Dynamic response of 2-D elastoplastic structures by a BEM/FEM scheme

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ABSTRACT

A combined boundary element / finite element method (BEM/FEM) in the time domain is developed and applied for solving two-dimensional dynamic elastoplastic problems. The BEM is used for the portion of the structural domain expected to remain elastic, while the FEM for the remaining expected to become elastoplastic during the deformation history. Thus, while an interior discretization is required for the FEM domain, only a boundary discretization is necessary for the BEM domain with obvious gains in efficiency. The two methods are coupled together at their interface through equilibrium and compatibility. The solution procedure is based on a step-by-step time integration algorithm of the Newmark type associated with an iterative scheme at every time step. The proposed method is illustrate by means of numerical examples, which also serve to demonstrate its merits.

INTRODUCTION

Realistic problems of dynamic analysis of inelastic structures can only be solved by numerical methods. The finite element method (FEM) is the most widely used method for solving these problems [1]. Recently, the boundary element method (BEM) in the time domain has also been successfully applied to dynamic inelastic problems as it is evident, e.g.,in the works of Ahmad and Banerjee [2] utilizing the elastodynamic fundamental solution and Carrer and Telles [3] and Kontoni and Beskos [4] utilizing the elastostatic fundamental solution.

In an effort to exploit the advantages of both the FEM and the BEM and reduce or completely eliminate their disadvantages, various BEM/FEM combined schemes have been proposed and successfully applied to problems in elastostatics, static inelasticity and elastodynamics. Usually, the FEM is used for that portion of the structure characterized by nonlinearities, anisotropies and

inhomogeneities, while the BEM for the remaining portion characterized by linear elastic behaviour, isotropy, homogeneity and infinite extend. The two methods are coupled together at their interface through equilibrium and compatibility. In most of the cases the resulting total influence matrices are nonsymmetric. Since the subject matter of this paper deals with dynamic inelastic analysis, one can mention here only references on static inelasticity and time domain elastodynamics. Thus, e.g., Beer [5], Eberhardsteiner et al [6] and Wearing et al [7] have successfully employed BEM/FEM schemes for two- and three-dimensional problems of static and quasi-static inelasticity (plasticity, viscoplasticity), while Karambalis and Beskos [8], Spyrakos and Beskos [9], Von Estorff et all [10-12] and Von Estorff [13] have successfully solved two- and three- dimensional elastodynamic soil-structure interaction problems by time domain BEM/FEM schemes. A comprehensive literature survey on BEM/FEM coupling for static inelastic and time domain elastodynamic analysis can be found in Pavlatos and Beskos [14].

The present paper describes a time domain BEM/FEM scheme for an accurate and efficient dynamic analysis of elastoplastic structures under plane stress or plane strain conditions. The time domain BEM in conjunction with the elastodynamic fundamental solution is used for the portion of the structure expected to remain elastic, while the time domain FEM is used for the remaining portion to become elastoplastic during the deformation history. Coupling of two methods is accomplished by imposing equilibrium and compatibility at their interface and the solution procedure employs a step-by-step time integration algorithm of the Newmark type in conjunction with iterations at every time step to obtain the response history of the structure. Numerical examples serve to illustrate the method and demonstrate its advantages.

DESCRIPTION OF THE METHOD

In a dynamically loaded elastoplastic structure, plastic deformation is usually confined only to a finite part of it, while the remaining part remains linearly elastic. Thus it is natural to try to analyze such a structure by a hybrid BEM/FEM scheme that employs the FEM for the part of the structure expected to become plastic and the BEM for the remaining part expected to stay elastic. Such a scheme combines the advantages of both methods and reduces or eliminates their disadvantages. Indeed, the elastic part will only require a boundary discretization, while the plastic part will be treated by a method ideally suited for nonlinear problems. The efficiency of this BEM/FEM scheme increases with the BEM to FEM volume ratio, i.e., for problems involving limited yielding and/or infinitely extending bodies.

