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DYNAMIC RESPONSE OF PLATES ON ELASTIC FOUNDATIONS DUE TO THE MOVING LOADS

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Abstract

This paper discusses the dynamic response of thin plates on the elastic foundations due to the moving loads by means of the variational calculus. In the text we take the mass of moving loads into account, treat a series of questions such as the forced oscillations, the influence surfaces of the flexions and the influence surfaces of the inner forces, resonance conditions and critical speed and so forth.

I. Introduction

The dynamic problems due to the moving loads, which takes the mass into consideration and which acts on the beam, were studied by R.Willis et al.[1-8]. In [6-8] the method of small parameters were applied. The dynamic questions for the plates and the shells caused by the moving loads were discussed by M.F. Dimentberg et al.[9-12], but they generally considered the moving loads as moving constant forces which are subjected on the plates or the shells and without taking the mass of the moving loads into account, reference[13] dissussed the approximate solutions of the forced vibrations of elastic shallow shell due to the moving mass. It is also necessary and are very important to take the mass of the moving loads into consideration for the studies of the forced vibrations, resonance conditions and critical speed of structures.

This paper will discuss the dynamic response of the plates on the elastic foundations due to the moving loads by means of the variational calculus.

II. Fundamental Equations

Now let us study the orthotropic elastic thin plate lying on the elastic foundations in moving state, its fundamental equations may be written in the variational form[16,22] as follows

$$\iint\left\{ D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}+\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}+K_1 w-K_2\nabla^2 w+K_3\frac{\partial w}{\partial t}\right.$$
$$\left.-Z\right\}\delta w dxdy=0 \tag{2.1}$$

This is a variational equation which contains a lateral flexions w as an unknown quantity. Where the coordinate axes xOy coincide with the medium plane of the plate, the deflections w and the z-axis downward are considered as the positive, the Z is the intensity of distributed lateral load, D_1 and D_2 are the flexural rigidity of the plates in the elastical principal directions, D_3 is the reduced rigidity, furthermore, we have[22]

$$D_1=\frac{E_1h^3}{12(1-\mu_1\mu_2)},\quad D_2=\frac{E_2h^3}{12(1-\mu_1\mu_2)},\quad D_3=D_1\mu_2+2D_k$$

in which D_k is the torsional rigidity of the thin plates in the elastical principal directions, h is the thickness of the plates, where E_1 and E_2 are the modulus of elasticity of the materials of the plates in tension or compression along the elastical principal directions, μ_1 and μ_2 are the shrinkage coefficients in transverse directions, and what is more[22]

$$D_k=\frac{1}{12}Gh^3,\quad \mu_2E_1=\mu_1E_2,\quad D_1\mu_2=D_2\mu_1$$

in which G is the modulus of elasticity in shear; t is the time, γ and g are density of the materials of the plates and the gravitational acceleration respectively; K_1 and K_2 are the basic coefficients of elastical foundations in the elastic semi-space[14], K_3 is a constant coefficient; δw is the variation of the lateral deflections of thin plates.

Let the thin plates be constant thickness, rectange, its boundaries are: $x=0, x=a; y=0, y=b$. The integration of (2.1) is all over the region of the medium plane of the plate over, namely: $0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b$

Besides, all the bending inner forces may repressented by the flexural functions[22] as follows

$$\left.\begin{aligned}&M_x=-D_1\left(\frac{\partial^2 w}{\partial x^2}+\mu_2\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D_2\left(\frac{\partial^2 w}{\partial y^2}+\mu_1\frac{\partial^2 w}{\partial x^2}\right)\\&M_{xy}=-2D_k\frac{\partial^2 w}{\partial x\partial y};\quad V_x=-D_1\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2}+\left(2\frac{D_3}{D_1}-\mu_2\right)\frac{\partial^2 w}{\partial y^2}\right]\\&V_y=-D_2\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2}+2\left(\frac{D_3}{D_2}-\frac{D_1}{D_2}\mu_2\right)\frac{\partial^2 w}{\partial y^2}\right]\end{aligned}\right\}\qquad(2.2)$$

in which the M_x and M_y are the bending moments, M_{xy} is the twisting moment, the V_x and V_y are resultant lateral shearing forces.

