Miao et al. *Advances in Difference Equations* (2017) 2017:226
DOI 10.1186/s13662-017-1289-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Dynamical analysis of a stochastic SIS epidemic model with nonlinear incidence rate and double epidemic hypothesis

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Abstract

In this paper, a stochastic SIS epidemic model with nonlinear incidence rate and double epidemic hypothesis is proposed and analysed. We explain the effects of stochastic disturbance on disease transmission. To this end, firstly, we investigated the dynamic properties of the system neglecting stochastic disturbance and obtained the threshold and the conditions for the extinction and the permanence of two kinds of epidemic diseases by considering the stability of the equilibria of the deterministic system. Secondly, we paid prime attention on the threshold dynamics of the stochastic system and established the conditions for the extinction and the permanence of two kinds of epidemic diseases. We found that there exists a significant difference between the threshold of the deterministic system and that of the stochastic system. Moreover, it has been established that the persistent of infectious disease analysed by use of deterministic system becomes extinct under the same conditions due to the stochastic disturbance. This implies that a stochastic disturbance has significant impact on the spread of infectious diseases and the larger stochastic disturbance leads to control the epidemic diseases. In order to illustrate the dynamic difference between the deterministic system and the stochastic system, there have been given a series of numerical simulations by using different noise disturbance coefficients.

MSC: 60H10; 65C30; 91B70

Keywords: stochastic SIS epidemic model; double epidemic hypothesis; Beddington-DeAngelis incidence rate; extinction; permanence

1 Introduction

Infectious disease is generally considered as the enemy of human health; in history, the epidemic of infectious diseases such as smallpox, cholera, AIDS and so on have brought great disaster to the national economy of a country and people's livelihood [1]. In order to control the spread of infectious diseases, researchers have built a great deal of mathematical models to study the dynamical behavior of infectious diseases [2–6]. The mathematical models of differential equations play a significant role in describing the dynamic behavior and have been widely used in biology [7–11], physics [12–16], medicine [17, 18], and so on [19–24]. An important mathematical model describing the evolution infectious diseases is called 'compartmental model' which was originally established by Kermack and

Mckendrick to study the spread of the infectious diseases such as Great Plague in London and the Plague of Mumbai [25]. In the compartmental system, the population is divided into three separate compartments, namely, the susceptible compartment S, the infected compartment I, and the removed compartment R. In the system, the susceptible person get infected and becomes an infected person making contact with an infected person, and the infected person can be recovered taking treatments, the individuals who reach this class have permanent immunity for the relevant disease. This type of model is called the *SIR* (susceptible-infected-removed) model, which can be mathematically expressed as (see p.47 in [25]):

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - \gamma I(t), \\ \dot{R}(t) = \gamma I(t), \end{cases} \tag{1}$$

where $S(t)$, $I(t)$ and $R(t)$ represents the number of susceptible, infected and removed individuals at time t, respectively, β is the contact rate, γ is the recovery rate, βSI is called bilinear infection rate. However, some diseases do not conform to the *SIR* system, such as influenza, infected individuals do not get permanent immunity for the disease although they take proper treatments, in this case, there exists a high possibility of recovered individuals to be re-infected. This type of model is called an SIS model, the mathematical system can be expressed in the following form (see p.62 in [26]):

$$\begin{cases} \dot{S}(t) = -dS(t) - \beta S(t)I(t) + \gamma I(t), \\ \dot{I}(t) = \beta S(t)I(t) - (d + \gamma)I(t), \end{cases} \tag{2}$$

where d is the natural death rate and γ is the recovery rate of the infective individuals. In systems (1) and (2) there has been used the bilinear infection rate, a common nonlinear infection rate. In the previous work, the researchers studied other types of nonlinear infection rates, for example, several epidemic models with saturated infection rates $\beta SI/(1 + \alpha S)$ were discussed by Xu *et al.* [27, 28] and Zhang *et al.* [29]. A nonlinear incidence rate $\lambda S^q I^p$ was proposed by Liu *et al.* [30, 31], while a nonlinear infection rate $\beta SI^p/(1 + \alpha S^q)$ was considered by Hethcote *et al.* [32]. And a special case $p = q = 2$ for $\beta SI^p/(1 + \alpha S^q)$ was investigated by Ruan and Wang [33]. The nonlinear infection rate of the form Beddington-DeAngelis functional response $\beta SI/(1+aS+bI)$ was studied by Chen *et al.* [34]. For a more general nonlinear incidence rate, we refer the reader to Wang [35].

It is well known that random noise factors play an important role in the transmission of infectious diseases. Therefore, many scholars [36–43] have studied the impact of the stochastic epidemic system, various stochastic perturbation approaches have been introduced into epidemic models and have obtained excellent results. For example, the authors of [44–49] have considered a stochastic epidemic model with a Markov transform. A class of epidemic model which shows the effect of the random white noise has been studied by the researchers in the articles [50–64]. Further, in [65–70], the authors studied a class of stochastic epidemic models, in which the stochastic white noise is assumed to be proportional to S, I and R. It can be seen following the literature that the authors of the articles [71, 72] analysed a stochastic epidemic model with two different kinds of perturbation. A stochastic epidemic model with Lévy jumps has been proposed and studied by

the researchers [73–75]; the authors investigated stochastic perturbation around the positive equilibria of deterministic models (see, for example, [40, 41, 76–78]). Although there were limited numbers of publications in the recent literature considering time delay and stochastic behavior, the authors in [77] paid attention on the stochastic epidemic model with time delay.

Recently, Meng *et al.* [50] constructed a nonlinear stochastic SIS epidemic model with double epidemic hypothesis, in which the saturated incidence rates $\frac{\beta_i SI_i}{a_i+I_1}$ $(i = 1, 2)$ is proposed. Then, based on the previous work, firstly, we propose a deterministic epidemic model with Beddington-DeAngelis nonlinear incidence rate and double epidemic hypothesis as follows:

$$\begin{cases} \dot{S}(t) = A - dS(t) - \frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - \frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} \\ \qquad + r_1I_1(t) + r_2I_2(t), \\ \dot{I}_1(t) = \frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - (d + \alpha_1 + r_1)I_1(t), \\ \dot{I}_2(t) = \frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} - (d + \alpha_2 + r_2)I_2(t), \end{cases} \tag{3}$$

