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Research Article

Dynamical Analysis of a Class of Prey-Predator Model with Beddington-DeAngelis Functional Response, Stochastic Perturbation, and Impulsive Toxicant Input

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A stochastic prey-predator system in a polluted environment with Beddington-DeAngelis functional response is proposed and analyzed. Firstly, for the system with white noise perturbation, by analyzing the limit system, the existence of boundary periodic solutions and positive periodic solutions is proved and the sufficient conditions for the existence of boundary periodic solutions and positive periodic solutions are derived. And then for the stochastic system, by introducing Markov regime switching, the sufficient conditions for extinction or persistence of such system are obtained. Furthermore, we proved that the system is ergodic and has a stationary distribution when the concentration of toxicant is a positive constant. Finally, two examples with numerical simulations are carried out in order to illustrate the theoretical results.

1. Introduction and Model Formulation

The Lotka-Volterra model [1–3] is a classical model in the study of biological mathematics, and the continuous Lotka-Volterra model which is modeled by ordinary differential equations and delay differential equations is widely used to characterize the dynamics of biological systems [4–13]. The functional response functions are important in the population ecological models [14]. In general, functional responses fall into two categories: one depends only on the density of the prey, such as Holling I–III [15–17]; the other depends on the density of both the prey and the predator, such as Beddington-DeAngelis type [18, 19]. Compared with the Holling II functional response, the Beddington-DeAngelis type functional response, $F = f_{12}xy/(b_2 + y + W_{12}x)$, has an additional term y in the denominator modeling mutual interference among predators. In other words, this type of functional response is affected by both predator and prey. Some biologists believe that if the predators compete with each other to obtain food, functional response should depend on the density of both the prey and the predator. Arditi et al. [20] and Jost et al. [21, 22] used the actual observation data to verify this point. In particular, having collected observation data from 19 predator-prey communities, Skalski and Gilliam [23] found that predator-dependent functional responses were in agreement with the observation data, and in many instances, the Beddington-DeAngelis type looked better than the others. The Beddington-DeAngelis functional response has been widely used in the modeling of ecosystems in which there is mutual interference among predators [24, 25]. In [19], DeAngelis et al. have extensively investigated the dynamical properties of the following prey-predator system:

$$\begin{aligned} &\dot{x}(t) \\ &\quad = x(t)\left(a_1 - \frac{f_{12}y(t)}{b_2 + y(t) + W_{12}x(t)} - d_1 - g_1 x(t)\right), \\ &\dot{y}(t) = y(t)\left(\frac{e_{12}f_{12}x(t)}{b_2 + y(t) + W_{12}x(t)} - d_2 - g_2 y(t)\right), \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ represent the density of the prey and the predator, respectively. a_1 is the intrinsic growth rate of the prey, f_{12}, b_2, and W_{12} are the consumption rate, the saturation constant, and the saturation constant for an alternative prey, respectively. e_{12} is the conversion rate of nutrients into the reproduction for the predator. The parameters d_i and g_i ($i = 1, 2$) are the nonpredatory loss rate and the interspecific competition rate. We refer the reader to [19] for more details.

In many ecosystems, predators tend to be omnivorous, they have wide variety of food sources. For example, the giant panda is omnivorous animal, since it can eat both meat and plant such as bamboos. In the lake ecosystem, some fishes not only prey on aquatic invertebrates, but also feed on algae and other aquatic plants. Polis and Strong in [26] and McCann and Hastings in [27] studied omnivorous nature of animals in the food chain in 1996 and 1997, respectively. Based on the above literature, we established a kind of omnivorous model as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r_1 - b_1 x(t) - \frac{\lambda y(t)}{a + my(t) + nx(t)}\right), \\ \dot{y}(t) &= y(t)\left(r_2 + \frac{\gamma x(t)}{a + my(t) + nx(t)} - b_2 y(t)\right), \end{aligned} \tag{2}$$

where r_2 represents the growth rate of y due to omnivorous nature and b_i ($i = 1, 2$) denote the density-dependent coefficient of the prey and the predator, respectively. λ, a, m, n, and γ are the consumption rate, the saturation constant, the predator interference, the saturation constant for an alternative prey, and the conversion rate, respectively. All parameters are positive in system (2).

It is well known that the biological population is inevitably affected by environment perturbation while the stochastic population model is more in line with the actual situation. Recently, various models based on stochastic differential equations (SDEs) have extensively been paid the attention of the researchers (see, e.g., [28–37]). Parameter perturbation induced by white noise is an important and common form to describe the effect of stochasticity (see, e.g., [37–48]). In this paper, we consider the white noise perturbation for the intrinsic growth rates of the prey and predator; that is, $r_1 \to r_1 + \sigma_1 \dot{B}_1(t)$ and $r_2 \to r_2 + \sigma_2 \dot{B}_2(t)$, where $B_1(t)$, $B_2(t)$ are mutually independent Brownian motions and σ_1, σ_2 denote the intensities of the white noise. On the other hand, it can be seen from the recent literature that the environmental pollution has an important effect on the population systems [49–60]. In 1983, Hallam et al. [61, 62] studied the influence of environmental pollution on the population and established a relationship model between environmental toxins and population. Subsequently, Hallam et al. [63, 64] studied the persistence and extinction of population in polluted environment. The mathematical model established by Hallam et al. considered only the toxins in the organism to cause a decrease in the birth rate or an individual death, which is reasonable in the case of lower concentration of the toxicant in the environment. When pollution is serious, the emission of pollutants may directly lead to the death of the species; see [65–69]. The authors in [68] added the environmental toxic term directly to the model; this is reasonable in the heavily polluted environment. For example, in a lake ecosystem, the discharge of large amounts of industrial waste water may directly lead to the death of fish, aquatic invertebrates, and so on. Therefore, we assume that the emission of pollutants to the environment is impulsive and directly affects the survival of the species in such an environment, so we get the following system:

$$\begin{aligned} \mathrm{d}x(t) &= x(t) \\ &\quad \cdot \left(r_1 - b_1 x(t) - \frac{\lambda y(t)}{a + my(t) + nx(t)} - \beta_1 c_e(t)\right)\mathrm{d}t \\ &\quad + \sigma_1(t) x(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}y(t) &= y(t) \\ &\quad \cdot \left(r_2 + \frac{\gamma x(t)}{a + my(t) + nx(t)} - b_2 y(t) - \beta_2 c_e(t)\right)\mathrm{d}t \\ &\quad + \sigma_2(t) y(t)\,\mathrm{d}B_2(t), \\ \frac{\mathrm{d}c_e(t)}{\mathrm{d}t} &= -hc_e(t), \\ &\qquad t \neq k\tau, \\ \Delta x(t) &= 0, \\ \Delta y(t) &= 0, \\ \Delta c_e(t) &= \mu, \\ &\qquad t = k\tau, \end{aligned} \tag{3}$$

where $\sigma_1(t)$, $\sigma_2(t)$ are positive, nonconstant, and continuous functions of period τ, $c_e(t)$ stands for the concentration of the toxicant in the environment, h denotes the loss rate of toxicant at time t, τ is the impulsive input period and μ is the impulsive input amount, and β_1 and β_2 represent the dose-response of the prey and predator to the environmental toxicant, respectively.

Furthermore, the prey-predator model may be perturbed by telegraph noise which is distinguished by factors such as rain falls and nutrition and can be represented by switching among two or more regimes of environment [40, 60, 70–80]. For example, population growth rates in different seasons are not the same. The intraspecific competition coefficient varies according to the changes in nutrition and food resources. Generally, the switching between different regimes is memoryless and the waiting time for the next switch is exponentially distributed [81, 82]. Therefore, it can be described by a continuous-time Markov chain $r(t)$ taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, m\}$. Taking into account the influences of white noise and telegraph noise, we propose the following stochastic differential system with impulsive toxicant input:

$$\begin{aligned} \mathrm{d}x(t) &= x(t)\Bigg(r_1(r(t)) \\ &\quad - \frac{\lambda(r(t))\, y(t)}{a(r(t)) + m(r(t))\, y(t) + n(r(t))\, x(t)} \\ &\quad - \beta_1(r(t))\, c_e(t) - b_1(r(t))\, x(t)\Bigg)\mathrm{d}t + \sigma_1(r(t)) \\ &\quad \cdot x(t)\,\mathrm{d}B_1(t), \end{aligned}$$

$$\begin{aligned} \mathrm{d}y(t) &= y(t)\Bigg(r_2(r(t)) \\ &\quad + \frac{\gamma(r(t))x(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} \\ &\quad - \beta_2(r(t))c_e(t) - b_2(r(t))y(t)\Bigg)\mathrm{d}t + \sigma_2(r(t)) \\ &\quad \cdot y(t)\,\mathrm{d}B_2(t), \\ \frac{\mathrm{d}c_e(t)}{\mathrm{d}t} &= -hc_e(t), \\ &\qquad\qquad t \neq k\tau, \\ \Delta x(t) &= 0, \\ \Delta y(t) &= 0, \\ \Delta c_e(t) &= \mu, \\ &\qquad\qquad t = k\tau. \end{aligned} \tag{4}$$

For any $k \in \mathbb{S}$, $r_i(k)$, $b_i(k)$, $\beta_i(k)$, $\sigma_i(k)$ $(i = 1, 2)$, $\lambda(k)$, $a(k)$, $m(k)$, $n(k)$, and $\gamma(k)$ are all positive constants. In model (4), the population is inevitably affected by severe stochastic interference such as drought; the parameter switches one state $r(t) = i$ into another state $r(t) = j$ and it will switch into the next regime until the next major environmental change.

The rest of this paper is organized as follows. In Section 2, we provide preliminaries which are used in the following sections. In Section 3, we show that system (3) admits a nontrivial positive τ-periodic solution by constructing Lyapunov function. In Section 4, we explore the sufficient conditions for extinction and permanence in mean of system (4). Finally, some examples with numerical simulations have been given to illustrate our theoretical results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\mathcal{F}_{t\geq 0}$ satisfying the usual conditions, $B_i(t)$ $(i = 1, 2)$ is one-dimensional Brownian motion on this space, and $r(t)$ is a right-continuous Markov chain and independent of the Brownian motion $B_i(t)$. The state space of this Markov chain is $\mathbb{S} = \{1, 2, \ldots, m\}$. Suppose that the generator matrix of $r(t)$ is $\Gamma = (q_{ij})_{1\leq i,j\leq m}$, where q_{ij} stands for the transition rate from state i to j and satisfies the following conditions:

$$\mathbb{P}(r(t+\Delta t) = j \mid r(t) = i) = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + q_{ij}\Delta t + o(\Delta t), & \text{if } i = j; \end{cases} \tag{5}$$

here, $q_{ij} \geq 0$ if $i \neq j$, while $q_{ii} = -\sum_{i\neq j} q_{ij}$, $i, j = 1, 2, \ldots, m$. As a standing hypothesis, we assume that the Markov chain $r(t)$ is irreducible, which means that system (4) can switch from one regime to another. Under this assumption, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ which is the solution of the system of linear equations $\pi\Gamma = 0$ subject to $\sum_{j=1}^{m} \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. Hence, for any vector $\varpi = (\varpi(1), \ldots, \varpi(m))^T$, we have that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varpi(r(s))\,ds = \sum_{k\in\mathbb{S}} \pi_k \varpi(k). \tag{6}$$

Let us consider the following stochastic differential equation with Markov conversion.

$$\begin{aligned} \mathrm{d}x(t) &= f(x(t), r(t))\,\mathrm{d}t + g(x(t), r(t))\,\mathrm{d}B(t), \\ x(0) &= x_0, \\ r(0) &= r_0, \end{aligned} \tag{7}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times d}$, and $B(t)$ is a d-dimensional Brownian motion defined on the underlying probability space. The $n\times n$ matrix

$$G(x,k) = g(x,k)\,g^T(x,k) = (G_{ij})_{n\times n} \tag{8}$$

is called the diffusion matrix. Let $V : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ be twice continuously differentiable and $\mathscr{L}V(x,k)$ which is defined as follows be the diffusion operator about $V(x,k)$:

$$\begin{aligned} \mathscr{L}V(x,k) &= \sum_{i=1}^{n} f_i(x,k)\frac{\partial V(x,k)}{\partial x_i} \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2 V(x,k)}{\partial x_i \partial x_j} \\ &\quad + \sum_{i\neq k\in\mathbb{S}} q_{ki}(V(x,i) - V(x,k)). \end{aligned} \tag{9}$$

Particularly, for one-dimensional stochastic system

$$\begin{aligned} \mathrm{d}x(t) &= x(t)\,[a(r(t)) - b(r(t))\,x(t)]\,\mathrm{d}t \\ &\quad + \alpha(r(t))\,x(t)\,\mathrm{d}B(t), \\ x(0) &= x_0, \\ r(0) &= r_0; \end{aligned} \tag{10}$$

the following two lemmas can be given from referring to the articles [72, 77].