Because when a moving load is considered which takes the mass into consideration, and which is moving on the plate lying on the elastic foundations, the relationship between the load and the deflections will be nonlinear, therefore now our questions are nonlinear. For the sake of finding the solutions for the previous variational equations (2.1), we try to choose the functions in the following form[15]

$$w(x,y,t)=A_{mn}(t)X_m(x)Y_n(y),\quad Z(x,y,t)=B_{mn}(t)X_m(x)Y_n(y)\qquad(2.3)$$

where $X_m(x)$ and $Y_n(y)$ are characteristic functions of the beam as the functions of the vibratory mode of thin plates, it is well known they possess orthogonality, we should choose it in advance to satisfy the boundary conditions of rectangular plates along the x-and the y-directions respectively. $A_{mn}(t)$ and $B_{mn}(t)$ are the coefficients of the flexural functions w and the intensity of the load Z which are expanded by the characteristic functions. The variations corressponding to the flexural functions w are

$$\delta w=X_m(x)Y_n(y)\delta A_{mn}\qquad(2.4)$$

substitute (2.3) and (2.4) into (2.1) and observe that the variations δA_{mn} of coefficients are arbitrary and independent, moreover the characteristic functions or the functions of the vibratory mode which possess the orthogonality may obtain the following differential equation

$$\ddot{A}_{mn}(t)+2\alpha\dot{A}_{mn}(t)+\omega_{mn}^2A_{mn}(t)=\frac{g}{\gamma h}B_{mn}(t) \tag{2.5}$$

in which

$$2\alpha=\frac{g}{\gamma h}K_3,\quad \omega_{mn}^2=\frac{g}{\gamma h}\frac{I_1}{I_2} \tag{2.6}$$

$$\left.\begin{aligned}I_1=&\iint\Big\{D_1\frac{\partial^4}{\partial x^4}\Big[X_m(x)Y_n(y)\Big]+2D_3\frac{\partial^4}{\partial x^2\partial y^2}\Big[X_m(x)Y_n(y)\Big]+D_2\frac{\partial^4}{\partial y^4}\\&\cdot\Big[X_m(x)Y_n(y)\Big]+K_1X_m(x)Y_n(y)-K_2\nabla^2\Big[X_m(x)Y_n(y)\Big]\Big\}X_m(x)Y_n(y)dxdy\\I_2=&\iint X_m^2(x)Y_n^2(y)dxdy\end{aligned}\right\} \tag{2.7}$$

α is the so-called coefficient of damping, ω_{mn} is the eigenfrequency of free vibrations for thin plates in the case of no damping. Equation (2.5) is a linear inhomogeneous ordinary differential equation in second order about coefficient A_{mn}.

When $\omega_{mn}^2>\alpha^2$, the solutions of equation (2.6) are

$$\begin{aligned}A_{mn}(t)=&e^{-\alpha t}(a_{mn}\sin\Omega_{mn}t+b_{mn}\cos\Omega_{mn}t)\\&+\frac{g}{\gamma h\Omega_{mn}}\int_0^t e^{-\alpha(t-\tau)}B_{mn}(\tau)\sin\Omega_{mn}(t-\tau)d\tau\end{aligned} \tag{2.8}$$

in which $\Omega_{mn}=\sqrt{\omega_{mn}^2-\alpha^2}$ is the so-called eigenfrequency of free vibrations with damping for thin plates. The first term in the right hand in formula (2.8) represents the free vibrations of thin plates, and the second term denoted the forced vibrations of thin plate. From (2.3) we may obtain the solution for the flexural functions w .

$$\begin{aligned}w(x,y,t)=&e^{-\alpha t}(a_{mn}\sin\Omega_{mn}t+b_{mn}\cos\Omega_{mn}t)X_m(x)Y_n(y)\\&+\frac{g}{\gamma h\Omega_{mn}}X_m(x)Y_n(y)\int_0^t e^{-\alpha(t-\tau)}B_{mn}(\tau)\sin\Omega_{mn}(t-\tau)d\tau\end{aligned} \tag{2.9}$$

in which a_{mn} and b_{mn} are constants which should be determined by the initial conditions of the moving, if the damping forces are not taken into account, then $\alpha=0$; if the initial moving is absent, then $a_{mn}=b_{mn}=0$. $B_{mn}(\tau)$ depends on the character of loads, now let us analyse it as follows.

