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Dynamic Programming Formulation of Periodic Event-Triggered Control: Performance Guarantees and Co-Design

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***Abstract*— While potential benefits of choosing the transmissions times in a networked control system based on state or event information have been advocated in the literature, few general methods are available that guarantee closed-loop improvements over traditional periodic transmission strategies. In this paper, we propose event-triggered controllers that guarantee better quadratic discounted cost performance than periodic control strategies using the same average transmission rate. Moreover, we show that the performance of a method in the line of previous Lyapunov based approaches is within a multiplicative factor of periodic control performance, while using less transmissions. Our approach is based on a dynamic programming formulation for the co-design problem of choosing both transmission decisions and control inputs in the context of periodic event-triggered control for linear systems. A numerical example illustrates the advantages of the proposed method over traditional periodic control.**

I. Introduction

Event-triggered control (ETC) has emerged in recent years due to the need to reduce the computation and communication burden in (networked) control systems. Traditional digital control loops are designed under the premise that measurement data is transmitted periodically to a controller at a fixed rate, at which control values are also computed and actuators updated. The premise of ETC is to balance the need to transmit data and/or compute control laws with the need to enforce closed-loop stability and performance.

One line of research on ETC [1]–[5] proposes transmissions and control computations to be triggered only when they are needed in order to guarantee (directly or indirectly) a certain decrease condition for a Lyapunov function. In [1], [2], this Lyapunov function is previously designed assuming that communication links are ideal, and therefore it is guaranteed to decrease if transmissions occur sufficiently frequently. In Periodic Event-Triggered Control (PETC) [3] a scheduler periodically samples the plant and decides whether or not to compute and transmit control data. The co-design problem of synthesizing simultaneously the transmission triggering rule and the controller is tackled in [4], [5]. Another line of research on ETC [6]–[10], formulates the event-triggered problem in an optimal control framework in which the optimization cost penalizes transmissions. An estimation average cost problem was originally tackled in [6], and performance bounds for approximate methods to the problem appeared in [7], [8]. The works [9] and [10] concern the existence of certainty equivalence controllers for finite-horizon control problems with state and output observations, respectively. See also [11].

In this paper, we follow the PETC approach and propose a dynamic programming formulation for the co-design problem of choosing both the scheduling/transmission decisions and the control inputs in the context of linear systems. We propose event-triggered controllers that guarantee better or equal quadratic discounted performance than periodic transmission strategies using the same average transmission rate. Our method is a variant of rollout strategies (see [12, Ch. 6]) described as follows. We consider a set of *scheduling times* spaced by a *scheduling period* which is an integer multiple of a *baseline period* at which transmissions are allowed to occur. However, to reduce transmissions, only a subset of possible transmission times are used. At scheduling times, the transmission decisions and corresponding control inputs within a scheduling period are chosen as the ones that would minimize the cost if a periodic transmission strategy would be used afterwards, which in general also only uses a subset of possible transmission times. By imposing a limit on the number of transmissions over the initial scheduling period equal to the average number of transmissions of the periodic strategy used in the construction of the method we assure that both strategies have the same average transmission rate, while we prove that our proposed strategy always performs better or equal than the periodic strategy.

Another contribution of the present work is to show that the performance of a Lyapunov based strategy in the line of previous approaches [1]–[5] is within a multiplicative factor $(1 + \theta)$ of periodic control discounted cost performance, while using less transmissions. The positive parameter θ allows to trade transmissions for performance. The key to achieve this result, is to interpret existing event-triggered control policies as in [1]–[5] as rollout sub-optimal strategies for an optimal control problem that penalizes transmissions proportionally to the running cost, i.e., a quadratic function of the state and input. This differs from the problem formulation in [6]–[10] in which transmissions are penalized with a constant cost.

A numerical example considering a mass-spring system demonstrates the performance improvements of our strategy

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in comparison to standard periodic strategies.

The remainder of the paper is organized as follows. Section II formulates the problem considered in this paper. Section III describes the new ETC method establishing our main performance improvement result, while Section IV gives a performance bound for a method related with Lyapunov-based approaches. An illustrative example is given in Section V while Section VI provides concluding remarks.

Notation : The transpose of a matrix A is denoted by $A^\intercal$. The notation $0_{n\times m}$ indicates a matrix of zeros with n columns and m rows, 0_n denotes a vector with n zeros, and I_n denotes the identity matrix of dimension n. Dimensions are omitted when no confusion arises.

