathbb{R}$, i.e., he will not accept to work for Principal unless the contract is such that his value function is above R.

The contract agreement holds during the time period $[t,T]$. Agent is only cares about the compensation ξ received from Principal at time T. Principal does not observe Agent's effort, only the output process. Consequently, the compensation ξ, which takes values in $\mathbb{R}$, can only be contingent on X, that is ξ is $\mathcal{F}_T-$measurable.

Random variable ξ is called a *contract* on $[t,T]$, and we write $\xi\in\mathcal{C}_t$ if the following integrability condition is satisfied:

$$\sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty. \quad (2.8)$$

We now introduce Agent's objective function:

$$J^A(t,x,\mathbb{P},\xi) \ := \ \mathbb{E}^{\mathbb{P}}\Big[\mathcal{K}^{\nu^{\mathbb{P}}}_{t,T}(X)\xi - \int_t^T \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X)c_s\big(X,\nu_s^{\mathbb{P}}\big)ds\Big],\ \ \mathbb{P}\in\mathcal{P}(t,x),\ \xi\in\mathcal{C}_t, \quad (2.9)$$

where

$$\mathcal{K}^\nu_{t,s}(X) \ := \ \exp\Big(-\int_t^s k_r(X,\nu_r)dr\Big),\quad s\in[t,T],$$

is a discount factor defined by means of a bounded measurable function

$$k:\mathbb{R}_+\times\Omega\times A\times B\longrightarrow\mathbb{R},\quad \text{with}\quad k(\cdot,u)\ \mathbb{F}-\text{progressive for all } u\in A\times B.$$

Notice that J^A is well-defined for all $(t,x)\in[0,T]\times\Omega$, $\xi\in\mathcal{C}_t$ and $\mathbb{P}\in\mathcal{P}(t,x)$. This is a consequence of the boundedness of k, the non-negativity of c, as well as the conditions (2.8) and (2.3).

Remark 2.3. *If Agent is risk-averse with utility function U_A, then we replace ξ with $\xi' = U_A(\xi)$ in J^A, and we replace ξ by $U_A^{-1}(\xi')$ in Principal's problem below. All the results remain valid.*

Remark 2.4. *Our approach can also accommodate an objective function for the agent of the form*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\mathrm{sgn}(U_A)\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X)c_s\big(X,\nu_s^{\mathbb{P}}\big)ds\right)\mathcal{K}_{t,T}^{\nu^{\mathbb{P}}}(X)U_A(\xi)\right],$$

for a utility function U_A having constant sign. In particular, our framework includes exponential utilities. Obviously, such case would require some modifications of our assumptions.

Agent's goal is to choose optimally the amount of effort, given the compensation contract ξ promised by Principal:

$$V^A(t,x,\xi) := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} J^A(t,x,\mathbb{P},\xi). \tag{2.10}$$

An admissible control $\mathbb{P}^\star \in \mathcal{P}(t,x)$ will be called optimal if

$$V^A(t,x,\xi) = J^A(t,x,\mathbb{P}^\star,\xi).$$

We denote by $\mathcal{P}^\star(t,x,\xi)$ the collection of all such optimal controls $\mathbb{P}^\star$.

In the economics literature, the dynamic value function V^A is called *continuation utility* or *promised utility*, and it turns out to play a crucial role as the state variable of Principal's optimization problem; see Sannikov (2008) for its use in continuous-time models for and further references.

2.4 Principal's problem

We now define the optimization problem of choosing the compensation contract ξ that Principal should offer to Agent.

At the maturity T, Principal receives the final value of the output X_T and pays the compensation ξ promised to Agent. Principal only observes the output resulting from Agent's optimal strategy. We restrict the contracts proposed by Principal to those that admit a solution to Agent's problem, i.e., we allow only the contracts ξ for which $\mathcal{P}^\star(t,x,\xi) \neq \emptyset$. Recall also that the participation of Agent is conditioned on having his value function above reservation utility R. For this reason, Principal is restricted to choose a contract from the set

$$\Xi(t,x) := \big\{\xi \in \mathcal{C}_t,\ \mathcal{P}^\star(t,x,\xi) \neq \emptyset,\ V^A(t,x,\xi) \geq R\big\}. \tag{2.11}$$

As a final ingredient, we need to fix Agent's optimal strategy in the case in which set $\mathcal{P}^\star(t,x,\xi)$ contains many solutions. Following the standard Principal-Agent literature, we assume that Agent, when indifferent between such solutions, implements the one that is the best for Principal.

In view of this, Principal's problem reduces to

$$V^P(t,x) \quad := \quad \sup_{\xi\in\Xi(t,x)} J^P(t,x,\xi), \tag{2.12}$$

where

$$J^P(t,x,\xi) := \sup_{\mathbb{P}^\star\in\mathcal{P}^\star(t,x,\xi)} \mathbb{E}^{\mathbb{P}^\star}\Big[\mathcal{K}^P_{t,T}(X)U\big(\ell(X_T)-\xi\big)\Big],$$

where function $U:\mathbb{R}\longrightarrow\mathbb{R}$ is a given non-decreasing and concave utility function, $\ell:\mathbb{R}^d\longrightarrow\mathbb{R}$ is a liquidation function, and

$$\mathcal{K}^P_{t,s}(X) \quad := \quad \exp\Big(-\int_t^s k^P_r(X)dr\Big), \quad s\in[t,T],$$

is a discount factor defined by means of a bounded measurable function

$$k^P:\mathbb{R}_+\times\Omega\longrightarrow\mathbb{R},$$

such that k^P is $\mathbb{F}-$progressive.

Remark 2.5. *Agent's and Principal's problems are non-standard. First, ξ is allowed to be of non-Markovian nature. Second, Principal's optimization is over ξ, and is a priori not in a control problem that may be approached by dynamic programming. The objective of this paper is to develop an approach that naturally reduces the problems to those that can be solved by dynamic programming. We used this approach in our previous paper [5], albeit in a much less general setting.*

3 Restricted contracts

In this section we identify a restricted family of contract payoffs for which the standard stochastic control methods can be applied.