III. Analyses of Moving Loads

If a moving concentrated load P, acts on the thin plate to a certain point $M(\xi,\eta)$, its mass must be P/g. Now we take the effect of the moving load on the lateral vibrations of the thin plate into account, therefore it is necessary to take the lateral inertia forces of the load on thin plate $-(P/g)(d^2w/dt^2)$ into consideration, the whole lateral pressure which applies on the $M(\xi,\eta)$ point of thin plate is

$$P^*=P-\frac{P}{g}\left(\frac{d^2w}{dt^2}\right)_{\substack{x=\xi\\y=\eta}}=P\left[1-\frac{1}{g}\left(\frac{d^2w}{dt^2}\right)_{\substack{x=\xi\\y=\eta}}\right] \tag{3.1}$$

The lateral loads Z in eqs. (2.1) and (2.3) may be regarded as the following intensity of distributed loads

$$Z=\frac{P^*}{\Delta x\Delta y},\quad \text{when}\begin{cases}\xi\leqslant x\leqslant\xi+\Delta x\\ \eta\leqslant y\leqslant\eta+\Delta y\end{cases}\qquad Z=0,\quad \text{elsewhere} \tag{3.2}$$

in accordance with (2.3) we expand (3.2) and get

$$B_{mn}(t)=\iint Z(x,y,t)X_m(x)Y_n(y)dxdy\cdot\left[\iint X_m^2(x)Y_n^2(y)dxdy\right]^{-1} \tag{3.3}$$

the integration in the above formula is all over the region of the medium plane of the plate over.

If we assume a load to move with uniform velocity v on the plate, its components of velocity in x-and y-directions are respectively

$$v_x=\frac{dx}{dt}=\text{const},\qquad v_y=\frac{dy}{dt}=\text{const} \tag{3.4}$$

We consider the flexsural functions w in formula (3.1) as the following functions

$$w=w(x(t),\ y(t)) \tag{3.5}$$

Observing formula (3.4), we have

$$\frac{dw}{dt}=v_x\frac{dw}{dx}+v_y\frac{dw}{dy},\quad \frac{d^2w}{dt^2}=v_x^2\frac{d^2w}{dx^2}+v_y^2\frac{d^2w}{dy^2} \tag{3.6}$$

substituting (3.6) into (3.1) we may get the general lateral pressure subjected at point $M(\xi,\eta)$ on the plate

$$P^*=P\left[1-\frac{1}{g}\left(v_x^2\frac{d^2w}{dx^2}+v_y^2\frac{d^2w}{dy^2}\right)_{\substack{x=\xi\\y=\eta}}\right] \tag{3.7}$$

from (3.2), (3.3), (3.7) we may obtain $B_{mn}(t)$

$$B_{mn}(t)=\frac{P\left[1-\frac{1}{g}\left(v_x^2\frac{d^2w}{dx^2}+v_y^2\frac{d^2w}{dy^2}\right)_{\substack{x=\xi\\y=\eta}}\right]X_m(\xi)Y_n(\eta)}{\iint X_m^2(x)Y_n^2(y)dxdy} \tag{3.8}$$

from (3.2), (3.3), (3.8) we knows that $B_{mn}(t)$ equals zero else-where except point $M(\xi,\eta)$ on the plate.

IV. Influence Surfaces of Flexions and Influence Surfaces of Internal Forces

If we don't consider the free vibrations of thin plate, but study the forced vibrations of thin plate only, then substituting (3.8) in (2.9) and let $\xi=v_xt,\ \eta=v_yt$, we may obtain the general formula of the dynamic deflections of thin plate

$$w(x,y,t)=\frac{Pg}{\gamma h}\frac{X_m(x)Y_n(y)}{\Omega_{mn}\iint X_m^2(x)Y_n^2(y)dxdy}\int_0^t e^{-\alpha(t-\tau)}\cdot\left[1-\frac{1}{g}\left(v_x^2\frac{d^2w}{dx^2}+v_y^2\frac{d^2w}{dy^2}\right)_{\substack{x=v_x\tau\\y=v_y\tau}}\right]\cdot X_m(v_x\tau)Y_n(v_y\tau)\cdot\sin\Omega_{mn}(t-\tau)d\tau \tag{4.1}$$