The BEM and the FEM are combined together through equilibrium and compatibility at their interface to create the proposed hybrid BEM/FEM scheme for dynamically analyzing elastoplastic structures under conditions of plane strain or plane stress. Thus the domain of the structure is divided into two parts, part F to be treated by the FEM and part B to be treated by the BEM, which are interfaced along line C, as shown in Fig. 1. Domain F is discretized
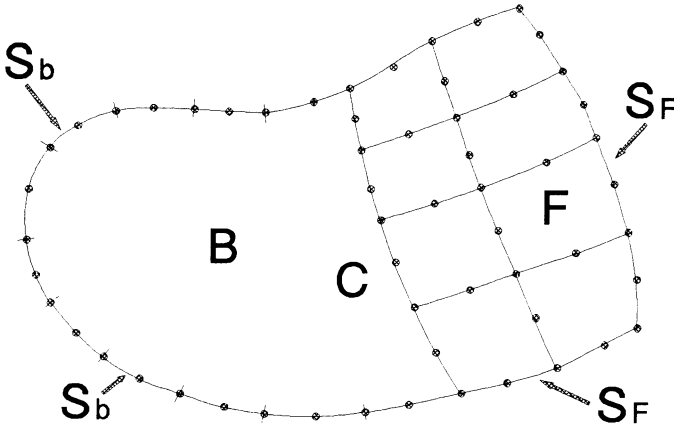
Figure 1. Discretization of plane body into Boundary and Finite Elements.

by 8-noded quadratic quadrilateral isoparametric finite elements, while domain B by 3-noded quadratic isoparametric boundary elements, as shown again in Fig.1. It should be noted that, even though domain B is shown to be finite in Fig. 1, this domain can also extended to infinity.

The matrix equation describing the motion of the B domain has the form

$$[\,T\,]_1\,\{\,u\,\}_{n+1}\ =\ [\,U\,]_1\,\{\,t\,\}_{n+1}\ +\ \{\,P\,\}_{1,n} \tag{1}$$

where subscript $n+1$ denotes the time station $t_n+\Delta t$., [T] and [U] are the influence matrices corresponding to Stoke's full space fundamental elastodynamic solution pair, $\{u\}$ and $\{t\}$ are the vectors of boundary nodal displacements and tractions, respectively and vector $\{P\}$ represents the effect of the previous time steps, i.e. Dominquez [15]. The above equations after employment of the boundary conditions rearrangement and partitioning receives the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{Bmatrix} x_B \\ u_C \\ t_C \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \tag{2}$$

where vector x_B represents the unknown nodal displacements u and traction t along the boundary S_b, vectors u_C and t_C the unknown nodal displacements and tractions, respectively, along the interface C and where the subscripts $n+1$ have been omitted for simplicity.

The matrix equation describing the motion of the F domain has the form

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$$[M] \{ \alpha \}_{n+1} + \{ p \}_{n+1} = \{ f \}_{n+1} \qquad (3)$$

where $[M], \{a\}, \{p\}$ and $\{f\}$ are the mass matrix, acceleration vector, internal force vector and applied force vector, respectively. The internal force vector is a function of the nodal displacements $\{d\}$ and such that at the element level $\partial\{p\}/\partial\{d\} = [K_T] =$ the tangent stiffness matrix, which depends on the deformation $\{d\}$ and the elastoplastic matrix $[D_{ep}]$, which is given explicitly for the yield criteria of Tresca , V. Mises, Mohr - Coulomb and Drucker - Prager of isotropic hardening plasticity, e.g., in Owen and Hinton [16]. Equation (3) can written in the effective static form

$$[K]_n \{ \Delta d \}_{n+1} = \{ F \}_{n+1}$$
$$\{ d \}_{n+1} = \{ d \}_n + \{ \Delta d \}_{n+1} \qquad (4)$$

where

$$[K]_n = \left(\frac{1}{\beta \; \Delta t^2}\right) [M] + [K_T (\{ d \}_n)] \qquad (5)$$

$$\{ F \}_{n+1} = \{ f \}_{n+1} - [M] \{ \alpha \}_{n+1} - \{ p \}_{n+1} \qquad (6)$$

Equation (4) is solved by a step-by-step algorithm of the Newmark type (β is a parameter of this algorithm) and during every time step iterations are performed to improve the value of $\{d\}_{n+1}$. Omitting the subscripts $n+1$ for simplicity one can rewrite equation (4) in the partitioned form

$$\begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & -I \end{bmatrix} \begin{Bmatrix} \Delta d_F \\ \Delta d_C \\ f_C \end{Bmatrix} = \begin{Bmatrix} F_F \\ -(M\alpha + p)_C \end{Bmatrix} \qquad (7)$$