Lemma 1. *System (10) has a unique continuous positive solution $x(t)$. When it exists, the solution is global and stochastically ultimately bounded.*

Lemma 2. *Suppose that $\chi = \sum_{k\in\mathbb{S}} \pi_k[a(k) - (1/2)\alpha^2(k)] \neq 0$; then*

(i) system (10) is stochastic permanent if and only if $\chi > 0$;

(ii) system (10) is extinct if and only if $\chi < 0$;

(iii) when $\chi > 0$, system (10) is ergodic and there exists a unique stationary distribution $\nu(\cdot,\cdot)$, such that

$$\chi = \sum_{k\in\mathbb{S}} b(k)\int_{\mathbb{R}_+} x\nu(\mathrm{d}x, k). \tag{11}$$

Next, we consider the following stochastic differential equation:

$$\mathrm{d}y(t) = f(t, y(t))\,\mathrm{d}t + g(t, y(t))\,\mathrm{d}B(t). \tag{12}$$

Lemma 3 (see [78]). *Suppose that the coefficients of (12) are τ-periodic in t and there exists a function $V(t, y) \in C^2$ which is τ-periodic in t, and $V(t, y)$ satisfies the following conditions:*

(i) $\inf_{|y|>R} V(t, y) \to \infty$ as $R \to \infty$.

(ii) $LV(t, y) \le -1$ outside some compact set.

Then there exists a solution for (12) which is a τ-periodic Markov process.

Furthermore, we introduce some results from [80, 83] in Lemmas 4 and 5, which will be used in next section.

Lemma 4 (see [80]). *Let $X(t) \in C[\Omega \times [0, +\infty), \mathbb{R}_+]$. Then*

(i) if there are two positive constants T and m_0 such that

$$\ln X(t) \le mt - m_0 \int_0^t X(s)\,\mathrm{d}s + \sum_{k=1}^{n} \omega_k B_k(t) \tag{13}$$

holds for all $t \ge T$ and constants ω_k $(k = 1, 2, \ldots, n)$, then

$$\begin{gathered}\limsup_{t\to+\infty} \frac{1}{t}\int_0^t X(s)\,\mathrm{d}s \le \frac{m}{m_0}, \quad a.s. \quad \text{if } m > 0, \\ \lim_{t\to+\infty} X(t) = 0, \quad a.s. \quad \text{if } m < 0,\end{gathered} \tag{14}$$

(ii) if there are three positive constants T, m, and m_0 such that

$$\ln X(t) \ge mt - m_0 \int_0^t X(s)\,\mathrm{d}s + \sum_{k=1}^{n} \omega_k B_k(t) \tag{15}$$

holds for any $t \ge T$, then

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t X(s)\,\mathrm{d}s \ge \frac{m}{m_0} \quad a.s. \tag{16}$$

Finally, we give some basic properties of the following subsystem of system (3),

$$\begin{aligned}\frac{\mathrm{d}c_e(t)}{\mathrm{d}t} &= -hc_e(t), \quad t \ne k\tau, \ k \in Z, \\ \Delta c_e(t) &= \mu, \quad t = k\tau, \ k \in Z.\end{aligned} \tag{17}$$

Lemma 5 (see [83]). *System (17) has a unique τ-periodic solution $c_e^*(t)$ which is globally asymptotically stable. Here $c_e^*(t) = \mu e^{-h(t-k\tau)}/(1 - e^{-h\tau})$, $t \in [k\tau, (k+1)\tau)$, $c_{\max} = \mu/(1 - e^{-h\tau})$, and $c_{\min} = \mu e^{-h\tau}/(1 - e^{-h\tau})$.*

For convenience and simplicity, define $\hat{\alpha} = \min_{i\in\mathbb{S}} \alpha_i$, $\check{\alpha} = \max_{i\in\mathbb{S}} \alpha_i$, and $\langle \varphi \rangle_\theta = (1/\theta)\int_0^\theta \varphi(s)\mathrm{d}s$, where $\varphi(t)$ is an integrable function on $[0, +\infty)$. If f is a bounded function on $[0, +\infty)$, define $f^u = \sup_{t\in[0,+\infty)} f(t)$.

3. Existence of Periodic Solutions of System (3)

In this section, we devote our attention to the investigation of the existence of periodic solutions of system (3). From Lemma 5, we know that system (17) has a globally asymptotically stable periodic solution $c_e^*(t)$; therefore, the limit system of (3) is

$$\begin{aligned}\mathrm{d}x(t) &= x(t) \\ &\quad \cdot \left(r_1 - b_1 x(t) - \frac{\lambda y(t)}{a + my(t) + nx(t)} - \beta_1 c_e^*(t)\right)\mathrm{d}t \\ &\quad + \sigma_1(t)\, x(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}y(t) &= y(t) \\ &\quad \cdot \left(r_2 + \frac{\gamma x(t)}{a + my(t) + nx(t)} - b_2 y(t) - \beta_2 c_e^*(t)\right)\mathrm{d}t \\ &\quad + \sigma_2(t)\, y(t)\,\mathrm{d}B_2(t), \\ x(0) &= x_0, \\ y(0) &= y_0,\end{aligned} \tag{18}$$

where $\sigma_1(t)$, $\sigma_2(t)$, and $c_e^*(t)$ are all positive and continuous functions of period τ.

Now, we discuss the existence of periodic solutions of system (18).

Define

$$h_i = \frac{1}{\tau}\int_0^\tau \left(r_i - \beta_i c_e^*(s) - \frac{1}{2}\sigma_i^2(s)\right)\mathrm{d}s, \quad i = 1, 2. \tag{19}$$

Then, we have the following theorem about periodic solutions of system (18).

Theorem 6. *If $h_1 < 0$ and $h_2 > 0$, there exists a prey extinction periodic solution $(0, y^*(t))$ of system (18).*

Proof. From the first equation of system (18), it is easy to see

$$\begin{aligned}\mathrm{d}x(t) &\le x(t)\left[r_1 - b_1 x(t) - \beta_1 c_e^*(t)\right]\mathrm{d}t \\ &\quad + \sigma_1(t)\, x(t)\,\mathrm{d}B_1(t).\end{aligned} \tag{20}$$

Applying Itô's formula and then integrating from 0 to t, we obtain

$$\begin{aligned}\ln x(t) - \ln x(0) &\le \int_0^t \left(r_1 - \beta_1 c_e^*(s) - \frac{1}{2}\sigma_1^2(s)\right)\mathrm{d}s \\ &\quad - \int_0^t b_1 x(s)\,\mathrm{d}s + M(t),\end{aligned} \tag{21}$$

where $M(t) = \int_0^t \sigma_1(s)\mathrm{d}B_1(s)$ is local martingale. From strong law of large numbers for martingales (see [84]), we have

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0, \quad \text{a.s.} \tag{22}$$

It then follows from (21) by dividing t on both sides and letting $t \to \infty$ that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le \left\langle r_1 - \beta_1 c_e^*(s) - \frac{1}{2}\sigma_1^2(s)\right\rangle_\tau = r_1 - \frac{\mu\beta_1}{\tau h} - \frac{1}{2}\left\langle \sigma_1^2(s)\right\rangle_\tau = h_1 < 0; \quad (23)$$

namely, $x(t)$ tends to zero exponentially almost surely.

Since $\lim_{t\to\infty}x(t) = 0$, a.s., from the second equation of system (18), its limit system is

$$\mathrm{d}y(t) = y(t)\left(r_2 - \beta_2 c_e^*(t) - b_2 y(t)\right) + \sigma_2(t)\,y(t)\,\mathrm{d}B_2(t). \quad (24)$$

According to Theorem 4.2 in [85], when $r_2 - \beta_2 c_e^*(t) > 0$ and

$$\left\langle r_2 - \beta_2 c_e^*(s) - \frac{1}{2}\sigma_2^2(s)\right\rangle_\tau = r_2 - \frac{\mu\beta_2}{\tau h} - \frac{1}{2}\left\langle \sigma_2^2(s)\right\rangle_\tau = h_2 > 0, \quad (25)$$

(24) has a unique positive τ-periodic solution $y^*(t)$.

Overall, when $h_1 < 0$ and $h_2 > 0$, there exists a prey extinction periodic solution $(0, y^*(t))$ of system (18).

The proof of this theorem is completed. □

In order to investigate the existence of a nontrivial positive τ-periodic solution for system (18), first of all, we assume following conditions hold.

(H_1) $h_i > 0$, $i = 1, 2$.

(H_2) $\lambda_1 = (r_1 - \beta_1 c_{\min} + r_1 - \beta_1 c_{\max})^2 - 4(r_1 - \beta_1 c_{\max})h_1 > 0$ and $\xi_1 > (\gamma^2/4b_1h_2)\lambda_1$.

(H_3) $\xi_2 = ab_2h_1 - \lambda(r_2 + \gamma/n - \beta_2 c_{\min}) > 0$ and $(\xi_2/ab_1)(r_1 - \beta_1 c_{\max}) > (b_2\xi_1/\gamma^2)[-h_2 + (r_2 + \gamma/n - \beta_2 c_{\min})]$.