If we demand $P=1$ in the above formula, then we get the formula of influence surface for deflections of thin plate in the case in which the load is moving with constant velocity. Over again after performing differentiating (4.1) and substituting it into the formulae of the bending inner forces (2.2), we may obtain the formula of influence surface for the internal forces of thin plate. Let the loads be started from $x=0$ and moved with constant velocity v_x only parallel to the x-axis along a straight line $y=\eta$ on the plate, then we may put $x=v_xt, y=\eta, v_y=0$, $P=1$ in the previous formula (4.1), and obtain the expressions of influence surface for the deflections which correspond to the above mentioned cases as follows

$$w(x,y,t)=\frac{g}{\gamma h}\frac{X_m(x)Y_n(y)}{\Omega_{mn}\iint X_m^2(x)Y_n^2(y)dxdy}\int_0^t e^{-\alpha(t-\tau)}\cdot\left[1-\frac{1}{g}v_x^2\left(\frac{d^2w}{dx^2}\right)_{\substack{x=v_x\tau\\ y=\eta}}\right]$$
$$\cdot X_m(v_x\tau)Y_n(\eta)\sin\Omega_{mn}(t-\tau)d\tau \tag{4.2}$$

Similarly assuming the load be started from $y=0$ and moved with a constant velocity v_y only parallel to the y-axis along a straight line $x=\xi$ on the plate, then we put $x=\xi$, $y=v_yt$, $v_x=0$, $P=1$ in formula (4.1) and may obtain the formula of influence surface for the flexions which correspond to this case as follows

$$w(x,y,t)=\frac{g}{\gamma h}\frac{X_m(x)Y_n(y)}{\Omega_{mn}\iint X_m^2(x)Y_n^2(y)dxdy}\int_0^t e^{-\alpha(t-\tau)}\cdot\left[1-\frac{1}{g}v_y^2\left(\frac{d^2w}{dy^2}\right)_{\substack{x=\xi\\ y=v_y\tau}}\right]$$
$$\cdot X_m(\xi)Y_n(v_y\tau)\sin\Omega_{mn}(t-\tau)d\tau \tag{4.3}$$

For the obtained formulae (4.1), (4.2), (4.3) of the influence surface which are suitable for the various rectangular plates and arbitrary boundary conditions, they possess of generality. The most substantial is to select the functions of the vibratory modes in formulae in accordance with the different boundary conditions of the plates.

The above obtained expressions, with respect to the influence surface, are just suitable only when the moving loads have not left the surface of the plate yet. If the moving loads have left the plate, then the vibrations of plate will be transformed from the forced vibrations into the free vibrations, by this time we may use the foregoing term in formula (2.9), which may be written as

$$w(x,y,t)=\exp[-\alpha(t-t_0)][a_{mn}\sin\Omega_{mn}(t-t_0)+b_{mn}\cos\Omega_{mn}(t-t_0)]X_m(x)Y_n(y) \tag{4.4}$$

constants a_{mn} and b_{mn} should be determined by the initial conditions which, according to the quantities, must be relative to an instant when the loads have left the thin plate, namely, using $w(x,y,t_0)$ and $(\partial w/\partial t)_{t=t_0}$ for the determinations. $t_0=a/v_x$ or $t_0=b/v_y$ are the time of the full course which have been passed by the moving loads on the plate along the directions in the x-axis or in the y-axis respectively

At present though we have got expressions (4.1) of the dynamic deflections and formulae (4.2) with (4.3) of the influence surfaces, yet there are flexural functions w contained under the integrating symbols in all the formulae, hence the problems is a nonlinear in quality. If in light of the method of R.Willis[1,17,18]: instead of the flexural functions w in (4.1), (4.2), (4.3) is expressed by a static deflection of the plate on which the concentrated force acts at point $M(\xi,\eta)$, then the questions mentioned earlier would be simplified and consequently we gain the approximate solutions of this problem.

V. Practical Applications

Let a rectangular orthotropic plate lie on the elastic foundations, simply supported at all edges, a and b are dimensions of thin plate along the x-and the y-axes directions respectively, K_1 and K_2 are the basic coefficients of elastical foundations in the elastic semi-space. Let the thickness of thin plate is constant and its magnitude is h.