3.1 Agent's dynamic programming equation

In view of the definition of Agent's problem in (2.10), it is natural to introduce the Hamiltonian functional, for all $(t,x)\in[0,T)\times\Omega$ and $(y,z,\gamma)\in\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d(\mathbb{R})$:

$$H_t(x,y,z,\gamma) \quad := \quad \sup_{u\in A\times B} h_t(x,y,z,\gamma,u), \tag{3.1}$$

$$h_t(x,y,z,\gamma,u) \quad := \quad -c_t(x,u)-k_t(x,u)y+\sigma_t(x,\beta)\lambda_t(x,\alpha)\cdot z+\frac{1}{2}(\sigma_t\sigma_t^\top)(x,\beta):\gamma, \tag{3.2}$$

for $u:=(\alpha,\beta)$.

Remark 3.1. (i) *The map H plays an important role in the theory of stochastic control of Markov diffusions, see e.g. Fleming and Soner (1993). Indeed, suppose that*

- *the coefficients $\lambda_t,\sigma_t,c_t,k_t$ depend on x only through the current value $x(t)$,*
- *the contract ξ depends on x only through the final value $x(T)$, i.e. $\xi(x)=g(x(T))$ for some function $g:\mathbb{R}^d\longrightarrow\mathbb{R}$.*

Then, under fairly general conditions, the value function of Agent's problem is identified by $V^A(t,x(t),\xi)=v(t,x(t))$, where the function $v:[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}$ can be characterized as the unique viscosity solution (with appropriate growth at infinity) of the dynamic programming equation

$$-\partial_t v(t,\mathrm{x})-H(t,\mathrm{x},v(t,\mathrm{x}),Dv(t,\mathrm{x}),D^2v(t,\mathrm{x}))=0,\ (t,\mathrm{x})\in[0,T)\times\mathbb{R}^d,\ v(T,\mathrm{x})=g(\mathrm{x}),\ \mathrm{x}\in\mathbb{R}^d.$$

(ii) *The recently developed theory of path-dependent partial differential equations extends the first item of this remark to the non-Markovian case. See Ekren, Touzi and Zhang (2014).*

Note that Agent's value function process V_t at the terminal time is equal to the contract payoff, $\xi = V_T$. This will motivate us to consider payoffs ξ of a specific form.
The main result of this section follows the line of the standard verification result in stochastic control theory. Fix some $(t,x) \in [0,T] \times \Omega$. Let

$$Z : [t,T] \times \Omega \longrightarrow \mathbb{R}^d \quad \text{and} \quad \Gamma : [t,T] \times \Omega \longrightarrow \mathcal{S}_d(\mathbb{R})$$

be $\mathbb{F}-$predictable processes with

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left[Z_s Z_s^\top : \widehat{\sigma}_s^2 + |\Gamma_s : \widehat{\sigma}_s^2|\right] ds\right)^{\frac{p}{2}}\right] < +\infty, \text{ for all } \mathbb{P} \in \mathcal{P}(t,x),$$

We denote by $\mathcal{V}(t,x)$ the collection of all such pairs of processes (Z,Γ).

Given an initial condition $Y_t \in \mathbb{R}$, define the $\mathbb{F}-$progressively measurable process $Y^{Z,\Gamma}$, $\mathbb{P}-$a.s., for all $\mathbb{P} \in \mathcal{P}(t,x)$ by

$$Y_s^{Z,\Gamma} := Y_t - \int_t^s H_r\left(X, Y_r^{Z,\Gamma}, Z_r, \Gamma_r\right) dr + \int_t^s Z_r \cdot dX_r + \frac{1}{2}\int_t^s \Gamma_r : d\langle X\rangle_r, \; s \in [t,T]. \quad (3.3)$$

Notice that $Y^{Z,\Gamma}$ is well-defined as a consequence of the Lipschitz property of H in y, resulting from the boundedness of k.

The next result follows the line of the classical verification argument in stochastic control theory, and requires the following condition.

Assumption 3.2. *For any $t \in [0,T]$, the map H has at least one measurable maximizer $u^\star = (\alpha^\star, \beta^\star) : [t,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R}) \longrightarrow A \times B$, i.e. $H(.) = h\big(., u^\star(.)\big)$. Moreover, for all $(t,x) \in [0,T] \times \Omega$ and for any $(Z,\Gamma) \in \mathcal{V}(t,x)$, the control process*

$$\nu_s^{\star,Z,\Gamma}(\cdot) := u_s^\star\big(\cdot, Y_s^{Z,\Gamma}(\cdot), Z_s(\cdot), \Gamma_s(\cdot)\big), \; s \in [t,T],$$

is admissible, that is $\nu^{\star,Z,\Gamma} \in \mathcal{U}(t,x)$.

We are now in a position to provide a subset of contracts which, when proposed by Principal, have a very useful property of revealing Agent's optimal effort. Under this set of revealing contracts, Agent's value function coincides with the above process $Y^{Z,\Gamma}$.

Proposition 3.3. *For $(t,x) \in [0,T] \times \Omega$, $Y_t \in \mathbb{R}$ and $(Z,\Gamma) \in \mathcal{V}(t,x)$, we have:*

(i) $Y_t \geq V^A\big(t,x,Y_T^{Z,\Gamma}\big)$.

(ii) *Assuming further that Assumption 3.2 holds true, we have $Y_t = V_A\big(t,x,Y_T^{Z,\Gamma}\big)$. Moreover, given a contract payoff $\xi = Y_T^{Z,\Gamma}$, any weak solution $\mathbb{P}^{\star,Y,Z}$ of the SDE* (2.1) *with control $\nu^{\star,Z,\Gamma}$ is optimal for the Agent problem, i.e. $\mathbb{P}^{\nu^{\star,Z,\Gamma}} \in \mathcal{P}^\star\big(t,x,Y_T^{Z,\Gamma}\big)$.*

Proof. (i) Fix an arbitrary $\mathbb{P} \in \mathcal{P}(t,x)$, and denote the corresponding control process $\nu^{\mathbb{P}} := (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$. Then, it follows from a direct application of Itô's formula that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}}} Y_T^{Z,\Gamma}\right] \;=\; & Y_t + \mathbb{E}^{\mathbb{P}}\Bigg[\int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}}}(X)\Big(-k_s(X,\nu_s^{\mathbb{P}})Y_s^{Z,\Gamma} - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) \\ & + Z_s \cdot \sigma_s(X,\beta_s^{\mathbb{P}})\lambda(X,\alpha_s^{\mathbb{P}}) + \frac{1}{2}\widehat{\sigma}_s^2 : \Gamma_s\Big) ds\Bigg],\end{aligned}$$

where we have used the fact that $(Z,\Gamma)\in\mathcal{V}(t,x)$, together with the fact that the stochastic integral $\int_t^\cdot \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X)Z_s\cdot\widehat{\sigma}^2_s dW^{\mathbb{P}}_s$ defines a martingale, by the boundedness of k and σ.

