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DYNAMIC RESPONSE ANALYSIS OF A COMBINED SYSTEM OF FRAMED TUBED, SHEAR CORE AND OUTRIGGER-BELT TRUSS

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ABSTRACT

In this paper, an analytical approach is presented for the determination of the natural frequencies of tall structures with a combined system of the framed tube, shear core and outrigger-belt truss. It has been assumed that the structure has variable stiffness and mass along the height. The framed tube is modeled as a cantilevered beam with variable box cross section and effects of outrigger-belt truss and shear core on the framed tube system are modeled as a concentrated moment applied at outrigger-belt truss locations. Through repetitive integrations, the governing partial differential equations are converted into weak form integral equations. By applying the boundary conditions, the integration constants are determined. The mode shape function is approximated by a power series. Substitution of the power series into weak form integral equations results in a system of linear algebraic equations. The natural frequencies are determined by calculation of the non-trivial solution for the resulting system of equations. Accuracy of the proposed method is verified through several numerical examples, in which the results of the analysis are compared with those obtained from other references.

Keywords: Tall structure; weak form integral equation; framed tube; shear core; outrigger-belt truss; natural frequency.

1. INTRODUCTION

The natural frequency of a tall structure is one of the most important parameters that effects on the response of the structure to earthquake excitation. Hence, it is necessary to develop new and simple methods for free vibration analysis and determination of the natural frequencies and mode shape functions. In existing real tall structures, the stiffness and mass of the structure changes along the height. Therefore, modeling of the tall structure by a cantilevered beam with variable stiffness and mass may provide a realistic distribution of mass and stiffness desired for accurate structural analysis. In recent years, tubular building has been accepted as

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an economical and developed structural system. This system consists of closely spaced exterior columns along the periphery of the structure interconnected by deep spandrel beams of each floor level. This produces a system of rigidly jointed orthogonal frame panels forming a rectangular tube which acts as a cantilever hollow box (Fig. 1).

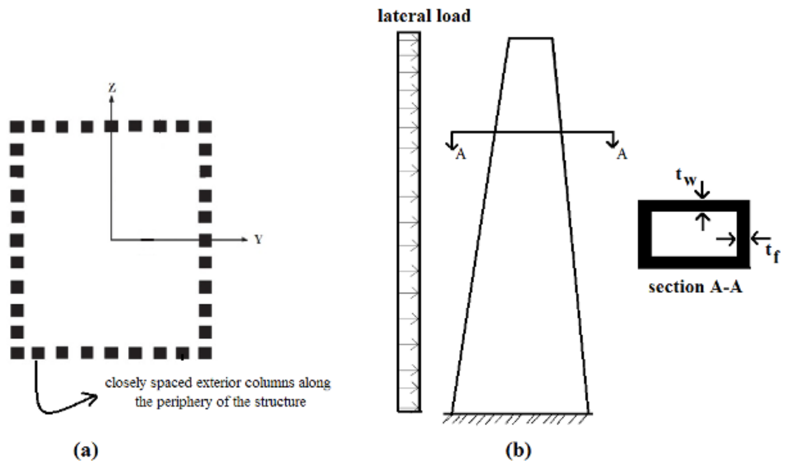

Figure 1. Equivalent structure of tubular building with variable stiffness and mass, (a): actual structure, (b): equivalent cantilevered beam with variable hollow box cross section

In tubular buildings, flexibility of spandrel beams produces shear lag phenomenon which has the effect of increasing axial stress in corner columns and decreasing the axial stress in the inner columns while reducing lateral stiffness of the structure [1]. Tubular buildings combined with shear core and outrigger-belt truss have shear-flexural behavior in vibration, hence provide sufficient rigidity and strength, limited lateral displacement, less material consumption and high flexibility provided in the internal space design in compare to commonly framed structures. It should be noted that when buildings taller than a certain limit are to be constructed, common structural systems will no longer be suitable. This is because of the fact that in tall buildings, rigidity and stability criteria become more important than the strength criterion [2].

Exact analysis of tall structure (3-D analysis) is expensive, but approximate methods presented for free vibration analysis of the structures are suitable solutions for preliminary design stage. Many researchers have investigated free vibration of the tall structures using various approaches [3-15]. An analytical model for the dynamic analysis of tall buildings with a shear wall–frame structural system has been proposed by Park *et al.* [16]. It has been shown that the deformed shape of the shear wall–frame structural system is the combination of flexural mode and shear mode. A modified theory on the premise that a frame-wall system, deforming in shear and flexural modes, can be separated into two substructures that lie above and below the point of counter-flexure in the base story columns has been developed by Kazaz *et al.* [17]. Rahgozar *et al.* [18] presented a dynamic analysis of the combined system of framed tube and shear walls by Galerkin method using B-spline functions. Rahgozar *et al.* [1] determined the optimum location of a belt truss reinforcing system on tall buildings such that the displacements due to lateral loadings would generate the least amounts of stress and strain in building's structural members. Malekinejad and rahgozar [2] calculated the natural frequencies and mode shapes of multistory buildings with combined system of framed tube, shear core and outrigger- belt trusses systems. Timoshenko's beam model, which considers the influence of shear and flexural deformation, has been used in modeling of framed tube structures by Kamgar *et al.* [19]. Lee [20] has presented an approximate solution procedure for free vibration analysis of tube-in-tube tall buildings using the power-series solution method. The first natural frequency of tall buildings including framed tube, shear core, belt truss and outrigger system with multiple jumped discontinuities in the cross section of framed tube and shear core has been determined by Kamgar and Saadatpour [21]. They partitioned the entire length of the tall building into uniform segments between each two successive discontinuity points and by applying the continuity conditions in conjunction with different segments obtained the unique mode shape for mentioned system. The optimal outrigger placement of tall structure using topology optimization has been determined by Lee *et al.*[22]. Saffari *et al.* [23], Mohammadnejad *et al.* [24], Saffari *et al.* [25] and Mohammadnejad [26] presented an analytical approach for determination of the natural frequencies of non-prismatic Beams. They converted the governing differential equations to its weak-form integral equations. The continuum approach has been used for conversion of the governing equations of wall-frame structures with outriggers into solvable equations by Lee *et al.,* [15]. They idealized the whole structure as a shear-flexural cantilever with rotational springs. Kwan [27] has proposed an approximate method for analysis of framed tube structures with the shear lag effects.

