

**Research Article**

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# Dynamical analysis of a Lotka Volterra commensalism model with additive Allee effect

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**Abstract:** We propose and analyze a Lotka-Volterra commensal model with an additive Allee effect in this article. First, we study the existence and local stability of possible equilibria. Second, the conditions for the existence of saddle-node bifurcations and transcritical bifurcations are derived by using Sotomayor's theorem. Third, we give sufficient conditions for the global stability of the boundary equilibrium and positive equilibrium. Finally, we use numerical simulations to verify the above theoretical results. This study shows that for the weak Allee effect case, the additive Allee effect has a negative effect on the final density of both species, with increasing Allee effect, the densities of both species are decreasing. For the strong Allee effect case, the additive Allee effect is one of the most important factors that leads to the extinction of the second species. The additive Allee effect leads to the complex dynamic behaviors of the system.

**Keywords:** Lotka-Volterra commensal model, additive Allee effect, bifurcation

**MSC 2020:** 92D25, 34D20

## 1 Introduction

Commensalism is a symbiotic relationship between two species in which one species benefits from another species, while the other species neither gains nor loses. In the past few decades, many scholars have done work on the dynamic behaviors of the commensalism model, and some essential progress has been obtained [1–42].

Sun and Wei [1] first time proposed and studied a two species commensalism symbiosis model:

$$\begin{cases} \frac{dx}{dt} = r_1x \left( \frac{k_1 - x + \alpha y}{k_1} \right), \\ \frac{dy}{dt} = r_2y \left( \frac{k_2 - y}{k_2} \right). \end{cases} \quad (1.1)$$

They investigated the local stability property of four equilibria, among which the boundary equilibria  $E_1(0, 0)$ ,  $E_2(k_1, 0)$ , and  $E_3(0, k_2)$  are unstable, and the unique positive equilibrium  $E_4(k_1 + \alpha k_2, k_2)$  is always

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locally stable. However, they did not conduct a further study on the global stability of the positive equilibrium  $E_4(k_1 + \alpha k_2, k_2)$ .

Han and Chen [2] proposed the following commensalism model:

$$\begin{cases} \frac{dx}{dt} = x(b_1 - a_{11}x) + a_{12}xy, \\ \frac{dy}{dt} = y(b_2 - a_{22}y). \end{cases} \quad (1.2)$$

They showed that the system admits a unique positive equilibrium, which is globally asymptotically stable. In addition, they added feedback control variables into the system (1.2) and found that the feedback control variable only changes the position of the positive equilibrium but still maintains its property of global stability.

When the populations have non-overlapping generations, the discrete-time models governed by difference equations are more appropriate than the continuous ones. Thus, Xie et al. [3] proposed the discrete commensal symbiosis model. Based on [3], Li et al. [4] proposed the discrete commensal symbiosis model with the Holling II functional response. They gave some sufficient conditions for the existence of positive periodic solution of the models they considered. Chen [5] and Yu et al. [6] studied the commensal symbiosis model with the Michaelis-Menten type harvesting. In [6], Yu et al. studied the global existence of positive periodic solutions of the system and gave sufficient conditions which ensure the global attractivity of the positive periodic solution.

On the other hand, in 1931, Allee [23] pointed out that when the population density is too low, individuals in the population will encounter difficulties in finding mates and resisting natural enemies, which will lead to a decrease in the birth rate and an increase in the death rate of the population. This phenomenon is called the Allee effect [24]. Since then, many scholars began to study the ecological model with the Allee effect. Bazykin [25] proposed a single model with multiple Allee effects for the first time as follows:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) (x - m), \quad (1.3)$$

where  $r$  represents the inherent per capita growth rate of the population and  $K$  represents the environmental carrying capacity. If  $0 < m < K$ , it shows the strong Allee effect when the population is lower than the threshold, the population growth is negative, and the population is at risk of extinction; otherwise, the population can survive. While if  $m \leq 0$ , it shows the weak Allee effect; the population growth slows down but there is no risk of extinction.

Furthermore, Dennis [28] proposed the model with the additive Allee effect for the first time as follows:

$$\frac{dx}{dt} = rx \left[ \left(1 - \frac{x}{K}\right) - \frac{m}{x+a} \right]. \quad (1.4)$$

Here, we denote the additive Allee effect by  $\frac{m}{x+a}$ ,  $m$  and  $a$  are both constants, and the additive Allee effect has the following properties:

- (1) If  $0 < m < a$ , then system (1.4) has the weak Allee effect.
- (2) If  $m > a$ , then system (1.4) has the strong Allee effect.

Merdan [29] proposed the following predator-prey model:

$$\begin{cases} \frac{dx}{dt} = r \left(1 - \frac{x}{K}\right) \frac{x}{\beta + x} - axy, \\ \frac{dy}{dt} = ay(x - y). \end{cases}$$

Merdan showed that the system subject to an Allee effect takes a much longer time to reach its stable steady-state solution; also, the Allee effect reduces the population densities of both predator and prey at the steady state. However, the Allee effect has no destabilizing role.

For more articles on the Allee effect, please see [23–36].

We mention here that in nature, one of the typical commensal relationships between epiphyte and plants with epiphyte, as shown in Figure 1, the plant (host) generally speaking, is huge, need more space to grow, and its density is sparse; this certainly increases the chance of the Allee effect on the plant. Indeed, recently, Jiao *et al.* [30] have shown that in a coastal wetland, the plant population does exhibit the Allee effect. Hence, it is natural to propose and study the commensalism model with the Allee effect.

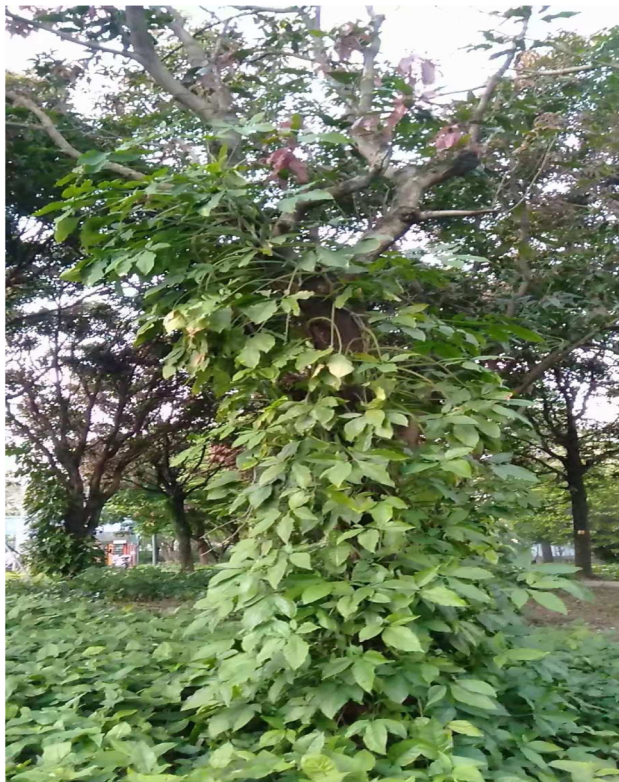
Recently, Wu *et al.* [7] added the Holling-type functional response and Merdan-type Allee effect (one could refer to [29] for more details) to the system (1.2), this leads to the following system:

$$\begin{cases} \frac{dx}{dt} = x \left( a_1 - b_1 x + \frac{c_1 y^p}{1 + y^p} \right), \\ \frac{dy}{dt} = y(a_2 - b_2 y) \frac{y}{u + y}. \end{cases} \quad (1.5)$$

They showed that the unique positive equilibrium is globally stable and the Allee effect has no influence on the final density of the species, and that the stronger the Allee effect ( $u$  become large), the system takes a longer time to reach its steady-state solution.