Equations (2) and (7) are coupled together at the interface C through equilibrium and compatibility. Prior to this, tractions in (2) are transformed into forces with the aid of

$$\{ f_C \} = \int_C [N]^T \{ t_C \} \, dS = [N^*] \{ t_C \} \qquad (8)$$

where $[N]$ is the matrix of the BEM shape functions. Thus vectors t_C in (2) are replaced by f_C and submatrices A_{13} and A_{23} are multiplied by $[N^*]^{-1}$ and become A'_{13} and A'_{23}, respectively. Employment of the equilibrium equations

$$f_C^F = -f_C^B \qquad (9)$$

and the compatibility equations

$$\Delta d_C = \Delta u_C \tag{10}$$

at the interface C, couples equations (2) and (7) and produces the combined stiffness equation

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{12} & K_{22} & 0 & -I \\ 0 & A_{12} & A_{11} & A'_{13} \\ 0 & A_{22} & A_{21} & A'_{23} \end{bmatrix} \begin{Bmatrix} \Delta d_F \\ \Delta u_C \\ x_B \\ f_C \end{Bmatrix} = \begin{Bmatrix} F_F \\ -(M\alpha + p)_C \\ B_1 \\ B_2 \end{Bmatrix} = \{Q\} \tag{11}$$

It should be noted that since the BEM works with u, while the FEM with Δu at every time step, at the interface C considered as part of the B domain, u_C is replaced by $(\Delta u)_C$ and this implies that B_1 and B_2 are modified accordingly.

The complete computational procedure pertaining to the above hybrid BEM/FEM scheme consists of the following steps:

1. Input data for geometry, boundary conditions, material properties, load and time integration parameters.
2. Compute matrix [M] and vector $\{a\}_o = [M]^{-1} \{f\}_o$.
3. Compute matrices $[T]_1$,$[U]_1$ form submatrices A_{ij} and A'_{ij} of equation (11).
4. Set time step counter $n = 1$
5. Set iteration cycle $i = 0$
6. Start 'predictor' phase (for the FEM only)

$$\begin{aligned} \{d\}^{[i]}_{n+1} &= \{d\}_n \\ \{u\}^{[i]}_{n+1} &= \{u\}_n \\ \{\alpha\}^{[i]}_{n+1} &= [\{d\}^{[i]}_{n+1} - \{\overline{d}\}_{n+1}]/(\Delta t^2 * \beta) \\ \{\overline{d}\}_{n+1} &= \{d\}_n + \Delta t\{u\}_n + \Delta t^2(1-2\beta)\{\alpha\}_n/2 \\ \{\overline{u}\}_{n+1} &= \{u\}_n + \Delta t(1-\gamma)\{\alpha\}_n \end{aligned} \tag{12}$$

7. Compute residual forces (for the FEM only)

$$\{F\}^{[i]}_{n+1} = \{f\}_{n+1} - [M]\{\alpha\}^{[i]}_{n+1} - \{p\}^{[i]}_{n+1} \tag{13}$$

8. Compute matrices $[T]_{n+1}$, $[U]_{n+1}$,$\{P\}_n$ and then vector $\{B\}_{n+1} = \{B_1,B_2\}^T_{n+1}$ for the first iteration only ($i = 0$) and form vector $\{Q\}_{n+1}$ of (11) from $\{F\}_{n+1}$ and $\{B\}_{n+1}$. For $n = 1$, calculation of $[T]_{n+1}$, $[U]_{n+1}$ and $\{P\}_n$ is not required.
9. For $n = 1$ and $i = 0$ or for $n > 1$ and $i \geq 0$, if updating of $[K_T]$ is desirable, form the effective static stiffness matrix

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$$[K]_n = \left(\frac{1}{\beta \, \Delta t^2}\right) [M] + [K_T (\{ d \}_N^{[i]})] \tag{14}$$

If the updating is not desirable then go to step 11.

10.   Form the combined stiffness matrix $[K_{BF}]$ of equation (11) using (14) and results of step 3. Decompose $[K_{BF}]$ using LU algorithm with pivoting for non-symmetric matrices [17].

11.   Solve the combined stiffness equation (11) or

$$[K_{BF}] \ \{ X \}_{n+1}^{[i]} = \{ Q \}_{n+1}^{[i]}$$

to obtain $\{d_F\}_{n+1}^{[i]}$, $\{u_C\}_{n+1}^{[i]}$, $\{f_C\}_{n+1}^{[i]}$, $\{u_B\}_{n+1}^{[i]}$ and $\{t_B\}_{n+1}^{[i]}$.