In this paper, a new and simple analytical approach is proposed for approximate analysis of tall structure with a combined system of framed tube, shear core and outrigger-belt truss. The tall structure is modeled by a cantilevered beam with variable stiffness and mass along the height, hence the governing partial differential equation with variable coefficients is solved in order to calculate the natural frequencies. The effects of outrigger-belt truss and shear core on the framed tube system are modeled as a concentrated moment applied at outrigger-belt truss locations. Behavior of the tall structure is equivalent to a cantilevered beam with variable hollow box cross section that has been constrained at belt truss locations with rotating springs (Figs. 1, 2).

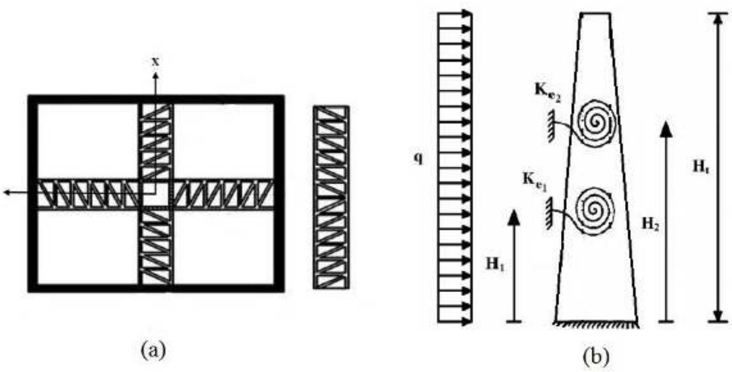

Figure 2. Constrained beam at belt truss locations with rotating springs, a: Plan of the original structure at location of Belt truss with outrigger b: Equivalent rotational springs including the effect of the belt truss and outrigger system on the framed tube

2. METHODOLOGY: WEAK FORM OF DIFFERENTIAL EQUATIONS

The governing differential equation for free vibration of a beam with variable stiffness and mass is a partial differential equation with variable coefficients. Many mathematical techniques may be employed to determine the numerical solution or the approximate solution for this equation. The presented approach in this paper for conversion of the governing partial differential equation into solvable one is based on the conversion of the governing equation into its weak form. A differential equation includes a function and its derivatives. The weak form of the differential equation is obtained through the repetitive integration of the initial equation. The integration continues till the resulting integral equation, includes only the function itself after the last integration stage; derivatives of the function will have been eliminated due to the integration. The solution of the weak form of the differential equation instead of the initial equation has many applications in the finite elements analysis [28].

3. FORMULATION AND SOLUTION

3.1 Conversion of the governing partial differential equation into its weak form

Consider a tall structure with a combined system of framed tube, shear core and outrigger-belt truss elements which has variable mass and stiffness along the height and subjected to the action of transverse loading, q, distributed along the its height. Accounting for total potential energy of the system and applying Hamilton's principle, the governing equation for equivalent beam is given as follows [2]:

$$\frac{\partial}{\partial x}[GA(\mathrm{x})\frac{\partial}{\partial x}y(\mathrm{x},t)]-\frac{\partial^2}{\partial x^2}[EI(\mathrm{x})\frac{\partial^2}{\partial x^2}y(\mathrm{x},t)]-m(\mathrm{x})\frac{\partial^2}{\partial t^2}y(\mathrm{x},t)+q(\mathrm{x},t)=0,$$
$$0<x<L \qquad (1)$$

In which $y(\mathrm{x},t), L, m(\mathrm{x}), EI(\mathrm{x})$ and $q(\mathrm{x},t)$ are the transverse displacement, height of structure, mass per unit length, bending stiffness and distributed applied load, respectively. $GA(\mathrm{x})$ is the shear stiffness which depends on both shear modulus of elasticity G and cross-sectional area $A(\mathrm{x})$. In order to treat free vibration of the structure, it is necessary to take $q(\mathrm{x},t)=0$. If motion is represented by a harmonic vibration, the transverse displacement is obtained using the following relation:

$$y(x,t)=\phi(x)e^{\mathrm{i}\omega t} \qquad (2)$$

where $\phi(x)$ and ω are the mode shape function and natural frequency of the beam, respectively. Substitution of relationship (2) into Eq. (1) leads to a single-variable equation in terms of location, as follows:

$$\frac{\mathrm{d}}{\mathrm{dx}}[GA(\mathrm{x})\frac{\mathrm{d}}{\mathrm{dx}}\phi(\mathrm{x})]-\frac{\mathrm{d}^2}{\mathrm{dx}^2}[EI(\mathrm{x})\frac{\mathrm{d}^2}{\mathrm{dx}^2}\phi(\mathrm{x})]+\omega^2 m(\mathrm{x})\phi(\mathrm{x})=0, \qquad 0<x<L \tag{3}$$

For further convenience, the following variables are introduced:

$$\xi=\frac{x}{L} \tag{4}$$

Substituting variable (4) into Eq. (3) leads to:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[L^2GA(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)]-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}[EI(\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)]+\omega^2L^4 m(\xi)\phi(\xi)=0, \qquad 0<\xi<1 \tag{5}$$