Later, Lin [8] considered adding the Merdan-type Allee effect in the first species of the system (1.2), and they studied the dynamic behaviors of the following system:

$$\begin{cases} \frac{dx}{dt} = x(b_1 - a_{11}x) \frac{x}{\beta + x} + a_{12}xy, \\ \frac{dy}{dt} = y(b_2 - a_{22}y), \end{cases} \quad (1.6)$$



**Figure 1:** *Syngonium podophyllum* Schott and host tree, the picture comes from Minjiang Park, a park that lies in Fuzhou city, P. R. China.

where  $b_i, a_{ii}, i = 1, 2, \beta$  and  $a_{12}$  are positive constants,  $F(x) = \frac{x}{\beta+x}$  represents the Allee effect of the first species. They observed that as the Allee effect increased, the final density of the species affected by Allee effect also increased. Moreover, the positive equilibrium of the system (1.6) is still globally stable.

Inspired by Wu et al. [7] and Lin [8], we consider replacing the Merdan-type Allee effect with additive Allee effect on the traditional Lotka-Volterra commensalism model, this leads to the following model:

$$\begin{cases} \frac{dx}{dt} = x(r - bx) + cxy, \\ \frac{dy}{dt} = y\left(d - ey - \frac{m}{y+a}\right), \end{cases} \tag{1.7}$$

where  $r, b, c, d, e, m,$  and  $a$  are all positive constants. We use the term  $F(y) = \frac{m}{y+a}$  to describe the additive Allee effect of the second species, and  $F(y) = \frac{m}{y+a}$  has the following properties:

- (a) If  $0 < m < ad$ , then the Allee effect in (1.7) is weak;
- (b) If  $m > ad$ , then the Allee effect in (1.7) is strong.

To the best of the authors' knowledge, this is the first time to propose and study the commensal model with the additive Allee effect. Our most important task is to find out the influence of the additive Allee effect on the system (1.7), especially on the  $y$  species. We also want to know if the system (1.7) has similar dynamic behaviors or any new properties compared with the systems considered in [2,7,8].

The rest of this article is arranged as follows: We investigate the existence of the equilibria in the next section and then study the local stability property of the equilibria in Section 3. In Section 4, we discuss the saddle-node bifurcations and transcritical bifurcations. In Section 5, we give sufficient conditions to ensure the global stability of the boundary equilibrium and the positive equilibrium, respectively. Finally, the article ends with some numeric simulations and a brief discussion.

## 2 Existence of equilibria

The equilibria of system (1.7) are given by the system:

$$\begin{cases} x(r - bx) + cxy = 0, \\ y\left(d - ey - \frac{m}{y+a}\right) = 0. \end{cases} \tag{2.1}$$

Obviously, system (2.1) always has two boundary equilibria given by  $E_0(0, 0)$  and  $E_r(\frac{r}{b}, 0)$ . In order to obtain the other equilibria, we simplify  $d - ey - \frac{m}{y+a} = 0$  to obtain the equation:

$$ey^2 + (ae - d)y + m - ad = 0, \tag{2.2}$$

For (2.2), let  $\Delta$  be its discriminant:

$$\Delta = (ae - d)^2 - 4e(m - da) = (ae + d)^2 - 4em,$$

and let  $m^*$  be the root of  $\Delta = 0$ , we have

$$m^* = \frac{(ae + d)^2}{4e} \geq ad.$$

Hence, we obtain that  $\Delta > 0$  if  $m < m^*$  and then (2.2) has two roots, denoted by  $y_1 = \frac{d - ae + \sqrt{\Delta}}{2e}, y_2 = \frac{d - ae - \sqrt{\Delta}}{2e}; \Delta = 0$  if  $m = m^*$  and hence (2.2) only has one root, denoted by  $y_3 = \frac{d - ae}{2e}; \Delta < 0$  if  $m > m^*$  and hence (2.2) has no real roots. Consequently, we can conclude that

- (1) If  $x = 0, y \neq 0$ , then system (1.7) has boundary equilibria  $E_i(0, y_i)$ , where  $y_i$  is the root of equation (2.2).
- (2) If  $x \neq 0, y \neq 0$ , then system (1.7) has positive equilibria  $E_i^*(x_i^*, y_i)$ , where  $y_i$  is the root of equation (2.2), and  $x_i^* = \frac{r + cy_i}{b}$ .

Through the above analysis, we know that system (1.7) always has two boundary equilibria given by  $E_0(0, 0)$  and  $E_r\left(\frac{r}{b}, 0\right)$ , and for the other possible equilibria, we have the following results:

**Theorem 2.1.** (The case of weak Allee effect, i.e.,  $m < ad$ ) System (1.7) has a boundary equilibrium  $E_1(0, y_1)$  and a positive equilibrium  $E_1^*(x_1^*, y_1)$ .

**Theorem 2.2.** (The case of  $m = ad$ )

- (1) When  $ae - d < 0$ , system (1.7) has a boundary equilibrium  $E_1(0, y_1)$  and a positive equilibrium  $E_1^*(x_1^*, y_1)$ .
- (2) When  $ae - d \geq 0$ , system (1.7) has no other equilibria.

**Theorem 2.3.** (The case of strong Allee effect, i.e.,  $m > ad$ )

- (1) When  $ae - d < 0$ ,
  - (a) if  $ad < m < m^*$ , then system (1.7) has two boundary equilibria  $E_1(0, y_1)$ ,  $E_2(0, y_2)$  and two positive equilibria  $E_1^*(x_1^*, y_1)$ ,  $E_2^*(x_2^*, y_2)$ .
  - (b) if  $m = m^*$ , then system (1.7) has a boundary equilibrium  $E_3(0, y_3)$  and a positive equilibrium  $E_3^*(x_3^*, y_3)$ .
  - (c) if  $m > m^*$ , then system (1.7) has no other equilibrium.
- (2) When  $ae - d \geq 0$ , system (1.7) has no other equilibrium.

### 3 Local stability of equilibria

In this section, we investigate the local stability of the equilibria. The Jacobian matrix of system (1.7) is calculated as

$$J(E) = \begin{pmatrix} r - 2bx + cy & cx \\ 0 & d - 2ey - \frac{ma}{(y+a)^2} \end{pmatrix}.$$

**Theorem 3.1.**

- (1)  $E_0(0, 0)$  is always unstable.
- (2) For  $E_r\left(\frac{r}{b}, 0\right)$ , we have
  - (a) if  $m > ad$ , then  $E_r\left(\frac{r}{b}, 0\right)$  is a stable node;
  - (b) if  $m < ad$ , then  $E_r\left(\frac{r}{b}, 0\right)$  is a saddle;
  - (c) if  $m = ad$ ,  $E_r\left(\frac{r}{b}, 0\right)$  is a stable node for  $ae = d$  and a saddle-node for  $ae \neq d$ .

**Proof.** (1) The Jacobian matrix of system (1.7) at  $E_0(0, 0)$  is

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & d - \frac{m}{a} \end{pmatrix},$$

whose eigenvalues are  $\lambda_1 = r > 0$  and  $\lambda_2 = d - \frac{m}{a}$ . If  $m > ad$ , then  $\lambda_2 < 0$  and hence  $E_0(0, 0)$  is a saddle; if  $m < ad$ , then  $\lambda_2 > 0$  and hence  $E_0(0, 0)$  is an unstable node; if  $m = ad$ , then  $\lambda_2 = 0$ , in this case, the local stability property of  $E_0$  is difficult to be judged directly from the characteristic root.