By the definition of the Hamiltonian H in (3.1), we may re-write the last equation as

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\mathcal{K}^{\nu^{\mathbb{P}}}_{t,T}Y^{Z,\Gamma}_T\right] &= Y_t+\mathbb{E}^{\mathbb{P}}\Big[\int_t^T \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X)\big(c_s(X,\nu^{\mathbb{P}})-H_s\left(X,Y^{Z,\Gamma}_s,Z_s,\Gamma_s\right)\\ &\qquad\qquad +h_s(X,Y^{Z,\Gamma}_s,Z_s,\Gamma_s,\nu^{\mathbb{P}}_s)\big)ds\Big]\\ &\le Y_t+\mathbb{E}^{\mathbb{P}}\left[\int_t^T \mathcal{K}^{\nu^{\mathbb{P}}}_{t,s}(X)c_s(X,\alpha^{\mathbb{P}}_s,\beta^{\mathbb{P}}_s)ds\right],\end{aligned}$$

and the result follows by arbitrariness of $\mathbb{P}\in\mathcal{P}(t,x)$.

(ii) Let $\nu^\star:=\nu^{\star,Z,\Gamma}$ for simplicity. Under Assumption 3.2, the exact same calculations as in (i) provide for any weak solution $\mathbb{P}^{\nu^\star}$

$$\mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\mathcal{K}^{\nu^{\mathbb{P}^{\nu^\star}},t,x}_{t,T}Y^{Z,\Gamma}_T\right]=Y_t+\mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_t^T\mathcal{K}^{\nu^{\mathbb{P}^{\nu^\star}}}_{t,s}(X)c_s(X,\alpha^{\mathbb{P}^{\nu^\star}}_s,\beta^{\mathbb{P}}_s)ds\right].$$

Together with (i), this shows that $Y_t=V_A\big(t,x,Y^{Z,\Gamma}_T\big)$, and $\mathbb{P}^{\nu^\star}\in\mathcal{P}^\star\big(t,x,Y^{Z,\Gamma}_T\big)$. □

3.2 Restricted Principal's problem

Recall the process $u^\star_t(x,y,z,\gamma)=(\alpha^\star,\beta^\star)_t(x,y,z,\gamma)$ introduced in Assumption 3.2. In this section, we denote

$$\lambda^\star_t(x,y,z,\gamma):=\lambda_t\big(x,\alpha^\star_t(x,y,z,\gamma)\big),\quad \sigma^\star_t(x,y,z,\gamma):=\sigma_t\big(x,\beta^\star_t(x,y,z,\gamma)\big). \tag{3.4}$$

Notice that Assumption 3.2 says that for all $(t,x)\in[0,T]\times\Omega$ and for all $(Z,\Gamma)\in\mathcal{V}(t,x)$, the stochastic differential equation, driven by a $n-$dimensional Brownian motion W

$$\begin{aligned}X^{t,x,u^\star}_s&=x(t)+\int_t^s\sigma^\star_r(X^{t,x,u^\star},Y^{Z,\Gamma}_r,Z_r,\Gamma_r)\big[\lambda^\star_r(X^{t,x,u^\star},Y^{Z,\Gamma}_r,Z_r,\Gamma_r)dr+dW_r\big],\ s\in[t,T],\\ X^{t,x,u^\star}_s&=x(s),\ s\in[0,t],\end{aligned}\tag{3.5}$$

has at least one weak solution $\mathbb{P}^{\star,Z,\Gamma}$. The following result on Principal's value function V^P when the contract payoff is $\xi=Y^{Z,\Gamma}_T$ is a direct consequence of Proposition 3.3.

Proposition 3.4. *For all $(t,x)\in[0,T]\times\Omega$, we have $V^P(t,x)\ge\sup_{Y_t\ge R}\underline{V}(t,x,Y_t)$, where, for $Y_t\in\mathbb{R}$:*

$$\underline{V}(t,x,Y_t):=\sup_{(Z,\Gamma)\in\mathcal{V}(t,x)}\ \sup_{\hat{\mathbb{P}}^{Z,\Gamma}\in\mathcal{P}^\star(t,x,Y^{Z,\Gamma}_T)}\mathbb{E}^{\mathbb{P}^{\star,Z,\Gamma}}\big[\mathcal{K}^P_{t,T}U\big(\ell(X_T)-Y^{Z,\Gamma}_T\big)\big]. \tag{3.6}$$

We continue this section by a discussion of the bound $\underline{V}(t,x,y)$ which represents Principal's value function when the contracts are restricted to the $\mathcal{F}_T-$measurable random variables $Y^{Z,\Gamma}_T$ with given initial condition Y_t. In the sections below, we identify conditions under which that restriction is without loss of generality.