where $S(t)$ denotes the number of the population susceptible to the disease, $I_1(t)$ and $I_2(t)$ are the total population of the infectives with virus V_A and V_B at time t, respectively. The recruitment to the susceptible population is to be considered as a constant A, β_1 and β_2 are the contact rates, d is natural mortality rate, α_1 and α_2 are the rates of disease-related death, r_1 and r_2 are the treatment cure rates of two diseases, respectively. a_i, b_i are the parameters that measure the inhibitory effect. The infection rate $\frac{\beta_i S(t)I_i(t)}{1+a_iS(t)+b_iI_i(t)}$ $(i = 1, 2)$ of susceptible individuals through their contacts with infectious, includes three forms: The first one is the bilinear incidence rate $\beta S(t)I_i(t)$ for the case $a_i = b_i = 0$; the second one is the saturated incidence rate for the susceptible with the form $\frac{\beta S(t)I_i(t)}{1+a_iS(t)}$ for the case $a_i > 0$, $b_i = 0$; and the third one is the saturated incidence rate for the infectives with the form $\frac{\beta S(t)I_i(t)}{1+b_iI(t)}$ for the case $a_i = 0$, $b_i > 0$. Thus, the nonlinear incidence rates $\frac{\beta_i S(t)I_i(t)}{1+a_iS(t)+b_iI_i(t)}$ $(i = 1, 2)$ are more general and realistic than the saturated incidence rate $\frac{\beta S(t)I_i(t)}{1+a_iS(t)}$ and $\frac{\beta S(t)I_i(t)}{1+b_iI(t)}$, because it takes into account the inhibition effect of the susceptible and the infectives.

Secondly, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the saturated response rate

$$\frac{\beta_i S(t)I_i(t)}{1 + a_iS(t) + b_iI_i(t)} \to \frac{\beta_i S(t)I_i(t)}{1 + a_iS(t) + b_iI_i(t)} + \frac{\sigma_i S(t)I_i(t)}{1 + a_iS(t) + b_iI_i(t)}\,\mathrm{d}B_i(t),$$

where $B(t) = (B_1(t), B_2(t))$ is the standard Brownian motion with intensity $\sigma_i > 0$ $(i = 1, 2)$. Finally, a stochastic version of system (3) is obtained as follows:

$$\begin{cases} \mathrm{d}S(t) = (A - dS(t) - \frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - \frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} \\ \qquad + r_1I_1(t) + r_2I_2(t))\,\mathrm{d}t - \frac{\sigma_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)}\,\mathrm{d}B_1(t) \\ \qquad - \frac{\sigma_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)}\,\mathrm{d}B_2(t), \\ \mathrm{d}I_1(t) = (\frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - (d + \alpha_1 + r_1)I_1(t))\,\mathrm{d}t \\ \qquad + \frac{\sigma_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)}\,\mathrm{d}B_1(t), \\ \mathrm{d}I_2(t) = (\frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} - (d + \alpha_2 + r_2)I_2(t))\,\mathrm{d}t \\ \qquad + \frac{\sigma_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)}\,\mathrm{d}B_2(t), \end{cases} \tag{4}$$

where the parameters of system (4) have the same biological meaning as in system (3). The rest of the paper, we will dedicate to study the deterministic system (3) and the stochastic system (4) with nonlinear incidence rate and double epidemic hypothesis. The main aims of this paper are (a) to establish a set of most suitable conditions such that diseases to be died out or to be persistent, and (b) to obtain the thresholds (based on the basic reproductive number) of the above two *SIS* epidemic models.

2 Preliminaries and lemmas

In this section, we will give some notations, definitions and some lemmas which will be used for analysing our main results.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.* it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets $R^3_+ = \{x_i > 0, i = 1,2,3\}$). The function $B(t)$ denotes a scalar Brownian motion defined on the complete probability space Ω. For an integrable function f on $[0,+\infty)$, define $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\,\mathrm{d}\theta$.

Definition 2.1 ([50, 68])

(i) The diseases $I_1(t)$ and $I_2(t)$ are said to be extinctive if $\lim_{t\to+\infty} I_1(t) = 0$ and $\lim_{t\to+\infty} I_2(t) = 0$.

(ii) The diseases $I_1(t)$ and $I_2(t)$ are said to be permanent in mean if there exist two positive constants λ_1 and λ_2 such that $\liminf_{t\to+\infty}\langle I_1(t)\rangle \geq \lambda_1$ and $\liminf_{t\to+\infty}\langle I_2(t)\rangle \geq \lambda_2$.

Lemma 2.1 *For any initial value* $(S(0), I_1(0), I_2(0)) \in R^3_+$, *there exists a unique solution* $(S(t), I_1(t), I_2(t))$ *to system* (4) *on* $t \geq 0$, *and the solution will remain in* R^3_+ *with probability* 1, *namely*, $(S(t), I_1(t), I_2(t)) \in R^3_+$ *for all* $t \geq 0$ *almost surely.*

Proof Since the coefficients of system (4) are locally Lipschitz continuous for any given initial value $(S(0), I_1(0), I_2(0)) \in R^3_+$, then by the work of Mao *et al.* [36], there is a unique local solution $(S(t), I_1(t), I_2(t))$ on $t \in [0, \tau)$, where τ is the explosion time (see [58]). To show that this solution is global, we need to show that $\tau_\infty = \infty$ almost surely. To do so, let $\varepsilon_0 \geq 0$ such that $S(0) > \varepsilon_0, I_1(0) > \varepsilon_0, I_2(0) > \varepsilon_0$. For any positive $\varepsilon \leq \varepsilon_0$, define the stopping time as follows:

$$\tau_\varepsilon = \inf\big\{t \in [0,\tau_e) : S(t) \leq \varepsilon \text{ or } I_1(t) \leq \varepsilon \text{ or } I_2(t) \leq \varepsilon\big\},$$

where throughout this paper, we set $\inf \varnothing = \infty$ (in the usual notation, $\varnothing$ denotes the empty set). Clearly, τ_ε is increasing as $\varepsilon \to 0$. Set $\tau_0 = \lim_{\varepsilon\to 0}\tau_\varepsilon$, whence $\tau_0 \leq \tau_\varepsilon$ almost surely. If we can show $\tau_0 = \infty$ almost surely, then $\tau_e = \infty$ and $(S(t), I_1(t), I_2(t)) \in R^3_+$ for all $t \geq 0$ almost surely. In other words, to complete the proof we only need to show that $\tau_0 = \infty$ almost surely.

If this statement is false, then there is a pair of constants $T > 0$ and $\delta \in (0,1)$ such that $P\{\tau_0 \leq T\} > \delta$. Hence, there is a positive constant $\varepsilon_1 \leq \varepsilon_0$ such that $P\{\tau_\varepsilon \leq T\}$ for any positive $\varepsilon \leq \varepsilon_1$.