Theorem 7. *Suppose that (H_1), (H_2), and (H_3) hold, then there exists a positive τ-periodic solution for system (18).*

Proof. Obviously, the coefficients of system (18) are continuous bounded positive periodic functions in t. Now, we show that conditions (i) and (ii) of Lemma 3 hold. Define a nonnegative C^2-function

$$V(t,x,y) = x - \frac{r_1 - \beta_1 c_{\max}}{b_1}\ln x + \frac{r_1 - \beta_1 c_{\max}}{b_1}\omega_1(t) - \frac{\xi_1}{\gamma^2}\ln y + \frac{\xi_1}{\gamma^2}\omega_2(t) + qy =: V_1 + V_2 + V_3, \quad (26)$$

where $V_1 = x - ((r_1 - \beta_1 c_{\max})/b_1)\ln x + ((r_1 - \beta_1 c_{\max})/b_1)\omega_1(t)$, $V_2 = -(\xi_1/\gamma^2)\ln y + (\xi_1/\gamma^2)\omega_2(t)$, $V_3 = qy$, $q = b_2(((r_1 - \beta_1 c_{\max})/b_1)h_1 + (\xi_1/\gamma^2)h_2)/(r_2 - \beta_2 c_{\min} + \gamma/n)^2$, and $\omega_i(t)$ is a function defined on $[0,\infty)$ satisfying $\omega_i'(t) = r_i - (1/2)\sigma_i^2(t) - \beta_i c_e^*(t) - h_i$ and $\omega_i(0) = 0$ ($i = 1, 2$). Obviously, $\omega_i(t)$ is a τ-periodic function on $[0,\infty)$. Therefore, the function $V(t,x,y)$ is τ-periodic in t and satisfies

$$\liminf_{k\to\infty,(x,y)\in\mathbb{R}_+^2\setminus U_k} V(t,x,y) = \infty, \quad (27)$$

where $U_k = (1/k, k) \times (1/k, k)$. Therefore, condition (i) of Lemma 3 holds. Next, we will prove that condition (ii) of Lemma 3 also holds.

Applying Itô's formula, one has

$$\begin{aligned}LV_1 &= x\left[r_1 - b_1 x - \frac{\lambda y}{a + my + nx} - \beta_1 c_e^*(t)\right] - \frac{r_1 - \beta_1 c_{\max}}{b_1}\left[r_1 - b_1 x - \frac{\lambda y}{a + my + nx} - \beta_1 c_e^*(t) - \frac{\sigma_1^2(t)}{2}\right] + \frac{r_1 - \beta_1 c_{\max}}{b_1}\omega_1'(t) \le -b_1 x^2 \\ &\quad + \left[(r_1 - \beta_1 c_{\min}) + (r_1 - \beta_1 c_{\max})\right]x + \frac{\lambda}{b_1 a}(r_1 - \beta_1 c_{\max})y - \frac{r_1 - \beta_1 c_{\max}}{b_1}h_1, \\ LV_2 &= -\frac{\xi_1}{\gamma^2}\left(r_2 + \frac{\gamma x}{a + my + nx} - b_2 y - \beta_2 c_e^*(t) - \frac{\sigma_2^2(t)}{2}\right) + \frac{\xi_1}{\gamma^2}\omega_2'(t) \le \frac{\xi_1 b_2}{\gamma^2}y - \frac{\xi_1}{\gamma^2}h_2, \\ LV_3 &= qy\left[r_2 + \frac{\gamma x}{a + my + nx} - b_2 y - \beta_2 c_e^*(t)\right] \le q\left(r_2 + \frac{\gamma}{n} - \beta_2 c_{\min}\right)y - qb_2y^2.\end{aligned} \quad (28)$$

Therefore,

$$\begin{aligned}LV &\le -b_1x^2 + \left[(r_1 - \beta_1 c_{\min}) + (r_1 - \beta_1 c_{\max})\right]x - \frac{r_1 - \beta_1 c_{\max}}{b_1}h_1 - qb_2y^2 + \left[\frac{\lambda(r_1 - \beta_1 c_{\max})}{b_1 a}\right. \\ &\quad \left. + q\left(r_2 + \frac{\gamma}{n} - \beta_2 c_{\min}\right) + \frac{\xi_1 b_2}{\gamma^2}\right]y - \frac{\xi_1}{\gamma^2}h_2.\end{aligned} \quad (29)$$

Define a bounded closed set

$$\mathscr{D} = \left\{(x,y) \in \mathbb{R}_+^2 : \varepsilon \le x \le \frac{1}{\varepsilon},\ \varepsilon \le y \le \frac{1}{\varepsilon}\right\}, \quad (30)$$

where $0 < \varepsilon < 1$ is a sufficiently small number such that

$$\begin{aligned}&\left[(r_1 - \beta_1 c_{\min}) + (r_1 - \beta_1 c_{\max})\right]\varepsilon \le \frac{1}{2}\frac{r_1 - \beta_1 c_{\max}}{b_1}h_1 + \frac{1}{2}\frac{\xi_1}{\gamma^2}h_2 \\ &\quad - \frac{1}{2}\cdot\frac{\left[(\lambda/b_1 a)(r_1 - \beta_1 c_{\max}) + \xi_1 b_2/\gamma^2 + q(r_2 - \beta_2 c_{\min} + \gamma/n)\right]^2}{4qb_2},\end{aligned} \quad (31)$$

$$\left[\frac{\lambda\left(r_1-\beta_1 c_{\max}\right)}{b_1 a}+q\left(r_2+\frac{\gamma}{n}-\beta_2 c_{\min}\right)+\frac{\xi_1 b_2}{\gamma^2}\right]\varepsilon < -\frac{1}{8b_1}\left[\left(r_1-\beta_1 c_{\min}+r_1-\beta_1 c_{\max}\right)^2-4\left(r_1-\beta_1 c_{\max}\right)h_1 - \frac{4b_1 h_2}{\gamma^2}\xi_1\right], \tag{32}$$

$$-\frac{b_1}{2\varepsilon^2}+K_3\le -1, \tag{33}$$

$$-\frac{b_2 q}{2\varepsilon^2}+K_4\le -1, \tag{34}$$

and K_3, K_4 are quantities to be determined in the rest of the proof.

Denote

$$\begin{aligned}\mathcal{D}_\varepsilon^1&=\left\{(x,y)\in\mathbb{R}_+^2: 0<x<\varepsilon\right\},\\ \mathcal{D}_\varepsilon^2&=\left\{(x,y)\in\mathbb{R}_+^2: 0<y<\varepsilon\right\},\\ \mathcal{D}_\varepsilon^3&=\left\{(x,y)\in\mathbb{R}_+^2: x>\frac{1}{\varepsilon}\right\},\\ \mathcal{D}_\varepsilon^4&=\left\{(x,y)\in\mathbb{R}_+^2: y>\frac{1}{\varepsilon}\right\}.\end{aligned} \tag{35}$$

Note that $\mathbb{R}_+^2\setminus\mathcal{D}=\mathcal{D}_\varepsilon^1\cup\mathcal{D}_\varepsilon^2\cup\mathcal{D}_\varepsilon^3\cup\mathcal{D}_\varepsilon^4$. Now, we prove $LV(t,x,y)\le -1, (x,y)\in\mathbb{R}_+^2\setminus\mathcal{D}$.

Case 1. If $(x,y)\in\mathcal{D}_\varepsilon^1$, from (29), it implies that

$$\begin{aligned}LV &\le -qb_2\Bigg(y - \frac{(\lambda/b_1 a)\left(r_1-\beta_1 c_{\max}\right)+\xi_1 b_2/\gamma^2+q\left(r_2-\beta_2 c_{\min}+\gamma/n\right)}{2qb_2}\Bigg)^2 \\ &\quad + \frac{\left[(\lambda/b_1 a)\left(r_1-\beta_1 c_{\max}\right)+\xi_1 b_2/\gamma^2+q\left(r_2-\beta_2 c_{\min}+\gamma/n\right)\right]^2}{4qb_2} \\ &\quad - \frac{r_1-\beta_1 c_{\max}}{b_1}h_1-\frac{\xi_1}{\gamma^2}h_2+\left[\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)\right]\varepsilon \\ &\le K_1,\end{aligned} \tag{36}$$

where $K_1=(1/2)\{[(\lambda/b_1 a)(r_1-\beta_1 c_{\max})+\xi_1 b_2/\gamma^2+q(r_2-\beta_2 c_{\min}+\gamma/n)]^2/4qb_2-((r_1-\beta_1 c_{\max})/b_1)h_1-(\xi_1/\gamma^2)h_2\}<0$. In fact, from condition (H_3), one can get

$$\begin{aligned}&\frac{\lambda}{b_1 a}\left(r_1-\beta_1 c_{\max}\right)+\frac{\xi_1 b_2}{\gamma^2}+q\left(r_2-\beta_2 c_{\min}+\frac{\gamma}{n}\right)-2\sqrt{q}\sqrt{b_2}\sqrt{\frac{r_1-\beta_1 c_{\max}}{b_1}h_1+\frac{\xi_1}{\gamma^2}h_2}\\ &\quad=\frac{\left(\xi_1 b_2/\gamma^2\right)\left[-h_2+\left(r_2+\gamma/n-\beta_2 c_{\min}\right)\right]-\left(\xi_2/ab_1\right)\left(r_1-\beta_1 c_{\max}\right)}{r_2+\gamma/n-\beta_2 c_{\min}}<0;\end{aligned} \tag{37}$$

that is to say, $K_1<0$.

Case 2. If $(x,y)\in\mathcal{D}_\varepsilon^2$, from (29) and (32), we can get

$$\begin{aligned}LV &\le -b_1\left[x-\frac{\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)}{2b_1}\right]^2 \\ &\quad+\frac{1}{4b_1}\left\{\left[\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)\right]^2 - 4\left(r_1-\beta_1 c_{\max}\right)h_1\right\}-\frac{\xi_1}{\gamma^2}h_2+\left[\frac{\lambda\left(r_1-\beta_1 c_{\max}\right)}{b_1 a}\right. \\ &\quad\left.+q\left(r_2+\frac{\gamma}{n}-\beta_2 c_{\min}\right)+\frac{\xi_1 b_2}{\gamma^2}\right]\varepsilon \\ &\le \frac{1}{4b_1}\left\{\left[\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)\right]^2 - 4\left(r_1-\beta_1 c_{\max}\right)h_1-\frac{4b_1\xi_1}{\gamma^2}h_2\right\} \\ &\quad+\left[\frac{\lambda\left(r_1-\beta_1 c_{\max}\right)}{b_1 a}+q\left(r_2+\frac{\gamma}{n}-\beta_2 c_{\min}\right)+\frac{\xi_1 b_2}{\gamma^2}\right]\varepsilon\le K_2,\end{aligned} \tag{38}$$

where $K_2=(1/8b_1)[(r_1-\beta_1 c_{\min}+r_1-\beta_1 c_{\max})^2-4(r_1-\beta_1 c_{\max})h_1-(4b_1 h_2/\gamma^2)\xi_1]$. Using condition (H_2), one can get $K_2<0$.

Case 3. If $(x,y)\in\mathcal{D}_\varepsilon^3$, then

$$\begin{aligned}LV &\le -\frac{b_1}{2}x^2-\frac{b_1}{2}x^2+\left[\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)\right] \\ &\quad\cdot x-\frac{r_1-\beta_1 c_{\max}}{b_1}h_1-b_2 q y^2+\left[\frac{\lambda\left(r_1-\beta_1 c_{\max}\right)}{ab_1}\right. \\ &\quad\left.+\frac{\xi_1 b_2}{\gamma^2}+q\left(r_2-\beta_2 c_{\min}+\frac{\gamma}{n}\right)\right]y-\frac{\xi_1}{\gamma_2}h_2\le -\frac{b_1}{2\varepsilon^2} \\ &\quad+K_3,\end{aligned} \tag{39}$$

where

$$K_3=\sup_{(x,y)\in\mathbb{R}_+^2}\left\{-\frac{b_1}{2}x^2+\left[\left(r_1-\beta_1 c_{\min}\right)+\left(r_1-\beta_1 c_{\max}\right)\right]x-\frac{r_1-\beta_1 c_{\max}}{b_1}h_1-b_2 q y^2\right.$$

$$+\left[\frac{\lambda(r_1-\beta_1 c_{\max})}{ab_1}+\frac{\xi_1 b_2}{\gamma^2}+q\left(r_2-\beta_2 c_{\min}+\frac{\gamma}{n}\right)\right]y-\frac{\xi_1}{\gamma_2}h_2\Bigg\}. \quad (40)$$

By (33), we have $LV \le -1$.