First, we expand system (1.7) in power series up to the third order around  $E_0(0, 0)$  and let  $d\tau = rdt$ :

$$\begin{cases} \frac{dx}{d\tau} = x + \frac{c}{r}xy - \frac{b}{r}x^2, \\ \frac{dy}{d\tau} = \frac{d-ae}{ar}y^2 - \frac{d}{a^2r}y^3 + \frac{d}{a^3r}y^4 - \frac{d}{a^4r}y^5. \end{cases} \quad (3.1)$$

By applying Theorem 7.1 of Chapter 2 in [37], we have

- (i) if  $ae = d$ , then  $m = 3$ ,  $a_3 = -\frac{d}{a^2r} < 0$ ; hence  $E_0(0, 0)$  of system (3.1) is a saddle, and then  $E_0(0, 0)$  of system (1.7) is also a saddle.
- (ii) if  $ae \neq d$ , then  $m = 2$ ,  $a_2 = \frac{1}{r}(-e + \frac{d}{a}) \neq 0$ ; hence  $E_0(0, 0)$  of system (3.1) is a saddle-node, and then  $E_0(0, 0)$  of system (1.7) is also a saddle-node.

Obviously,  $E_0(0, 0)$  is always unstable.

(2) The Jacobian matrix of system (1.7) at  $E_r(\frac{r}{b}, 0)$  is

$$J(E_r) = \begin{pmatrix} -r & \frac{cr}{b} \\ 0 & d - \frac{m}{a} \end{pmatrix},$$

whose eigenvalues are  $\lambda_1 = -r < 0$  and  $\lambda_2 = d - \frac{m}{a}$ . One could easily see that if  $m > ad$ , then  $\lambda_2 < 0$ , and thus  $E_r(\frac{r}{b}, 0)$  is a stable node; if  $m < ad$ , then  $\lambda_2 > 0$ , and thus  $E_r(\frac{r}{b}, 0)$  is a saddle; if  $m = ad$ , then  $\lambda_2 = 0$ . In this case,  $E_r(\frac{r}{b}, 0)$  is difficult to be judged directly from the characteristic root.

We first transform equilibrium  $E_r(\frac{r}{b}, 0)$  to the origin by setting  $(X, Y) = (x - \frac{r}{b}, y)$  and then expand the new system in power series around the origin:

$$\begin{cases} \frac{dX}{dt} = -rX + \frac{cr}{b}Y - bX^2 + cXY, \\ \frac{dY}{dt} = \frac{d - ae}{a}Y^2 - \frac{d}{a^2}Y^3 + \frac{d}{a^3}Y^4 - \frac{d}{a^4}Y^5. \end{cases} \tag{3.2}$$

Now, we apply the transformation:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{cr}{b} & -r \\ r & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

and introduce the new time variable  $d\tau_1 = -rdt$ , we have

$$\begin{cases} \frac{dU}{d\tau_1} = \frac{ae - d}{a}U^2 + \frac{dr}{a^2}U^3 - \frac{dr^2}{a^3}U^4 + \frac{dr^3}{a^4}U^5, \\ \frac{dV}{d\tau_1} = V - bV^2 + cUV + \frac{c(ae - d)}{ab}U^2 + Q_1(U, V), \end{cases} \tag{3.3}$$

where  $Q_1(U, V)$  is a power series in  $(U, V)$  with terms  $U^iV^j$  satisfying  $i + j \geq 3$ .

From the first equation of (3.3), we have

$$\frac{dU}{d\tau_1} = \frac{ae - d}{a}U^2 + \frac{dr}{a^2}U^3 - \frac{d^2}{a^3}U^4 + \frac{dr^3}{a^4}U^5 + \dots .$$

By applying Theorem 7.1 of Chapter 2 in [37], we have

- (i) If  $ae = d$ , then  $m = 3$ ,  $a_3 = \frac{dr}{a^2} > 0$ , hence  $E_r(\frac{b}{a}, 0)$  of system (3.3) is an unstable node. Since we use the transformation  $d\tau_1 = -rdt$  and  $-r < 0$ , the orbits with time go in the opposite direction, so  $E_r(\frac{b}{a}, 0)$  of system (1.7) is a stable node.
- (ii) If  $ae \neq d$ , then  $m = 2$ , hence  $E_r(\frac{b}{a}, 0)$  of system (3.3) is a saddle-node, and then  $E_r(\frac{b}{a}, 0)$  of system (1.7) is also a saddle-node.

This ends the proof of Theorem 3.1. □

**Theorem 3.2.** For  $i = 1, 2, 3$ , if  $E_i(0, y_i)$  exists, then  $E_i(0, y_i)$  are all unstable.

**Proof.** For  $i = 1, 2, 3$ , the Jacobian matrix of system (1.7) at  $E_i(0, y_i)$  is

$$J(E_i) = \begin{pmatrix} r + cy_i & 0 \\ 0 & y_i \left[ \frac{m}{(y_i + a)^2} - e \right] \end{pmatrix}.$$

The eigenvalues of  $J(E_i)$  are

$$\lambda_1 = r + cy_i > 0, \quad \lambda_2 = y_i \left[ \frac{m}{(y_i + a)^2} - e \right].$$

From Theorems 2.1–2.3, we know that if  $E_1$  and  $E_2$  exist, then  $m < m^*$ ; if  $E_3$  exists, then  $m = m^*$ . Next, we will discuss the eigenvalues of the three equilibria.

(1) For  $E_1(0, y_1)$ ,  $y_1 = \frac{d - ae + \sqrt{\Delta}}{2e}$ ,

$$\lambda_2 = y_1 \left[ \frac{m}{(y_1 + a)^2} - e \right] = y_1 \left[ \frac{me}{\frac{(d + ae + \sqrt{\Delta})^2}{4e}} - e \right] < ey_1 \left( \frac{m}{m^*} - 1 \right) < 0.$$

Consequently,  $E_1$  is a saddle if  $E_1$  exists.

(2) For  $E_2(0, y_2)$ ,  $y_2 = \frac{d - ae - \sqrt{\Delta}}{2e}$ ,

$$\lambda_2 = y_2 \left[ \frac{m}{(y_2 + a)^2} - e \right] = ey_2 \left[ \frac{4me}{(d + ae - \sqrt{\Delta})^2} - 1 \right] = ey_2 \frac{2\sqrt{\Delta}(\sqrt{\Delta + 4em} - \sqrt{\Delta})}{(d + ae - \sqrt{\Delta})^2} > 0.$$

Consequently,  $E_2$  is an unstable node if  $E_2$  exists.

(3) For  $E_3(0, y_3)$ ,  $y_3 = \frac{d - ae}{2e}$ ,

$$\lambda_2 = y_3 \left[ \frac{m}{(y_3 + a)^2} - e \right] = ey_3 \left[ \frac{m}{\frac{(d + ae)^2}{4e}} - 1 \right] = ey_3 \left( \frac{m}{m^*} - 1 \right) = 0.$$

In this case,  $E_3$  is difficult to be judged directly from the characteristic root.

We first shift  $E_3(0, y_3)$  to the origin by the transformation  $x = X_3$ ,  $y = Y_3 + y_3$  and then expand the new system in power series up to the third order around the origin:

$$\begin{cases} \frac{dX_3}{dt} = HX_3 - bX_3^2 + cX_3Y_3, \\ \frac{dY_3}{dt} = \frac{e(ae - d)}{ae + d}Y_3^2 - \frac{4ae^3}{(ae + d)^2}Y_3^3 + Q_2(X_3, Y_3), \end{cases} \quad (3.4)$$

where  $H = \left[ r + \frac{c(d - ae)}{2e} \right] > 0$ ,  $Q_2(X_3, Y_3)$  is a power series in  $(X_3, Y_3)$  with terms  $X_3^i Y_3^j$  satisfying  $i + j \geq 4$ .