Besides, for $t < \tau_e$, we see that

$$\begin{aligned} d\big(S(t) + I_1(t) + I_2(t)\big) &= \big[A - d\big(S(t) + I_1(t) + I_2(t)\big) - \alpha_1 I_1(t) - \alpha_2 I_2(t)\big]\,\mathrm{d}t \\ &\leq \big[A - d\big(S(t) + I_1(t) + I_2(t)\big)\big]\,\mathrm{d}t \end{aligned} \tag{5}$$

and

$$S(t) + I_1(t) + I_2(t) \le \max\left\{S(0) + I_1(0) + I_2(0), \frac{A}{d}\right\} := C_1. \tag{6}$$

Define a function

$$V\big(S(t), I_1(t), I_2(t)\big) = -\ln\frac{S(t)}{C_1} - \ln\frac{I_1(t)}{C_1} - \ln\frac{I_2(t)}{C_1}.$$

Obviously, V is positive definite. Using the Itô formula, we get

$$\mathrm{d}V = LV\,\mathrm{d}t + \frac{\sigma_1(I_1(t) - S(t))}{1 + a_1S(t) + b_1I_1(t)}\,\mathrm{d}B_1(t) + \frac{\sigma_2(I_2(t) - S(t))}{1 + a_2S(t) + b_2I_2(t)}\,\mathrm{d}B_2(t),$$

where

$$\begin{aligned} LV = {} & -\frac{A + r_1I_1(t) + r_2I_2(t)}{S(t)} + 3d + \alpha_1 + r_1 + \alpha_2 + r_2 \\ & + \frac{\beta_1(I_1(t) - S(t))}{1 + a_1S(t) + b_1I_1(t)} + \frac{\beta_2(I_2(t) - S(t))}{1 + a_2S(t) + b_2I_2(t)} \\ & + \frac{\sigma_1^2(I_1(t)^2 + S(t)^2)}{2(1 + a_1S(t) + b_1I_1(t))^2} + \frac{\sigma_2^2(I_2(t)^2 + S(t)^2)}{2(1 + a_2S(t) + b_2I_2(t))^2}. \end{aligned}$$

By using (6), we can obtain

$$\begin{aligned} LV &\le 3d + \alpha_1 + r_1 + \alpha_2 + r_2 + \frac{\beta_1I_1(t)}{1 + a_1S(t) + b_1I_1(t)} + \frac{\beta_2I_2(t)}{1 + a_2S(t) + b_2I_2(t)} \\ &\quad + \frac{\sigma_1^2(I_1(t)^2 + S(t)^2)}{2(1 + a_1S(t) + b_1I_1(t))^2} + \frac{\sigma_2^2(I_2(t)^2 + S(t)^2)}{2(1 + a_2S(t) + b_2I_2(t))^2} \\ &\le 3d + \alpha_1 + r_1 + \alpha_2 + r_2 + \beta_1C_1 + \beta_2C_1 + \sigma_1^2C_1^2 + \sigma_2^2C_1^2 := C_2. \end{aligned}$$

Therefore,

$$\mathrm{d}V \le C_2\,\mathrm{d}t + \frac{\sigma_1(I_1(t) - S(t))}{1 + a_1S(t) + b_1I_1(t)}\,\mathrm{d}B_1(t) + \frac{\sigma_2(I_2(t) - S(t))}{1 + a_2S(t) + b_2I_2(t)}\,\mathrm{d}B_2(t).$$

Integrating both sides from 0 to $\tau_\varepsilon \wedge T$, by considering expectations, yields

$$EV\big(S(\tau_\varepsilon \wedge T), I_1(\tau_\varepsilon \wedge T), I_2(\tau_\varepsilon \wedge T)\big) \le V\big(S(0), I_1(0), I_2(0)\big) + C_2T.$$

Set $\Omega_\varepsilon = \{\tau_\varepsilon \le T\}$ for any positive $\varepsilon \le \varepsilon_1$ and then $P(\Omega_\varepsilon > \delta)$. Note that, for every $\omega \in \Omega_\varepsilon$, there is at least one of $S(\tau_\varepsilon, \omega)$, $I_1(\tau_\varepsilon, \omega)$, $I_2(\tau_\varepsilon, \omega)$ equal to ε, then

$$V\big(S(\tau_\varepsilon), I_1(\tau_\varepsilon), I_2(\tau_\varepsilon)\big) \ge -\ln\frac{\varepsilon}{C_1}.$$

Consequently,

$$\begin{aligned} V\big(S(0),I_1(0),I_2(0)\big) + C_2T &\geq E\big[I_{\Omega_\varepsilon} V\big(S(\tau_\varepsilon \wedge T), I_1(\tau_\varepsilon \wedge T), I_2(\tau_\varepsilon \wedge T)\big)\big] \\ &= P(\Omega_\varepsilon) V\big(S(\tau_\varepsilon), I_1(\tau_\varepsilon), I_2(\tau_\varepsilon)\big) \\ &> -\delta \ln \frac{\varepsilon}{C_1}, \end{aligned}$$

where I_{Ω_ε} is the indicator function of Ω_ε. Letting $\varepsilon \to 0$ leads to the contradiction

$$\infty > V\big(S(0),I_1(0),I_2(0)\big) + C_2T = \infty.$$

Therefore, we must have $\tau_0 = \infty$ almost surely. The proof of Lemma 2.1 is completed. □

Lemma 2.2 *Denote* $\Gamma = \{(S(t),I_1(t),I_2(t)) \in R^3_+ : S(t),I_1(t),I_2(t) \leq \frac{A}{d}, t \geq 0\}$, *then* Γ *is an invariant set on system* (3) *or* (4).

Proof From system (3) or the system (4), we have

$$\frac{\mathrm{d}(S(t) + I_1(t) + I_2(t))}{\mathrm{d}t} \leq A - d\big(S(t) + I_1(t) + I_2(t)\big).$$

This implies that

$$S(t) + I_1(t) + I_2(t) \leq \frac{A}{d} + \left(S(0) + I_1(0) + I_2(0) - \frac{A}{d}\right)e^{-dt}.$$

Then, if we denote $\Gamma = \{(S(t),I_1(t),I_2(t)) \in R^3_+ : S(t),I_1(t),I_2(t) \leq \frac{A}{d}, t \geq 0\}$, we have $S(t) + I_1(t) + I_2(t) \leq \frac{A}{d}$. Thus, the region Γ is positively invariant. □

By Lemma 2.2 and the strong law of large numbers for martingales [36], we can obtain the following lemma.