12.   Start 'corrector' phase

$$\{ d \}_{n+1}^{[i+1]} = \{ d \}_{n+1}^{[i]} + \{ \Delta d \}_{n+1}^{[i]}$$

$$\{ \alpha \}_{n+1}^{[i+1]} = [\{ d \}_{n+1}^{[i+1]} - \{ \bar{d} \}_{n+1}] / (\Delta t^2 \beta) \tag{15}$$

$$\{ u \}_{n+1}^{[i+1]} = \{ \bar{u} \}_{n+1} + \Delta t \, \gamma \, \{ \alpha \}_{n+1}^{[i+1]}$$

which is restricted only to the FEM domain.

13.   If $\{\Delta d\}_{n+1}^{[i+1]}$ does not satisfy the convergence criterion (for FEM only), then set $i = i+1$ and go to step 7 otherwise continue.

14.   Set

$$\{ d \}_{n+1} = \{ d \}_{n+1}^{[i+1]}$$

$$\{ u \}_{n+1} = \{ u \}_{n+1}^{[i+1]} \tag{16}$$

$$\{ \alpha \}_{n+1} = \{ \alpha \}_{n+1}^{[i+1]}$$

for use in the next time steps.

Also set $n = n+1$, $i = 0$, save the results and start next time step.

NUMERICAL EXAMPLES

The following two numerical examples serve to illustrate the proposed BEM/FEM scheme and demonstrate its advantages

Example 1

Consider a simply supported deep beam subjected to a suddenly applied uniform loading of magnitude p under conditions of plane stress. The material of the beam is ideally elastoplastic obeying the V. Mises yield condition. The geometric, material and load parameters are length $L = 24$, height $h = 6$, Young Modulus $E = 100$, Poisson ratio $\nu = 0.333$, yield stress $\sigma_y = 0.16$, mass density $\rho = 1.50$ and $p = 0.75 p_o = 0.015$ where $p_o = 2\sigma_y h^2/L^2$ is the static collapse load. Figure 2 shows the history of the vertical displacement at point A for elastic and elastoplastic material behaviour as obtained by i) the

proposed BEM/FEM scheme with the discretization of Fig. 3a, ii) the DR/BEM of Kontoni and Beskos [8] with the discretization of Fig. 3b, and iii) the FEM as a special case of the proposed BEM/FEM scheme with the discretization of Fig. 3c. The elastic case has been treated by the DR/BEM of [8] and the BEM as a special case of the proposed BEM/FEM scheme. The agreement between all the methods is very good to excellent.

Example 2

    Consider a thick circular cylinder of inner and outer radii r_i and r_o respectively, made of an ideally elastoplastic V. Mises material subjected to an internal suddenly applied uniform pressure of magnitude p under conditions of plane strain. The geometric, material and load parameters have the values $r_i = 100$ mm, $r_o = 200$ mm, Young modulus $E = 21 \times 10^4$ N/mm^2, Poison ratio $v = 0.3$, yield stress $\sigma_y = 355$ N/mm^2, mass density $\rho = 7.85 \times 10^{-6}$ Kg/mm^3 and $p = 185$ N/mm^2. Figure 4 depicts the history of the radial displacement at point A or elastic and elastoplastic material behaviour as obtained by i) the proposed BEM/FEM scheme with the discretization of Fig. 5a and ii) the FEM as a special case of the proposed BEM/FEM scheme with the discretization of the Fig. 5b. The results of the two methods are in very good agreement.

CONCLUSIONS

This paper describes the development of a hybrid BEM/FEM scheme for the dynamic analysis of elastoplastic structures under plane stress or plane strain conditions. This scheme works in the time domain and combines the BEM, used to model the part of the structure expected to remain elastic, with the FEM used to model the remaining part of the structure expected to become elastoplastic. Quadratic boundary line elements and quadratic quadrilateral finite elements are employed for increased accuracy. The solution is obtained by using a Newmark's type step-by-step time integration algorithm in conjunction with iterations at each time step. The efficiency of the method increases with the BEM to FEM volume ratio. This it is ideal for problems of dynamic fracture mechanics and soil-structure interaction, especially under three-dimensional conditions.