Eq. (5) is, in fact, the free vibration equation of the tall structure based on the non-dimensional variable ξ . In order to transform Eq. (5) to its weak form, both sides of Eq. (5) are integrated twice with respect to ξ within the range 0 to ξ . The results are the integral equations as follows:

$$L^2GA(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)-\frac{\mathrm{d}}{\mathrm{d}\xi}[EI(\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)]+\omega^2L^4\int_0^\xi m(s)\phi(s)ds=C_1 \tag{6}$$

$$L^2GA(\xi)\phi(\xi)+\int_0^\xi[\omega^2L^4(\xi-s)m(s)-L^2GA'(s)]\phi(s)ds$$

$$-EI(\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)=C_1\xi+C_2 \tag{7}$$

Further, integration from both sides of Eq. (7) twice with respect to ξ from 0 to ξ yields:

$$\int_0^\xi[L^2GA(s)-L^2G(\xi-s)A'(s)-EI''(s)+\frac{\omega^2L^4}{2}(\xi-s)^2m(s)]\phi(s)ds$$

$$-EI(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)+EI'(\xi)\phi(\xi)=\frac{C_1}{2}\xi^2+C_2\xi+C_3 \tag{8}$$

$$\int_0^\xi[L^2G(\xi-s)A(s)-\frac{L^2G}{2}(\xi-s)^2A'(s)+2EI'(s)-(\xi-s)EI''(s)+\frac{\omega^2L^4}{6}(\xi-s)^3m(s)]\phi(s)ds$$

$$-EI(\xi)\phi(\xi)=\frac{C_1}{6}\xi^3+\frac{C_2}{2}\xi^2+C_3\xi+C_4 \tag{9}$$

Eq. (9) is the integral equation of the weak form of Eq. (5). As can be seen, Eq. (5) includes a fourth order derivative of the mode shape function and after four successive integrations, Eq. (9) includes only the mode shape function itself. In Eq. (9) C_1, C_2, C_3 and C_4 are the integration constants which are determined through boundary conditions of both ends of the beam. Eqs. (6)-(9) are applicable for determination of the integration constants.

3.2 Boundary conditions

Since the original structure is modeled by an equivalent cantilevered beam, the following boundary conditions must be considered for the cantilevered beam:

$$\begin{cases} x=0 \quad y(0,\mathrm{t})=0 \quad or \quad \xi=0 \quad \phi(0)e^{i\omega t}=0 \rightarrow \quad \phi(0)=0 & (10) \\ x=0 \quad \dfrac{\partial y}{\partial x}(0,\mathrm{t})=0 \quad or \quad \xi=0 \quad \dfrac{\mathrm{d}\phi}{L\mathrm{d}\xi}(0)e^{i\omega t}=0 \rightarrow \quad \dfrac{\mathrm{d}\phi}{\mathrm{d}\xi}(0)=0 & (11) \end{cases}$$

Also, the following boundary conditions are established for a tall structure with a combined system of framed tube, shear core and belt truss elements:

$$\begin{cases} \left[-GA(\mathrm{x})\dfrac{\partial}{\partial x}y(\mathrm{x},t)+\dfrac{\partial}{\partial x}[EI(\mathrm{x})\dfrac{\partial^2}{\partial x^2}\mathrm{y}(\mathrm{x},t)]\right]_{x=L}=0 \\ or \\ \left[-L^2GA(\xi)\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)+\dfrac{\mathrm{d}}{\mathrm{d}\xi}[EI(\xi)\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)]\right]_{\xi=1}=0 \end{cases} \tag{12}$$

$$\begin{cases} \left[-EI(\mathrm{x})\dfrac{\partial^2}{\partial x^2}y(\mathrm{x},t)-2k_{e_1}\dfrac{\partial}{\partial x}y(L_1,t)-2k_{e_2}\dfrac{\partial}{\partial x}y(L_2,t)\right]_{x=L}=0 \\ or \\ \left[-EI(\xi)\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right]_{\xi=1}=2k_{e_1}L\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi_1)+2k_{e_2}L\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi_2) \end{cases} \tag{13}$$

In the above relations, k_{e_1} and k_{e_2} are the equivalent stiffness values of rotational springs placed at $x=L_1$ and $x=L_2$ respectively. L_1 and L_2 are location of belt truss. In Eq. (13) $\xi_1=\dfrac{L_1}{L}$ and $\xi_2=\dfrac{L_2}{L}$ are applied. The effect of belt truss and shear core on the framed tube structure are modeled as a concentrated moment applied at belt truss locations; which acts in the opposite direction of rotation created by the lateral loads(Fig. 2). Application of the condition (10) at Eq. (9) and conditions (10) and (11) at Eq. (8) leads to:

$$C_3=0, \quad C_4=0 \tag{14}$$

Also, Application of the condition (12) at Eq. (6) leads to:

$$\int_0^1 [\omega^2 L^4 m(s)]\phi(s)ds = C_1 \tag{15}$$

Substitution of the condition (13) and relation (15) into Eq. (7) yields:

$$\int_0^1 [\omega^2 L^4 (1-s)m(s) - L^2 GA'(s) - \omega^2 L^4 m(s)]\phi(s)ds$$
$$+ L^2 GA(1)\phi(1) + 2k_{e_1} L \frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_1) + 2k_{e_2} L \frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_2) = C_2 \tag{16}$$

It is easily found that in Eq. (16), $\phi(1), \frac{d\phi}{d\xi}(\xi_1)$ and $\frac{d\phi}{d\xi}(\xi_2)$ are also unknown. As a consequence, two other independent equations are needed for uniquely determining C_2. Substitution of C_1, C_3 and C_4 into Eq. (9) results in an equation which can be used for calculation of $\phi(\xi)$ as follows:

$$\phi(\xi) = \int_0^{\xi} [\frac{L^2 G}{EI(\xi)}(\xi - s)A(s) - \frac{L^2 G}{2EI(\xi)}(\xi - s)^2 A'(s) + \frac{2I'(s)}{I(\xi)} - \frac{(\xi - s)I''(s)}{I(\xi)}$$
$$+ \frac{\omega^2 L^4}{6EI(\xi)}(\xi - s)^3 m(s)]\phi(s)ds - \int_0^1 [\frac{\omega^2 L^4}{6EI(\xi)}\xi^3 m(s)\big]\phi(s)ds - \frac{C_2}{2EI(\xi)}\xi^2 \tag{17}$$