Case 4. If $(x,y)\in\mathscr{D}_\varepsilon^4$, then

$$\begin{aligned}LV &\le -\frac{b_2 q}{2}y^2-b_1x^2+\left[(r_1-\beta_1 c_{\min})+(r_1-\beta_1 c_{\max})\right]x-\frac{r_1-\beta_1 c_{\max}}{b_1}h_1\\&-\frac{b_2 q}{2}y^2+\left[\frac{\lambda(r_1-\beta_1 c_{\max})}{ab_1}+\frac{\xi_1 b_2}{\gamma^2}+q\left(r_2-\beta_2 c_{\min}+\frac{\gamma}{n}\right)\right]\\&\cdot y-\frac{\xi_1}{\gamma^2}h_2\le-\frac{qb_2}{2\varepsilon^2}+K_4,\end{aligned} \quad (41)$$

where

$$\begin{aligned}K_4 = \sup_{(x,y)\in\mathbb{R}_+^2}&\Bigg\{-b_1x^2+\left[(r_1-\beta_1 c_{\min})+(r_1-\beta_1 c_{\max})\right]x-\frac{r_1-\beta_1 c_{\max}}{b_1}h_1-\frac{qb_2}{2}y^2\\&+\left[\frac{\lambda(r_1-\beta_1 c_{\max})}{ab_1}+\frac{\xi_1 b_2}{\gamma^2}+q\left(r_2-\beta_2 c_{\min}+\frac{\gamma}{n}\right)\right]y-\frac{\xi_1}{\gamma^2}h_2\Bigg\}.\end{aligned} \quad (42)$$

By (34), we obtain $LV \le -1$.

Thus,

$$LV \le \min\{K_1,K_2,-1\},\quad (x,y)\in\mathbb{R}_+^2\setminus\mathscr{D}. \quad (43)$$

Therefore, the proof of Theorem 7 is completed. □

4. Extinction and Persistence in Mean of System (4)

In this section, we investigate the long-term dynamic behaviors of the prey-predator system (4) with white noise and telegraph noise in a polluted environment and then discuss the extinction and average persistence of prey and predator. According to Lemma 5, the periodic solution $c_e^*(t)$ of the toxicant input is globally asymptotically stable, so the limit system of (4) is

$$\begin{aligned}dx(t)&=x(t)\Bigg(r_1(r(t))-b_1(r(t))x(t)-\frac{\lambda(r(t))y(t)}{a(r(t))+m(r(t))y(t)+n(r(t))x(t)}\\&\quad-\beta_1(r(t))c_e^*(t)\Bigg)dt+\sigma_1(r(t))x(t)\,dB_1(t),\\dy(t)&=y(t)\Bigg(r_2(r(t))+\frac{\gamma(r(t))x(t)}{a(r(t))+m(r(t))y(t)+n(r(t))x(t)}\\&\quad-b_2(r(t))y(t)-\beta_2(r(t))c_e^*(t)\Bigg)dt+\sigma_2(r(t))\cdot y(t)\,dB_2(t),\\x(0)&=x_0,\\y(0)&=y_0,\\r(0)&=r_0.\end{aligned} \quad (44)$$

In order to obtain the threshold conditions of persistence and extinction of system (44), we assume that

(A_1): $\eta_1=\sum_{k=1}^m\pi_k(r_1(k)-(1/2)\sigma_1^2(k)-c_{\min}\beta_1(k))<0$,

(A_2): $\eta_2=\sum_{k=1}^m\pi_k(r_2(k)+\gamma(k)/n(k)-(1/2)\sigma_2^2(k)-c_{\min}\beta_2(k))<0$,

(A_3): $\eta_3=\sum_{k=1}^m\pi_k(r_2(k)-(1/2)\sigma_2^2(k)-c_{\min}\beta_2(k))<0$,

(A_4): $\eta_4=\sum_{k=1}^m\pi_k(r_1(k)-\lambda(k)/m(k)-(1/2)\sigma_1^2(k)-c_{\max}\beta_1(k))>0$,

(A_5): $\eta_5=\sum_{k=1}^m\pi_k(r_1(k)-(1/2)\sigma_1^2(k)-c_{\max}\beta_1(k))>0$.

Theorem 8. *Given initial value $(x(0),y(0),r(0))\in\mathbb{R}_+^2\times\mathbb{S}$ for system (44), then*

(i) if (A_1) is established, the prey population will be extinct,

(ii) if (A_2) is established, the predator population will be extinct,

(iii) if (A_1) and (A_3) are established, both the prey and the predator will die out.

Proof. (i) By Itô's formula, we get

$$\begin{aligned} \mathrm{d}\ln x(t) = & \left(r_1(r(t)) - b_1(r(t))x(t) \right. \\ & - \frac{\lambda(r(t))y(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} \\ & \left. - \frac{\sigma_1^2(r(t))}{2} - \beta_1(r(t))c_e^*(t) \right) \mathrm{d}t \\ & + \sigma_1(r(t))\,\mathrm{d}B_1(t) \le \left(r_1(r(t)) - \frac{\sigma_1^2(r(t))}{2} \right. \\ & \left. - \beta_1(r(t))c_{\min} \right) \mathrm{d}t + \sigma_1(r(t))\,\mathrm{d}B_1(t); \end{aligned} \quad (45)$$

then,

$$\begin{aligned} & \frac{\ln x(t) - \ln x(0)}{t} \\ & \le \frac{1}{t}\int_0^t \left(r_1(r(t)) - \frac{\sigma_1^2(r(t))}{2} - \beta_1(r(t))c_{\min} \right) \mathrm{d}t \\ & + \frac{1}{t}\int_0^t \sigma_1(r(t))\,\mathrm{d}B_1(t). \end{aligned} \quad (46)$$

By the ergodic theory of the Markov chain and the strong law of large number, we have

$$\begin{aligned} & \limsup_{t\to\infty} \frac{\ln x(t)}{t} \\ & \le \sum_{k=1}^m \pi_k \left(r_1(k) - \frac{1}{2}\sigma_1^2(k) - c_{\min}\beta_1(k) \right) = \eta_1, \\ & \qquad \text{a.s.}; \end{aligned} \quad (47)$$

from (A_1), we know

$$\lim_{t\to\infty} x(t) = 0. \quad \text{a.s.} \quad (48)$$

(ii) Similarly, from the second equation of system (44), we have

$$\begin{aligned} \mathrm{d}\ln y(t) = & \left(r_2(r(t)) - b_2(r(t))y(t) \right. \\ & + \frac{\gamma(r(t))x(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} \\ & \left. - \frac{\sigma_2^2(r(t))}{2} - \beta_2(r(t))c_e^*(t) \right) \mathrm{d}t \\ & + \sigma_2(r(t))\,\mathrm{d}B_2(t); \end{aligned} \quad (49)$$

then,

$$\begin{aligned} \frac{\ln y(t) - \ln y(0)}{t} \le & \frac{1}{t}\int_0^t \left(r_2(r(t)) + \frac{\gamma(r(t))}{n(r(t))} \right. \\ & \left. - \frac{\sigma_2^2(r(t))}{2} - \beta_2(r(t))c_{\min} \right) \mathrm{d}t + \frac{1}{t} \\ & \cdot \int_0^t \sigma_2(r(t))\,\mathrm{d}B_2(t); \end{aligned} \quad (50)$$

further,

$$\begin{aligned} & \limsup_{t\to\infty} \frac{\ln y(t)}{t} \\ & \le \sum_{k=1}^m \pi_k \left(r_2(k) + \frac{\gamma(k)}{n(k)} - \frac{1}{2}\sigma_2^2(k) - c_{\min}\beta_2(k) \right) \\ & = \eta_2, \quad \text{a.s.}; \end{aligned} \quad (51)$$

from (A_2), we know

$$\lim_{t\to\infty} y(t) = 0. \quad \text{a.s.} \quad (52)$$

(iii) By the condition (A_1), one can get $\lim_{t\to\infty} x(t) = 0$ a.s., so that the limit system of the second equation of system (44) is

$$\begin{aligned} \mathrm{d}y(t) = & \, y(t) \\ & \cdot \left(r_2(r(t)) - b_2(r(t))y(t) - \beta_2(r(t))c_e^*(t) \right) \mathrm{d}t \\ & + \sigma_2(r(t))y(t)\,\mathrm{d}B_2(t) \le y(t) \\ & \cdot \left(r_2(r(t)) - b_2(r(t))y(t) - \beta_2(r(t))c_{\min} \right) \mathrm{d}t \\ & + \sigma_2(r(t))y(t)\,\mathrm{d}B_2(t). \end{aligned} \quad (53)$$

Through Lemma 2, if (A_3) holds, we obtain

$$\lim_{t\to\infty} y(t) = 0. \quad \text{a.s.} \quad (54)$$

This completes the proof of the theorem. □

Remark 9. If $r_i(k)$ $(i = 1,2)$ remains unchanged and $\sigma_i(k)$ $(i = 1,2)$ or $c_{\min}$ increases so that $\eta_1 < 0$ or $\eta_2 < 0$, then condition (A_1) or (A_2) is established. That is to say, if the intrinsic growth rate and the predation intensity are relatively fixation, the increase of white noise intensity or pollutant concentration will lead to the extinction of the biological population.

Next, we will discuss the persistence of system (44). Applying Itô's formula to the first equation of system (44), one can get

$$\begin{aligned} \mathrm{d}\ln x(t) \ge & \left(r_1(r(t)) - b_1(r(t))x(t) - \frac{\lambda(r(t))}{m(r(t))} \right. \\ & \left. - \frac{\sigma_1^2(r(t))}{2} - \beta_1(r(t))c_{\max} \right) \mathrm{d}t \\ & + \sigma_1(r(t))\,\mathrm{d}B_1(t); \end{aligned} \quad (55)$$

then,

$$\frac{\ln x(t)-\ln x(0)}{t} \geq \frac{1}{t}\int_0^t\left(r_1(r(t))-b_1(r(t))x(t) - \frac{\lambda(r(t))}{m(r(t))}-\frac{\sigma_1^2(r(t))}{2}-\beta_1(r(t))c_{\max}\right)\mathrm{d}t + \frac{1}{t}\int_0^t\sigma_1(r(t))\,\mathrm{d}B_1(t), \tag{56}$$

when t is large enough, we derive

$$\frac{\ln x(t)}{t} \geq \sum_{k=1}^m \pi_k\left(r_1(k)-\frac{\lambda(k)}{m(k)}-\frac{1}{2}\sigma_1^2(k)-c_{\max}\beta_1(k)\right) - \frac{\check{b}_1}{t}\int_0^t x(t)\,\mathrm{d}t-\varepsilon, \tag{57}$$

where ε is a sufficiently small positive number. In view of (A_4) and Lemma 4, we deduce

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(t)\,\mathrm{d}t \geq \frac{\eta_4}{\check{b}_1}, \quad \text{a.s.} \tag{58}$$

That is, the prey population of system (44) will be persistence in mean under condition (A_4).