Clearly, $\underline{V}$ is the value function of a standard stochastic control problem with control processes $(Z,\Gamma)\in\mathcal{V}(t,x)$, and controlled state process $(X,Y^{Z,\Gamma})$, the controlled dynamics of X being given (in weak formulation) by (3.5), and those of $Y^{Z,\Gamma}$ given by (3.3):

$$dY_s^{Z,\Gamma}=\Big(Z_s\cdot\sigma_s^\star\lambda_s^\star+\frac{1}{2}\Gamma_s:\sigma_s^\star(\sigma_s^\star)^\top-H_s\Big)(X,Y_s^{Z,\Gamma},Z_s,\Gamma_s)ds+Z_s\cdot\sigma_s^\star(X,Y_s^{Z,\Gamma},Z_s,\Gamma_s)dW_s^{\mathbb{P}}. \tag{3.7}$$

In view of the controlled dynamics (3.5)-(3.7), the relevant optimization term for the dynamic programming equation corresponding to the control problem $\underline{V}$ is defined for $(t,\mathrm{x},y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$ by:

$$\begin{aligned}&G(t,\mathrm{x},y,p,M)\\&:=\sup_{(z,\gamma)\in\mathbb{R}\times\mathcal{S}_d(\mathbb{R})}\Big\{(\sigma_t^\star\lambda_t^\star)(\mathrm{x},y,z,\gamma)\cdot p_x+\Big(z\cdot(\sigma_t^\star\lambda_t^\star)+\frac{1}{2}\gamma:\sigma_t^\star(\sigma_t^\star)^\top-H_t\Big)(\mathrm{x},y,z,\gamma)p_y\\&\qquad+\frac{1}{2}(\sigma_t^\star(\sigma_t^\star)^\top)(\mathrm{x},y,z,\gamma):\big(M_{\mathrm{xx}}+zz^\top M_{\mathrm{yy}}\big)+(\sigma_t^\star(\sigma_t^\star)^\top)(\mathrm{x},y,z,\gamma)z\cdot M_{\mathrm{xy}}\Big\},\end{aligned}$$

where $M=:\begin{pmatrix}M_{\mathrm{xx}} & M_{\mathrm{xy}}\\ M_{\mathrm{xy}}^\top & M_{\mathrm{yy}}\end{pmatrix}\in\mathcal{S}_{d+1}(\mathbb{R})$, $M_{xx}\in\mathcal{S}_d(\mathbb{R})$, $M_{yy}\in\mathbb{R}$, $M_{xy}\in\mathcal{M}_{d,1}(\mathbb{R})$ and $p=:\begin{pmatrix}p_x\\ p_y\end{pmatrix}\in\mathbb{R}^d\times\mathbb{R}$.

Theorem 3.5. *Let $\varphi_t(x,.)=\varphi_t(x_t,.)$ for $\varphi=k,k^P,\lambda^\star,\sigma^\star,H$, and let Assumption 3.2 hold true. Assume further that the map $G:[0,T)\times\mathbb{R}^d\times\mathbb{R}^{d+1}\times\mathbb{S}_{d+1}(\mathbb{R})\longrightarrow\mathbb{R}$ is upper semicontinuous. Then, $\underline{V}(t,x,y)$ is a viscosity solution of the dynamic programming equation:*

$$\begin{cases}(\partial_t v-k^P v)(t,\mathrm{x},y)+G\big(t,\mathrm{x},v(t,\mathrm{x},y),Dv(t,\mathrm{x},y),D^2v(t,\mathrm{x},y)\big)=0,\ (t,\mathrm{x},y)\in[0,T)\times\mathbb{R}^d\times\mathbb{R},\\ v(T,\mathrm{x},y)=U(\ell(\mathrm{x})-y),\ (\mathrm{x},y)\in\mathbb{R}^d\times\mathbb{R}.\end{cases}$$

The last statement is formulated in the Markovian case, *i.e.* when the model coefficients are not path-dependent. A similar statement can be formulated in the path dependent case, by using the notion of viscosity solutions of path-dependent PDEs introduced in Ekren, Keller, Touzi & Zhang (2014), and further developed in [10, 11, 25, 26]. However, one then faces the problem of unboundedness of the controls (z,γ), which typically leads to a non-Lipschitz G in terms of the variables (Dv,D^2v), unless additional conditions on the coefficients are introduced.

4 Comparison with the unrestricted case

In this section we find conditions under which equality holds in Proposition 3.4, i.e. the value function of the restricted Principal's problem of Section 3.2 coincides with Principal's value function with unrestricted contracts. We start with the case in which the diffusion coefficient is not controlled.

4.1 Fixed volatility of the output

We consider here the case in which Agent is only allowed to control the drift of the output process:

$$B=\{\beta^o\}\quad\text{for some fixed}\quad\beta^o\in\overline{\mathcal{U}}(t,x). \tag{4.1}$$

Let $\mathbb{P}^{\beta^o}$ be any weak solutions of the corresponding SDE (2.4).

The main tool for our main result is the use of Backward SDEs. This requires introducing filtration $\mathbb{F}_+^{\mathbb{P}^{\beta^o}}$, defined as the $\mathbb{P}^{\beta^o}-$completion of the right-limit of $\mathbb{F}$,[4] under which the predictable martingale representation property holds true.

In the present setting, all probability measures $\mathbb{P} \in \mathcal{P}(t,x)$ are equivalent to $\mathbb{P}^{\beta^o}$. Consequently, the expression of the process Y in (3.3) only needs to be solved under $\mathbb{P}^{\beta^0}$, and reduces to

$$Y_s^Z := Y_s^{Z,0} = Y_t - \int_t^s F_r^0\left(X, Y_r^{Z,\Gamma}, Z_r\right) dr + \int_t^s Z_r \cdot dX_r,\ s \in [t,T],\ \mathbb{P}^{\beta^o} - a.s., \tag{4.2}$$

where the dependence on the process Γ simplifies immediately, and

$$F_t^0(x,y,z) \quad := \quad \sup_{\alpha \in A} \big\{ - c_t(x,\alpha,b) - k_t(x,\alpha,b)y + \sigma_t\big(x, \beta_t^0(x)\big)\lambda_t(x,\alpha)\cdot z \big\}. \tag{4.3}$$

Theorem 4.1. *Let Assumption 3.2 hold. In the setting of* (4.1)*, assume in addition that* $(\mathbb{P}^{\beta^o}, \mathbb{F}_+^{\mathbb{P}^{\beta^o}})$ *satisfies the predictable martingale representation property and the Blumenthal zero-one law. Then,*

$$V^P(t,x) \;=\; \sup_{y \geq R} \underline{V}(t,x,y), \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Proof. For all $\xi \in \Xi(t,x)$, we observe that condition (2.8) guarantees that $\xi \in \mathbb{L}^p(\mathbb{P}^{\beta^0})$. To prove that the required equality holds, it is sufficient to show that all such ξ can be represented in terms of a controlled diffusion $Y^{Z,0}$. However, we have already seen that F is uniformly Lipschitz-continuous in (y,z), since k, σ and λ are bounded, and by definition of admissible contracts, we have also that

$$\mathbb{E}^{\mathbb{P}^{\beta^o}}\left[\int_0^T |F_t^0(X,0,0)|^p\right] \quad < \quad \infty,$$