Let  $d\tau_2 = \left[ r + \frac{c(d - ae)}{2e} \right] dt$ , where  $\tau_2$  is a new time variable. Then, system (3.4) becomes

$$\begin{cases} \frac{dX_3}{d\tau_2} = X_3 - \frac{b}{H}X_3^2 + \frac{c}{H}X_3Y_3, \\ \frac{dY_3}{d\tau_2} = \frac{e(ae - d)}{H(ae + d)}Y_3^2 - \frac{4ae^3}{H(ae + d)^2}Y_3^3 + Q_3(X_3, Y_3), \end{cases} \quad (3.5)$$

where  $Q_3(X_3, Y_3)$  are power series in  $(X_3, Y_3)$  with terms  $X_3^i Y_3^j$  satisfying  $i + j \geq 4$ .

From the second equation of (3.5), we have

$$\frac{dY_3}{d\tau_2} = \frac{e(ae - d)}{H(ae + d)}Y_3^2 - \frac{4ae^3}{H(ae + d)^2}Y_3^3 + Q_3(X_3, Y_3) + \dots,$$

we know that  $ae < d$ , so  $a_2 = \frac{(ae - d)e}{H(ae + d)} < 0$ . By applying Theorem 7.1 of Chapter 2 in [37], the equilibrium  $E_3(0, y_3)$  of system (3.5) is a saddle-node; therefore,  $E_3(0, y_3)$  of system (1.7) is also a saddle-node.

In summary,  $E_i(i = 1, 2, 3)$  are all unstable.

This ends the proof of Theorem 3.2. □

**Theorem 3.3.** *For the positive equilibrium, we have the following conclusions:*

- (1) *When  $E_1^*(x_1^*, y_1)$  exists, it is a stable node.*
- (2) *When  $E_2^*(x_2^*, y_2)$  exists, it is a saddle.*
- (3) *When  $E_3^*(x_3^*, y_3)$  exists, it is a saddle-node.*

**Proof.** The Jacobian matrix of system (1.7) at  $E_i^*(x_i^*, y_i)$  is

$$J(E_i^*) = \begin{pmatrix} -bx_i^* & cx_i^* \\ 0 & y_i^* \left[ \frac{m}{(y_i + a)^2} - e \right] \end{pmatrix}.$$

The eigenvalues of  $J(E_i^*)$  are  $\lambda_1 = -bx_i^* < 0$ ,  $\lambda_2 = y_i^* \left[ \frac{m}{(y_i + a)^2} - e \right]$ .

From the proof of Theorem 3.2 we know that

- (1) If  $E_1^*$  exist, for the eigenvalue  $\lambda_2$  of  $J(E_1^*)$ , we have  $\lambda_2 < 0$ , so  $E_1^*$  is a stable node.
- (2) If  $E_2^*$  exist, for the eigenvalue  $\lambda_2$  of  $J(E_2^*)$ , we have  $\lambda_2 > 0$ , so  $E_2^*$  is a saddle.
- (3) If  $E_3^*$  exist, for the eigenvalue  $\lambda_2$  of  $J(E_3^*)$ , we have  $\lambda_2 = 0$ , we can easily obtain that  $E_3^*$  is a saddle-node (this proof is similar to Theorem 3.2 (3)).

This ends the proof of Theorem 3.3. □

We use Table 1 to sum up the above conclusions.

## 4 Bifurcation analysis

From Theorems 2.1 to 2.3, we conjecture that system (1.7) may have saddle-node bifurcations at  $E_3$  and  $E_3^*$ , and transcritical bifurcations at the equilibria  $E_0$  and  $E_r$ , respectively. Indeed, we have the following results.

**Table 1:** Equilibria of system (1.7) in finite planes

Parameters	Location of equilibria	Types and stability
$m < ad$	$E_0, E_r, E_1, E_1^*$	$E_0$ unstable node, $E_r$ saddle, $E_1$ saddle, $E_1^*$ stable node
$ae < d$	$E_0, E_r, E_1, E_1^*$	$E_0, E_r$ saddle-node, $E_1$ saddle, $E_1^*$ stable node
$m = ad$	$E_0, E_r$	$E_0$ saddle, $E_r$ saddle-node
$ae > d$	$E_0, E_r$	$E_0, E_r$ saddle-node
$ad < m < m^*$	$E_0, E_r, E_1, E_2, E_1^*, E_2^*$	$E_0, E_1, E_2^*$ saddle, $E_r, E_1^*$ saddle-node, $E_2$ unstable node
$ae > d$	$E_0, E_r$	$E_0$ saddle, $E_r$ saddle-node
$m = m^*$	$E_0, E_r, E_3, E_3^*$	$E_0$ saddle, $E_r$ saddle-node, $E_3, E_3^*$ saddle-node
$ae \geq d$	$E_0, E_r$	$E_0$ saddle, $E_r$ saddle-node
$m > m^*$	$E_0, E_r$	$E_0$ saddle, $E_r$ saddle-node



**Theorem 4.1.** When  $ae < d$ , system (1.7) undergoes a saddle-node bifurcation around  $E_3$  with respect to the parameter  $m$  if  $m = m_{SN} = \frac{(ae+d)^2}{4e}$ .

**Proof.** The Jacobian matrix at  $E_3$  is

$$J(E_3) = \begin{pmatrix} r + cy_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is obvious that the matrix has a zero eigenvalue, named  $\lambda_1$ . Let  $V$  and  $W$  represent the eigenvectors corresponding to the eigenvalue  $\lambda_1$  for matrices  $J_{E_3}$  and  $J_{E_3}^T$ . By calculation, we can obtain:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.1)$$

Define

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x(r - bx) + cxy \\ y\left(d - ey - \frac{m}{y+a}\right) \end{pmatrix},$$

then

$$F_m(E_3; m_{SN}) = \begin{pmatrix} 0 \\ ae - d \\ ae + d \end{pmatrix}, \quad (4.2)$$

$$D^2F(E_3; m_{SN})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_3; m_{SN})} = \begin{pmatrix} 0 \\ \frac{2e(ae - d)}{ae + d} \end{pmatrix}. \quad (4.3)$$

From (4.1)–(4.3), it follows that

$$W^T F_m(E_3; m_{SN}) = \frac{ae - d}{ae + d} \neq 0,$$

$$W^T [D^2F(E_3; m_{SN})(V, V)] = \frac{2e(ae - d)}{ae + d} \neq 0.$$

So, according to Sotomayor's theorem in [38], system (1.7) undergoes a saddle-node bifurcation around  $E_3$  at  $m = m_{SN}$ .

This ends the proof of Theorem 4.1.  $\square$

**Theorem 4.2.** When  $ae < d$ , system (1.7) undergoes a saddle-node bifurcation around  $E_3^*$  with respect to the parameter  $m$  if  $m = m_{SN} = \frac{(ae+d)^2}{4e}$ .

**Proof.** The Jacobian matrix at  $E_3^*$  is

$$J(E_3^*) = \begin{pmatrix} -bx_3^* & cx_3^* \\ 0 & 0 \end{pmatrix}.$$

It is obvious that the matrix has a zero eigenvalue, named  $\lambda_1$ . Let  $V$  and  $W$  represent the eigenvectors corresponding to the eigenvalue  $\lambda_1$  for matrices  $J_{E_3^*}$  and  $J_{E_3^*}^T$ . By calculation, we can obtain:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.4)$$

Define

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x(r - bx) + cxy \\ y\left(d - ey - \frac{m}{y + a}\right) \end{pmatrix},$$

then

$$F_m(E_3^*; m_{SN}) = \begin{pmatrix} 0 \\ \frac{ae - d}{ae + d} \end{pmatrix}, \tag{4.5}$$

$$D^2F(E_3^*; m_{SN})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_3^*; m_{SN})} = \begin{pmatrix} 0 \\ \frac{2e(ae - d)}{ae + d} \end{pmatrix}. \tag{4.6}$$

From (4.4)–(4.6), it follows that

$$W^T F_m(E_3^*; m_{SN}) = \frac{ae - d}{ae + d} \neq 0,$$

$$W^T [D^2F(E_3^*; m_{SN})(V, V)] = \frac{2e(ae - d)}{ae + d} \neq 0.$$

So, according to Sotomayor’s theorem in [38], system (1.7) undergoes a saddle-node bifurcation around  $E_3^*$  at  $m = m_{SN}$ .