Furthermore, the persistent property of the predator species of system (44) can be investigated as follows.

From the first equation of system (44), we have

$$\mathrm{d}x(t) \leq x(t)\cdot\left(r_1(r(t))-b_1(r(t))x(t)-\beta_1(r(t))c_e^*(t)\right)\mathrm{d}t + \sigma_1(r(t))x(t)\,\mathrm{d}B_1(t). \tag{59}$$

Consider the following stochastic differential equations:

$$\begin{aligned}\mathrm{d}\phi(t) &= \phi(t)\cdot\left(r_1(r(t))-b_1(r(t))\phi(t)-\beta_1(r(t))c_e^*(t)\right)\mathrm{d}t + \sigma_1(r(t))\phi(t)\,\mathrm{d}B_1(t),\\ \phi(0) &= x(0),\end{aligned} \tag{60}$$

$$\begin{aligned}\mathrm{d}\psi(t) &= \psi(t)\cdot\left(r_1(r(t))-b_1(r(t))\psi(t)-\beta_1(r(t))c_{\max}\right)\mathrm{d}t + \sigma_1(r(t))\psi(t)\,\mathrm{d}B_1(t),\\ \psi(0) &= x(0).\end{aligned} \tag{61}$$

Obviously, $x(t)\leq\phi(t)$, $\psi(t)\leq\phi(t)$. Using Lemma 2, if $\eta_5>0$, system (61) is ergodic and there exists a unique stationary distribution μ_ψ, such that

$$\sum_{k=1}^m b_1(k)\int_{\mathbb{R}^+}x\mu_\psi(\mathrm{d}x,k) = \sum_{k=1}^m\pi_k\left(r_1(k)-\frac{1}{2}\sigma_1^2(k)-c_{\max}\beta_1(k)\right). \tag{62}$$

Applying Itô's formula to (60) and then integrating from 0 to t, we get

$$\frac{\ln\phi(t)-\ln\phi(0)}{t} = \frac{1}{t}\int_0^t\left(r_1(r(t))-\frac{\sigma_1^2(r(t))}{2} - \beta_1(r(t))c_e^*(t)-b_1(r(t))\phi(t)\right)\mathrm{d}t+\frac{1}{t} \cdot\int_0^t\sigma_1(r(t))\,\mathrm{d}B_1(t). \tag{63}$$

From the first equation of system (44), it yields that

$$\begin{aligned}\frac{\ln x(t)-\ln x(0)}{t} &= \frac{1}{t}\int_0^t\left(r_1(r(t))-\frac{\sigma_1^2(r(t))}{2}-\beta_1(r(t))c_e^*(t)\right)\mathrm{d}t\\ &\quad-\frac{1}{t}\int_0^t\frac{\lambda(r(t))y(t)}{a(r(t))+m(r(t))y(t)+n(r(t))x(t)}\mathrm{d}t\\ &\quad-\frac{1}{t}\int_0^t b_1(r(t))x(t)\,\mathrm{d}t+\frac{1}{t}\int_0^t\sigma_1(r(t))\,\mathrm{d}B_1(t).\end{aligned} \tag{64}$$

Due to $x(t)\leq\phi(t)$, one can get

$$\frac{1}{t}\int_0^t(\phi(t)-x(t))\,\mathrm{d}t \leq \frac{1}{t}\int_0^t\frac{\check{\lambda}}{\hat{a}\hat{b}_1}y(t)\,\mathrm{d}t. \quad \text{a.s.} \tag{65}$$

From the second equation of system (44) we have

$$\begin{aligned}\frac{\ln y(t)}{t} &= \frac{\ln y(0)}{t}+\frac{1}{t}\int_0^t\left(r_2(r(t))-\frac{\sigma_2^2(r(t))}{2} - \beta_2(r(t))c_e^*(t)\right)\mathrm{d}t-\frac{1}{t}\int_0^t b_2(r(t))y(t)\,\mathrm{d}t\\ &\quad+\frac{1}{t}\int_0^t\frac{\gamma(r(t))\phi(t)}{a(r(t))+n(r(t))\phi(t)}\mathrm{d}t-\frac{1}{t}\end{aligned}$$

$$\cdot \int_0^t \left(\frac{\gamma(r(t))\phi(t)}{a(r(t)) + n(r(t))\phi(t)} - \frac{\gamma(r(t))x(t)}{a(r(t)) + n(r(t))x(t)} \right) \mathrm{d}t - \frac{1}{t} \cdot \int_0^t \left(\frac{\gamma(r(t))x(t)}{a(r(t)) + n(r(t))x(t)} - \frac{\gamma(r(t))x(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} \right) \mathrm{d}t + \frac{1}{t}\int_0^t \sigma_2(r(t))\,\mathrm{d}B_2(t) \geq \frac{\ln y(0)}{t} + \frac{1}{t} \cdot \int_0^t \left(r_2(r(t)) - \frac{\sigma_2^2(r(t))}{2} - \beta_2(r(t))c_{\max} \right) \mathrm{d}t + \frac{1}{t}\int_0^t \frac{\gamma(r(t))\psi(t)}{a(r(t)) + n(r(t))\psi(t)} \mathrm{d}t - \left(\check{b}_2 + \frac{\check{\gamma}\check{\lambda}}{\hat{a}^2\hat{b}_1} + \frac{\check{\gamma}\check{m}}{\hat{a}\hat{n}} \right) \frac{1}{t}\int_0^t y(t)\,\mathrm{d}t + \frac{1}{t}\int_0^t \sigma_2(r(t))\,\mathrm{d}B_2(t). \tag{66}$$

By Lemma 4, if

(A_6): $\eta_6 = \sum_{k=1}^m \pi_k (r_2(k) - (1/2)\sigma_2^2(k) - c_{\max}\beta_2(k)) + \sum_{k=1}^m \pi_k \int_{\mathbb{R}^+} (\gamma(k)x/(a(k) + n(k)x))\mu_\psi(\mathrm{d}x, k) > 0$

holds, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(t)\,\mathrm{d}t \geq \frac{\eta_6}{\check{b}_2 + \check{\gamma}\check{\lambda}/\hat{a}^2\hat{b}_1 + \check{\gamma}\check{m}/\hat{a}\hat{n}} \quad \text{a.s.} \tag{67}$$

In summary, one gets the following.

Theorem 10. *Given initial value $(x(0), y(0), r(0)) \in \mathbb{R}_+^2 \times \mathbb{S}$ for system (44), then*

(i) if (A_4) is established, the prey population will be persistent in mean,

(ii) if conditions (A_5) and (A_6) are satisfied, the predator population will be persistent in mean.

Remark 11. (i) It can be seen from η_4, in the case where the intrinsic growth rate and the predation intensity are relatively constant, only by reducing the intensity of white noise or pollutant concentration, so that (A_4) can be established to ensure the lasting survival of the prey population.

(ii) Obviously, $\eta_4 < \eta_5$; if the prey population is persistent, the predator population is persistent as long as the white noise interference intensity or the toxin concentration is small enough, such that (A_6) is established. As can be seen from condition (A_6), the omnivorous nature of y contributes to its permanence.

In system (4), if the concentration of the toxicant in the environment remains unchanged, that is, $c_e(t) = c$ is a positive constant, then the system can be converted into the following system:

$$\begin{aligned}
\mathrm{d}x(t) &= x(t)\Big(r_1(r(t)) - b_1(r(t))x(t) - \frac{\lambda(r(t))y(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} - c\beta_1(r(t)) \Big)\mathrm{d}t + \sigma_1(r(t))x(t)\,\mathrm{d}B_1(t), \\
\mathrm{d}y(t) &= y(t)\Big(r_2(r(t)) + \frac{\gamma(r(t))x(t)}{a(r(t)) + m(r(t))y(t) + n(r(t))x(t)} - b_2(r(t))y(t) - c\beta_2(r(t)) \Big)\mathrm{d}t + \sigma_2(r(t)) \cdot y(t)\,\mathrm{d}B_2(t), \\
x(0) &= x_0, \\
y(0) &= y_0, \\
r(0) &= r_0.
\end{aligned} \tag{68}$$

Lemma 12 (see [77]). *System (7) is ergodic and positive recurrent if the following conditions are satisfied:*

(i) For $i \neq j$, $q_{ij} > 0$.

(ii) For each $k \in \mathbb{S}$,

$$\sigma \|\xi\|^2 \leq \xi^T G(x,k)\xi \leq \sigma^{-1}\|\xi\|^2 \tag{69}$$

for all $\xi \in \mathbb{R}^n$, with some constant $\sigma \in (0,1]$ for all $x \in \mathbb{R}^n$.

(iii) There exists a bounded open set $\mathscr{D} \subset \mathbb{R}^n$ with a smooth boundary satisfying that, for each $k \in \mathbb{S}$, there is a twice continuously differentiable nonnegative function $V(\cdot,\cdot) : \mathscr{D}^c \to \mathbb{R}$ and that for some $\zeta > 0$, $\mathscr{L}V(x,k) \leq -\zeta$, for any $(x,k) \in \mathscr{D}^c \times \mathbb{S}$.

Moreover, the Markov process $(x(t), r(t))$ has a unique ergodic stationary distribution $\mu(\cdot,\cdot)$. Hence, for any Borel measurable function $H(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}$, if $\sum_{k\in\mathbb{S}} \int_{\mathbb{R}^n} |H(x,k)|\mu(x,k)\mathrm{d}x < \infty$, then

$$P\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t H(x(s), r(s))\,\mathrm{d}s = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}^n} H(x,k)\mu(x,k)\,\mathrm{d}x \right) = 1. \tag{70}$$

Recently, the ergodicity and stationary distribution have been explored by many authors. In the following, we give sufficient conditions for the existence of stationary distribution of system (68) and prove the following theorem by showing that system (68) satisfies the three conditions in Lemma 12.

Theorem 13. *Assume that for $i \neq j$, $q_{ij} > 0$ and*

(B_1): $\sum_{k\in\mathbb{S}} \pi_k(r_1(k) - \lambda(k)/m(k) - \sigma_1^2(k)/2 - c\beta_1(k)) > 0$,

(B_2): $\sum_{k\in\mathbb{S}} \pi_k(r_2(k) - \sigma_2^2(k)/2 - c\beta_2(k)) > 0$

hold; then the stochastic process $(x(t), y(t), r(t))$ of system (68) is ergodic and has a unique stationary distribution in $\mathbb{R}_+^2 \times \mathbb{S}$.