Then, the standard theory (see for instance [24]) guarantees that the BSDE

$$Y_t \quad = \quad \xi + \int_t^T F_r^0(X, Y_r, Z_r)dr - \int_t^T Z_r \cdot \sigma_r(X, \beta_r^o) dW_r^{\mathbb{P}^{\beta^o}},$$

is well-posed, because we also have that $\mathbb{P}^{\beta^o}$ satisfies the predictable martingale representation property. Moreover, we then have automatically $(Z,0) \in \mathcal{V}(t,x)$. This implies that ξ can indeed be represented by the process Y which is of the form (4.2). □

4.2 The general case

The purpose of this section is to extend Theorem 4.1 to the case in which Agent controls both the drift and the diffusion of the output process X. Similarly to the previous section, the critical tool is the theory of Backward SDEs, but suitable for path-dependent stochastic control problems.

[4]For a semimartingale probability measure $\mathbb{P}$, we denote by $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$ its right-continuous limit, and by $\mathcal{F}_{t+}^{\mathbb{P}}$ the corresponding completion under $\mathbb{P}$. The completed right-continuous filtration is denoted by $\mathbb{F}_+^{\mathbb{P}}$.

The additional control of the volatility requires to invoke the recent extension of Backward SDEs to the second order case. This needs additional notation, as follows. Let $\mathbb{M}$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. The universal filtration $\mathbb{F}^U = \big(\mathcal{F}_t^U\big)_{0\le t\le T}$ is defined by

$$\mathcal{F}_t^U \ := \ \cap_{\mathbb{P}\in\mathbb{M}} \ \mathcal{F}_t^{\mathbb{P}}, t\in[0,T],$$

and we denote by $\mathbb{F}_+^U$, the corresponding right-continuous limit. Moreover, for a subset $\mathcal{P}\subset\mathbb{M}$, we introduce the set of $\mathcal{P}-$polar sets $\mathcal{N}^{\mathcal{P}} := \big\{N\subset\Omega : N\subset A$ for some $A\in\mathcal{F}_T$ with $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}(A)=0\big\}$, and we introduce the $\mathcal{P}-$completion of $\mathbb{F}$

$$\mathbb{F}^{\mathcal{P}} := \big(\mathcal{F}_t^{\mathcal{P}}\big)_{t\in[0,T]}, \text{ with } \mathcal{F}_t^{\mathcal{P}} := \mathcal{F}_t^U \vee \sigma\left(\mathcal{N}^{\mathcal{P}}\right), \ t\in[0,T],$$

together with the corresponding right-continuous limit $\mathbb{F}_+^{\mathcal{P}}$.

Finally, for technical reasons, we work under the ZFC set-theoretic axioms, as well as the axiom of choice and the continuum hypothesis[5].

4.2.1 2BSDE characterization of Agent's problem

We now provide a representation of Agent's value function by means of the so-called second order BSDEs, or 2BSDEs as introduced by Soner, Touzi and Zhang (2011), [30] (see also Cheridito, Soner, Touzi and Victoir (2007)). Furthermore, we use crucially recent results of Possamaï, Tan and Zhou (2015) to bypass the regularity conditions in [30].

We first re-write the map H in (3.1) as:

$$\begin{aligned}
H_t(x,y,z,\gamma) &= \sup_{\beta\in B}\Big\{F_t\big(x,y,z,\Sigma_t(x,\beta)\big)+\frac{1}{2}\Sigma_t(x,\beta):\gamma\Big\},\\
F_t(x,y,z,\Sigma) &:= \sup_{(\alpha,\beta)\in A\times B_t(x,\Sigma)}\big\{-c_t(x,\alpha,\beta)-k_t(x,\alpha,\beta)y+\sigma_t(x,\beta)\lambda_t(x,\alpha)\cdot z\big\}.
\end{aligned}$$

We consider a reformulation of Assumption 3.2 in this setting:

Assumption 4.2. *The map F has at least one measurable maximizer $u^\star=(\alpha^\star,\beta^\star):[0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d^+\longrightarrow A\times B$, i.e. $F(\cdot,y,z,\Sigma)=-c(\cdot,\alpha^\star,\beta^\star)-k(\cdot,\alpha^\star,\beta^\star)y+\sigma(\cdot,\beta^\star)\lambda(\cdot,\alpha^\star)\cdot z$. Moreover, for all $(t,x)\in[0,T]\times\Omega$, and for all admissible controls $\beta\in\overline{\mathcal{U}}(t,x)$, the control process*

$$\nu_s^{\star,Y,Z,\beta} := (\alpha_s^\star,\beta_s^*)\big(Y_s,Z_s,\Sigma_s(\beta_s)\big), \quad s\in[t,T],$$

is admissible, that is $\nu^{\star,Y,Z,\beta}\in\mathcal{U}(t,x)$.

We also need the following condition.

Assumption 4.3. *For any $(t,x,\beta)\in[0,T]\times\Omega\times B$, the matrix $(\sigma_t\sigma_t^\top)(x,\beta)$ is invertible, with a bounded inverse.*

The following lemma shows that the sets $\overline{\mathcal{P}}(t,x)$ satisfy natural properties.

[5] Actually, we do not need the continuum hypothesis, *per se.* Indeed, we only want to be able to use the main result of Nutz (2012), which only requires axioms ensuring the existence of medial limits. We make this choice here for ease of presentation.

Lemma 4.4. *The family $\{\overline{\mathcal{P}}(t,x),(t,x)\in[0,T]\times\Omega\}$ is saturated, and satisfies the dynamic programming requirements of Assumption 2.1 in [24].*

Proof. Consider some $\mathbb{P}\in\overline{\mathcal{P}}(t,x)$ and some $\mathbb{P}^{'}$ under which X is a martingale, and which is equivalent to $\mathbb{P}$. Then, the quadratic variation of X under $\mathbb{P}^{'}$ is the same as its quadratic variation under $\mathbb{P}$, that is $\int_t^{\cdot}(\sigma_s\sigma_s^\top)(X,\beta_s)ds$. By definition, $\mathbb{P}^{'}$ is therefore a weak solution to (2.4) and belongs to $\overline{\mathcal{P}}(t,x)$.