II. Problem Formulation

Consider a continuous-time plant modeled by the following stochastic differential equation

$$dx_C = (A_C x_C + B_C u_C)dt + B_\omega d\omega,\ \ x_C(0)=x_0,\ \ t\in\mathbb{R}_{\geq 0}, \tag{1}$$

where $x_C(t)\in\mathbb{R}^{n_x}$ is the state, $u_C(t)\in\mathbb{R}^{n_u}$ is the control input and ω is an n_w dimensional Wiener process with incremental covariance $I_{n_w}dt$ (cf. [13]); the pair (A_C, B_C) is assumed to be controllable. Performance is measured by the following discounted cost to be minimized

$$\mathbb{E}[\int_0^\infty e^{-\alpha_C t} g_C(x_C(t), u_C(t))dt], \tag{2}$$

where $g_C(x,u) := x^\intercal Q_C x + u^\intercal R_C u$, for positive semi-definite matrices Q_C and R_C, and $\alpha_C \in \mathbb{R}_{\geq 0}$. The pair $(A_C, Q_C^{\frac{1}{2}})$ is assumed to be observable. For the undiscounted case $\alpha_C = 0$ we assume that (1) is not driven by disturbances ($B_\omega = 0$) since otherwise (2) would be unbounded.

We assume that a scheduler-controller pair is collocated with the plant sensors and that it is connected to the actuators by a communication network. The scheduler-controller periodically samples the state of the plant x_C and decides whether or not to compute and transmit control and measurement data over a network to the actuators. Denote the sampling times by t_k, $k\in\mathbb{N}_0$, spaced by a baseline period τ, i.e., $t_k = k\tau$, $k\in\mathbb{N}_0$. Assuming that the actuation is held constant between sampling times and that there are no transmission delays we have

$$u_C(t) = u_C(t_k), \quad t\in[t_k, t_{k+1}).$$

Let $\{\sigma_k | k\in\mathbb{N}_0\}$ be the transmission scheduling sequence defined as

$$\sigma_k := \begin{cases} 1, & \text{if a transmission occurs at } t_k, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $x_k := x_C(t_k)$, $\hat{u}_k := u_C(t_k)$, $\xi_k := (x_k, \hat{u}_k)$, and u_k be the control input sent by the controller to the actuators at times t_k, $k\in\mathbb{N}_0$, that satisfy $\sigma_k = 1$, and $u_k := 0_{n_u}$ at times t_k, $k\in\mathbb{N}_0$, that satisfy $\sigma_k = 0$. Then, we can write

$$\xi_{k+1} = A_{\sigma_k}\xi_k + B_{\sigma_k}u_k + w_k, \quad k\in\mathbb{N}_0, \tag{3}$$

where

$$A_j := \begin{bmatrix} \bar{A}_\tau & (1-j)\bar{B}_\tau \\ 0 & (1-j)I \end{bmatrix}, \quad B_j := \begin{bmatrix} j\bar{B}_\tau \\ jI \end{bmatrix}, \quad j\in\{0,1\},$$

$\bar{A}_\tau := e^{A_C\tau}$, $\bar{B}_\tau := \int_0^\tau e^{A_C s}ds B_C$, and $\{w_k | k\in\mathbb{N}_0\}$ is a sequence of zero-mean independent Gaussian vectors with covariance $\mathbb{E}[w_k w_k^\intercal] = \Phi_\tau^w$, $\forall k\in\mathbb{N}_0$, where

$$\Phi_\tau^w := \begin{bmatrix} \bar{\Phi}_\tau^w & 0 \\ 0 & 0_{n_u\times n_u} \end{bmatrix}, \quad \bar{\Phi}_\tau^w := \int_0^\tau e^{A_C s} B_\omega B_\omega^\intercal e^{A_C^\intercal s} ds.$$

The expression for Φ_τ^w can be obtained from the arguments provided in [13, p. 82-85]. We assume that τ is non-pathological (see [14, p. 45]), and therefore the pair $(\bar{A}_\tau, \bar{B}_\tau)$ is controllable.