This ends the proof of Theorem 4.2. □

**Theorem 4.3.** *When  $ae < d$ , system (1.7) undergoes a transcritical bifurcation around  $E_0$  with respect to the parameter  $m$  if  $m = m_{TC} = ad$ .*

**Proof.** The Jacobian matrix at  $E_0$  is

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}.$$

It is obvious that the matrix has a zero eigenvalue, named  $\lambda_1$ . Let  $V$  and  $W$  represent the eigenvectors corresponding to the eigenvalue  $\lambda_1$  for matrices  $J_{E_0}$  and  $J_{E_0}^T$ . By calculation, we can obtain:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.7}$$

Define

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x(r - bx) + cxy \\ y\left(d - ey - \frac{m}{y + a}\right) \end{pmatrix},$$

then

$$F_m(E_0; m_{TC}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.8}$$

$$DF_m(E_0; m_{TC})V = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{a} \end{pmatrix}, \tag{4.9}$$

$$D^2F(E_0; m_{TC})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_0; m_{TC})} = \begin{pmatrix} 0 \\ \frac{2(d - ae)}{a} \end{pmatrix}. \quad (4.10)$$

From (4.7)–(4.10), it follows that

$$\begin{aligned} w^T F_m(E_0; m_{TC}) &= 0, \\ w^T [DF_m(E_0; m_{TC})V] &= -\frac{1}{a} \neq 0, \\ w^T [D^2F(E_0; m_{TC})(V, V)] &= \frac{2(d - ae)}{a} \neq 0. \end{aligned}$$

So, according to Sotomayor's theorem in [38], system (1.7) undergoes a transcritical bifurcation around  $E_0$  at  $m = m_{TC}$ .

This ends the proof of Theorem 4.3.  $\square$

**Theorem 4.4.** *When  $ae < d$ , system (1.7) undergoes a transcritical bifurcation around  $E_r$  with respect to the parameter  $m$  if  $m = m_{TC} = ad$ .*

**Proof.** The Jacobian matrix at  $E_r$  is

$$J(E_r) = \begin{pmatrix} -r & cr \\ 0 & 0 \end{pmatrix}.$$

It is obvious that the matrix has a zero eigenvalue, named  $\lambda_1$ . Let  $V$  and  $W$  represent the eigenvectors corresponding to the eigenvalue  $\lambda_1$  for matrices  $J_{E_r}$  and  $J_{E_r}^T$ . By calculation, we can obtain:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}, \quad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.11)$$

Define

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x(r - bx) + cxy \\ y\left(d - ey - \frac{m}{y + a}\right) \end{pmatrix},$$

then

$$F_m(E_r; m_{TC}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4.12)$$

$$DF_m(E_r; m_{TC})V = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{b}{a} \end{pmatrix}, \quad (4.13)$$

$$D^2F(E_r; m_{TC})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2 \frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2 \frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_r; m_{TC})} = \begin{pmatrix} 0 \\ \frac{2b^2(d - ae)}{a} \end{pmatrix}. \quad (4.14)$$

From (4.11)–(4.14), it follows that

$$\begin{aligned} w^T F_m(E_r; m_{TC}) &= 0, \\ w^T [DF_m(E_r; m_{TC})V] &= -\frac{b}{a} \neq 0, \\ w^T [D^2F(E_r; m_{TC})(V, V)] &= \frac{2b^2(d - ae)}{a} \neq 0. \end{aligned}$$

So, according to Sotomayor’s theorem in [38], system (1.7) undergoes a transcritical bifurcation around  $E_r$  at  $m = m_{TC}$ .

This ends the proof of Theorem 4.4. □

## 5 Global stability of equilibria

In Theorem 3.3, we have proved that  $E_1^*$  is locally asymptotically stable if  $E_1^*$  exists. In Theorem 3.1, we have shown that  $E_r$  is locally asymptotically stable if  $m > ad$ . In this section, we will provide some sufficient conditions for the global stability of  $E_1^*$  and  $E_r$ .

**Lemma 5.1.** [39] *If  $a > 0, b > 0$ , and  $\frac{dx}{dt} \geq x(b - ax)$ , then when  $t > 0$  and  $x(0) > 0$  we have*

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b}{a}.$$

*If  $a > 0, b > 0$ , and  $\frac{dx}{dt} \leq x(b - ax)$ , then when  $t > 0$  and  $x(0) > 0$  we have*

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b}{a}.$$

**Theorem 5.1.** *The positive equilibrium  $E_1^*$  of system (1.7) is globally asymptotically stable if one of the following conditions holds.*

- (1)  $m < ad$ ;
- (2)  $m = ad$  and  $ae < d$ .

**Proof.** From Table 1, we find that in addition to  $E_0$  and  $E_r$ , system (1.7) also has a boundary equilibrium  $E_1$  and a positive equilibrium  $E_1^*$  when (1) or (2) holds. Under these conditions,  $E_0, E_r$ , and  $E_1$  are all unstable, but  $E_1^*$  is locally asymptotically stable. Obviously, all  $\{(x, 0) | x \geq 0\}$ ,  $\{(0, y) | x \geq 0\}$ , and  $\{(x, y) | x > 0, y > 0\}$  (the interior of  $R_+^2$ ) are positively invariant subsets of the system (1.7). If we prove that there are no closed orbits in the interior of  $R_+^2$ , then we can obtain that  $E_1^*$  is globally asymptotically stable. Now, let us consider the Dulac function  $B(x, y) = \frac{1}{xy^2}$ . Then

$$D = \frac{\partial(BF_1)}{\partial x} + \frac{\partial(BF_2)}{\partial y} = -\frac{(a + y)^2(bx + d) - 2my - am}{xy^2(y + a)^2} = -\frac{(a + y)^2bx + dy^2 + (ad - m)(a + 2y)}{xy^2(y + a)^2} < 0,$$

where

$$F_1 = x(r - bx) + cxy,$$

$$F_2 = y\left(d - ey - \frac{m}{y + a}\right).$$

According to the Bendixson-Dulac discriminant [38], system (1.7) has no limit cycle in the first quadrant, so  $E_1^*$  is globally asymptotically stable.

This ends the proof of Theorem 5.1. □

**Remark 5.1.** Theorem 5.1 shows that for the weak Allee effect case, the stability of  $E_1^*$  is not affected, that is, systems (1.2) and (1.7) admit a positive equilibrium  $E_1^*$ , which is globally asymptotically stable. We also find that the values of  $x_1^*$  and  $y_1$  depend on the value of  $a$  and  $m$ , which means that the additive Allee effect has an effect on the final density of the species.

From

$$\frac{dy_1(m)}{dm} = -\frac{1}{\sqrt{(ae+d)^2 - 4em}} < 0,$$

$$\frac{dy_1(a)}{da} = \frac{1}{2} \left[ -1 + \frac{(ae+d)}{\sqrt{(ae+d)^2 - 4em}} \right] > 0,$$

and

$$x_1^* = \frac{r + cy_1}{b},$$

we conclude that as  $m$  increases and  $a$  decreases, the Allee effect is increasing, and the final density of both species are decreasing.