Proof. By the assumption $q_{ij} > 0$ for $i \neq j$ in Theorem 13, condition (i) in Lemma 12 is satisfied. Let $g(x, y, k) = \operatorname{diag}(\sigma_1(k)x, \sigma_2(k)y)$; then

$$G(x, y, k) = g(x, y, k) g^T(x, y, k) = \operatorname{diag}\left(\sigma_1^2(k) x^2, \sigma_2^2(k) y^2\right). \tag{71}$$

Define a bounded open subset as follows:

$$U = \left(\varepsilon, \frac{1}{\varepsilon}\right) \times \left(\varepsilon, \frac{1}{\varepsilon}\right) \subset \mathbb{R}_+^2, \tag{72}$$

where $0 < \varepsilon < 1$ is a constant. Let $\sigma = \min\{1, M_1, 1/M_2\}$; here $M_1 = \min_{(x,y,k)\in\overline{U}\times\mathbb{S}}\{\sigma_1^2(k)x^2 + \sigma_2^2(k)y^2\}$ and $M_2 = \max_{(x,y,k)\in\overline{U}\times\mathbb{S}}\{\sigma_1^2(k)x^2 + \sigma_2^2(k)y^2\}$. For $(x, y, k) \in U \times \mathbb{S}$, we have

$$\sigma\|\xi\|^2 \le \xi^T G(x, y, k)\xi = \sigma_1^2(k) x^2 \xi_1^2 + \sigma_2^2(k) y^2 \xi_2^2 \le \sigma^{-1}\|\xi\|^2 \tag{73}$$

for all $\xi \in \mathbb{R}^2$. Thus condition (ii) in Lemma 12 holds. Therefore, it remains for us to verify condition (iii) in Lemma 12.

Define a C^2-function on $\mathbb{R}_+^2 \times \mathbb{S}$,

$$V(x, y, k) = (1 - \upsilon\eta_k) x^{-\upsilon} + (1 - \upsilon\delta_k) y^{-\upsilon} + (x + 2)(y + M) =: V_4 + V_5, \tag{74}$$

where $V_4 = (1 - \upsilon\eta_k)x^{-\upsilon} + (1 - \upsilon\delta_k)y^{-\upsilon}$, $V_5 = (x + 2)(y + M)$, υ is a sufficiently small positive constant satisfying $\upsilon < \min(1/\eta_k, 1/\delta_k)$, and $M \ge 1 + (1/4\widehat{b}_1\widehat{b}_2)(\check{r}_1 - c\widehat{\beta}_1 + \check{r}_2 - c\widehat{\beta}_2 + \check{\gamma}/\widehat{n})^2$. η_k, δ_k are quantities to be determined below.

An application of the operator $\mathscr{L}$ to V_4 yields

$$\begin{aligned}
\mathscr{L}V_4(x, y, k) &= -\upsilon(1 - \upsilon\eta_k) x^{-\upsilon}\left(r_1(k) - b_1(k)x - \frac{\lambda(k)y}{a(k) + m(k)y + n(k)x} - c\beta_1(k)\right) + \frac{1}{2}\upsilon(1 + \upsilon)(1 - \upsilon\eta_k)\sigma_1^2(k)x^{-\upsilon} - \upsilon\sum_{i\neq k} q_{ki}(\eta_i - \eta_k)x^{-\upsilon} \\
&\quad - \upsilon(1 - \upsilon\delta_k) y^{-\upsilon}\left(r_2(k) + \frac{\gamma(k)x}{a(k) + m(k)y + n(k)x} - b_2(k)y - c\beta_2(k)\right) \\
&\quad + \frac{1}{2}\upsilon(1 + \upsilon)(1 - \upsilon\delta_k)\sigma_2^2(k)y^{-\upsilon} - \upsilon\sum_{i\neq k} q_{ki}(\delta_i - \delta_k)y^{-\upsilon} \le \upsilon(1 - \upsilon\eta_k) \\
&\quad \cdot\left[-\left(r_1(k) - \frac{\lambda(k)}{m(k)} - c\beta_1(k) - \frac{\sigma_1^2(k)}{2}\right) - \sum_{i\neq k} q_{ki}(\eta_i - \eta_k) + \frac{1}{2}\upsilon\sigma_1^2(k) - \frac{\upsilon\eta_k}{1 - \upsilon\eta_k}\sum_{i\neq k} q_{ki}(\eta_i - \eta_k)\right] x^{-\upsilon} + \upsilon(1 - \upsilon\delta_k) \\
&\quad \cdot\left[-\left(r_2(k) - c\beta_2(k) - \frac{1}{2}\sigma_2^2(k)\right) - \sum_{i\neq k} q_{ki}(\delta_i - \delta_k) + \frac{1}{2}\upsilon\sigma_2^2(k) - \frac{\upsilon\delta_k}{1 - \upsilon\delta_k}\sum_{i\neq k} q_{ki}(\delta_i - \delta_k)\right] y^{-\upsilon} + \upsilon(1 \\
&\quad - \upsilon\eta_k) b_1(k) x^{1-\upsilon} + \upsilon(1 - \upsilon\delta_k) b_2(k) y^{1-\upsilon}.
\end{aligned} \tag{75}$$

Define the vectors $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ and $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_m)^T$ with $\xi_k = -(r_1(k) - \lambda(k)/m(k) - c\beta_1(k) - (\sigma_1^2/2)(k))$, $\vartheta_k = -(r_2(k) - c\beta_2(k) - (1/2)\sigma_2^2(k))$. As the generator matrix Γ is irreducible, for each ξ_k and ϑ_k, there exists $\eta = (\eta_1, \eta_2, \ldots, \eta_m)^T$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_m)^T$, respectively, which is a solution of the Poisson system [78]

$$\begin{aligned}
(\Gamma\eta)_k - \xi_k &= -\sum_{j=1}^{m} \pi_j \xi_j, \\
(\Gamma\delta)_k - \vartheta_k &= -\sum_{j=1}^{m} \pi_j \vartheta_j.
\end{aligned} \tag{76}$$

Therefore we have

$$\begin{aligned}
&-\left(r_1(k) - \frac{\lambda(k)}{m(k)} - c\beta_1(k) - \frac{\sigma_1^2}{2}(k)\right) - \sum_{i\neq k} q_{ki}(\eta_i - \eta_k) \\
&\quad = -\sum_{k\in\mathbb{S}} \pi_k\left(r_1(k) - \frac{\lambda(k)}{m(k)} - \frac{\sigma_1^2(k)}{2} - c\beta_1(k)\right),
\end{aligned} \tag{77}$$

$$\begin{aligned}
&-\left(r_2(k) - c\beta_2(k) - \frac{1}{2}\sigma_2^2(k)\right) - \sum_{i\neq k} q_{ki}(\delta_i - \delta_k) \\
&\quad = -\sum_{k\in\mathbb{S}} \pi_k\left(r_2(k) - \frac{\sigma_2^2(k)}{2} - c\beta_2(k)\right).
\end{aligned} \tag{78}$$

Combining (75), (77), and (78), we obtain

$$\begin{aligned}\mathscr{L}V_4(x,y,k) &\le v(1-v\eta_k)\\&\cdot\left[-\sum_{k\in\mathbb{S}}\pi_k\left(r_1(k)-\frac{\lambda(k)}{m(k)}-\frac{\sigma_1^2(k)}{2}-c\beta_1(k)\right)\right.\\&\left.+\frac{1}{2}v\sigma_1^2(k)-\frac{v\eta_k}{1-v\eta_k}\sum_{i\neq k}q_{ki}(\eta_i-\eta_k)\right]x^{-v}+v(1\\&-v\delta_k)\left[-\sum_{k\in\mathbb{S}}\pi_k\left(r_2(k)-\frac{\sigma_2^2(k)}{2}-c\beta_2(k)\right)\right.\\&\left.+\frac{1}{2}v\sigma_2^2(k)-\frac{v\delta_k}{1-v\delta_k}\sum_{i\neq k}q_{ki}(\delta_i-\delta_k)\right]y^{-v}+v(1\\&-v\eta_k)b_1(k)x^{1-v}+v(1-v\delta_k)b_2(k)y^{1-v}.\end{aligned}\tag{79}$$

Similarly, for $V_5(x,y)$, we calculate

$$\begin{aligned}\mathscr{L}V_5 &= (y+M)x\left(r_1(k)-b_1(k)x\right.\\&\left.-\frac{\lambda(k)y}{a(k)+m(k)y+n(k)x}-c\beta_1(k)\right)+(x+2)\\&\cdot y\left(r_2(k)+\frac{\gamma(k)x}{a(k)+m(k)y+n(k)x}-b_2(k)y\right.\\&\left.-c\beta_2(k)\right)\le-\widehat{b}_1x^2+M\left(\check{r}_1-c\widehat{\beta_1}\right)x-\widehat{b}_2y^2\\&+2\left(\check{r}_2-c\widehat{\beta}_2+\frac{\check{\gamma}}{\widehat{n}}\right)y.\end{aligned}\tag{80}$$

From conditions (B_1) and (B_2), we can choose v sufficiently small such that

$$\begin{aligned}&-\sum_{k\in\mathbb{S}}\pi_k\left(r_1(k)-\frac{\lambda(k)}{m(k)}-\frac{\sigma_1^2(k)}{2}-c\beta_1(k)\right)\\&\quad+\frac{1}{2}v\sigma_1^2(k)-\frac{v\eta_k}{1-v\eta_k}\sum_{i\neq k}q_{ki}(\eta_i-\eta_k)<0,\\&-\sum_{k\in\mathbb{S}}\pi_k\left(r_2(k)-\frac{\sigma_2^2(k)}{2}-c\beta_2(k)\right)+\frac{1}{2}v\sigma_2^2(k)\\&\quad-\frac{v\delta_k}{1-v\delta_k}\sum_{i\neq k}q_{ki}(\delta_i-\delta_k)<0.\end{aligned}\tag{81}$$

So $\mathscr{L}V=\mathscr{L}V_4+\mathscr{L}V_5$ can be estimated as follows:

$$\begin{aligned}\mathscr{L}V &\le v(1-v\eta_k)\\&\cdot\left[-\sum_{k\in\mathbb{S}}\pi_k\left(r_1(k)-\frac{\lambda(k)}{m(k)}-\frac{\sigma_1^2(k)}{2}-c\beta_1(k)\right)\right.\\&\left.+\frac{1}{2}v\sigma_1^2(k)-\frac{v\eta_k}{1-v\eta_k}\sum_{i\neq k}q_{ki}(\eta_i-\eta_k)\right]x^{-v}+v(1\\&-v\eta_k)b_1(k)x^{1-v}-\widehat{b}_1x^2+M\left(\check{r}_1-c\widehat{\beta_1}\right)x+v(1\\&-v\delta_k)\left[-\sum_{k\in\mathbb{S}}\pi_k\left(r_2(k)-\frac{\sigma_2^2(k)}{2}-c\beta_2(k)\right)\right.\\&\left.+\frac{1}{2}v\sigma_2^2(k)-\frac{v\delta_k}{1-v\delta_k}\sum_{i\neq k}q_{ki}(\delta_i-\delta_k)\right]y^{-v}+v(1\\&-v\delta_k)b_2(k)y^{1-v}-\widehat{b}_2y^2+2\left(\check{r}_2-c\widehat{\beta}_2+\frac{\check{\gamma}}{\widehat{n}}\right)y\\&=f_1(x)+f_2(y).\end{aligned}\tag{82}$$

It is easy to see that

$$\begin{aligned}\mathscr{L}V &\le f_1(x)+f_2(y)\\&\le\begin{cases}f_1(x)+f_2^u(y)\longrightarrow-\infty, & \text{if } x\longrightarrow 0 \text{ or } x\longrightarrow+\infty,\\ f_1^u(x)+f_2(y)\longrightarrow-\infty, & \text{if } y\longrightarrow 0 \text{ or } y\longrightarrow+\infty.\end{cases}\end{aligned}\tag{83}$$

Consequently, we derive that, for a sufficiently small ε,

$$\mathscr{L}V(x,y,k)\le-1,\quad\forall(x,y,k)\in U^c\times\mathbb{S}.\tag{84}$$

Using Lemma 12, we obtain the conclusion that $(x(t), y(t), r(t))$ is ergodic and positive recurrent; that is, system (68) is positive recurrent and has a unique stationary distribution.