$\phi(1)$ can be calculated by setting $\xi = 1$ into Eq. (17). The result is as follows:

$$\phi(1) = \int_0^1 [\frac{L^2 G}{EI(1)}(1-s)A(s) - \frac{L^2 G}{2EI(1)}(1-s)^2 A'(s) + \frac{2I'(s)}{I(1)} - \frac{(1-s)I''(s)}{I(1)} + \frac{\omega^2 L^4}{6EI(1)}(1-s)^3 m(s)$$
$$- \frac{\omega^2 L^4}{6EI(1)} m(s)\big]\phi(s)ds - \frac{C_2}{2EI(1)} \tag{18}$$

Also, substitution of C_1, C_3 and Eq. (17) into Eq. (8) results in the following equation which can be used for calculation of $\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)$:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)=\int_0^{\xi}[\frac{f_1(\xi,s)}{EI(\xi)}+\frac{I'(\xi)}{EI^2(\xi)}f_2(\xi,s)]\phi(s)ds-\int_0^1[\frac{\omega^2L^4I'(\xi)}{6EI^2(\xi)}\xi^3m(s)+\frac{\omega^2L^4}{2EI(\xi)}\xi^2m(s)]\phi(s)ds$$
$$-[\frac{I'(\xi)\xi^2}{2EI^2(\xi)}+\frac{\xi}{EI(\xi)}]C_2 \tag{19}$$

In which $f_1(\xi,s)$ and $f_2(\xi,s)$ are obtained as follows:

$$\begin{cases} f_1(\xi,s)=L^2GA(s)-L^2G(\xi-s)A'(s)-EI''(s)+\dfrac{\omega^2L^4}{2}(\xi-s)^2m(s) \\ f_2(\xi,s)=L^2G(\xi-s)A(s)-\dfrac{L^2G}{2}(\xi-s)^2A'(s)+2EI'(s)-(\xi-s)EI''(s)+\dfrac{\omega^2L^4}{6}(\xi-s)^3m(s) \end{cases} \tag{20}$$

$\frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_1)$ and $\frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_2)$ can be calculated by setting $\xi=\xi_1$ and $\xi=\xi_2$ into Eq. (19) respectively. The results are as follows:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_1)=\int_0^{\xi_1}[\frac{f_1(\xi_1,s)}{EI(\xi_1)}+\frac{I'(\xi_1)}{EI^2(\xi_1)}f_2(\xi_1,s)]\phi(s)ds-\int_0^1[\frac{\omega^2L^4I'(\xi_1)}{6EI^2(\xi_1)}\xi_1^3m(s)+\frac{\omega^2L^4}{2EI(\xi_1)}\xi_1^2m(s)]\phi(s)ds$$
$$-[\frac{I'(\xi_1)\xi_1^2}{2EI^2(\xi_1)}+\frac{\xi_1}{EI(\xi_1)}]C_2 \tag{21}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi}(\xi_2)=\int_0^{\xi_2}[\frac{f_1(\xi_2,s)}{EI(\xi_2)}+\frac{I'(\xi_2)}{EI^2(\xi_2)}f_2(\xi_2,s)]\phi(s)ds-\int_0^1[\frac{\omega^2L^4I'(\xi_2)}{6EI^2(\xi_2)}\xi_2^3m(s)+\frac{\omega^2L^4}{2EI(\xi_2)}\xi_2^2m(s)]\phi(s)ds$$
$$-[\frac{I'(\xi_2)\xi_2^2}{2EI^2(\xi_2)}+\frac{\xi_2}{EI(\xi_2)}]C_2 \tag{22}$$

By substitution of $\phi(1)$, $\frac{d\phi}{d\xi}(\xi_1)$ and $\frac{d\phi}{d\xi}(\xi_2)$ into Eq. (16), integration constant C_2 is calculated. Substitution of the integration constants C_1, C_2, C_3 and C_4 into Eq. (9) results in an integral equation in $\phi(\xi)$ as follows:

$$\int_0^{\xi}f_2(\xi,s)\phi(\mathrm{s})\,\mathrm{ds}+\int_0^1 f_3(\xi,s)\phi(\mathrm{s})\,\mathrm{ds}+\int_0^{\xi_1}f_4(\xi,s)\phi(\mathrm{s})\,\mathrm{ds}+\int_0^{\xi_2}f_5(\xi,s)\phi(\mathrm{s})\,\mathrm{ds}-EI(\xi)\phi(\xi)=0 \tag{23}$$

In which:

$$\begin{cases} f_3(\xi,s) = \dfrac{\xi^2}{2\mathrm{K}} M_1(s) - \dfrac{\omega^2 L^4}{6}\xi^3 m(s) \\ f_4(\xi,s) = \dfrac{\xi^2}{2\mathrm{K}} M_2(s) \\ f_5(\xi,s) = \dfrac{\xi^2}{2\mathrm{K}} M_3(s) \end{cases} \tag{24}$$

where:

$$\begin{cases} M_1(s) = \left[\omega^2 L^4(1-s) - \omega^2 L^4 - \dfrac{\omega^2 L^6 GA(1)}{6EI(1)} - \dfrac{k_{e_1} L^5 \omega^2 \xi_1^3 I'(\xi_1)}{3EI^2(\xi_1)} - \dfrac{k_{e_1} L^5 \omega^2 \xi_1^2}{EI(\xi_1)} - \dfrac{k_{e_2} L^5 \omega^2 \xi_2^3 I'(\xi_2)}{3EI^2(\xi_2)} - \dfrac{k_{e_2} L^5 \omega^2 \xi_2^2}{EI(\xi_2)}\right] m(s) \\ \qquad - L^2 GA'(s) + \dfrac{L^2 GA(1)}{EI(1)} f_2(1,s) \\ M_2(s) = \dfrac{2k_{e_1} L}{EI(\xi_1)} f_1(\xi_1,s) + \dfrac{2k_{e_1} L I'(\xi_1)}{EI^2(\xi_1)} f_2(\xi_1,s) \\ M_3(s) = \dfrac{2k_{e_2} L}{EI(\xi_2)} f_1(\xi_2,s) + \dfrac{2k_{e_2} L I'(\xi_2)}{EI^2(\xi_2)} f_2(\xi_2,s) \\ \mathrm{K} = -\dfrac{L^2 GA(1)}{2EI(1)} - \dfrac{2K_{e_1} L I'(\xi_1)}{2EI^2(\xi_1)}\xi_1^2 - \dfrac{2K_{e_1} L \xi_1}{EI(\xi_1)} - \dfrac{2K_{e_2} L I'(\xi_2)}{2EI^2(\xi_2)}\xi_2^2 - \dfrac{2K_{e_2} L \xi_2}{EI(\xi_2)} \end{cases} \tag{25}$$

3.3 Solution of the resulting integral equation

In preceding section, the governing partial differential equation for free vibration of the tall structure with a combined system of framed tube, shear core and outrigger-belt truss is converted into the integral equation (23) as:

$$\int_0^{\xi} f_2(\xi,s)\phi(\mathrm{s})\,\mathrm{ds} + \int_0^{1} f_3(\xi,s)\phi(\mathrm{s})\,\mathrm{ds} + \int_0^{\xi_1} f_4(\xi,s)\phi(\mathrm{s})\,\mathrm{ds} + \int_0^{\xi_2} f_5(\xi,s)\phi(\mathrm{s})\,\mathrm{ds} - EI(\xi)\phi(\xi) = 0$$

The mode shape function $\phi(s)$ is the only unknown parameter in the obtained integral equation. In order to solve the integral equation (23) and to determine the natural frequencies, the mode shape function is approximated by the following power series:

$$\phi(\xi) = \sum_{r=0}^{R} c_r \xi^r \tag{26}$$

where C_r are unknown coefficients to be determined and R is a given positive integer, which is adopted such that the accuracy of the results are sustained. Introducing Eq. (26) into integral equation (23) leads to:

$$\sum_{r=0}^{R}\left[\int_0^{\xi} f_2(\xi,s)s^r \mathrm{d}s + \int_0^{1} f_3(\xi,s)s^r \mathrm{d}s + \int_0^{\xi_1} f_4(\xi,s)s^r \mathrm{d}s + \int_0^{\xi_2} f_5(\xi,s)s^r \mathrm{d}s - EI(\xi)\xi^r\right]c_r = 0 \qquad (27)$$

Both sides of Eq. (27) are multiplied by ξ^m and integrated subsequently with respect to ξ between 0 and 1. This results in a system of linear algebraic equations in C_r :

$$\sum_{r=0}^{R}\left[F_2(m,r)+F_3(m,r)+F_4(m,r)+F_5(m,r)+G(m,r)\right]c_r = 0 \qquad m = 0,1,2,...,R \qquad (28)$$

In which functions $F_i(m,r)$, $(i = 2,3,4,5)$ and $G(m,r)$ are expressed as follows:

$$\begin{cases} F_2(m,r) = \int_0^1\int_0^{\xi} f_2(\xi,s)s^r\xi^m \mathrm{d}s\mathrm{d}\xi \\ F_3(m,r) = \int_0^1\int_0^{1} f_3(\xi,s)s^r\xi^m \mathrm{d}s\mathrm{d}\xi \\ F_4(m,r) = \int_0^1\int_0^{\xi_1} f_4(\xi,s)s^r\xi^m \mathrm{d}s\mathrm{d}\xi v \\ F_5(m,r) = \int_0^1\int_0^{\xi_2} f_5(\xi,s)s^r\xi^m \mathrm{d}s\mathrm{d}\xi \\ G(m,r) = \int_0^1 \xi^{r+m} EI(\xi)\mathrm{d}\xi \end{cases} \qquad (29)$$

The system of linear algebraic equations (28) may be expressed in matrix notations as follows:

$$[A]_{(R+1)\times(R+1)}[C]_{(R+1)\times 1} = [0]_{(R+1)\times 1} \qquad (30)$$

In which $[A]$ and $[C]$ are coefficients matrix and unknowns vector respectively. The only unknown parameter in the coefficients matrix $[A]$ is the natural frequency of the tall structure ω. $[C]=0$ is a trivial solution for the resulting system of equations introduced in (30). The natural frequencies are determined through calculation of a non-trivial solution for resulting system of equations. To achieve this, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation in ω (which is a polynomial function of the order $2(R+1)$) is introduced. The roots of the frequency equation are the natural frequencies of tall structure. Given the fact that the mode shape function is approximated by the power series of (26), the results accuracy is improved if more number of the series sentences is taken into account.