Lemma 2.3 *Let* $(S(t),I_1(t),I_2(t))$ *be a solution of system* (4) *with initial value* $(S(0),I_1(0),I_2(0)) \in R^3_+$. *Then*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\sigma_i S(\tau)}{1 + a_iS(\tau) + b_iI_i(\tau)}\,\mathrm{d}B_i(\tau) = 0, \qquad \lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i S(\tau)\,\mathrm{d}B_i(\tau) = 0, \quad i = 1,2.$$

3 Dynamics of deterministic system (3)

In this section, we will qualitatively analyse the dynamics of deterministic system (3). Firstly, to find the equilibria of system (3), we consider the following set of equations:

$$\begin{cases} f_1 := A - dS(t) - \frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - \frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} \\ \qquad + r_1I_1(t) + r_2I_2(t) = 0, \\ f_2 := \frac{\beta_1 S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)} - (d + \alpha_1 + r_1)I_1(t) = 0, \\ f_3 := \frac{\beta_2 S(t)I_2(t)}{1+a_2S(t)+b_2I_2(t)} - (d + \alpha_2 + r_2)I_2(t) = 0. \end{cases} \tag{7}$$

By direct calculation, there can be obtained the following equilibria for system (3):

$$E_0: \quad \left(\frac{A}{d}, 0, 0\right),$$

E_1: $\left(S_1^*, I_1^*, 0\right)$ with

$$S_1^* = \frac{(d+\alpha_1+r_1)(1+b_1I_1^*)}{\beta_1-(d+\alpha_1+r_1)a_1},$$

$$I_1^* = \frac{A\beta_1-(Aa_1+d)(d+\alpha_1+r_1)}{d(d+\alpha_1+r_1)b_1+(d+\alpha_1)(\beta_1-(d+\alpha_1+r_1)a_1)},$$

E_2: $\left(S_2^*, 0, I_2^*\right)$ with

$$S_2^* = \frac{(d+\alpha_2+r_2)(1+b_2I_2^*)}{\beta_2-(d+\alpha_2+r_2)a_2},$$

$$I_2^* = \frac{A\beta_2-(Aa_2+d)(d+\alpha_2+r_2)}{d(d+\alpha_2+r_2)b_2+(d+\alpha_2)(\beta_2-(d+\alpha_2+r_2)a_2)},$$

E^*: $\left(S^*, I_1^{**}, I_2^{**}\right)$ with

$$S^* = \frac{(Ab_1b_2+C_1b_2+C_2b_1)B_1B_2}{db_1b_2B_1B_2+C_1b_2B_2(\beta_1-B_1a_1)+C_2b_1B_1(\beta_2-B_2a_2)},$$

$$I_1^{**} = \frac{S^*(\beta_1-B_1a_1)-B_1}{B_1b_1},$$

$$I_2^{**} = \frac{S^*(\beta_2-B_2a_2)-B_2}{B_2b_2},$$

where

$$B_1 = d+\alpha_1+r_1, \qquad B_2 = d+\alpha_2+r_2, \qquad C_1 = d+\alpha_1, \qquad C_2 = d+\alpha_2.$$

For convenience, let us denote following expressions as Q_1, Q_2 and Q_3 and the combination of these three conditions are denoted by a single character H,

$$Q_1 = (Ab_2a_1+db_2+C_2a_1)(R_1-1)-C_2a_2(R_2-1)+C_2\frac{d}{A}(R_1-R_2) > 0,$$

$$Q_2 = (Ab_1a_2+db_1+C_1a_2)(R_2-1)-C_1a_1(R_1-1)+C_1\frac{d}{A}(R_2-R_1) > 0,$$

$$Q_3 = \left(db_1b_2B_1B_2+C_1b_2B_2(\beta_1-B_1a_1)+C_2b_1B_1(\beta_2-B_2a_2)\right) > 0.$$

Therefore, there exist a unique positive equilibrium E^* for system (3) if H holds. Thus, if let

$$\mathcal{R}_1 = \frac{\beta_1A}{(Aa_1+d)(d+\alpha_1+r_1)}, \qquad \mathcal{R}_2 = \frac{\beta_2A}{(Aa_2+d)(d+\alpha_2+r_2)}, \tag{8}$$

then we have the following theorem.

Theorem 3.1 *For system* (3), *the following conclusions are true*:

(i) *if* $\mathcal{R}_1 < 1$ *and* $\mathcal{R}_2 < 1$, *then both diseases go extinct and system* (3) *has a unique stable 'diseases-extinction' equilibrium* E_0;

(ii) *if* $\mathcal{R}_1 > 1$ *and* $\mathcal{R}_2 < 1$, *then the disease* I_2 *goes extinct and system* (3) *has a unique stable equilibrium* E_1;

(iii) *if* $\mathcal{R}_1 < 1$ *and* $\mathcal{R}_2 > 1$, *then the disease* I_1 *goes extinct and system* (3) *has a unique stable equilibrium* E_2; *and*

(iv) *when condition H holds, and if* $\mathcal{R}_1 > 1$ *and* $\mathcal{R}_2 > 1$, *then* E^* *is a unique stable equilibrium, which implies both diseases of system* (3) *are permanent*.

Proof Let S, I_1, I_2 be an arbitrary equilibrium of system (3), then the Jacobian matrix associating to the corresponding equilibrium of system (3) is

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Here

$$\begin{aligned}
a_{11} &= -d - \frac{\beta_1(I_1 + b_1 I_1^2)}{(1 + a_1 S + b_1 I_1)^2} - \frac{\beta_2(I_2 + b_2 I_2^2)}{(1 + a_2 S + b_2 I_2)^2}, \\
a_{12} &= r_1 - \frac{\beta_1(S + a_1 S^2)}{(1 + a_1 S + b_1 I_1)^2}, \\
a_{13} &= r_2 - \frac{\beta_2(S + a_2 S^2)}{(1 + a_2 S + b_2 I_2)^2}, \\
a_{21} &= \frac{\beta_1(I_1 + b_1 I_1^2)}{(1 + a_1 S + b_1 I_1)^2}, \\
a_{22} &= \frac{\beta_1(S + a_1 S^2)}{(1 + a_1 S + b_1 I_1)^2} - (d + \alpha_1 + r_1), \\
a_{23} &= 0, \\
a_{31} &= \frac{\beta_2(I_2 + b_2 I_2^2)}{(1 + a_2 S + b_2 I_2)^2}, \\
a_{32} &= 0, \\
a_{33} &= \frac{\beta_2(S + a_2 S^2)}{(1 + a_2 S + b_2 I_2)^2} - (d + \alpha_2 + r_2).
\end{aligned}$$

The stability of the 'diseases-extinction' equilibrium $(\frac{A}{d}, 0, 0)$ of system (3) is determined by the Jacobian matrix

$$J_0 = \begin{pmatrix} -d & r_1 - \frac{\beta_1 A}{d} & r_2 - \frac{\beta_2 A}{d} \\ 0 & \frac{\beta_1 S}{1 + a_1 S} - (d + \alpha_1 + r_1) & 0 \\ 0 & 0 & \frac{\beta_2 S}{1 + a_2 S} - (d + \alpha_2 + r_2) \end{pmatrix},$$

which has following eigenvalues:

$$\lambda_1 = -d < 0, \qquad \lambda_2 = \frac{\beta_1 A}{A a_1 + d} - (d + \alpha_1 + r_1), \qquad \lambda_3 = \frac{\beta_2 A}{A a_2 + d} - (d + \alpha_2 + r_2).$$

According to stability theory, $(\frac{A}{d}, 0, 0)$ is stable if $\lambda_2 < 0$ and $\lambda_3 < 0$, *i.e.*, $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$.