We are interested in a *co-design* problem of finding a policy, i.e., a set of functions

$$\pi = \{(\mu_0^\sigma(\mathsf{I}_0), \mu_0^u(\mathsf{I}_0)), (\mu_1^\sigma(\mathsf{I}_1), \mu_1^u(\mathsf{I}_1)), \dots, \},$$

for jointly designing the scheduling and control inputs

$$(\sigma_k, u_k) = (\mu_k^\sigma(\mathsf{I}_k), \mu_k^u(\mathsf{I}_k)), \quad k\in\mathbb{N}_0,$$

based on the information available to the scheduler-controller at time t_k,

$$\mathsf{I}_k := \{\xi_\ell, \sigma_\ell | 0\leq \ell < k\}\cup\{\xi_k\}, \quad k\in\mathbb{N}_0.$$

By keeping track of previous data in I_k, the scheduler-controller can, e.g., make decisions based on the number of previous transmissions up to time t_k or based on previous state values. The possible scheduling and control inputs belong to the set $\mathsf{U}_k(\mathsf{I}_k)$, i.e., $(\mu_k^\sigma(\mathsf{I}_k), \mu_k^u(\mathsf{I}_k)) \in \mathsf{U}_k(\mathsf{I}_k)$, $k\in\mathbb{N}_0$, where

$$\text{Co-design: } \mathsf{U}_k(\mathsf{I}_k) = \big(\{0\}\times\{0_{n_u}\}\big)\cup\big(\{1\}\times\mathbb{R}^{n_u}\big),$$

but our results can also be adapted to an emulation framework (see, e.g., [1], [3]), in which the control policy is fixed to a static state feedback law $u_k = Fx_k$ for $k\in\mathbb{N}_0$ that satisfy $\sigma_k = 1$, by considering

$$\text{Emulation: } \mathsf{U}_k(\mathsf{I}_k) = \big(\{0\}\times\{0_{n_u}\}\big)\cup\big(\{1\}\times\{Fx_k\}\big).$$

The discounted cost (2) for policy π can be shown to be given, apart from an additive constant term, by

$$J_\pi(\xi_0) := \mathbb{E}[\sum_{k=0}^\infty \alpha_\tau^k g(\xi_k, \mu_k^u(\mathsf{I}_k), \mu_k^\sigma(\mathsf{I}_k))], \tag{4}$$

where $\alpha_\tau := e^{-\alpha_C\tau}$, $g(\xi,u,j) := \xi^\intercal Q_j\xi + 2\xi^\intercal S_j u + u^\intercal R_j u$, and, for $j\in\{0,1\}$,

$$Q_j := \begin{bmatrix} \bar{Q}_\tau & (1-j)\bar{S}_\tau \\ (1-j)\bar{S}_\tau^\intercal & (1-j)\bar{R}_\tau \end{bmatrix}, \quad S_j := \begin{bmatrix} j\bar{S}_\tau \\ 0 \end{bmatrix}, \quad R_j := j\bar{R}_\tau,$$

where

$$\begin{bmatrix} \bar{Q}_\tau & \bar{S}_\tau \\ \bar{S}_\tau^\intercal & \bar{R}_\tau \end{bmatrix} := \int_0^\tau e^{\begin{bmatrix} A_C & B_C \\ 0 & 0 \end{bmatrix}^\intercal s} \begin{bmatrix} Q_C & 0 \\ 0 & R_C \end{bmatrix} e^{\begin{bmatrix} A_C & B_C \\ 0 & 0 \end{bmatrix} s} ds.$$

We omit the dependency of J_π on ξ_0 and for two policies π and ρ we use $J_\pi \leq J_\rho$ to denote $J_\pi(\xi_0) \leq J_\rho(\xi_0)$ for every

$\xi_0 \in \mathbb{R}^{n_x+n_u}$. The average transmission rate of policy π is defined as

$$R_\pi := \frac{1}{\tau}\limsup_{K\to\infty}\frac{1}{K}\mathbb{E}[\sum_{k=0}^{K-1}\mu_k^\sigma(\mathsf{I}_k)]. \tag{5}$$

A typical event-triggered control policy depends on some parameter, denoted here by γ, which allows to tune the trade-off between average transmission rate and performance (see, e.g, [1], [9]), and can therefore be considered a family of policies π_γ rather than a single policy. Consider the curve in the plane

$$(R_{\pi_\gamma}, J_{\pi_\gamma}), \tag{6}$$

parameterized by γ belonging to $[\underline{\gamma}, \bar{\gamma}]$ such that R_{π_γ} lies in a region of interest $[\underline{R}, \bar{R}]$. For example, the traditional periodic strategy can be parameterized by $\gamma = \tau$ and, under the observability and controllability assumptions for the problem stated above, is described by (cf. [12], [13]) $\pi_\tau = \{(\mu^\sigma_{\tau,0}, \mu^u_{\tau,0}), (\mu^\sigma_{\tau,1}, \mu^u_{\tau,1}), \dots\}$,

$$(\mu^\sigma_{\tau,k}(\mathsf{I}_k), \mu^u_{\tau,k}(\mathsf{I}_k)) = (1, \bar{K}_\tau x_k), \quad k \in \mathbb{N}_0,$$