**Theorem 5.2.** *The equilibrium  $E_r$  of the system (1.7) is globally asymptotically stable if one of the following conditions holds*

- (1)  $m = ad$  and  $ae = d$ ;
- (2)  $ad < m < m^*$  and  $ae > d$ ;
- (3)  $m = m^*$  and  $ae \geq d$ ;
- (4)  $m > m^*$ .

**Proof.** From Table 1, we find that the system (1.7) has two boundary equilibria  $E_1$  and  $E_r$  when system (1.7) satisfies one of conditions (1)–(4). Under these conditions,  $E_0$  is always unstable and  $E_r$  is locally asymptotically stable. Next, we will prove that  $E_r$  is globally asymptotically stable.

First, let us consider the system

$$\frac{dy}{dt} = y \left( d - ey - \frac{m}{y+a} \right), \quad (5.1)$$

we will show that under the assumption of Theorem 5.2, the equilibrium  $y = 0$  of the system (5.1) is globally asymptotically stable. Indeed, define the Lyapunov function  $V = y$ , it is obvious that the function  $V$  is zero at  $y = 0$  and is positive for all other positive values of  $y$ . The time derivative of  $V$  along the trajectories of (5.1) is

$$\frac{dV}{dt} = y \left( d - ey - \frac{m}{y+a} \right) = \frac{y}{y+a} [-ey^2 + (d - ae)y + ad - m].$$

When system (1.7) satisfies one of the conditions (1)–(4), we always have  $\frac{dV}{dt} \leq 0$  for all  $y \geq 0$ , and  $\frac{dV}{dt} = 0$  if and only if  $y = 0$ . Therefore,  $V$  satisfies Lyapunov's asymptotic stability theorem [40], so  $y = 0$  of the system (5.1) is globally asymptotically stable.

Noting that the second equation of the system (1.7) is only related to  $y$ , and independent of  $x$ . Therefore, under the assumption of Theorem 5.2, we can conclude that

$$\lim_{t \rightarrow +\infty} y(t) = 0. \quad (5.2)$$

Hence, for any sufficiently small  $\varepsilon > 0$ , there exists an integer  $T > 0$  such that

$$-\frac{\varepsilon}{c} < y(t) < \frac{\varepsilon}{c}, \quad t \geq T.$$

Then, it follows from the first equation of system (1.7):

$$x(r - bx - \varepsilon) \leq \frac{dx}{dt} \leq x(r - bx + \varepsilon), \quad t \geq T.$$

Applying Lemma 5.1 to the above inequality leads to

$$\frac{r - \varepsilon}{b} \leq \liminf_{t \rightarrow +\infty} x(t) \leq \limsup_{t \rightarrow +\infty} x(t) \leq \frac{r + \varepsilon}{b}.$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{t \rightarrow +\infty} x(t) = \frac{r}{b}. \quad (5.3)$$

From (5.2) and (5.3), we can conclude that

$$\lim_{t \rightarrow +\infty} (x(t), y(t)) = \left( \frac{r}{b}, 0 \right).$$

Consequently,  $E_r$  is globally asymptotically stable.

This ends the proof of Theorem 5.2.  $\square$

## 6 Numeric simulations

In this section, we use numerical simulations to verify the above theorem.

**Example 6.1.** We consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= x(0.5 - x) + xy, \\ \frac{dy}{dt} &= y \left( d - y - \frac{m}{y + a} \right). \end{aligned} \quad (6.1)$$

In this system, corresponding to the system (1.7), we take  $b = c = e = 1$ ,  $r = 0.5$ .

- (1) For  $a = 0.3$ ,  $d = 2$ ,  $m = 0.2$ , we obtain  $m < ad$ ; then  $E_0$  is unstable,  $E_r$  and  $E_1$  are saddle points, and  $E_1^*$  is a stable node (Figure 2).
- (2) For  $a = 0.1$ ,  $d = 2$ ,  $m = 0.2$ , we obtain  $m = ad$  and  $ae < d$ ; then  $E_0$  is unstable,  $E_r$  is a saddle-node,  $E_1$  is a saddle,  $E_1^*$  is a stable node (Figure 3(a)). For  $a = 0.3$ ,  $d = 0.3$ ,  $m = 0.09$ , we obtain  $m = ad$  and  $ae = d$ ; then  $E_0$  is a saddle, and  $E_r$  is a stable node (Figure 3(b)). For  $a = 2$ ,  $d = 0.2$ ,  $m = 0.4$ , we obtain  $m = ad$  and  $ae > d$ ; then  $E_0$  and  $E_r$  are saddle-nodes (Figure 3(c)).
- (3) For  $a = 0.1$ ,  $d = 1$ ,  $m = 0.2$ , we obtain  $ad < m < m^*$  and  $ae < d$ ; then  $E_0$ ,  $E_1$  and  $E_2^*$  are saddle points,  $E_r$  and  $E_1^*$  are stable nodes, and  $E_2$  is an unstable node (Figure 4(a)). For  $a = 0.2$ ,  $d = 0.1$ ,  $m = 1$ , we obtain  $ad < m < m^*$  and  $ae > d$ , then  $E_0$  is a saddle,  $E_r$  is a stable node (Figure 4(b)).
- (4) For  $a = 0.5$ ,  $d = 1.5$ ,  $m = 1$ , we obtain  $m = m^*$  and  $ae < d$ ; then  $E_0$  is a saddle,  $E_r$  is a stable node, and  $E_3$  and  $E_3^*$  are saddle-nodes (Figure 4(c)). For  $a = 1.5$ ,  $d = 0.5$ ,  $m = 1$ , we obtain  $m = m^*$  and  $ae > d$ , then  $E_0$  is a saddle,  $E_r$  is a stable node (Figure 4(d)).
- (5) For  $a = 0.5$ ,  $d = 1.5$ ,  $m = 1.2$ , we obtain  $m > m^*$ ; then  $E_0$  is a saddle,  $E_r$  is a stable node (Figure 4(e)).

**Example 6.2.** We consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= x(0.4 - 0.4x) + xy, \\ \frac{dy}{dt} &= y \left( d - y - \frac{m}{y + a} \right). \end{aligned} \quad (6.2)$$

In this system, corresponding to system (1.7), we take  $c = e = 1$ ,  $b = 0.4$ ,  $r = 0.4$ .

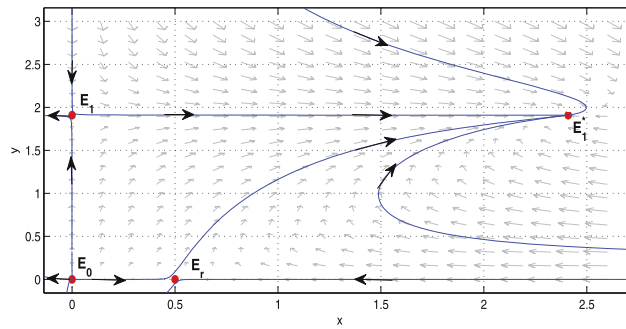
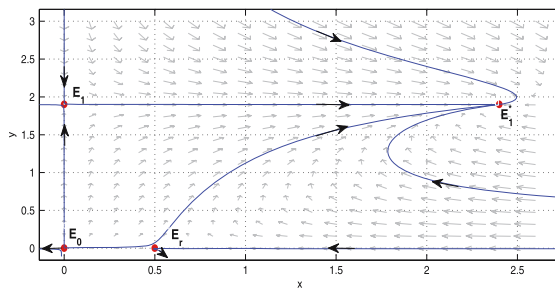
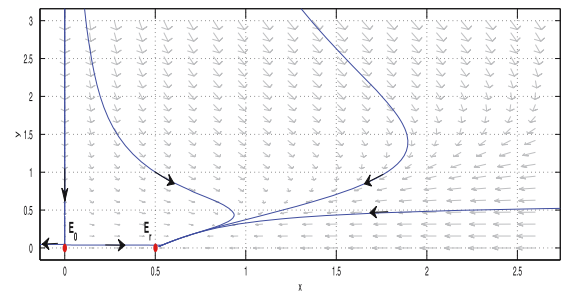


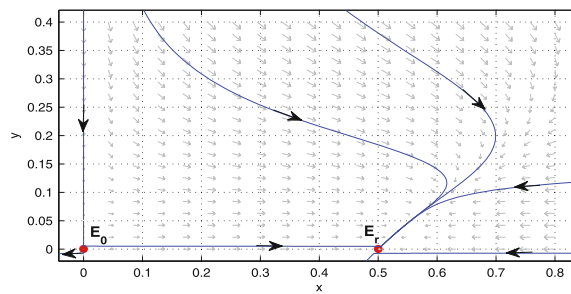
Figure 2: The phase portraits of system (1.7) when  $m < ad$ .