The dynamic programming requirements of Assumption 2.1 in [24] follow from the more general results given in [13, 14]. □

Given an admissible contract ξ, we consider the following saturated 2BSDE (in the sense of Section 5 of [24]):

$$Y_t \;=\; \xi+\int_t^T F_s(X,Y_s,Z_s,\widehat{\sigma}_s^2)ds-\int_t^T Z_s\cdot dX_s+\int_t^T dK_s, \tag{4.4}$$

where Y is $\mathbb{F}_+^{\overline{\mathcal{P}}_0}-$progressively measurable process, Z is an $\mathbb{F}^{\overline{\mathcal{P}}_0}-$predictable process, with appropriate integrability conditions, and K is an $\mathbb{F}^{\overline{\mathcal{P}}_0}-$optional non-decreasing process with $K_0=0$, and satisfying the minimality condition

$$K_t \;=\; \underset{\mathbb{P}^{'}\in\overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)}{\operatorname{essinf}^{\mathbb{P}}}\;\mathbb{E}^{\mathbb{P}^{'}}\big[K_T\big|\mathcal{F}_t^{\mathbb{P}+}\big],\quad 0\le t\le T,\quad \mathbb{P}-\text{a.s. for all}\quad \mathbb{P}\in\overline{\mathcal{P}}_0. \tag{4.5}$$

Notice that, in contrast with the 2BSDE definition in [30, 24], we are using here an aggregated non-decreasing process K. This is a consequence of the general aggregation result of stochastic integrals in [23].

Since k, σ, λ are bounded, and $\sigma\sigma^\top$ is invertible with a bounded inverse, it follows from the definition of admissible controls that F satisfies the integrability and Lipschitz continuity assumptions required in [24], that is for some $\kappa\in[1,p)$ and for any $(s,x,y,y',z,z',a)\in[0,T]\times\Omega\times\mathbb{R}^2\times\mathbb{R}^{2d}\times\mathcal{S}_d^+$

$$|F_s(x,y,z,a)-F_s(x,y',z',a)|\le C\left(|y-y'|+|a^{1/2}(z-z')|\right),$$

$$\sup_{\mathbb{P}\in\overline{\mathcal{P}}_0}\mathbb{E}^{\mathbb{P}}\left[\underset{0\le t\le T}{\operatorname{essup}^{\mathbb{P}}}\left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T|F_s(X,0,0,\widehat{\sigma}_s^2)|^\kappa\middle|\mathcal{F}_{t+}\right]\right)^{\frac{p}{\kappa}}\right]<+\infty.$$

Then, in view of Lemma 4.4, the well-posedness of the saturated 2BSDE (4.4) is a direct consequence of Theorems 4.1 and 5.1 in [24].

We use 2BSDEs (4.4) because of the following representation result.

Proposition 4.5. *Let Assumptions 4.2 and 4.3 hold. Then, we have*

$$V^A(t,x,\xi) \;=\; \sup_{\mathbb{P}\in\mathcal{P}(t,x)}\mathbb{E}^{\mathbb{P}}\left[Y_t\right].$$

Moreover, $\xi\in\Xi(t,x)$ if and only if there is an $\mathbb{F}-$adapted process $\beta^\star$ with values in B, such that $\nu^\star:=\big(a_\cdot^\star(X,Y_\cdot,Z_\cdot,\Sigma_\cdot(X,\beta_\cdot^\star)),\beta_\cdot^\star\big)\in\mathcal{U}(t,x)$, and

$$K_T \;=\; 0,\;\; \mathbb{P}^{\beta^\star}-\text{a.s.}$$

for any associated weak solution $\mathbb{P}^{\beta^\star}$ of (2.4).

Proof. By Theorem 4.2 of [24], we know that we can write the solution of the 2BSDE (4.4) as a supremum of solutions of BSDEs, that is

$$Y_t = \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)} \mathcal{Y}_t^{\mathbb{P}'}, \quad \mathbb{P}-\text{a.s. for all} \quad \mathbb{P}\in\overline{\mathcal{P}}_0,$$

where for any $\mathbb{P}\in\overline{\mathcal{P}}_0$ and any $s\in[0,T]$,

$$\mathcal{Y}_s^{\mathbb{P}} = \xi + \int_s^T F_s(X,\mathcal{Y}_r,\mathcal{Z}_r,\widehat{\sigma}_r^2)dr - \int_s^T \mathcal{Z}_r^{\mathbb{P}}\cdot dX_r - \int_s^T d\mathcal{M}_r^{\mathbb{P}},\ \mathbb{P}-\text{a.s.}$$

with a càdlàg $(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})-$martingale $\mathcal{M}^{\mathbb{P}}$ orthogonal to $W^{\mathbb{P}}$.

For any $\mathbb{P}\in\overline{\mathcal{P}}_0$, let $\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})$ denote the collection of all control processes β with $\beta\in B_t(X,\widehat{\sigma}_t^2)$, $dt\otimes\mathbb{P}-$a.e. For all $(\mathbb{P},\alpha)\in\overline{\mathcal{P}}_0\times\mathcal{A}$, and $\beta\in\mathcal{B}(X,\widehat{\sigma}^2,\mathbb{P})$, we next introduce the backward SDE

$$\begin{aligned}\mathcal{Y}_s^{\mathbb{P},\alpha,\beta} \quad = \quad & \xi + \int_s^T \big(-c_r(X,\alpha_r,\beta_r) - k_r(X,\alpha_r,\beta_r)\mathcal{Y}_r^{\mathbb{P},\alpha,\beta} + \sigma_r(X,\beta_r)\lambda_r(X,\alpha_r)\cdot\mathcal{Z}_r^{\mathbb{P},\alpha,\beta}\big)\,dr \\ & - \int_s^T \mathcal{Z}_r^{\mathbb{P},\alpha,\beta}\cdot dX_r - \int_s^T d\mathcal{M}_r^{\mathbb{P},\alpha,\beta},\ \mathbb{P}-\text{a.s.}\end{aligned}$$