4. MODE SHAPE FUNCTIONS

After calculation of the natural frequencies according to presented approach, we can calculate the mode shape functions of the vibration. Using Eq. (26) the mode shape function of ith mode $\phi_i(\xi)$ corresponding to natural frequency of ith mode ω_i is obtained as follows:

$$\phi_i(\xi) = \sum_{r=0}^{R} (c_r)_i \, \xi^r = (c_0)_i + (c_1)_i \, \xi + (c_2)_i \, \xi^2 + ... + (c_R)_i \, \xi^R \tag{31}$$

In which $(C_r)_i$ (r=0,1,…R) are the unknown coefficients of the power series corresponding to ith mode. To calculate the mode shape function $\phi_i(\xi)$, the unknown coefficients of power series $(C_r)_i$ should be calculated independently. System of linear algebraic equations (28) has the matrix form $[A]_{(R+1,R+1)}[C_r]_{(R+1,1)} = 0$. The natural frequency ω is the unknown parameter in the coefficients matrix $[A]_{(R+1,R+1)}$. By substitution of the natural frequency of ith mode ω_i calculated in the preceding section, into the coefficients matrix $[A]_{(R+1,R+1)}$, the system of linear algebraic equations (28) takes the following form:

$$[\mathrm{A}_i]_{(R+1,R+1)}[C_{r_i}]_{(R+1,1)} = 0 \tag{32}$$

In which $[\mathrm{A}_i]_{(R+1,R+1)}$ and $[C_{r_i}]_{(R+1,1)}$ are the coefficients matrix and unknowns vector corresponding to ith mode, respectively. By setting $(C_0)_i = 1$ into Eq. (32) and solving the obtained equation, the coefficients $(C_r)_i$ $i = 1,2,...,R$ are calculated.

5. NUMERICAL EXAMPLES

In order to verify the accuracy and efficiency of the proposed analytical approach, five numerical examples that have been examined by previous researcher are investigated in this section.

5.1 Numerical example 1

In this example, the numerical example presented by Malekinejad and Rahgozar [2] is investigated. Analyses has been performed for three high-rise 40, 55 and 70-storey reinforced concrete buildings for which the horizontal resistance to wind loading is provided by systems of (i) framed tube, (ii) framed tube and shear core, (iii) framed tube, shear core and two belt trusses. Geometric and material characteristics for the equivalent structure are given in Table 1. The first two natural frequencies for each building are calculated according to presented approach in this paper. The results are then compared to results of Malekinejad and Rahgozar [2] and those obtained using finite element analysis (SAP2000, Advanced V12, Computers and Structures, Berkeley, California, USA). The results are listed in Tables 2 and 3.In these tables, parameters $\zeta = L^2\sqrt{\frac{m}{EI}}$, $\bar{k}_{e_1} = \frac{2k_{e_1}}{EI}L$ and $\bar{k}_{e_2} = \frac{2k_{e_2}}{EI}L$ are applied.

Table 1: Geometric and material characteristics for the equivalent structure

Type of building	Properties of equivalent structure	
	EI (kg.m^2)	GA (kg)
Framed tube	1.0115×10^{13}	2.651×10^{9}
Framed tube with shear core	1.0151×10^{13}	4.651×10^{9}
Combined system with double belt trusses	1.0151×10^{13}	4.651×10^{9}

Table 2: Comparison of the first two natural frequencies of 40, 55 and 70-storey buildings

Type of building	$L\,(m)$	ζ	$\omega\,(\frac{\text{rad}}{\text{sec}})$ proposed method		$\omega\,(\frac{\text{rad}}{\text{sec}})$ Malekinejad and Rahgozar [2]		$\omega\,(\frac{\text{rad}}{\text{sec}})$ SAP-2000	
			ω_1	ω_2	ω_1	ω_2	ω_1	ω_2
Framed tube	120	2.7591	1.9327	8.9239	1.923	8.928	1.986	8.376
	165	5.1962	1.2344	5.1334	1.236	5.137	1.372	4.575
	210	8.4394	0.891	3.4478	0.893	3.451	0.976	3.270
Framed tube with shear core	120	2.7983	2.2394	9.4248	2.230	9.425	2.035	9.590
	165	5.2708	1.4666	5.6139	1.467	5.614	1.391	5.744
	210	8.5598	1.0742	3.8807	1.075	3.881	0.989	3.524

Table 3: Comparison of first two natural frequencies of 40, 55 and 70-storey buildings with double outrigger-belt trusses

H_1	H_2	$L\,(m)$	ζ	$\bar{k}_{e_1}$	$\bar{k}_{e_2}$	Proposed method		Malekinejad and Rahgozar [2]		SAP-2000	
						ω_1	ω_2	ω_1	ω_2	ω_1	ω_2
$\frac{L}{6}$	$\frac{L}{4}$	120	2.9536	0.1192	0.1185	2.1407	8.868	2.141	8.869	2.295	8.886
		165	5.4479	0.1632	0.1618	1.4288	5.3936	1.429	5.394	1.499	4.937
		210	8.7596	0.2066	0.2045	1.055	3.7677	1.056	3.768	1.029	3.388
$\frac{L}{6}$	$\frac{L}{2}$	120	2.9536	0.1192	0.1164	2.1435	8.9079	2.144	8.908	2.238	8.699
		165	5.4479	0.1632	0.1580	1.4296	5.4192	1.430	5.421	1.463	4.856