(a)



(b)



(c)

Figure 3: The phase portraits of system (1.7) when  $m = ad$ .

- (1) For  $a = 0.5, d = 0.6, m = 0.32$ , we obtain  $ad < m^* < m$  and  $ae < d$ ; then system (1.7) has two different boundary equilibria  $E_0$  and  $E_r$  (Figure 5(a)).
- (2) For  $a = 0.5, d = 0.8, m = 0.4$ , we obtain  $ad < m = m^*$  and  $ae < d$ ; then system (1.7) has four different equilibria  $E_0, E_r, E_3$  and  $E_3^*$  (Figure 5(b)).
- (3) For  $a = 0.1, d = 1, m = 0.3$ , we obtain  $ad < m < m^*$  and  $ae < d$ ; then system (1.7) has six different equilibria  $E_0, E_r, E_1, E_2, E_1^*$ , and  $E_2^*$  (Figure 5(c)).

Figure 5 shows that system (1.7) undergoes saddle-node bifurcations at  $E_3$  and  $E_3^*$ , respectively.

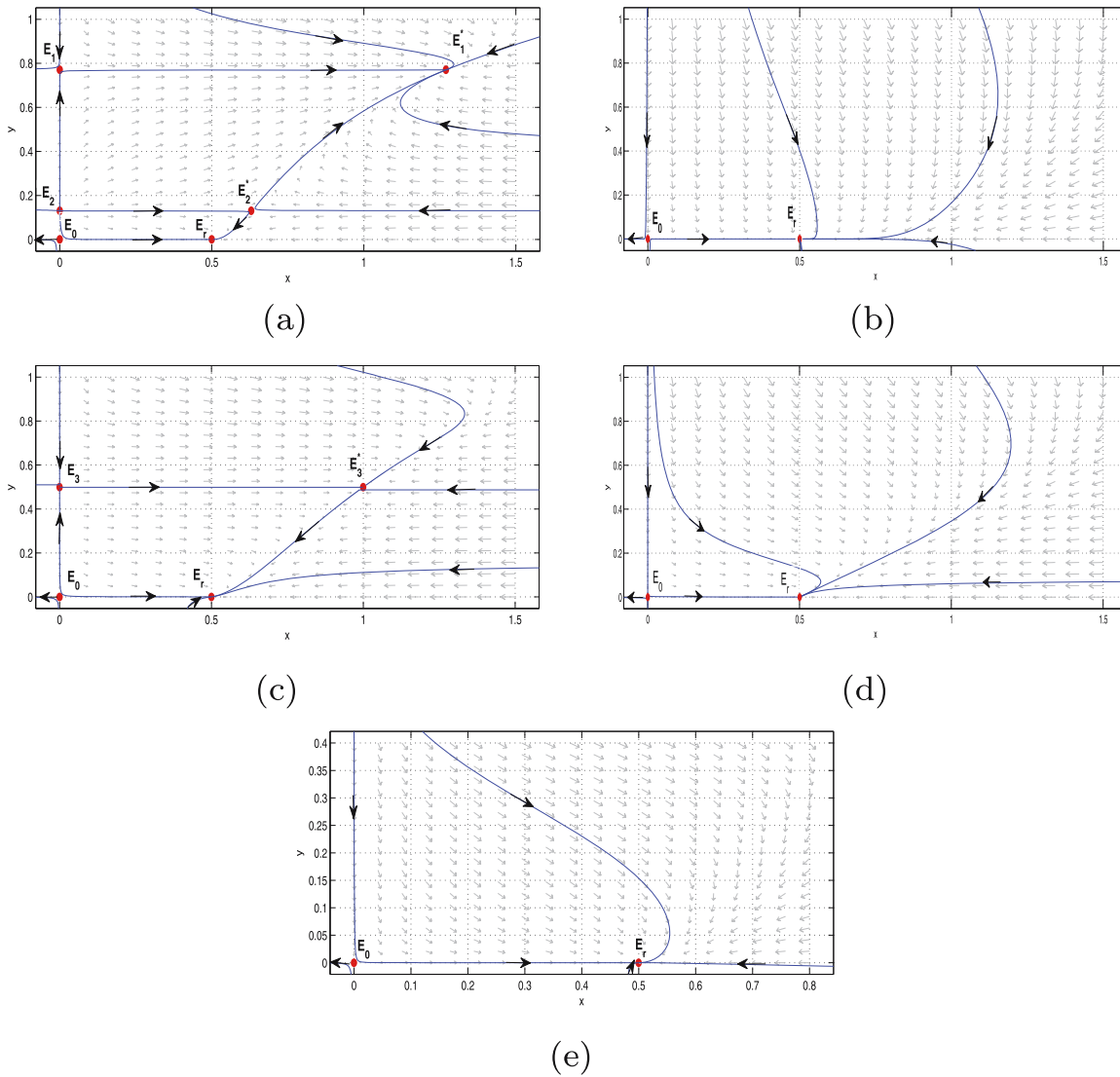


Figure 4: The phase portraits of system (1.7) when  $m > ad$ .

**Example 6.3.** We consider the following system:

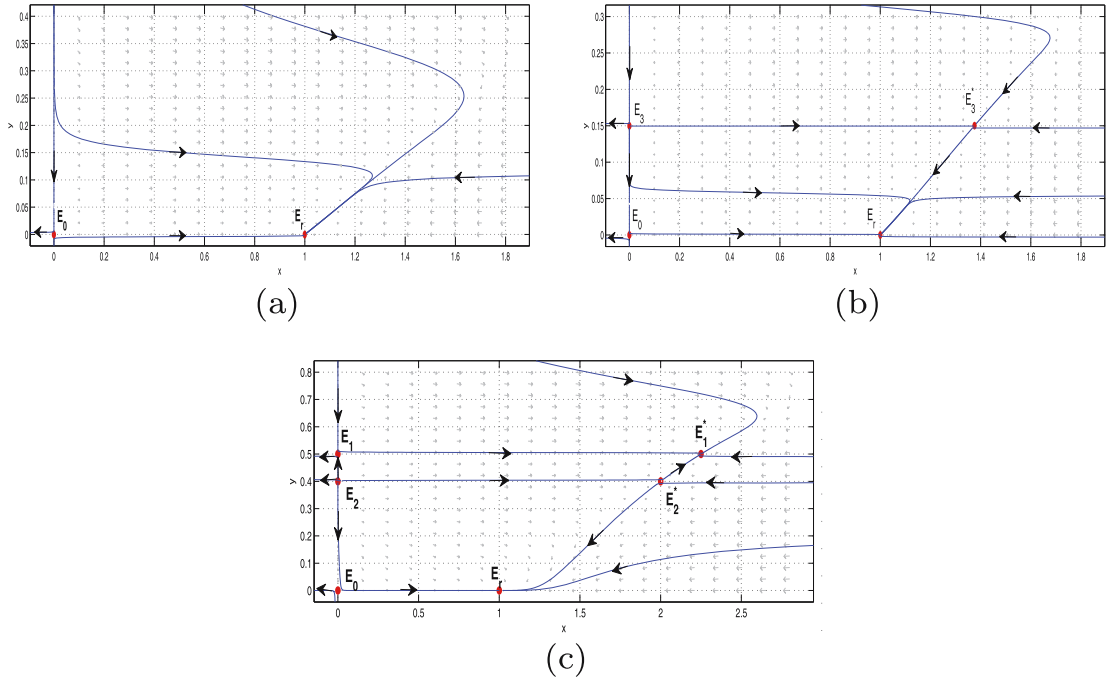
$$\begin{aligned} \frac{dx}{dt} &= x(0.3 - 0.3x) + xy, \\ \frac{dy}{dt} &= y\left(d - y - \frac{m}{y+a}\right). \end{aligned} \tag{6.3}$$