Now choose the functions of the vibratory mode

$$X_m(x)=\sin\lambda_m x,\quad Y_n(y)=\sin\mu_n y \tag{5.1}$$

in which

$$\lambda_m=\frac{m\pi}{a},\quad \mu_n=\frac{n\pi}{b}$$

then the flexural functions w of formulae (2.3) may satisfy the boundary conditions of the plate, i.e.

$$\left.\begin{array}{ll}\text{On the edges } x=0 \text{ and } x=a: & w=\dfrac{\partial^2 w}{\partial x^2}=0\\[2ex] \text{On the edges } y=0 \text{ and } y=b: & w=\dfrac{\partial^2 w}{\partial y^2}=0\end{array}\right\} \tag{5.2}$$

We know that the static deflections at any point for the rectangular plate on elastic foundations by simply supported at all edges to be applied by a concentrated force P at point $M(\xi,\eta)$ is

$$w(x,y)=\frac{4Pg}{ab\gamma h}\sum_m\sum_n\frac{1}{\omega_{mn}^2}\sin\lambda_m\xi\cdot\sin\mu_m\eta\cdot\sin\lambda_m x\cdot\sin\mu_n y$$
$$(m,\ n=1,2,3,\cdots,\infty) \tag{5.3}$$

in the above formula the ω_{mn}^2 which is under the sum can be calculated from (2.6), (2.7) and its value is

$$\omega_{mn}^2=\frac{g}{\gamma h}[D_1\lambda_m^4+2D_3\lambda_m^2\mu_n^2+D_2\mu_n^4+K_1+K_2(\lambda_m^2+\mu_n^2)]$$
$$(m,\ n=1,2,3,\cdots,\infty) \tag{5.4}$$

Substituting w of (5.3) into (4.1) and obtaining the approximate solutions of this problem consequently the general formula of the dynamic deflections may be represented as follows

$$w(x,y,t)=\frac{4Pg}{ab\gamma h\Omega_{mn}}\left\{\int_0^t e^{-\alpha(t-\tau)}\cdot\left[1+\frac{4P}{ab\gamma h}\sum_m\sum_n\frac{1}{\omega_{mn}^2}(\lambda_m^2v_x^2+\mu_n^2v_y^2)\right.\right.$$
$$\left.\left.\cdot\sin^2\lambda_m v_x\tau\cdot\sin^2\mu_n v_y\tau\right]\sin\lambda_m v_x\tau\cdot\sin\mu_n v_y\tau\cdot\sin\Omega_{mn}(t-\tau)d\tau\right\} \tag{5.5}$$
$$\cdot\sin\lambda_m x\cdot\sin\mu_n y$$

If we demand $P=1$ in the above formula, then we obtain the most general formula of the flexural influence surface of thin plate on which the moving loads are subjected.

If we don't take the resistance into account in the previous formula, then $\alpha=0$, therefore $\Omega_{mn}=\omega_{mn}$. We will discuss it in two cases as follows.

(1) If we demand $v_y=0$, $v_y\tau=\eta$, and perform integrating for the above formula, then we

may obtain the expressions of dynamic flexions of plate when the moving loads are started from $x=0$ and moved with a constant velocity v_x only parallel to the x-axis along a straight line $y=\eta$

$$w=\frac{4Pg}{ab\gamma h}\frac{\sin\mu_n\eta}{\lambda_m^2 v_x^2-\omega_{mn}^2}\left(\frac{\lambda_m v_x}{\omega_{mn}}\sin\omega_{mn}t-\sin\lambda_m v_x t\right)\sin\lambda_m x\cdot\sin\mu_n y$$
$$+\left(\frac{4P}{ab}\right)^2\frac{g}{(\gamma h)^2}\left\{\sum_m\sum_n\frac{(\lambda_m v_x)^2}{\omega_{mn}^2}\sin^3\mu_n\eta\left[-\frac{3}{4}\frac{\omega_{mn}\sin\lambda_m v_x t}{\lambda_m^2 v_x^2-\omega_{mn}^2}\right.\right.$$
$$\left.\left.+\frac{1}{4}\frac{\omega_{mn}\sin 3\lambda_m v_x t}{9\lambda_m^2 v_x^2-\omega_{mn}^2}+\frac{6\lambda_m^3 v_x^3\sin\omega_{mn}t}{(\lambda_m^2 v_x^2-\omega_{mn}^2)(9\lambda_m^2 v_x^2-\omega_{mn}^2)}\right]\right\}\sin\lambda_m x\cdot\sin\mu_n y \quad (5.6)$$