where

$$\bar{K}_\tau := -(\bar{R}_\tau + \alpha_\tau \bar{B}_\tau^\intercal \bar{P}_\tau \bar{B}_\tau)^{-1}(\alpha_\tau \bar{B}_\tau^\intercal \bar{P}_\tau \bar{A}_\tau + \bar{S}_\tau^\intercal), \tag{7}$$

and $\bar{P}_\tau$ is the unique positive semi-definite solution to the algebraic Ricatti equation

$$\begin{aligned}&\bar{P}_\tau = \alpha_\tau \bar{A}_\tau^\intercal \bar{P}_\tau \bar{A}_\tau + \bar{Q}_\tau - \\ &(\alpha_\tau \bar{A}_\tau^\intercal \bar{P}_\tau \bar{B}_\tau + \bar{S}_\tau)(\bar{R}_\tau + \alpha_\tau \bar{B}_\tau^\intercal \bar{P}_\tau \bar{B}_\tau)^{-1}(\alpha_\tau \bar{B}_\tau^\intercal \bar{P}_\tau \bar{A}_\tau + \bar{S}_\tau^\intercal).\end{aligned} \tag{8}$$

This policy yields a curve (6) described by

$$(\frac{1}{\tau}, J_{\text{per},\tau}), \tag{9}$$

where

$$J_{\text{per},\tau} = x_0^\intercal \bar{P}_\tau x_0 + \frac{\alpha_\tau}{1-\alpha_\tau}\text{tr}(\bar{P}_\tau \bar{\Phi}^w_\tau). \tag{10}$$

We are interested in finding a family of polices that achieves a better trade-off performance versus average transmission rate than traditional periodic control in the sense that the corresponding curve (6) lies below the traditional periodic curve (9). This can be expressed in terms of the following problem statement.

Problem 1: Find a family of policies π_γ for which the following holds

$$J_{\pi_\gamma} \leq J_{\text{per},R_{\pi_\gamma}}, \tag{11}$$

for $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ such that the average transmission rate belongs to a given region of interest $R_{\pi_\gamma} \in [\underline{R}, \bar{R}]$.

□

III. Proposed Method and Main Result

Let $\mathcal{T}$ denote the set of transmission scheduling sequences with m transmissions in the first h time steps, where h is an integer multiple of m, and that corresponds to periodic transmission with period $q := \frac{h}{m}$ in the remaining time steps. Formally, there are $n_\mathcal{T} := \frac{h!}{(h-m)!m!}$ scheduling sequences $\{\sigma_k^i | k \in \mathbb{N}_0\} \in \mathcal{T}$, $i \in \mathcal{M}$, $\mathcal{M} := \{1, \dots, n_\mathcal{T}\}$, characterized by a set of schedules in the interval $0 \leq k \leq h-1$, denoted by

$$\nu^i = (\nu_0^i, \dots, \nu_{h-1}^i) \in \mathcal{I}, \quad i \in \mathcal{M},$$

where $\mathcal{I} := \{\nu \in \{0,1\}^h | \sum_{k=0}^{h-1}\nu_k = m\}$ and for $k \geq h$, for all $i \in \mathcal{M}$,

$$\sigma_k^i = \begin{cases} 1, \text{ if } k \text{ is an integer multiple of } q, \\ 0, \text{ otherwise.}\end{cases}$$

Our proposed method is based on solutions to optimal control subproblems in which the transmission scheduling sequence is fixed and belongs to the set $\mathcal{T}$. Hence we start by describing the optimal control input policy that minimizes (4) for a fixed scheduling sequence in $\mathcal{T}$ labeled by $i \in \mathcal{M}$, which can be derived by standard optimal control arguments (cf. [12], [13]).

Let P_i be the first matrix $P_i = W_{0,i}$ of the backward recursion

$$\begin{aligned} W_{h,i} &= \begin{bmatrix} \bar{P}_{q\tau} & 0 \\ 0 & 0_{n_u\times n_u}\end{bmatrix}, \\ W_{\kappa,i} &= F_{\nu_\kappa^i}(W_{\kappa+1,i}), \quad 0 \leq \kappa \leq h-1,\end{aligned}$$

where $\bar{P}_{q\tau}$ can be obtained from the solution to (8) (with τ replaced by $q\tau$) and

$$\begin{aligned}&F_0(P) := \alpha_\tau A_0^\intercal P A_0 + Q_0, \\ &F_1(P) := \alpha_\tau A_1^\intercal P A_1 + Q_1 \\ &-(S_1 + \alpha_\tau A_1^\intercal P B_1)(R_1 + \alpha_\tau B_1^\intercal P B_1)^{-1}(\alpha_\tau B_1^\intercal P A_1 + S_1^\intercal).\end{aligned}$$