		210	8.7596	0.2066	0.1987	1.055	3.7849	1.056	3.785	1.006	3.319
$\frac{L}{6}$	$\frac{3L}{4}$	120	2.9536	0.1192	0.1145	2.1422	8.9487	2.142	8.949	2.120	8.213
		165	5.4479	0.1632	0.1546	1.4283	5.4422	1.429	5.443	1.415	5.035
		210	8.7596	0.2066	0.1929	1.0545	3.7977	1.055	3.799	0.989	3.442
$\frac{L}{6}$	L	120	2.9536	0.1192	0.1128	2.1403	8.9530	2.141	8.954	2.043	8.648
		165	5.4479	0.1632	0.1513	1.4270	5.4422	1.428	5.443	1.388	4.774
		210	8.7596	0.2066	0.1880	1.0537	3.7965	1.054	3.797	0.969	3.277
$\frac{L}{4}$	$\frac{L}{2}$	120	2.9536	0.1185	0.1164	2.1461	8.9083	2.147	8.909	2.217	8.510
		165	5.4479	0.1618	0.1580	1.4308	5.4201	1.431	5.422	1.458	4.707
		210	8.7596	0.2045	0.1987	1.056	3.7859	1.057	3.787	1.002	3.276
$\frac{L}{4}$	$\frac{3L}{4}$	120	2.9536	0.1185	0.1145	2.1446	8.9492	2.145	8.950	2.103	8.939
		165	5.4479	0.1618	0.1546	1.4294	5.4431	1.430	5.444	1.411	4.848
		210	8.7596	0.2045	0.1929	1.0551	3.7991	1.056	3.800	0.985	3.389
$\frac{L}{4}$	L	120	2.9536	0.1185	0.1128	2.1427	8.9535	2.143	8.954	2.028	8.431
		165	5.4479	0.1618	0.1513	1.4281	5.4430	1.429	5.444	1.384	5.615
		210	8.7596	0.2045	0.1880	1.0543	3.7974	1.055	3.798	0.966	3.732
$\frac{L}{2}$	$\frac{3L}{4}$	120	2.9536	0.1164	0.1145	2.1474	8.9893	2.147	8.99	2.303	9.156
		165	5.4479	0.1580	0.1546	1.4302	5.469	1.431	5.470	1.372	4.804
		210	8.7596	0.1987	0.1929	1.0552	3.8165	1.056	3.817	0.970	3.359
$\frac{L}{2}$	L	120	2.9536	0.1164	0.1185	2.1459	8.9954	2.146	8.996	2.281	8.798
		165	5.4479	0.1580	0.1513	1.4289	5.4682	1.429	5.470	1.348	5.529
		210	8.7596	0.1987	0.1880	1.0544	3.8142	1.055	3.815	0.952	3.694
$\frac{3L}{4}$	L	120	2.9536	0.1145	0.1185	2.1446	9.0346	2.144	9.036	1.913	9.597
		165	5.4479	0.1546	0.1513	1.4276	5.4902	1.428	5.491	1.308	4.995
		210	8.7596	0.1929	0.1880	1.0535	3.8268	1.054	3.828	0.941	3.568

5.2 Numerical example 2

The numerical example presented by Malekinejad and Rahgozar [6] is investigated here. A high-rise 60-storey reinforced concrete building, which consisted of framed tube, shear core and multi-outrigger–belt truss, is analyzed. Flexural and shear stiffness and mass per unit of length of equivalent beam for framed tube are $EI = 1.0115\times10^{13}$ kg.m^2, $GA = 2.651\times10^{9}$ kg and $m = 377593.5\frac{kg}{m}$. Also, the framed tube with shear core has $EI = 1.0151\times10^{13}$ kg.m^2, $GA = 4.651\times10^{9}$ kg and $m = 389666.1\frac{kg}{m}$. The results of first four frequencies of framed tube with and without shear core are presented in Table 4.

Table 4: first four frequencies of framed tube with and without shear core

	Present				Malekinejad and Rahgozar [6]				SAP-2000			
	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4
Framed tube	1.0846	4.3872	10.6703	20.0987	1.09	4.39	10.67	20.10	1.17	3.98	9.99	19.5
Framed tube with shear core	1.2947	4.8463	11.082	20.379	1.29	4.85	11.08	20.38	1.24	4.46	10.72	19.06

The first four frequencies of building with combined system of framed tube, shear core and double-belt trusses are calculated and presented in Table 5. The mass per unit of length of the building is $413467.6\frac{kg}{m}$. Flexural and shear stiffness of this building are the same what was stated for framed tube with shear core. In Table 5 $\bar{k}_{e_i} = \frac{2k_{e_i}}{EI}L \quad i = 1,2$ is applied.

Table 5: First four frequencies of building with combined system of framed tube, shear core and double-belt trusses

Position of Belt truss from Base of structure		$\bar{k}_{e_1}$	$\bar{k}_{e_2}$	Present				Malekinejad and Rahgozar [6]				SAP-2000			
				ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4
L/6	L/4	0.178	0.176	1.264	4.6726	10.786	19.787	1.27	4.67	10.79	19.79	1.32	4.32	10.03	18.90
L/6	L/2	0.178	0.172	1.2654	4.6946	10.748	19.769	1.27	4.69	10.75	19.77	1.30	4.24	9.84	18.73
L/4	L/2	0.176	0.172	1.2663	4.6956	10.735	19.798	1.27	4.69	10.73	19.79	1.29	4.32	9.74	18.82
L/4	3L/4	0.176	0.178	1.2653	4.7160	10.783	19.790	1.26	4.71	10.78	19.79	1.26	4.28	9.97	18.90
L/2	L	0.172	0.164	1.2645	4.7353	10.765	19.825	1.26	4.74	10.77	19.83	1.20	4.37	9.89	19.18

5.3 Numerical example 3

In this example a 25-story tube-in-tube structure that has been examined by previous researchers is investigated. The flexural stiffness of the outer tube is $(EI)_o = 35.2872\times10^9$ kN.m^2,the flexural stiffness of the inner tube is $(EI)_i = 7.5538\times10^9$ kN.m^2. The total stiffness of equivalent beam of the tube-in-tube is $(EI)_t = (EI)_i + (EI)_o$. The shear stiffness is $GA = 3.9888\times10^7\ kN$, mass per unit length is $m = 3385.728\ \frac{\text{kg}}{m}$ and building height is L=75.9 m. The first two natural frequencies are calculated and compared with those obtained by previous researchers. The results are presented in Table 6.

Table 6: Comparison of first two frequencies of the 25-story tube-in-tube tall building.

Methods	Proposed method	Malekinejad and Rahgozar [3]	Youlin, [5] (Top displacem et method)	Youlin, [5] (Mode superpositio n method)	Wang, [7]	Wang, [8]	Lashkari [9]	Wang, [11]	Lee, [20]
ω_1 (rad/ sec)	3.7056	3.705	3.157	3.279	3.462	3.461	3.715	3.462	3.518
ω_2 (rad/ sec)	16.1326	16.127	-	17.921	21.52 5	19.23 9	21.200	21.200	20.763

5.4 numerical example 4: Tall structure with variable properties

In this example, the numerical example presented by Kamgar and Saadatpour [21] which is a high-rise 40-storey reinforced concrete building consisting of framed tube, shear core, belt truss and outrigger system is analyzed. The flexural stiffness, shear stiffness and mass per unit length of the structure change along the height of the structure. Variation of the structure properties along the height are presented in Table 7.