(2) In the same manner, from (5.5) we may obtain the formula of dynamic deflections when the moving loads on the plate are moved with a constant velocity v_y only parallel to the y-axis along a straight line $x=\xi$

$$w=\frac{4Pg}{ab\gamma h}\frac{\sin\lambda_m\xi}{\mu_n^2 v_y^2-\omega_{mn}^2}\left(\frac{\mu_n v_y}{\omega_{mn}}\sin\omega_{mn}t-\sin\mu_n v_y t\right)\sin\lambda_m x\cdot\sin\mu_n y$$
$$+\left(\frac{4P}{ab}\right)^2\frac{g}{(\gamma h)^2}\left\{\sum_m\sum_n\frac{(\mu_n v_y)^2}{\omega_{mn}^2}\sin^3\lambda_m\xi\left[-\frac{3}{4}\frac{\omega_{mn}\sin\mu_n v_y t}{\mu_n^2 v_y^2-\omega_{mn}^2}\right.\right.$$
$$\left.\left.+\frac{1}{4}\frac{\omega_{mn}\sin 3\mu_n v_y t}{9\mu_n^2 v_y^2-\omega_{mn}^2}+\frac{6\mu_n^3 v_y^3\sin\omega_{mn}t}{(\mu_n^2 v_y^2-\omega_{mn}^2)(9\mu_n^2 v_y^2-\omega_{mn}^2)}\right]\right\}\sin\lambda_m x\cdot\sin\mu_n y \quad (5.7)$$

If in formulae (5.6) and (5.7) we demand $P=1$, we may obtain the expressions of the flexural influence surfaces for the above two cases. After differentiating formulae (5.6) and (5.7) and substituting them into formulae (2.2) of the bending internal forces we may obtain the expressions of the dynamic internal forces for this practical problems. If in the obtained expressions of the dynamic internal forces we put $P=1$, we may get the formulae of the influence surface of internal forces.

It must be pointed out that in the obtained expressions (5.6) and (5.7) the relationship between the dynamic deflections and load P is not a linear one; they are related to the square of load P. From the two preceding formulae we can see that the first term represents the linear and the second term represents the nonlinear, which is caused by taking the mass of the moving load into account.

From the former two expressions (5.6) and (5.7) it may be seen also, while $v_x=\omega_{mn}/\lambda_m$ and $v_x=\omega_{mn}/3\lambda_m$ or while $v_y=\omega_{mn}/\mu_n$ and $v_y=\omega_{mn}/3\mu_n$ the denominators in both of the two expressions vanish, then the flexions w will be increased to infinity, naturally at this time the bending inner forces are also increased to infinity at the same time, hence at present the resonance of the plate takes place. the resonance conditions are

$$\lambda_m^2 v_x^2-\omega_{mn}^2=0 \quad \text{and} \quad 9\lambda_m^2 v_x^2-\omega_{mn}^2=0 \quad (5.8)$$

or

$$\mu_n^2 v_y^2-\omega_{mn}^2=0 \quad \text{and} \quad 9\mu_n^2 v_y^2-\omega_{mn}^2=0$$
$$(m,n=1,2,3,\cdots) \quad (5.9)$$

The corresponding moving velocities v_x or v_y of the moving loads in the present case are called the critical speed and are represented by $(v_x)_{cr}$ or $(v_y)_{cr}$. From (5.8) and by using (5.4) we may gain

$$(v_x)_{cr}=\frac{1}{\lambda_m}\sqrt{\frac{g}{\gamma h}}\sqrt{D_1\lambda_m^4+2D_3\lambda_m^2\mu_n^2+D_2\mu_n^4+K_1+K_2(\lambda_m^2+\mu_n^2)} \qquad (5.10)$$

and

$$(v_x)_{cr}=\frac{1}{3\lambda_m}\sqrt{\frac{g}{\gamma h}}\sqrt{D_1\lambda_m^4+2D_3\lambda_m^2\mu_n^2+D_2\mu_n^4+K_1+K_2(\lambda_m^2+\mu_n^2)} \qquad (5.11)$$

in the same way,from(5.9) and by applying (5.4) we may get $(v_y)_{cr}=\omega_{mn}/\mu_n$ and $(v_y)_{cr}=\omega_{mn}/3\mu_n$