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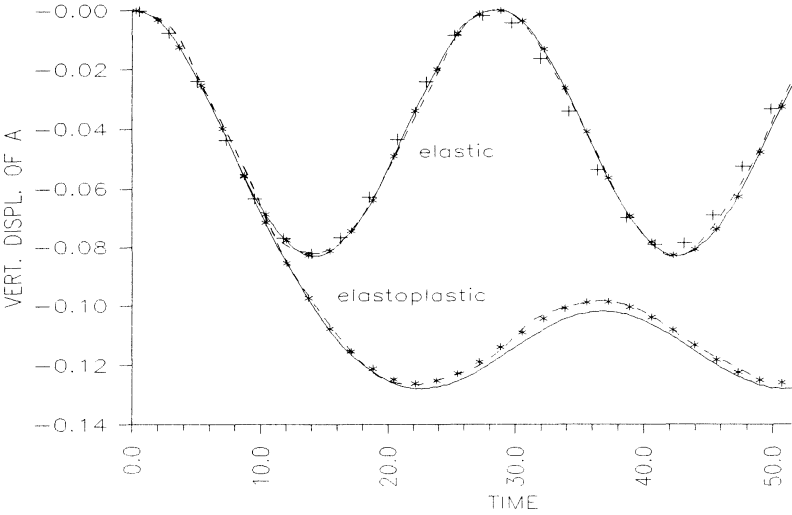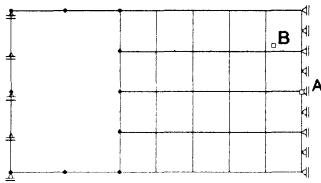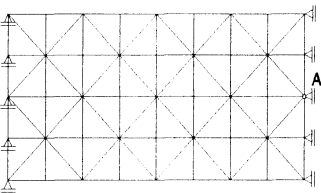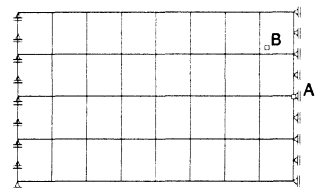
Figure 2.   Elastic and elastoplastic vertical displacement of point A of Example 1. :--- BEM/FEM, *** FEM of BEM/FEM, - - - DR/BEM, + + + + BEM of BEM/FEM

-a-
BEM/FEM   -present method-
Δt=0.75 sec

-b-
DR/BEM   -Kontoni & Beskos-
BEM   -present method-
Δt=0.56 sec

-c-
FEM   -present method-
Δt=0.75 sec

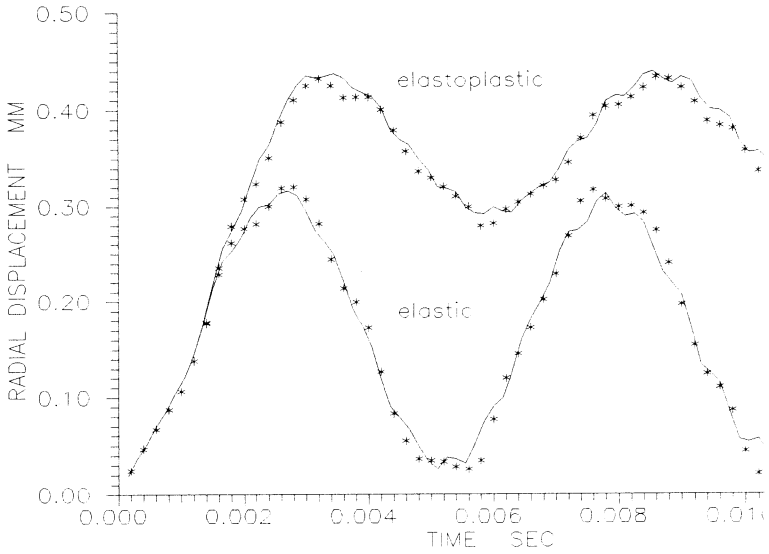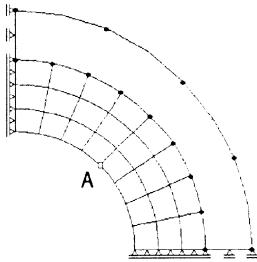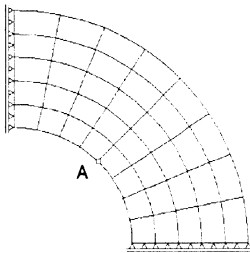Figure 3. Boundary Element and Finite Element discretization of Example 1.

Figure 4.   Elastic and elastoplastic radial displacement of point A of Example 2.:----BEM/FEM, ***** FEM of BEM/FEM

-a-

BEM/FEM   -present method-
$\Delta t = 1.5 \ 10^{-4}$ sec

-b-

FEM   -present method -
$\Delta t = 2.0 \ 10^{-4}$ sec

Figure 5. Boundary Element and Finite Element discretization of Example 2.

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