In this system, corresponding to system (1.7), we take  $c = e = 1, b = 0.3, r = 0.3$ .

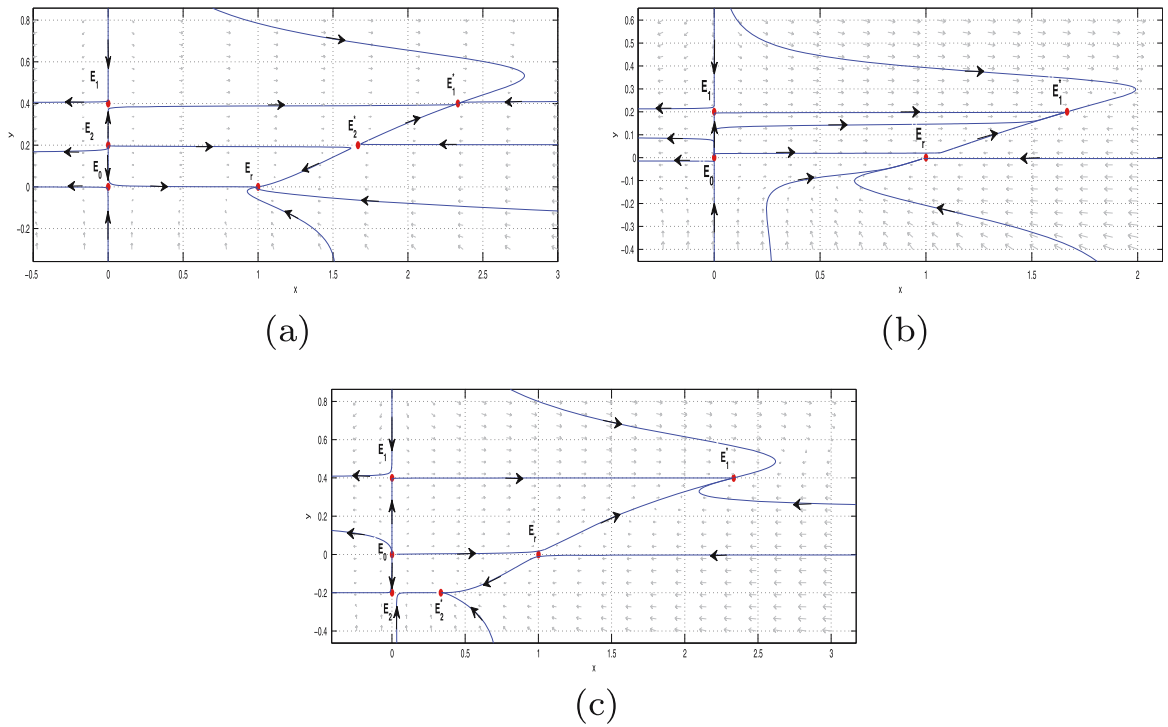
- (1) For  $a = 0.4, d = 1, m = 0.48$ , we obtain  $ad < m < m^*$  and  $ae < d$ ; then system (1.7) has six different boundary equilibria  $E_0, E_r, E_1, E_2, E_1^*,$  and  $E_2^*$  (Figure 6(a)).
- (2) For  $a = 0.6, d = 0.8, m = 0.48$ , we obtain  $ad = m < m^*$  and  $ae < d$ ; then system (1.7) has four different equilibria  $E_0, E_r, E_1,$  and  $E_1^*$  (Figure 6(b)).
- (3) For  $a = 0.6, d = 0.8, m = 0.4$ , we obtain  $m^* > ad > m$  and  $ae < d$ ; then system (1.7) has four different equilibria  $E_0, E_r, E_1,$  and  $E_1^*$  (Figure 6(c)).

Figure 6 shows that system (1.7) undergoes transcritical bifurcations at  $E_0$  and  $E_r$ , respectively.





**Figure 5:** The saddle-node bifurcation of system (1.7). (a)  $m > m^*$ , (b)  $m < m^*$ , and (c)  $m = m^*$ .



**Figure 6:** The transcritical bifurcation of system (1.7). (a)  $ad < m < m^*$ , (b)  $ad = m < m^*$ , and (c)  $m < ad < m^*$ .

## 7 Conclusion

In this article, we proposed and studied a commensalism model with the additive Allee effect. We study the dynamics behaviors under three conditions, i.e.,  $m < ad$ ,  $m = ad$ , and  $m > ad$ .

For the case  $m < ad$ , system (1.7) has four equilibria, of which three boundary equilibria are always unstable, and the unique positive equilibrium  $E_1^*$  is globally asymptotically stable. Compared with system (1.2), the weak Allee effect in the system (1.7) has no influence on its stability but changes the position of the equilibria, when the Allee effect increases, the final density of  $x$  and  $y$  species are decreasing.

For the case  $m = ad$ , if  $ae < d$ , then the situation is the same as  $m < ad$ ; if  $ae = d$ , system (1.7) has two boundary equilibria  $E_0$  and  $E_r$ , in which  $E_0$  is unstable and  $E_r$  is globally asymptotically stable, which means that the second species will be driven to extinction. If  $ae > d$ , system (1.7) has two boundary equilibria  $E_0$  and  $E_r$ , both of them are unstable.

For the case  $m > ad$ , we have two new findings. The first one is that system (1.7) has at least two boundary equilibria and at most six equilibria, this means that the additive Allee effect affects the number of equilibria and their stability. The other is that  $E_r$  is always stable, and  $E_r$  is globally asymptotically stable under some sufficient conditions, this shows that the additive Allee effect will cause the extinction of the second species.

In addition, from Theorems 4.1 to 4.4, we also proved that system (1.7) has saddle-node bifurcations at  $E_3$  and  $E_3^*$ , respectively, and transcritical bifurcations at  $E_0$  and  $E_r$  under some suitable assumptions, respectively.

Through the above analysis, we can conclude that when the additive Allee effect is weak, both species  $x$  and  $y$  can survive, and the additive Allee effect only affects the position of the equilibria. However, when the additive Allee effect presents as a strong Allee effect, the dynamic behaviors of two species have changed, and the second species even faces the risk of possible extinction, which is quite different from the findings in [2,7,8]. Moreover, in some conditions, system (1.7) has saddle-node bifurcations and transcritical bifurcations, which are also not found in [2,7,8].

It seems that different types of Allee effect expression may make results in different dynamic behaviors, it seems interesting to investigate the commensalism model with additive Allee effect and functional response; we leave this for future investigation.

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