For the isotropic plates, in the previous discussion we may take

$$D_1=D_2=D_3=D=\frac{Eh^3}{12(1-\mu^2)}$$

consequently, every formula in the above mentioned may be obtained in more simplified form, for instance, formulae (5.10) and (5.11) may be written

$$(v_x)_{cr}=\frac{1}{\lambda_m}\sqrt{\frac{g}{\gamma h}}\sqrt{D(\lambda_m^2+\mu_n^2)^2+K_1+K_2(\lambda_m^2+\mu_n^2)} \qquad (5.12)$$

and

$$(v_x)_{cr}=\frac{1}{3\lambda_m}\sqrt{\frac{g}{\gamma h}}\sqrt{D(\lambda_m^2+\mu_n^2)^2+K_1+K_2(\lambda_m^2+\mu_n^2)} \qquad (5.13)$$

When the mass of the moving loads is not considered we will only obtain one group of the resonance conditions; $(v_x)_{cr}=\omega_{mn}/\lambda_m$ or $(v_y)_{cr}=\omega_{mn}/\mu_n$, while we take the mass of the moving load into account, another group of the smaller critical speeds will be got: $(v_x)_{cr}=\omega_{mn}/3\lambda_m$ or $(v_y)_{cr}=\omega_{mn}/3\mu_n$

Since the mass of the moving load is not considered, the nonlinear terms in (5.6) and (5.7) will disappear, the resonance conditions gain one group only:

$$\lambda_m^2v_x^2-\omega_{mn}^2=0 \quad \text{or} \quad \mu_n^2v_y^2-\omega_{mn}^2=0 \qquad (m,\ n=1,2,3,\cdots) \qquad (5.14)$$

If the mass of the moving load is considered, then to (5.5), (5.6) and (5.7) we will be add additional nonlinear terms in which the terms of square of load P are contained, hence due to these terms we gain the additional resonance conditions which exist in the denominators of (5.6) and (5.7):

$$9\lambda_m^2v_x^2-\omega_{mn}^2=0 \quad \text{or} \quad 9\mu_n^2v_y^2-\omega_{mn}^2=0 \qquad (m,\ n=1,2,3,\cdots) \qquad (5.15)$$

Comparing (5.14) with (5.15) we may see that the gained critical speeds from (5.15) will be one third of the critical speeds got by (5.14). This case is quite probable in fact and therefore it is more important. If the mass of the moving load is not taken into account, then these results cannot be got, hence by using expressions (5.15) may determine the minimum critical speeds when the resonance of the plate, on which the moving load is applied, takes place.

VI. Numerical Examples

Let us illustrate a concrete numerical example. Set a concrete rectangular orthotropic plate on elastic foundations, given $E_1=3\times10^6\text{t/m}^2$, $E_2=6\times10^6\text{t/m}^2$, $G=1.47\times10^6\text{t/m}^2$, $\mu_1=0.14$, $\mu_2=0.28$, $a=80\text{m}$, $b=20\text{m}$, $h=0.4\text{m}$, $\gamma=2.5\text{t/m}^3$, by calculating we may obtain

$D_1=133.33$t-m, $D_2=266.67$t-m, $D_3=162.77$t-m, $D_4=62.72$t-m, taking $g=9.81$ m/sec², $K_1=3000$t/m³, $K_2=500$t/m.

By computation the eigenfrequencies $\bar{\omega}_{mn}$, which correspond to the different vibratory types of thin plates, the corresponding critical speeds $(v_n)_{cr}$ and their results are given in the following table

Table 1

Calculating results	Case of having elastic foundations			Case of no elastic foundations		
	ω_{mn}	$(v_x)_{cr}=\frac{\omega_{mn}}{\lambda_m}$	$(v_x)_{cr}=\frac{\omega_{mn}}{3\lambda_m}$	ω_{mn}	$(v_x)_{cr}=\frac{\omega_{mn}}{\lambda_m}$	$(v_x)_{cr}=\frac{\omega_{mn}}{3\lambda_m}$
m, n	rad./sec.	m/sec.	m/sec.	rad./sec.	m/sec.	m/sec.
1,1	171.9285	4378.0441	1459.3480	1.3093	33.3420	11.1140
1,2	173.0486	4406.6358	1468.8795	5.0985	129.8327	43.2776
1,3	177.5891	4522.2585	1507.4195	11.4085	290.4456	96.8152
2,1	171.9930	2189.8785	729.9595	1.4589	18.5750	6.1917
2,2	173.1188	2204.2126	734.7375	5.2417	66.7390	22.2463
2,3	175.1602	2230.2045	743.4015	11.5509	147.0701	49.0234
3,1	172.1122	1460.9297	486.9766	2.2818	19.3665	6.4555
3,2	173.2419	1470.5188	490.1729	5.6899	48.2972	16.0991
3,3	224.3787	1904.5803	634.8601	11.8901	100.9257	33.6419