Then the optimal control input policy for the scheduling sequence $\{\sigma_k^i | k \in \mathbb{N}_0\}$ is described in the interval $0 \leq \kappa \leq h-1$ by

$$u_k = \begin{cases} K_{k,i}x_k, \text{ if } \nu_k^i = 1, \\ 0_{n_u}, \text{ otherwise,}\end{cases}$$

where for $0 \leq \kappa \leq h-1$ such that $\nu_k^i = 1$,

$$K_{\kappa,i} := -(R_1 + \alpha_\tau B_1^\intercal W_{\kappa+1,i}B_1)^{-1}(\alpha_\tau B_1^\intercal W_{\kappa+1,i}A_1 + S_1^\intercal), \tag{12}$$

and for $k \geq h$,

$$u_k = \begin{cases} \bar{K}_{q\tau}x_k, \text{ if } k \text{ is an integer multiple of } q, \\ 0_{n_u}, \text{ otherwise.}\end{cases}$$

This policy yields the following discounted cost (4)

$$\xi_0^\intercal P_i \xi_0 + c_i + b, \quad i \in \mathcal{M}, \tag{13}$$

where $b := \frac{\alpha_\tau^h \alpha_{q\tau}}{1-\alpha_{q\tau}}\text{tr}(\bar{P}_{q\tau}\bar{\Phi}^w_{q\tau})$ and

$$c_i := \sum_{\kappa=1}^{h}\alpha_\tau^\kappa \text{tr}(W_{\kappa,i}\Phi^w_\tau), \quad i \in \mathcal{M}.$$

The proposed ETC method, described next, finds at each scheduling time the scheduling sequence in $\mathcal{T}$ that would optimize (4) if this scheduling sequence would be used thereafter along with a corresponding optimal policy for the control input.

Algorithm 2: (i) At *scheduling times* $\ell = jh$, $j \in \mathbb{N}_0$, choose the scheduling sequence from the set $\mathcal{T}$, labeled by $\iota \in \mathcal{M}$, that would lead to the smallest cost (4) if this fixed scheduling was used from time $\ell = jh$ onwards and an associated optimal policy was chosen for the control input, i.e., choose

$$\iota(\xi_\ell) = \text{argmin}_{r\in\mathcal{M}} \xi_\ell^\intercal P_r \xi_\ell + c_r. \tag{14}$$

(ii) For times $jh \leq k < (j+1)h$ pick the schedules $\sigma_k = \nu_{k-jh}^{\iota(\xi_\ell)}$ and the control inputs

$$u_k = \begin{cases} K_{k-jh,\iota(\xi_\ell)} x_k, & \text{if } \sigma_k = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and repeat (i) and (ii) at time $(j+1)h$.

□

In the terminology of Section II, Algorithm (2) corresponds to a family of policies parameterized by τ and described by $\rho_\tau = \{(\mu^\sigma_{\tau,0}, \mu^u_{\tau,0}), (\mu^\sigma_{\tau,1}, \mu^u_{\tau,1}), \dots\}$,

$$\begin{aligned}(\mu^\sigma_{\tau,k}(\mathsf{I}_k), \mu^u_{\tau,k}(\mathsf{I}_k)) = (\nu_{k-jh}^{\iota(\xi_{jh})}, K_{k-jh,\iota(\xi_{jh})} x_k), \\ jh \leq k < (j+1)h, \quad j \in \mathbb{N}_0.\end{aligned} \tag{15}$$

It is clear, from construction of Algorithm (2), that this policy yields an average transmission rate (5) equal to $\frac{1}{q\tau}$. The next result establishes that this policy performances better (or at least equally well) than the traditional periodic strategy with a corresponding transmission rate $\frac{1}{q\tau}$, and hence this policy provides a solution to Problem 1. Let J_{ρ_τ}, denote the cost (4) for policy ρ_τ.

Theorem 3: The following holds for every $\tau \in \mathbb{R}_{>0}$

$$J_{\rho_\tau} \leq J_{\text{per},q\tau}. \tag{16}$$

□

Note that Theorem (3) does not guarantee a strict performance improvement in (16) for our proposed method. However, this is typically the case in practice as we shall illustrate in the example of Section V.