At equilibrium E_1, the Jacobian matrix can be expressed as

$$J_1 = \begin{pmatrix} a_{11}^* & a_{12}^* & a_{13}^* \\ a_{21}^* & a_{22}^* & a_{23}^* \\ a_{31}^* & a_{32}^* & a_{33}^* \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}^* &= -d - \frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}, \\
a_{12}^* &= r_1 - \frac{\beta_1(S_1^* + a_1 S_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}, \\
a_{13}^* &= r_2 - \frac{\beta_2 S_1^*}{(1 + a_2 S_1^*)}, \\
a_{21}^* &= -\frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}, \\
a_{22}^* &= \frac{\beta_1(S_1^* + a_1 S_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2} - (d + \alpha_1 + r_1), \\
a_{23}^* &= 0, \\
a_{31}^* &= 0, \\
a_{32}^* &= 0, \\
a_{33}^* &= \frac{\beta_2 S_1^*}{1 + a_2 S_1^*} - (d + \alpha_2 + r_2),
\end{aligned}$$

and one of three eigenvalues of matrix J_1 is given by

$$\lambda_1 = a_{33}^* = \frac{\beta_2 S_1^*}{1 + a_2 S_1^*} - (d + \alpha_2 + r_2) < \frac{(\frac{d}{A} S_1^* - 1)(d + \alpha_2 + r_2)}{1 + a_2 S_1^*} < 0,$$

where $A - dS_1^* = (d + \alpha_1) I_1^* > 0$ is used. The other two eigenvalues λ_2 and λ_3 of matrix J_1 are the roots of the following equation:

$$\lambda^2 + \left(a_{11}^* + a_{22}^*\right)\lambda + a_{11}^* a_{22}^* - a_{21}^* a_{12}^* = 0.$$

Obviously, $a_{11}^* + a_{22}^* = \frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2} + d + \frac{\beta_1 b_1 S_1^* I_1^*}{(1 + a_1 S_1^* + b_1 I_1^*)^2} > 0$ and

$$\begin{aligned}
&a_{11}^* a_{22}^* - a_{21}^* a_{12}^* \\
&\quad = \left(\frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2} + d\right)\left((d + \alpha_1 + r_1) - \frac{\beta_1(S_1^* + a_1 S_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}\right) \\
&\qquad - \frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}\left(r_1 - \frac{\beta_1(S_1^* + a_1 S_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2}\right) \\
&\quad = (d + \alpha_1)\frac{\beta_1(I_1^* + b_1 I_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2} + d(d + \alpha_1 + r_1) - d\frac{\beta_1(S_1^* + a_1 S_1^{*2})}{(1 + a_1 S_1^* + b_1 I_1^*)^2} \\
&\quad = \frac{\beta_1(I_1^* + b_1 I_1^{*2})(d + \alpha_1) + d\beta_1 S_1^* b_1 I_1^*}{(1 + a_1 S_1^* + b_1 I_1^*)^2} > 0,
\end{aligned}$$

then λ_2 and λ_3 have negative real parts, thus the equilibrium E_1 is stable.

Similarly, we can show that if $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 > 1$, then the equilibrium E_2 of system (3) is stable.

Now, let us prove that the positive equilibrium E^* is stable as $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$.

Denote the Jacobian matrix of (3) at the positive equilibrium E^* by $J = (J_{ij})$. Then $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where $(x_j) = (S, I_1, I_2)$. More precisely,

$$J^* = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where

$$\begin{aligned}
J_{11} &= -\frac{\beta_1(I_1^{**} + b_1 I_1^{**2})}{(1 + a_1 S^* + b_1 I_1^{**})^2} - \frac{\beta_2(I_2^{**} + b_2 I_2^{**2})}{(1 + a_2 S^* + b_2 I_2^{**})^2} - d \\
&= -J_{21} - J_{31} - d, \\
J_{12} &= r_1 - \frac{\beta_1(S^* + a_1 S^{*2})}{(1 + a_1 S^* + b_1 I_1^{**})^2} = -\frac{\beta_1(S^* + a_1 S^{*2})}{(1 + a_1 S^* + b_1 I_1^{**})^2} - (d + \alpha_1) \\
&= -J_{22} - (d + \alpha_1), \\
J_{13} &= r_2 - \frac{\beta_2(S^* + a_2 S^{*2})}{(1 + a_2 S^* + b_2 I_2^{**})^2} = -\frac{\beta_2(S^* + a_2 S^{*2})}{(1 + a_2 S^* + b_2 I_2^{**})^2} - (d + \alpha_2) \\
&= -J_{33} - (d + \alpha_2), \\
J_{21} &= \frac{\beta_1(I_1^{**} + b_1 I_1^{**2})}{(1 + a_1 S^* + b_1 I_1^{**})^2}, \\
J_{22} &= \frac{\beta_1(S^* + a_1 S^{*2})}{(1 + a_1 S^* + b_1 I_1^{**})^2} - (d + \alpha_1 + r_1) = -\frac{\beta_1 b_1 S^* I_1^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}, \\
J_{23} &= 0, \\
J_{31} &= \frac{\beta_2(I_2^{**} + b_2 I_2^{**2})}{(1 + a_2 S^* + b_2 I_2^{**})^2}, \\
J_{32} &= 0, \\
J_{33} &= \frac{\beta_2(S^* + a_2 S^{*2})}{(1 + a_2 S^* + b_2 I_2^{**})^2} - (d + \alpha_2 + r_2) = -\frac{\beta_2 b_2 S^* I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2}.
\end{aligned}$$

At the positive equilibrium E^* we have the following characteristic equation:

$$\lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 = 0,$$

where

$$\begin{aligned}
p_2 &= -(J_{11} + J_{22} + J_{33}), \\
p_1 &= J_{11} J_{22} + J_{11} J_{33} + J_{22} J_{33} - J_{13} J_{31} - J_{12} J_{21}, \\
p_0 &= J_{12} J_{21} J_{33} + J_{13} J_{31} J_{22} - J_{11} J_{22} J_{33}.
\end{aligned}$$

Then the equilibrium E^* is stable if $p_j > 0$ $(j = 1, 2, 3)$ and $p_2 p_1 > p_0$.