This completes the proof of Theorem 13. □

5. Conclusions and Numerical Simulations

In this article, we discussed the dynamics of stochastic prey-predator models with Beddington-DeAngelis functional response in polluted environment.

Firstly, for system (3), there are the following properties:

(1) If $h_1<0$ and $h_2>0$, the limit system of (3) has a prey extinction periodic solution $(0, y^*(t))$.

(2) If conditions (H_1), (H_2), and (H_3) are established, the limit system of (3) has a positive periodic solution.

Secondly, system (4) possesses the following properties:

(1) If $\eta_1<0$, the prey population $x(t)$ will be extinct.

(2) If $\eta_2<0$, the predator population $y(t)$ will be extinct.

(3) If $\eta_1<0$ and $\eta_3<0$, then prey population and predator population will die out.

(4) If $\eta_4>0$, the prey population $x(t)$ will be persistent in mean.

(5) If $\eta_5>0$ and $\eta_6>0$, the predator population $y(t)$ will be persistent in mean.

To verify the correctness of the theoretical analysis, numerical simulations are employed in the following examples.

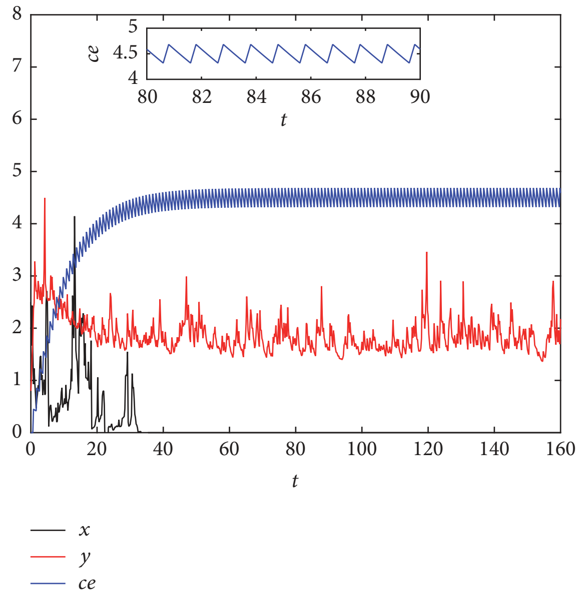


FIGURE 1: Sample paths of $x(t)$ and $y(t)$ with initial conditions $x(0) = 2.5$ and $y(0) = 0.8$.

Assume that the Markov chain $r(t)$ take values in $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{85}$$

By the linear equations $\pi\Gamma = 0$, we can see $(\pi_1, \pi_2) = (2/3, 1/3)$ which is the stationary distribution of $r(t)$. Furthermore, in the following examples, we suppose $\tau = 0.5$ and $h = 0.1$, consistently.

5.1. The Existence of Periodic Solutions of System (3)

Example 14. Assume $r_1 = 1.2$, $r_2 = 2.3$, $\beta_1 = 0.18$, $\beta_2 = 0.22$, $b_1 = 0.5$, $b_2 = 0.8$, $\lambda = 0.3$, $\gamma = 0.5$, $a = 1.4$, $m = 0.8$, and $n = 1.8$.

Case 1. We choose the density of white noise as the following: $\sigma_1(t) = 0.09 + 0.8\sin((2\pi/\tau)t)$, $\sigma_2(t) = 0.1 + 0.01\sin((2\pi/\tau)t)$, and let $\mu = 0.45$.

Note that $h_1 = -0.5841 < 0$, $h_2 = 0.3150 > 0$. The conditions of Theorem 6 hold, so there exists a boundary periodic solution $(0, y^*(t))$ of system (3) (see Figure 1).

Case 2. We change the density of the white noise to $\sigma_1(t) = 0.02 + 0.2\sin((2\pi/\tau)t)$, $\sigma_2(t) = 0.07 + 0.1\sin((2\pi/\tau)t)$, and $\mu = 0.13$. This gives $h_1 = 0.7218 > 0$, $h_2 = 1.7230 > 0$, $\lambda_1 = 0.0634 > 0$, and $\xi_2 = 0.2024 > 0$; choose $\xi_1 = 0.1$ according to $\xi_1 > (\gamma^2/4b_1h_1)\lambda_1$ and $(b_2\xi_1/\gamma^2)[-h_2 + (r_2 + \gamma/n - \beta_2 c_{\min})] < (\xi_2/ab_1)(r_1 - \beta_1 c_{\max})$.

From Theorem 7, we know that there exists a positive τ-periodic solution of system (3) (see Figure 2).

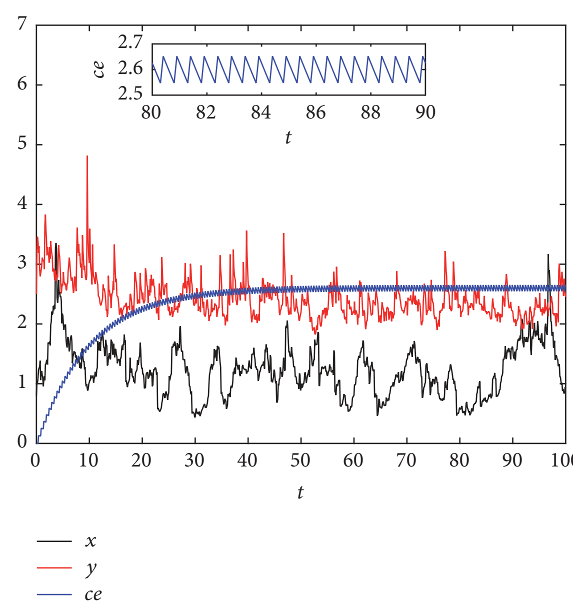
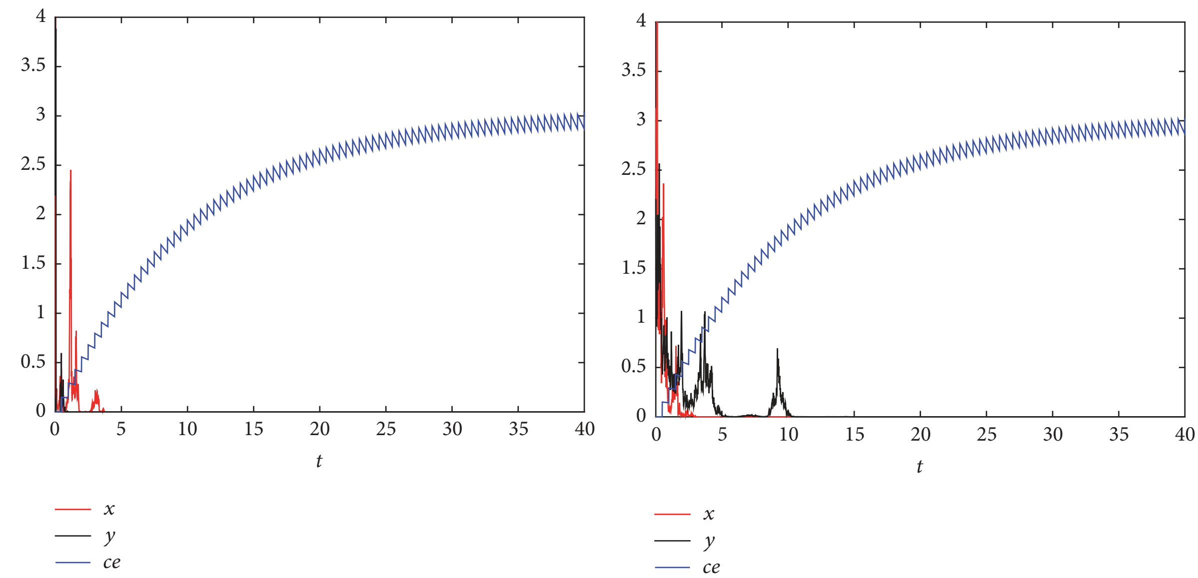


FIGURE 2: Sample paths of $x(t)$ and $y(t)$ with initial conditions $x(0) = 0.8$ and $y(0) = 2.5$.

5.2. The Extinction and Persistence of System (4)

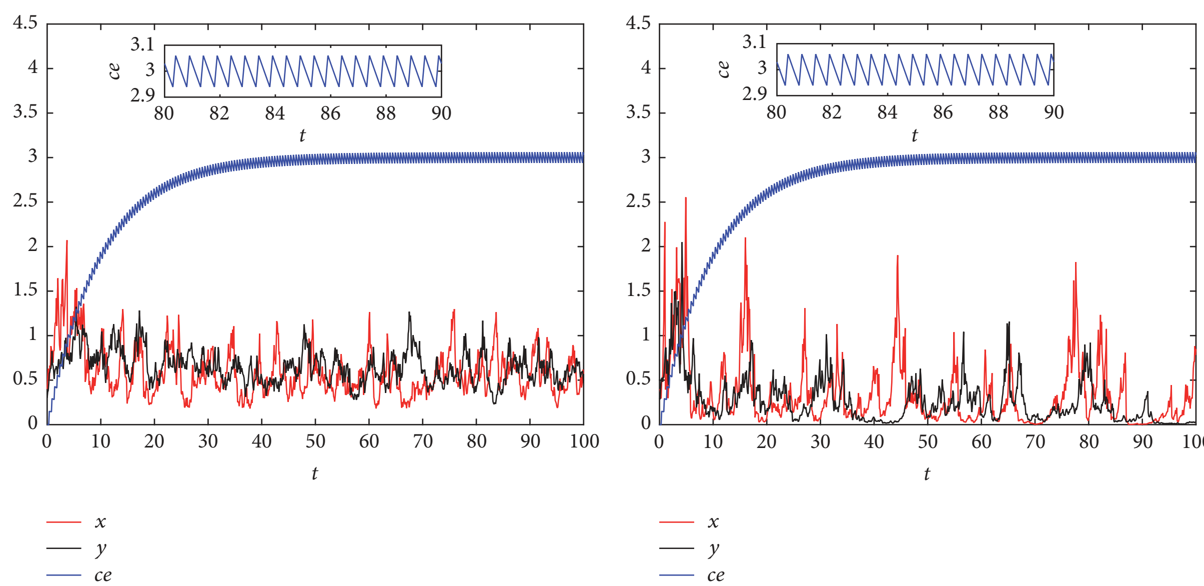
Example 15. Choose parameters $(r_1(1), r_2(1)) = (2.3, 1.5)$, $\lambda(1) = 0.06$, $\gamma(1) = 0.3$, $(b_1(1), b_2(1)) = (1.8, 1.6)$, $a(1) = 0.4$, $m(1) = 0.6$, $n(1) = 0.3$, and $(\beta_1(1), \beta_2(1)) = (0.4, 0.2)$, if $k = 1$, and $(r_1(2), r_2(2)) = (2.5, 1.3)$, $\lambda(2) = 0.08$, $\gamma(2) = 0.2$, $(b_1(2), b_2(2)) = (1.8, 1.6)$, $a(2) = 0.4$, $m(2) = 0.8$, $n(2) = 0.4$, and $(\beta_1(2), \beta_2(2)) = (0.5, 0.3)$, if $k = 2$.