Let $\mathbb{P}^{\alpha,\beta}$ be the probability measure, equivalent to $\mathbb{P}$, defined by

$$\frac{d\mathbb{P}^{\alpha,\beta}}{d\mathbb{P}} \quad := \quad \mathcal{E}\Big(\int_t^T R_s(X,\beta_s)\lambda_s(X,\alpha_s)\cdot dW_s^{\mathbb{P}}\Big).$$

Then, the solution of the last linear backward SDE is given by:

$$\mathcal{Y}_t^{\mathbb{P},\alpha,\beta} \quad = \quad \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha 1_s,\beta_s\big)ds\,\middle|\,\mathcal{F}_t^+\right],\quad \mathbb{P}-\text{a.s.}$$

Following El Karoui, Peng & Quenez (1997), it follows from the comparison result for BSDEs that the processes $\mathcal{Y}^{\mathbb{P},\alpha,\beta}$ induce a stochastic control representation for $\mathcal{Y}^{\mathbb{P}}$ (see also Lemma A.3 in [24]). This is justified by Assumption 4.2, and we now obtain that:

$$\mathcal{Y}_t^{\mathbb{P}} \quad = \quad \operatorname*{essup}^{\mathbb{P}}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P},\alpha,\beta},\ \mathbb{P}-\text{a.s., for any } \mathbb{P}\in\overline{\mathcal{P}}_0.$$

This implies that

$$\mathcal{Y}_t^{\mathbb{P}} = \operatorname*{essup}^{\mathbb{P}}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P})} \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha_s,\beta_s\big)ds\,\middle|\,\mathcal{F}_t^+\right],$$

and therefore for any $\mathbb{P}\in\overline{\mathcal{P}}_0$, we have $\mathbb{P}-$a.s.

$$\begin{aligned}Y_t &= \operatorname*{essup}^{\mathbb{P}}_{(\mathbb{P}',\alpha,\beta)\in\overline{\mathcal{P}}_0(t,\mathbb{P},\mathbb{F}^+)\times\mathcal{A}\times\mathcal{B}(\widehat{\sigma}^2,\mathbb{P}')} \mathbb{E}^{\mathbb{P}'^{\alpha,\beta}}\left[\mathcal{K}_{t,T}^{(\alpha,\beta)}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{(\alpha,\beta)}(X)c_s\big(X,\alpha_s,\beta_s\big)ds\,\middle|\,\mathcal{F}_t^+\right] \\ &= \operatorname*{essup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}^+)} \mathbb{E}^{\mathbb{P}'}\left[\mathcal{K}_{t,T}^{\nu^{\mathbb{P}'}}(X)\xi - \int_t^T \mathcal{K}_{t,s}^{\nu^{\mathbb{P}'}}(X)c_s\big(X,\alpha_s^{\mathbb{P}'},\beta_s^{\mathbb{P}'}\big)ds\,\middle|\,\mathcal{F}_t^+\right],\end{aligned}$$

where we have used the connection between $\mathcal{P}_0$ and $\overline{\mathcal{P}}_0$ recalled at the end of Section 2.1. The desired result follows by classical arguments similar to the ones used in the proofs of Lemma 3.5 and Theorem 5.2 of [24].

By the above equalities, together with Assumption 4.2, it is clear that a probability measure $\mathbb{P} \in \mathcal{P}(t,x)$ is in $\mathcal{P}^\star(t,x,\xi)$ if and only if

$$\nu^\star_\cdot = (a^\star_\cdot, \beta^\star_\cdot)(X, Y_\cdot, Z_\cdot, \Sigma^\star_\cdot),$$

where $\Sigma^\star$ is such for any associated weak solution $\mathbb{P}^{\beta^\star}$ to (2.4), we have

$$K_T^{\mathbb{P}^{\beta^\star}} = 0, \quad \mathbb{P}^{\beta^\star} - \text{a.s.}$$

□

4.3 The main result

Theorem 4.6. *Let Assumptions 3.2, 4.2, and 4.3 hold true. Then*

$$V^P(t,x) \; = \; \sup_{y \geq R} \underline{V}(t,x,y) \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Proof. The inequality $V^P(t,x) \leq \sup_{y\geq R} \underline{V}(t,x,y)$ was already stated in Proposition 3.4. To prove the converse inequality we consider an arbitrary $\xi \in \Xi(t,x)$ and we intend to prove that Principal's objective function $J^P(t,x,\xi)$ can be approximated by $J^P(t,x,\xi^\varepsilon)$, where $\xi^\varepsilon = Y_T^{Z^\varepsilon,\Gamma^\varepsilon}$ for some $(Z^\varepsilon, \Gamma^\varepsilon) \in \mathcal{V}(t,x)$.

Step 1: Let (Y, Z, K) be the solution of the 2BSDE (4.4)

$$Y_t \;=\; \xi + \int_t^T F(s, X_\cdot, Y_s, Z_s, \widehat{\sigma}_s^2) ds - \int_t^T Z_s \cdot dX_s + \int_t^T dK_s,$$

where we recall again that the aggregated process K exists as a consequence of the aggregation result of Nutz [23], see Remark 4.1 in [24]. By Proposition 4.5, we know that for every $\mathbb{P}^\star \in \mathcal{P}(t,x,\xi)$, we have

$$K_T \;=\; 0, \;\; \mathbb{P}^\star - \text{a.s.}$$

For all $\varepsilon > 0$, define the absolutely continuous approximation of K:

$$K_t^\varepsilon \;:=\; \frac{1}{\varepsilon} \int_{(t-\varepsilon)\wedge 0}^t K_s ds, \quad t \in [0,T].$$

Clearly, K^ε is $\mathbb{F}^{\overline{\mathcal{P}}_0}-$predictable, non-decreasing $\overline{\mathcal{P}}_0-$q.s. and

$$K_T^\varepsilon = 0, \quad \mathbb{P}^\star - \text{a.s. for all} \quad \mathbb{P}^\star \in \mathcal{P}(t,x,\xi). \tag{4.6}$$

We next define for any $t \in [0,T]$ the process

$$Y_t^\varepsilon \;:=\; Y_0 - \int_0^t F_s(X, Y_s^\varepsilon, Z_s, \widehat{\sigma}_s^2) ds + \int_0^t Z_s \cdot dX_s - \int_0^t dK_s^\varepsilon, \tag{4.7}$$

and verify that $(Y^\varepsilon, Z, K^\varepsilon)$ solves the 2BSDE (4.4) with terminal condition $\xi^\varepsilon := Y_T^\varepsilon$ and generator F. This requires to check that K^ε satisfies the required minimality condition, which is obvious by (4.6).