IV. Lyapunov Based Strategy

In Section IV-A we review a Lyapunov Based ETC approach which can be found in [3] and which follows the line of work of [1]. We relate it to a suboptimal strategy for a special dynamic programming problem in an emulation context. We then adapt these ideas in Section IV-B to a co-design context providing another triggering rule with guaranteed discounted cost.

A. Lyapunov Based Approach in an Emulation Context

In the framework of Section II, we consider a noise free plant ($B_w = 0$), an emulation based approach, and the undiscounted case $\alpha_C = 0$. Let $e_k := \hat{u}_{k-1} - F x_k, k \in \mathbb{N}_0$, $\hat{u}_{-1} := 0_{n_u}$, and $z_k := [x_k^\intercal \;\; e_k^\intercal]^\intercal$. Then,

$$x_{k+1} = (\bar{A}_\tau + \bar{B}_\tau F) x_k + (1 - \sigma_k) \bar{B}_\tau e_k. \tag{17}$$

For a given τ, we assume that $(\bar{A}_\tau + \bar{B}_\tau F)$ is Schur, which is equivalent to the following condition: for every $Q \succ 0$ there exists $P \succ 0$ such that

$$(\bar{A}_\tau + \bar{B}_\tau F)^\intercal P (\bar{A}_\tau + \bar{B}_\tau F) - P = -Q. \tag{18}$$

We pick a matrix $Q \succ 0$ for which a matrix $P \succ 0$ is obtained. If we define a Lyapunov function

$$V(x) := x^\intercal P x, \tag{19}$$

we obtain that (18) is equivalent to the following condition

$$\forall_{x_k \in \mathbb{R}^n}, \;\; V(x_{k+1}) - V(x_k) = -x_k^\intercal Q x_k, \tag{20}$$

when $\sigma_k = 1$. Suppose that in order to reduce transmissions, we are willing to loose some of this guaranteed decrease of this Lyapunov function by relaxing (20) to

$$V(x_{k+1}) - V(x_k) \leq -\eta x_k^\intercal Q x_k, \tag{21}$$

for some $0 < \eta < 1$. Then, the scheduler should only demand a control update if (21) is not met when $\sigma_k = 0$, which, using (17), (18), is equivalent to

$$(\sigma_k, u_k) = \begin{cases} (1, F x_k) \text{ if } h(z_k) > 0 \\ (0, 0_{n_u}) \text{ otherwise} \end{cases}, \tag{22}$$

where, for $z \in \mathbb{R}^{n_x + n_u}$,

$$h(z) := z^\intercal \begin{bmatrix} -(1-\eta) Q & (\bar{A}_\tau + \bar{B}_\tau F)^\intercal P \bar{B}_\tau \\ \bar{B}_\tau^\intercal P (\bar{A}_\tau + \bar{B}_\tau F) & \bar{B}_\tau^\intercal P \bar{B}_\tau \end{bmatrix} z.$$

One can easily check that the condition $h(z_k) \leq 0$, which means that (21) holds, is implied by $\|e_k\| \leq r\|x_k\|$ for sufficiently small $r > 0$, which is a simple condition often associated with ETC (cf. [1]). Moreover, noticing that we can pick $\alpha > 0$ such that $Q \succ \alpha P$, we have that $\hat{h}(z_k) > 0$ implies that $h(z_k) > 0$, where

$$\hat{h}(z) := z^\intercal \begin{bmatrix} -(1-\eta) \alpha P & (\bar{A}_\tau + \bar{B}_\tau F)^\intercal P \bar{B}_\tau \\ \bar{B}_\tau^\intercal P (\bar{A}_\tau + \bar{B}_\tau F) & \bar{B}_\tau^\intercal P \bar{B}_\tau \end{bmatrix} z.$$

Condition $\hat{h}(z_k) > 0$ matches one of the triggering conditions provided in [3].

We reinterpret triggering rule (22) as a suboptimal policy to a problem of minimizing a cost similar to (4), which can be described as follows. At each time step t_ℓ, $t_{k+1} - t_k = \tau$, $k \in \mathbb{N}_0$, the scheduler decides to transmit or not based on which decision would lead to the smallest cost, if from then on the scheduler would decide to transmit at every time step t_k, $k > \ell$. This can be seen as a rollout policy similar to Algorithm 2 with a unitary decision horizon $h = 1$ and with a different set of scheduling options $\mathcal{I} = \{0, 1\}$. Since this set of scheduling options does no longer impose rate constraints we encourage the scheduler to save transmissions by penalizing them in (4) proportionally to the running cost

$$g(\xi, u, j) = (1 + \theta j)\big(\xi^\intercal Q_j \xi + 2\xi^\intercal S_j u + u^\intercal R_j u\big). \tag{23}$$

In this section, to directly relate this approach with the Lyapunov based approach we consider a simpler cost

$$g(\xi, u, j) = (1 + \theta j) x^\intercal Q x \tag{24}$$

where Q is the matrix choosen in (18), and consider (23) only in the next section.