Note that

$$
\begin{aligned}
p_2 &= -(J_{11} + J_{22} + J_{33}) \\
&= J_{21} - J_{22} + J_{31} - J_{33} + d \\
&= \frac{\beta_1 I_1^{**}(1 + b_1 I_1^{**} + b_1 S^*)}{(1 + a_1 S^* + b_1 I_1^{**})^2} + \frac{\beta_2 I_2^{**}(1 + b_2 I_2^{**} + b_2 S^*)}{(1 + a_2 S^* + b_2 I_2^{**})^2} + d \\
&= p_{21} + p_{22} + d > 0, \\
p_1 &= -(J_{31} + d)J_{22} - (J_{21} + d)J_{33} + J_{22}J_{33} + (d + \alpha_2)J_{31} + (d + \alpha_1)J_{21} \\
&= J_{22}J_{33} - J_{22}J_{31} - J_{21}J_{33} + J_{21}(d + \alpha_1) - dJ_{22} + J_{31}(d + \alpha_2) - dJ_{33} \\
&= \frac{\beta_1 \beta_2 S^* I_1^{**} I_2^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}\big(b_1 b_2 S^* + b_1\big(1 + b_2 I_2^{**}\big) + b_2\big(1 + b_1 I_1^{**}\big)\big) \\
&\quad + \frac{\beta_1 I_1^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}\big(\big(1 + b_1 I_1^{**}\big)(d + \alpha_1) + d b_1 S^*\big) \\
&\quad + \frac{\beta_2 I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2}\big(\big(1 + b_2 I_2^{**}\big)(d + \alpha_2) + d b_2 S^*\big) \\
&= p_{11} + p_{12} + p_{13} > 0, \\
p_0 &= J_{12}J_{21}J_{33} + J_{13}J_{31}J_{22} - J_{11}J_{22}J_{33} \\
&= -\big(J_{22} + (d + \alpha_1)\big)J_{21}J_{33} - \big(J_{33} + (d + \alpha_2)\big)J_{31}J_{22} + (J_{21} + J_{31} + d)J_{22}J_{33} \\
&= -(d + \alpha_1)J_{21}J_{33} - (d + \alpha_2)J_{22}J_{31} + dJ_{22}J_{33} \\
&= (d + \alpha_1)\frac{\beta_1(I_1^{**} + b_1 I_1^{**2})}{(1 + a_1 S^* + b_1 I_1^{**})^2}\frac{\beta_2 b_2 S^* I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2} \\
&\quad + (d + \alpha_2)\frac{\beta_1 b_1 S^* I_1^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}\frac{\beta_2(I_2^{**} + b_2 I_2^{**2})}{(1 + a_2 S^* + b_2 I_2^{**})^2} \\
&\quad + d\frac{\beta_1 b_1 S^* I_1^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}\frac{\beta_2 b_2 S^* I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2} > 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
p_2 p_1 - p_0 &= \big[(p_{21} + p_{22})p_{11} + p_{21}p_{12} + p_{22}p_{13} + dp_1\big] \\
&\quad + (p_{21}p_{13} + p_{22}p_{12} - p_0) \\
&= p + q.
\end{aligned}
$$

It is easy to see that $p > 0$ when $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$. Let us now verify that

$$
\begin{aligned}
q &= p_{21}p_{13} + p_{22}p_{12} - p_0 \\
&= \frac{\beta_1 I_1^{**}(1 + b_1 I_1^{**} + b_1 S^*)}{(1 + a_1 S^* + b_1 I_1^{**})^2}\frac{\beta_2 I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2}\big(\big(1 + b_2 I_2^{**}\big)(d + \alpha_2) + d b_2 S^*\big) \\
&\quad + \frac{\beta_2 I_2^{**}(1 + b_2 I_2^{**} + b_2 S^*)}{(1 + a_2 S^* + b_2 I_2^{**})^2}\frac{\beta_1 I_1^{**}}{(1 + a_1 S^* + b_1 I_1^{**})^2}\big(\big(1 + b_1 I_1^{**}\big)(d + \alpha_1) + d b_1 S^*\big) \\
&\quad - (d + \alpha_1)\frac{\beta_1(I_1^{**} + b_1 I_1^{**2})}{(1 + a_1 S^* + b_1 I_1^{**})^2}\frac{\beta_2 b_2 S^* I_2^{**}}{(1 + a_2 S^* + b_2 I_2^{**})^2}
\end{aligned}
$$

$$
\begin{aligned}
&- (d+\alpha_2)\frac{\beta_1 b_1 S^* I_1^{**}}{(1+a_1S^*+b_1I_1^{**})^2}\frac{\beta_2(I_2^{**}+b_2I_2^{**2})}{(1+a_2S^*+b_2I_2^{**})^2}\\
&- d\frac{\beta_1 b_1 S^* I_1^{**}}{(1+a_1S^*+b_1I_1^{**})^2}\frac{\beta_2 b_2 S^* I_2^{**}}{(1+a_2S^*+b_2I_2^{**})^2}\\
=\ &\frac{\beta_1\beta_2 I_1^{**}I_2^{**}}{(1+a_1S^*+b_1I_1^{**})^2(1+a_2S^*+b_2I_2^{**})^2}\big(\big(1+b_1I_1^{**}\big)\big(d+\alpha_2+db_2I_2^{**}\\
&+db_2S^*+\alpha_2b_2I_2^{**}\big)+\big(1+b_2I_2^{**}\big)\big(d+\alpha_1+db_1I_1^{**}+\alpha_1b_1I_1^{**}+db_1S^*\\
&+db_1b_2S^{*2}\big)\big)>0,
\end{aligned}
$$

which implies that $p_2p_1 > p_0$. Then it can be concluded that E^* is stable when it exists. The proof is completed. □

4 Dynamics of stochastic system (4)

4.1 Extinction

In this section, we are going to explore the conditions which lead to the extinction of two infectious diseases mentioned in the system (4) under a white noise stochastic disturbance.

Theorem 4.1 *If*

$$
\sigma_i > \frac{\beta_i}{\sqrt{2(d+\alpha_i+r_i)}},\quad i=1,2,
$$

then two infectious diseases of system (4) *go to extinction almost surely.*

Proof Let $(S(t),I_1(t),I_2(t))$ be a solution of system (4) with initial value $(S(0),I_1(0),I_2(0)) \in R_+^3$. Applying Itô's formula to system (4) results in

$$
\begin{aligned}
\mathrm{d}\ln I_i(t) &= \left(\frac{\beta_i S(t)}{1+a_iS(t)+b_iI_i(t)} - (d+\alpha_i+r_i) - \frac{\sigma_i^2S^2(t)}{2(1+a_iS(t)+b_iI_i(t))^2}\right)\mathrm{d}t\\
&\quad + \frac{\sigma_iS(t)}{1+a_iS(t)+b_iI_i(t)}\,\mathrm{d}B_i(t),\quad i=1,2.
\end{aligned}
\tag{9}
$$

Integrating both sides of (9) from 0 to t gives

$$
\begin{aligned}
\ln I_i(t) &= -\frac{\sigma_i^2}{2}\int_0^t\left(\frac{S(\tau)}{1+a_iS(\tau)+b_iI_i(\tau)}-\frac{\beta_i}{\sigma_i^2}\right)^2\mathrm{d}\tau-(d+\alpha_i+r_i)t\\
&\quad+\frac{\beta_i^2}{2\sigma_i^2}t+M_i(t)+\ln I_i(0)\\
&\le -(d+\alpha_i+r_i)t+\frac{\beta_i^2}{2\sigma_i^2}t+M_i(t)+\ln I_i(0),
\end{aligned}
\tag{10}
$$

where $M_i(t)=\int_0^t\frac{\sigma_iS(\tau)}{1+a_iS(\tau)+b_iI_i(\tau)}\,\mathrm{d}B_i(\tau)$, $i=1,2$.