Step 2: For $(t, x, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, notice that the map $\gamma \longmapsto H_t(x, y, z, \gamma) - F_t(x, y, z, \widehat{\sigma}_t^2(x)) - \frac{1}{2}\widehat{\sigma}_t^2(x) : \gamma$ is valued in $\mathbb{R}_+$, convex, continuous on the interior of its domain, attains the value 0 by Assumption 3.2, and is coercive by the boundedness of λ, σ, k. Then, this map is surjective on $\mathbb{R}_+$. Let $\dot{K}^\varepsilon$ denote the density of the absolutely continuous process K^ε with respect to the Lebesgue measure. Applying a classical measurable selection argument, we may deduce the existence of an $\mathbb{F}-$predictable process Γ^ε such that

$$\dot{K}_s^\varepsilon \;=\; H_s(X, \bar{Y}_s^\varepsilon, \bar{Z}_s, \bar{\Gamma}_s^\varepsilon) - F_s(X, \bar{Y}_s^\varepsilon, \bar{Z}_s, \widehat{\sigma}_s^2) - \frac{1}{2}\widehat{\sigma}_s^2 : \bar{\Gamma}_s^\varepsilon.$$

Substituting in (4.7), it follows that the following representation of Y_t^ε holds:

$$Y_t^\varepsilon \;=\; Y_0 - \int_0^t H_s(X, Y_s^\varepsilon, Z_s, \Gamma_s^\varepsilon)ds + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\int_0^t \Gamma_s^\varepsilon : d\langle X\rangle_s.$$

Step 3: The contract $\xi^\varepsilon := Y_T^\varepsilon$ takes the required form (3.3), for which we know how to solve Agent's problem, *i.e.* $V^A(t, x, \xi^\varepsilon) = Y_t$, by Proposition 3.3. Moreover, it follows from (4.6) that

$$\xi \;=\; \xi^\varepsilon, \quad \mathbb{P}^\star - \text{a.s.}$$

Consequently, for any $\mathbb{P}^\star \in \mathcal{P}^\star(t, x, \xi)$, we have

$$\mathbb{E}^{\mathbb{P}^\star}\big[\mathcal{K}_{t,T}^P U(\ell(X_T) - \xi^\varepsilon)\big] \;=\; \mathbb{E}^{\mathbb{P}^\star}\big[\mathcal{K}_{t,T}^P U(\ell(X_T) - \xi)\big],$$

which implies that $J^P(t, x, \xi) = J^P(t, x, \xi^\varepsilon)$. □

4.4 Example: coefficients independent of X

In this section, we consider the particular case in which

$$\sigma,\ \lambda,\ c,\ k, \text{ and } k^P \quad \text{are independent of} \quad x. \tag{4.8}$$

In this case, the Hamiltonian H introduced in (3.1) is also independent of x, and we re-write the dynamics of the controlled process $Y^{Z,\Gamma}$ as:

$$Y_s^{Z,\Gamma} \;:=\; Y_t - \int_t^s H_r\left(Y_r^{Z,\Gamma}, Z_r, \Gamma_r\right)dr + \int_t^s Z_r \cdot dX_r + \frac{1}{2}\int_t^s \Gamma_r : d\langle X\rangle_r, \;\; s \in [t, T].$$

By classical comparison result of stochastic differential equation, this implies that the flow $Y_s^{Z,\Gamma}$ is increasing in terms of the corresponding initial condition Y_t. Thus, optimally, Principal will provide Agent with the minimum reservation utility R he requires. In other words, we have the following simplification of Principal's problem, as a direct consequence of Theorem (4.6).

Proposition 4.7. *Let Assumptions 3.2, 4.2, and 4.3 hold true. Then in the context of* (4.8)*, we have:*

$$V^P(t, x) \;=\; \underline{V}(t, x, R) \quad \textit{for all} \quad (t, x) \in [0, T] \times \Omega.$$

We conclude the paper with two additional simplifications of interest.

Example 4.8 (Exponential utility)**.** (i) *Let* $U(y) := -e^{-\eta y}$*, and consider the case* $k \equiv 0$*. Then under the conditions of Proposition 4.7, it follows that*

$$V^P(t,x) \;=\; e^{\eta R}\underline{V}(t,x,0) \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5, corresponding to $\underline{V}$*, may be reduced to* $[0,T] \times \mathbb{R}^d$ *by applying the change of variables* $v(t,x,y) = e^{\eta y} f(t,x)$*.*
(ii) *Assume in addition that the liquidation function* $\ell(x) = h \cdot x$ *is linear for some* $h \in \mathbb{R}^d$*. Then, it follows that*

$$V^P(t,x) \;=\; e^{-\eta(h\cdot x - R)}\underline{V}(t,0,0) \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5, corresponding to $\underline{V}$*, may be reduced to an ODE on* $[0,T]$ *by applying the change of variables* $v(t,x,y) = e^{-\eta(h\cdot x - R)} f(t)$*.*

Example 4.9 (Risk-neutral Principal)**.** *Let* $U(x) := x$*, and consider the case* $k \equiv 0$*. Then under the conditions of Proposition 4.7, it follows that*

$$V^P(t,x) \;=\; -R + \underline{V}(t,x,0) \quad \textit{for all} \quad (t,x) \in [0,T] \times \Omega.$$

Consequently, the HJB equation of Theorem 3.5, corresponding to $\underline{V}$*, may be reduced to* $[0,T] \times \mathbb{R}^d$ *by applying the change of variables* $v(t,x,y) = -y + f(t,x)$*.*

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