Using (18), we can conclude that the cost (4), (24) for the policy $(\sigma_k, u_k) = (1, Fx_k)$, $\forall k \in \mathbb{N}_0$, is given by $(1+\theta)x_0^\intercal P x_0$. Thus, at time ℓ the cost of option $\sigma_\ell = 1$ (assuming that from this iteration onwards transmissions occur at every time step) is given by

$$h_1(z_\ell) = (1+\theta)x_\ell^\intercal P x_\ell$$

whereas the cost of option $\sigma_\ell = 0$ is given by

$$h_0(z_\ell) = x_\ell^\intercal Q x_\ell + \\ ((\bar{A}_\tau + \bar{B}_\tau F)x_\ell + \bar{B}_\tau e_\ell)^\intercal (1+\theta) P ((\bar{A}_\tau + \bar{B}_\tau F)x_\ell + \bar{B}_\tau e_\ell).$$

Thus, the rollout policy described above boils down to

$$(\sigma_k, u_k) = \begin{cases} (1, Fx_k), \text{ if } h_1(z_k) < h_0(z_k) \\ (0, 0_{n_u}), \text{ otherwise.} \end{cases} \tag{25}$$

If we make

$$\frac{1}{1+\theta} = \eta, \tag{26}$$

we obtain, using (17), (18), that

$$h_1(z_k) < h_0(z_k) \Leftrightarrow h(z_k) > 0$$

and therefore (25) is equivalent to (22).

Condition (26) has the following interpretation. If the penalty on transmissions θ is close to zero then η is close to one, and (21) may be rarely satisfied when $\sigma_k = 0$, which means that we may expect many transmissions. On the other hand, if we significantly penalize transmissions by making θ large then η is close to zero and (22) may be often satisfied when $\sigma_k = 0$, which means that we may expect less transmissions.

B. Co-design with Performance Guarantees

We adapt the interpretation given in Section IV-A for the Lyapunov based strategy (22) to generalize it to a co-design context in the general framework of Section II, i.e., we allow for the plant (1) to be disturbed by white noise and consider a general $\alpha_C \geq 0$. To this effect, we consider that at each time step t_ℓ, the scheduler-controller pair decides to transmit or not based on which decision would lead to the smallest cost, if from then on the scheduler would decide to transmit at every time step t_k, $k > \ell$, and an optimal control policy would be used for the control input. Considering cost (4), (23), the decision of transmitting $\sigma_\ell = 1$ yields a cost proportional to (10), which is given by

$$f_1(\xi_\ell) := (1+\theta)x_\ell^\intercal \bar{P}_\tau x_\ell + a$$

where $a := (1+\theta)\frac{\alpha_\tau}{1-\alpha_\tau}\mathrm{tr}(\bar{P}_\tau \bar{\Phi}_\tau^w)$ and the corresponding control input is given by

$$u_k = \bar{K}_\tau x_k.$$

where $\bar{K}_\tau$ is described by (7). The decision of not transmitting $\sigma_\ell = 0$ (and transmitting for $j \geq \ell$) yields the cost

$$f_0(\xi_\ell) := x_\ell^\intercal \bar{Q}_\tau x_\ell + 2x_\ell^\intercal \bar{S}_\tau \hat{u}_\ell + \hat{u}_\ell^\intercal \bar{R}_\tau \hat{u}_\ell \\ + (\bar{A}_\tau x_\ell + \bar{B}_\tau \hat{u}_\ell)^\intercal (1+\theta)\alpha_\tau \bar{P}_\tau (\bar{A}_\tau x_\ell + \bar{B}_\tau \hat{u}_\ell) + a.$$

Thus, this policy can be described by

$$(\sigma_k, u_k) = \begin{cases} (1, \bar{K}_\tau x_k), \text{ if } f_1(\xi_k) < f_0(\xi_k) \\ (0, 0_{n_u}), \text{ otherwise.} \end{cases} \tag{27}$$

Note that although (27) takes a similar form to the policy obtained in the emulation context (25), here the gain $\bar{K}_\tau$ in (27) is uniquely defined, whereas in an emulation context the gain F can be taken as any gain that yields the closed-loop stable.

The next theorem gives a bound on the performance of this strategy. Let J_{δ_τ} denote the cost (4) (without penalizing transmissions) of the family of policies (27), parameterized by τ.