Dividing both sides of (10) by t, we have

$$
\frac{\ln I_i(t)}{t} \le -\left(d+\alpha_i+r_i-\frac{\beta_i^2}{2\sigma_i^2}\right)+\frac{M_i(t)}{t}+\frac{\ln I_i(0)}{t},\quad i=1,2.
\tag{11}
$$

The function $M_i(t)$ $(i = 1, 2)$ is also known as the local continuous martingale with $M_i(0) = 0$, and by Lemma 2.3, we have

$$\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0, \quad i = 1, 2.$$

Since $\sigma_i > \frac{\beta_i}{\sqrt{2(d+\alpha_i+r_i)}}$ for $i = 1, 2$, taking the limit superior of both sides of (11) leads to

$$\limsup_{t\to+\infty} \frac{\ln I_i(t)}{t} \le -\left(d + \alpha_i + r_i - \frac{\beta_i^2}{2\sigma_i^2}\right) < 0,$$

which implies that $\lim_{t\to+\infty} I_i(t) = 0$. This completes the proof of Theorem 4.1. □

Remark 4.1 Theorem 4.1 shows that when $\sigma_i > \frac{\beta_i}{\sqrt{2(d+\alpha_i+r_i)}}$, $i = 1, 2$, two infectious diseases of system (4) die out almost surely, that is to say, large white noise stochastic disturbance can lead to the two epidemics to be extinct. Therefore, we always assume that the white noise stochastic disturbance is not too large in the rest of this paper.

Let

$$\begin{aligned}
\mathcal{R}_1^* &= \frac{\beta_1 A}{(Aa_1 + d)(d + \alpha_1 + r_1)} - \frac{\sigma_1^2 A^2}{2(Aa_1 + d)^2(d + \alpha_1 + r_1)} \\
&= \mathcal{R}_1 - \frac{\sigma_1^2 A^2}{2(Aa_1 + d)^2(d + \alpha_1 + r_1)}, \\
\mathcal{R}_2^* &= \frac{\beta_2 A}{(Aa_2 + d)(d + \alpha_2 + r_2)} - \frac{\sigma_2^2 A^2}{2(Aa_2 + d)^2(d + \alpha_2 + r_2)} \\
&= \mathcal{R}_2 - \frac{\sigma_2^2 A^2}{2(Aa_2 + d)^2(d + \alpha_2 + r_2)},
\end{aligned}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the threshold of the deterministic system (3) given in (8). Then we have the following results mentioned in the theorem.

Theorem 4.2 *Let* $(S(t), I_1(t), I_2(t))$ *be a solution of system* (4) *with initial value* $(S(0), I_1(0), I_2(0)) \in R_+^3$. *Then if*

$$\mathcal{R}_i^* < 1 \quad \textit{and} \quad \sigma_i \le \sqrt{\frac{\beta_i(Aa_i + d)}{A}}, \quad i = 1, 2,$$

hold, two infectious diseases of system (4) *go to extinction almost surely, i.e.*

$$\lim_{t\to+\infty} I_i(t) = 0, \quad i = 1, 2.$$

Moreover, $\lim_{t\to+\infty} S(t) = \frac{A}{d}$, *almost surely.*

Proof For both sides of (9), integrating from 0 to t first and dividing by t yields

$$\begin{aligned}\frac{\ln I_i(t)}{t} &= \frac{1}{t}\int_0^t \left(\frac{\beta_i S(\tau)}{1+a_iS(t)+b_iI_i(t)} - (d+\alpha_i+r_i)\right.\\ &\quad \left. - \frac{\sigma_i^2 S^2(\tau)}{2(1+a_iS(t)+b_iI_i(t))^2}\right)\mathrm{d}\tau + \frac{M_i(t)}{t} + \frac{\ln I_i(0)}{t}\\ &\le \left(\frac{\beta_i A}{Aa_i+d} - (d+\alpha_i+r_i) - \frac{\sigma_i^2A^2}{2(Aa_i+d)^2}\right) + \frac{M_i(t)}{t} + \frac{\ln I_i(0)}{t}\\ &= (d+\alpha_i+r_i)\left(\frac{\beta_i A}{(Aa_i+d)(d+\alpha_i+r_i)} - \frac{\sigma_i^2A^2}{2(Aa_i+d)^2(d+\alpha_i+r_i)} - 1\right)\\ &\quad + \frac{M_i(t)}{t} + \frac{\ln I_i(0)}{t}. \end{aligned} \tag{12}$$

Taking the superior limit of both sides of (12) leads to

$$\limsup_{t\to+\infty} \frac{\ln I_i(t)}{t} \le (d+\alpha_i+r_i)\big(\mathcal{R}_i^* - 1\big) < 0,$$

which implies that $\lim_{t\to+\infty} I_i(t) = 0$, $i = 1, 2$.

Without loss of generality, we assume that $0 < I_i(t) < \varepsilon_i$ $(i = 1, 2)$ for all $t \ge 0$, by the first equation of system (4), we have

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} \ge A - \big(d + \beta_1\varepsilon_1 + \beta_2\varepsilon_2 + \sigma_1\varepsilon_1\big|\dot{B}_1(t)\big| + \sigma_2\varepsilon_2\big|\dot{B}_2(t)\big|\big)S(t). \tag{13}$$

As $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, taking the inferior limit of both sides of (13) yields

$$\liminf_{t\to+\infty} S(t) \ge \frac{A}{d}. \tag{14}$$

By the proof of Lemma 2.2, we have

$$\limsup_{t\to+\infty} S(t) \le \frac{A}{d}. \tag{15}$$

From (14) and (15), we have

$$\lim_{t\to+\infty} S(t) = \frac{A}{d},$$

almost surely. This completes the proof of Theorem 4.2. □

Remark 4.2 Theorem 4.1 and Theorem 4.2 show that two diseases will die out if the white noise disturbance is sufficiently larger or $\mathcal{R}_i^* < 1$ and the white noise disturbance is not large. Note that the expressions for $\mathcal{R}_i^*$ for $i = 1, 2$ which are the threshold values of system (4) are strictly different compared with the thresholds $\mathcal{R}_i$ of system (3), This implies that the conditions which are needed to have $I_i(t)$ for $i = 1, 2$ gone in extinction in deterministic system (3) are stronger than in the corresponding stochastic system (4).