From the above table it may be seen that in the case of having an elastic foundations the minimum critical speed, which corresponds to the eigenfrequency ω_{31} of thin plate, is 486.9766 m/sec., that is greater in value, but this case would be impossible. In general, the calculations show that in formulae (5.10)–(5.13) of $(v_x)_{cr}$, the percentage occupied by the coefficients of elastic foundations is larger, hence the differences of the neighboring eigenfrequencies correspond to the various vibrating modes of thin plates, and the differences of the corresponding critical speeds are not very obvious.

For the case of absent elastic foundations, the calculating results for the eigenfrequencies ω_{mn} correspond to the different vibratory types of thin plates, and the corresponding critical speeds $(v_x)_{cr}$ are given in the same table for the sake of comparison. The minimum critical speeds is 6.1917 m/sec. which corresponds to the eigenfrequency ω_{21} in the table, this case is guite probable, and what is more, the distinctions of the neighboring eigenfrequencies, which correspond to the different vibratory modes of thin plates, and the distinctions of corresponding critical speeds are larger.

VII. Concluding Remarks

1. This paper found the general expressions (4.1) of the dynamic deflections for the thin plate which lies on the elastic foundations and is subjected to the moving loads possessing the mass.

2. It solved the flexural influence surface formulae (4.2) and (4.3) for the thin plate which lies on the elastic foundations and due to the moving loads possessing the mass when the load moves with a

constant velocity parallel to the x-axis and y-axis respectively, consequently by differentiations we may obtain the influence surface formulae of inner forces also.

3. It obtained the general expressions (5.5) of the dynamic deflections for the rectangular thin plate which is simply supported at all edges and lies on the elastic foundations applied by the moving loads and it obtained formnlae (5.6) and (5.7) of the dynamic flexions or influence surface when the load moves with constant velocity parallel to the x-axis and y-axis directions respectively. These formulae are nonlinear and they are related to load P and the square of load P.

4. It gained the resonance conditions (5.8) with (5.9) for the rectangular thin plate which is simply supported at all edges and lies on the elastic foundations acted on by the moving loads and gained formulae (5.10) – (5.13) for the critical sppeds.

5. From the numerical examples in the text it may be seen that the value of coefficients of the elastic foundations plays a decisive role in calculations for the eigenfrequencies which correspond to the different oscillating modes of thin plates and for the corresponding critical speeds, since in the computations the percentage occupied by these coefficients is greater, the differences of the neighboring eigenfrequencies correspond to the various vibratory types of thin plates, and the differences of the corresponding critical velocities are not distinct, but yet for the case which don't have elastical bases, its differences are larger.

6. From formulae (5.6) or (5.7) we may gain two groups of the resonance conditions (5.8) or (5.9) for the plate, which lies on the elastic foundations and is simply supported. hence we may also obtain the critical speeds of two groups (5.10) and (5.11). One of them is least.

7. If the mass of the moving loads is not considered we will only obtain one group of the resonance conditions. While we take the mass of the moving loads into account, we will get two groups of the resonance conditions, hence the less critical speed is obtained, the ratio of the critical speeds between the front case and the back case is three to one.

8. Provided we select the suitable functions (2.3) of the vibratory modes in advance to satisfy all the boundary conditions, namely, the boundary conditions of geometry and internal forces, in such a manner, the results, which were obtained by this paper, are suitable in applications for the rectangular plates with various supports on boundaries.

9. The method which is mentioned in this paper is a kind of approximate solutions by simplification, but it has not yet arrived at perfection, besides, we may use any other ways which may lead to the study of the problems of the nonlinear parametric resonance, such as the troublesome Mathieu's equations[19-21] by the method of successive iterations and the method of small parameter or the perturbation method[6-8], etc.

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