Case 1. Let $\mu = 0.15$, $(\sigma_1(1), \sigma_2(1)) = (1.7, 2.1)$, and $(\sigma_1(2), \sigma_2(2)) = (1.4, 0.9)$; we note $\eta_1 = -0.1911 < 0$,



(a) $k = 1$ (b) $k = 2$

Figure 3: Sample paths of $x(t)$, $y(t)$ with initial value $x(0) = 2.8$ and $y(0) = 2.2$.



(a) $k = 1$ (b) $k = 2$

Figure 4: Sample paths of $x(t)$, $y(t)$ with initial value $x(0) = 0.4$ and $y(0) = 0.3$.

$\eta_2 = -0.0210 < 0$, and $\eta_3 = -0.8543 < 0$. The conditions of Theorem 8 are satisfied, so the prey and predator are both extinct (see Figure 3).

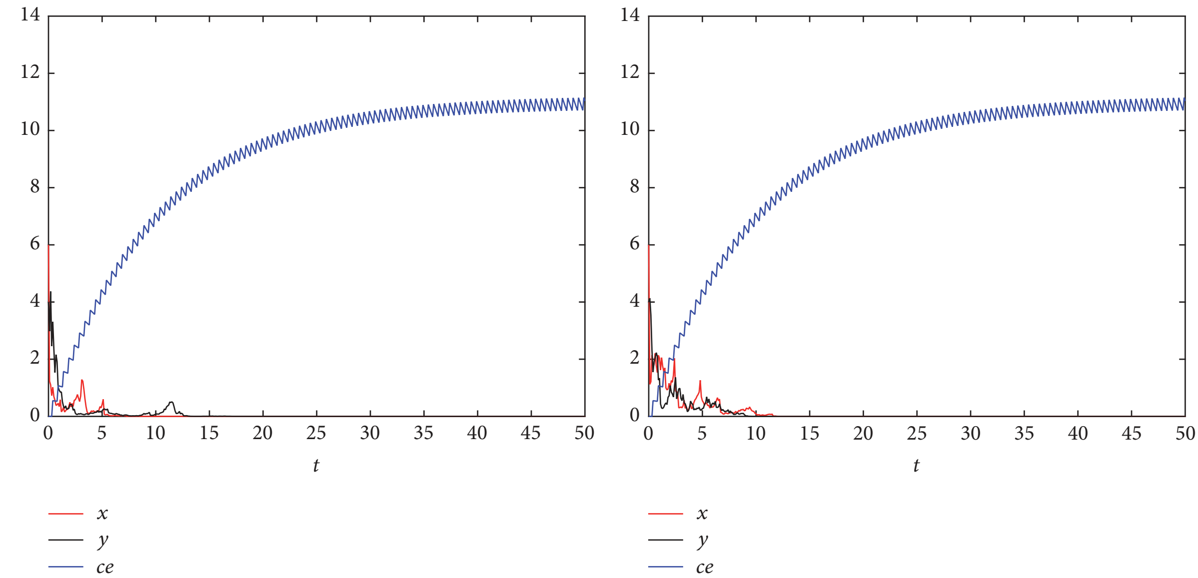
Next we only change the density of the white noise to $(\sigma_1(1), \sigma_2(1)) = (0.3, 0.2)$ and $(\sigma_1(2), \sigma_2(2)) = (0.5, 0.4)$ and keep $\mu = 0.15$. Simple calculation shows that $\eta_4 = 0.8622 > 0$, $\eta_5 = 0.9622 > 0$, and $\eta_6 > 0.6757 > 0$. The conditions of Theorem 10 are satisfied, so the prey and predator are persistent (see Figure 4).

It is easy to see from Figures 3 and 4 that the increase of the intensity of white noise can result in the extinction of prey and predator.

Case 2. Let $\mu = 0.55$, $(\sigma_1(1), \sigma_2(1)) = (0.6, 0.5)$, and $(\sigma_1(2), \sigma_2(2)) = (0.45, 0.55)$, which gives $\eta_1 = -2.4356 < 0$, $\eta_2 = -0.3701 < 0$, and $\eta_3 = -1.2035 < 0$. $x(t)$ and $y(t)$ are extinct (see Figure 5).

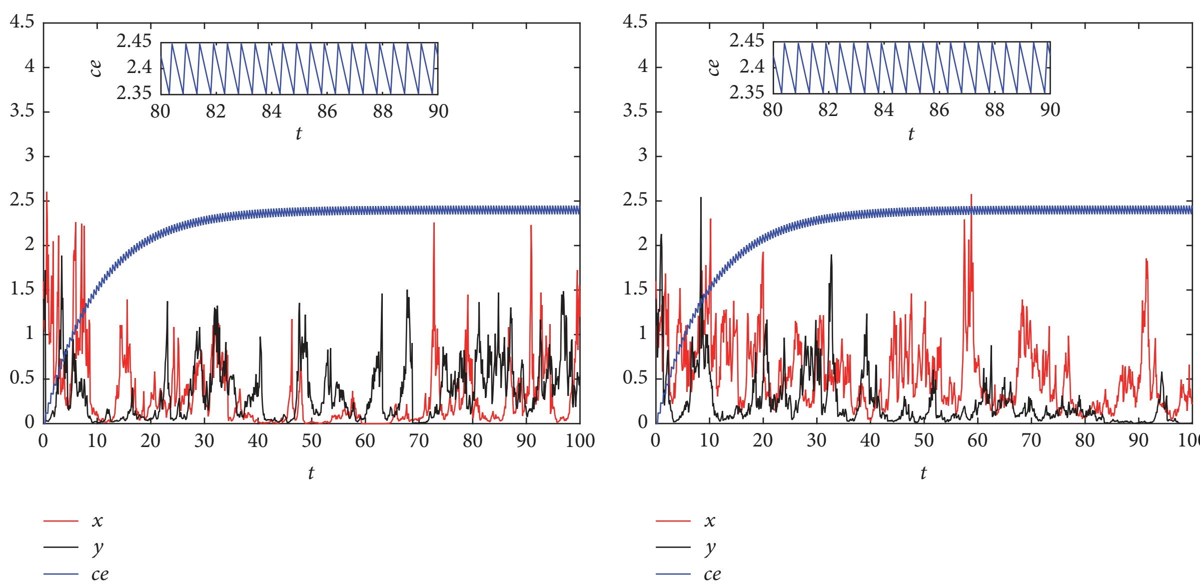
Next we only change the amount of toxicant to $\mu = 0.12$ and keep $(\sigma_1(1), \sigma_2(1)) = (0.6, 0.5)$ and $(\sigma_1(2), \sigma_2(2)) = (0.45, 0.55)$, We note $\eta_4 = 1.0467 > 0$, $\eta_5 = 1.1467 > 0$, and $\eta_6 > 0.7255 > 0$. Thus $x(t)$ and $y(t)$ are persistent in mean (see Figure 6).

Figures 5 and 6 show that the increase of the amount of toxicant can also result in the extinction of the prey and predator.



(a) $k = 1$ (b) $k = 2$

Figure 5: Sample paths of $x(t)$, $y(t)$ with initial value $x(0) = 6$ and $y(0) = 4$.



(a) $k = 1$ (b) $k = 2$

Figure 6: Sample paths of $x(t)$, $y(t)$ with initial value $x(0) = 1.6$ and $y(0) = 1.4$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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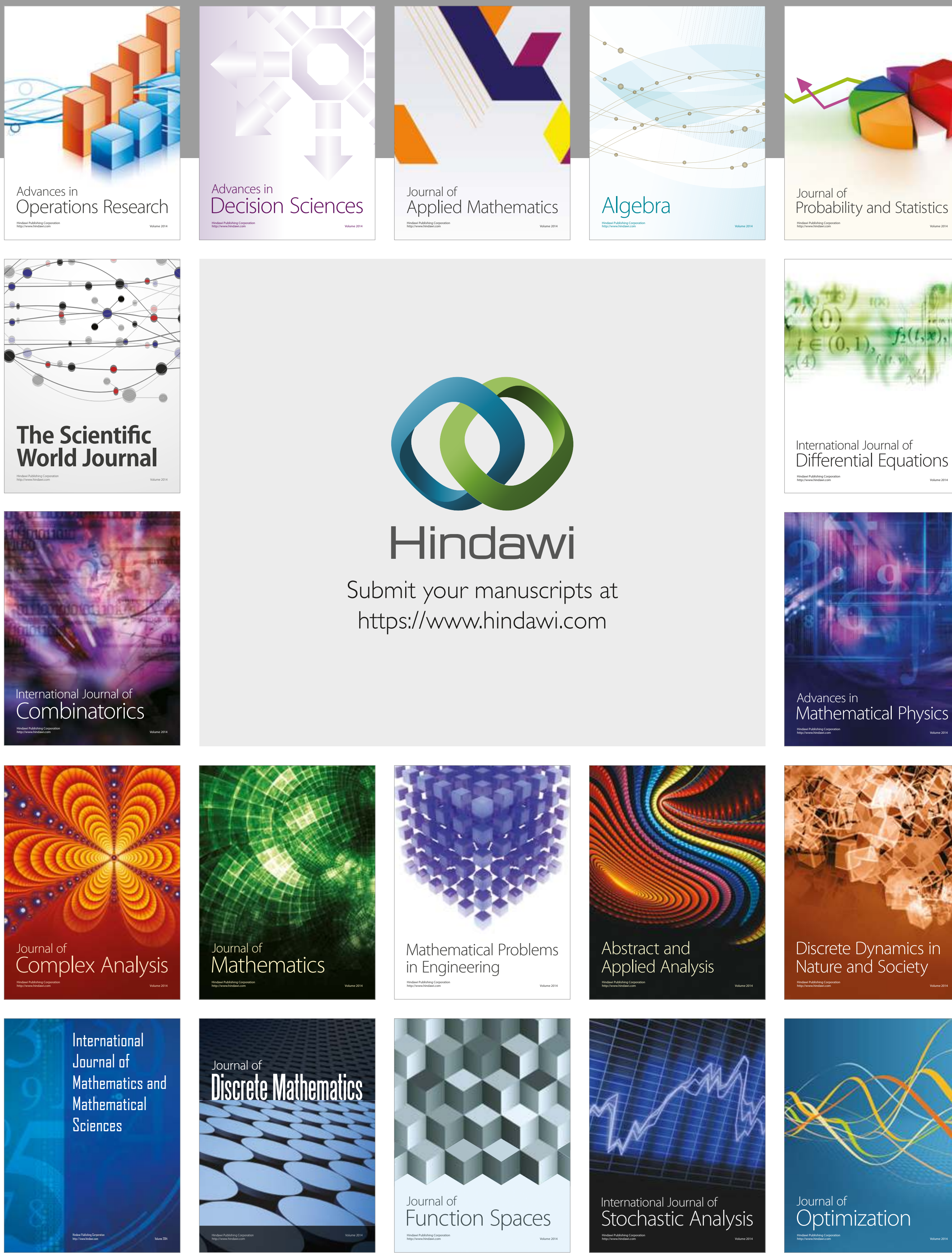
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