Table 7: Equivalent properties of combined system consisting of framed tube, shear core, belt truss and outrigger system

No. story	Height from the base of the structure (m)	Shear Stiffness AG (kg)	Flexural Stiffness EI (kg.m^2)	Mass per unit length m (kg/ m)
10	30	43.803×10^{8}	1.0548×10^{13}	378576
21	63	43.803×10^{8}	1.0548×10^{13}	410432
40	120	23.129×10^{8}	5.9091×10^{12}	343968

In order to calculate the natural frequency of the structure with variable properties according to presented approach in this paper, we need to interpolate the functions which describe the variation of the structure properties along the height. Using data presented in Table 7, the interpolated functions for Shear stiffness $GA(\xi)$ and flexural Stiffness $EI(\xi)$ are obtained as follows:

$$EI(\xi) = -4.608\times10^{11}\xi^2 - 4.17708\times10^{12}\xi + 1.0548\times10^{13} \quad \text{kg.m}^2 \qquad (0 \le \xi \le 1)$$

$$GA(\xi) = -20.16\times10^{7}\xi^2 - 186.012\times10^{7}\xi + 43.803\times10^{8} \quad \text{kg} \qquad (0 \le \xi \le 1)$$

$$m(\xi) \equiv m_{ave} = \frac{378576 + 410432 + 343968}{3} = 377658 \quad kg\ /\ m$$

Shear correction factor $\kappa = 0.86623$ and the equivalent stiffness of the rotational spring including the effect of the belt truss and outrigger system on the framed tube $k_e = 5.0115\times10^{9}\ kg.m$ are applied. The height of the structure is 120 m. The location of belt truss and outrigger system is 30 m from the base of the structure (or $\xi_1 = 0.25$). By neglecting axial force effects, the first natural frequency of the structure is calculated and compared with the result of Kamgar and Saadatpour [21] and result of analysis by SAP-2000. The results are presented in Table 8.

Table 8: the first natural frequency of 40-storey structure of example 5.4 with combined system and variable properties

Method	present	Kamgar and Saadatpour [21]	SAP-2000
ω_1 (rad/ sec)	1.9393	1.855	1.8034

5.5 Numerical example 5: Tall structure with variable properties

Another numerical example that has been presented by Kamgar and Saadatpour [21] are investigated here. A high-rise 50-storey reinforced concrete building consisting of framed tube, shear core, belt truss and outrigger system is analyzed. Table 9 presents variation of the structure properties along the height. Shear correction factor $\kappa = 0.86623$ and the equivalent stiffness of the rotational spring $k_e = 5.0115\times10^{9}\ kg.m$ are applied. The height of the structure is 150 m. The location of belt truss and outrigger system is 24 m from the base of the structure (or $\xi_1 = 0.16$).

Table 9: Equivalent properties of combined system of example 5.5

No. story	Height from the base of the structure (m)	Shear Stiffness AG (kg)	Flexural Stiffness EI (kg.m^2)	Mass per unit length m (kg/ m)
10	30	1.5327×10^{10}	2.4084×10^{13}	482238.72
20	60	8.4894×10^{9}	1.6589×10^{13}	444240
30	90	4.3803×10^{9}	1.0548×10^{13}	380646.4
40	120	2.3129×10^{9}	5.9091×10^{12}	327868.8
50	150	1.7745×10^{9}	2.6340×10^{12}	290304

Using data presented in Table 9, the interpolated functions for Shear stiffness $GA(\xi)$, flexural Stiffness $EI(\xi)$ and mass per unit length are obtained as follows:

$$EI(\xi) = 1.4258\times10^{14}\xi^5 - 4.0219\times10^{14}\xi^4 + 4.3272\times10^{14}\xi^3 - 2.0189\xi^2 + 7.3203\times10^{12}\xi + 2.4084\times10^{13} \quad \text{kg.m}^2 \quad (0 \le \xi \le 1)$$

$$GA(\xi) = 1.2728\times10^{11}\xi^5 - 3.8579\times10^{11}\xi^4 + 4.2816\times10^{11}\xi^3 - 1.8983\times10^{11}\xi^2 + 6.6285\times10^{9}\xi + 1.5327\times10^{10} \quad \text{kg} \quad (0 \le \xi \le 1)$$

$$m(\xi) \equiv m_{ave} = 385059.5 \quad kg\ /\ m$$

By neglecting axial force effects, the first natural frequency of the structure is calculated and compared with result of Kamgar and Saadatpour [21] and result of analysis by SAP-2000. The results are presented in Table 10.

Table 10: The first natural frequency of 50-storey structure of example 5.5 with combined system and variable properties

Method	Present	Kamgar and Saadatpour [21]	SAP-2000
ω_1 (rad/ sec)	1.7208	1.551	1.6175

6. CONCLUSION

Application of the weak form integral equations for free vibration analysis of tall structures with a combined system of framed tube, shear core and outrigger-belt truss that have variable stiffness and mass along the height has been presented. Through repetitive integrations, the governing partial differential equations with variable coefficients have been converted into weak form integral equations. In order to solve the resulting integral equations, the mode shape function of the vibration has been approximated by a power series and substitution of the power series into weak form integral equations transformed them into a system of linear algebraic equations. The natural frequencies of tall structure have been calculated by determination of a non-trivial solution for system of linear algebraic equations. Presented approach has been also used for determination of the mode shape functions of the vibration and internal forces of the structure. The accuracy, simplicity and reliability of the proposed method are verified thorough several numerical examples.

Differences between natural frequencies of proposed method and previous published works are in acceptable ranges.

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