Theorem 4: The following holds for every $\tau \in \mathbb{R}_{>0}$

$$J_{\delta_\tau} \leq (1+\theta)J_{\mathrm{per},\tau}. \tag{28}$$

□

Note that contrarily to the policy provided by Algorithm 2 we do not quantify the average transmission rate of policy (27). We can only state that, by construction, it leads to less transmissions than traditional periodic control with rate $\frac{1}{\tau}$. Note also that Theorem 4 subsumes that the family of policies (27) is parameterized by τ for a fixed θ, which is different from parameterizing such policies by θ, while keeping τ fixed.

V. EXAMPLE

Consider two unitary masses on a frictionless surface connected by an ideal spring and moving along a one dimensional axis. The control input is a force acting on the first mass. The state vector is $x_C = \begin{bmatrix} x_1 & x_2 & v_1 & v_2 \end{bmatrix}^\intercal$, where x_i, v_i are the displacements and velocities of the mass $i \in \{1, 2\}$, respectively, and the plant model (1) is described by

$$A_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa_m & \kappa_m & 0 & 0 \\ \kappa_m & -\kappa_m & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{29}$$

where κ_m is the spring coefficient. We set the initial state to $x_0 = [-1\,1\,0\,0]^\intercal$, meaning that the masses start with zero velocity and at opposite distances from their equilibrium values. We assume no disturbances act on the plant, i.e., $B_\omega = 0$. Consider the following performance index,

$$\int_0^\infty x_1(t)^2 + x_2(t)^2 + 0.1 u_C(t)^2 dt \tag{30}$$

which takes the form (2) with $\alpha_C = 0$. The matrix A_C has two eigenvalues at zero and two complex conjugates eigenvalues at $\pm\sqrt{2k_m}i$. We set $k_m = 2\pi^2$, implying that the sampling period must be different from pathological sampling periods 0.5κ, $\kappa \in \mathbb{N}$, so that the discretization of the plant is controllable [14]. Fig. 1 plots the (normalized) performance (30) obtained with the traditional periodic control strategy and with the rollout strategy described by Algorithm 2 with parameters $h = 6$, $m = 2$, $q = 3$, for several values of the average transmission period $q\tau$ in

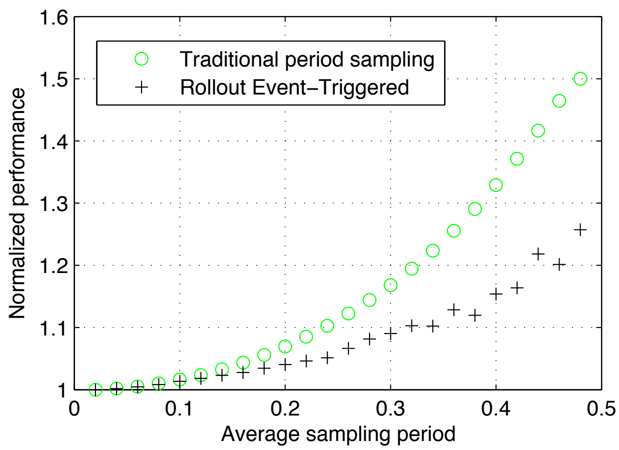

Fig. 1. Comparison of ETC and periodic policies for mass-spring example

the range $[0, 0.5]$. The performance (30) is normalized with respect to the optimal LQR performance achievable by a continuous-time controller. For small average transmission periods the methods perform very closely. In fact, periodic control approaches the optimal performance (2) achievable by a continuous-time controller when the sampling period tends to zero (cf. [14]) so there is little room for improvements. However, for larger transmission periods the rollout strategy obtains significant performance improvements over traditional periodic control.

VI. Concluding Remarks

We proposed a class of event-triggered strategies that can guarantee better quadratic performance than traditional periodic strategies. The key idea relies on a variant of rollout algorithms [12]. It is interesting to note that the general property of rollout algorithm discussed in [12, Vol 1,p. 341], which states that rollout algorithms achieve a substantial performance improvement over the base heuristics at the expense of extra computation, can be translated into an ETC context through our proposed method. In fact, our method achieves performance improvements over traditional periodic sampling (which plays the role of a base heuristic) at the expense of extra computations for scheduling decisions.

Future work includes: (i) considering an average cost (see [12]) instead of the discounted cost considered here; (ii) output-feedback problem, in which only a subset of the state variables are available to the scheduler-controller pair; and (iii) non-Gaussian stochastic models for the disturbances acting on the plant.

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