4.2 Permanence in mean

Theorem 4.3 *Let* $(S(t), I_1(t), I_2(t))$ *be the solution of system* (4) *with initial value* $(S(0), I_1(0), I_2(0)) \in \Gamma$, *then we have the following.*

(i) *If* $\mathcal{R}_1^* > 1$, $\mathcal{R}_2^* < 1$ *and* $\sigma_2 \le \sqrt{\frac{\beta_2(Aa_2+d)}{A}}$, *then the disease* I_2 *goes extinct and the disease* I_1 *is permanent in mean, moreover,* I_1 *satisfies*

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \ge \frac{(Aa_1+d)(d+\alpha_1+r_1)}{\beta_1(d+\alpha_1)+b_1d(d+\alpha_1+r_1)}\big(\mathcal{R}_1^*-1\big).$$

(ii) *If* $\mathcal{R}_2^* > 1$, $\mathcal{R}_1^* < 1$ *and* $\sigma_1 \le \sqrt{\frac{\beta_1(Aa_1+d)}{A}}$, *then the disease* I_1 *goes extinct and the disease* I_2 *is permanent in mean, moreover,* I_2 *satisfies*

$$\liminf_{t\to+\infty}\langle I_2(t)\rangle \ge \frac{(Aa_2+d)(d+\alpha_2+r_2)}{\beta_2(d+\alpha_2)+b_2d(d+\alpha_2+r_2)}\big(\mathcal{R}_2^*-1\big).$$

(iii) *If* $\mathcal{R}_1^* > 1$ *and* $\mathcal{R}_2^* > 1$, *then two infectious diseases* I_1 *and* I_2 *are permanent in mean, moreover,* I_1 *and* I_2 *satisfy*

$$\liminf_{t\to+\infty}\langle I_1(t)+I_2(t)\rangle \ge \frac{1}{\Delta_{\max}}\sum_{i=1}^{2}a_i(d+\alpha_i+r_i)\big(\mathcal{R}_i^*-1\big),$$

where

$$\Delta_{\max} = \sum_{i=1}^{2}\left[\frac{\beta_1+\beta_2}{d}(d+\alpha_i)+b_i(d+\alpha_i+r_i)\right].$$

Proof Part (i). By Theorem 3.1, since $\mathcal{R}_2^* < 1$ and $\sigma_2 \le \sqrt{\frac{\beta_2(Aa_2+d)}{A}}$, we have $\lim_{t\to+\infty} I_2(t) = 0$. Since $\mathcal{R}_1^* > 1$, for ε small enough, such that $0 < I_2(t) < \varepsilon$ for all t large enough we have

$$\frac{\beta_1(A-(d+\alpha_1)\varepsilon)}{(Aa_1+d)(d+\alpha_1+r_1)} - \frac{\sigma_1^2A^2}{2(Aa_1+d)^2(d+\alpha_1+r_1)} > 1.$$

Integrating from 0 to t and dividing by $t > 0$ on both sides of system (4) yields

$$\begin{aligned}\Theta(t) &\triangleq \frac{S(t)-S(0)}{t} + \frac{I_1(t)-I_1(0)}{t} + \frac{I_2(t)-I_2(0)}{t}\\ &= A - d\langle S(t)\rangle - (d+\alpha_1)\langle I_1(t)\rangle - (d+\alpha_2)\langle I_2(t)\rangle\\ &\ge A - d\langle S(t)\rangle - (d+\alpha_1)\langle I_1(t)\rangle - (d+\alpha_2)\varepsilon,\end{aligned}$$

then one can get

$$\langle S(t)\rangle \ge \frac{A-(d+\alpha_2)\varepsilon}{d} - \frac{d+\alpha_1}{d}\langle I_1(t)\rangle - \frac{\Theta(t)}{d}.$$

Applying Itô's formula gives

$$
\begin{aligned}
&\mathrm{d}\left(\left(1+a_1\frac{A}{d}\right)\ln I_1(t)+b_1I_1(t)\right)\\
&\quad=\left[\frac{(1+a_1\frac{A}{d})\beta_1S(t)}{1+a_1S(t)+b_1I_1(t)}-\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)-\frac{(1+a_1\frac{A}{d})\sigma_1^2S^2(t)}{1+a_1S(t)+b_1I_1(t)}\right]\mathrm{d}t\\
&\qquad+b_1\left[\frac{\beta_1S(t)I_1(t)}{1+a_1S(t)+b_1I_1(t)}-(d+\alpha_1+r_1)I_1(t)\right]\mathrm{d}t\\
&\qquad+\frac{(1+a_1\frac{A}{d})\sigma_1S(t)\mathrm{d}B_1(t)}{1+a_1S(t)+b_1I_1(t)}+\frac{b_1\sigma_1S(t)I_1(t)\mathrm{d}B_1(t)}{1+a_1S(t)+b_1I_1(t)}\\
&\quad\geq\left[\frac{(1+a_1\frac{A}{d})\beta_1S(t)}{1+a_1\frac{A}{d}+b_1I_1(t)}-\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)-\frac{(1+a_1\frac{A}{d})\sigma_1^2S^2(t)}{1+a_1S(t)+b_1I_1(t)}\right]\mathrm{d}t\\
&\qquad+b_1\left[\frac{\beta_1S(t)I_1(t)}{1+a_1\frac{A}{d}+b_1I_1(t)}-(d+\alpha_1+r_1)I_1(t)\right]\mathrm{d}t\\
&\qquad+\frac{(1+a_1\frac{A}{d})\sigma_1S(t)\mathrm{d}B_1(t)}{1+a_1\frac{A}{d}+b_1I_1(t)}+\frac{b_1\sigma_1S(t)I_1(t)\mathrm{d}B_1(t)}{1+a_1\frac{A}{d}+b_1I_1(t)}\\
&\quad\geq\left[\beta_1S(t)-\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)-b_1(d+\alpha_1+r_1)I_1(t)-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})}\right]\mathrm{d}t\\
&\qquad+\sigma_1S(t)\,\mathrm{d}B_1(t).
\end{aligned}
\tag{16}
$$

Integrating from 0 to t and dividing by $t>0$ on both sides of (16) yields

$$
\begin{aligned}
&\left(1+a_1\frac{A}{d}\right)\frac{(\ln I_1(t)-\ln I_1(0))}{t}+b_1\frac{I_1(t)-I_1(0)}{t}\\
&\quad\geq\beta_1\langle S(t)\rangle-\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)-b_1(d+\alpha_1+r_1)\langle I_1(t)\rangle\\
&\qquad-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})}+\frac{M(t)}{t}\\
&\quad\geq\beta_1\left[\frac{A-(d+\alpha_2)\varepsilon}{d}-\frac{d+\alpha_1}{d}\langle I_1(t)\rangle-\frac{\Theta(t)}{d}\right]\\
&\qquad-\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)\\
&\qquad-b_1(d+\alpha_1+r_1)\langle I_1(t)\rangle-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})}+\frac{M(t)}{t}\\
&\quad=\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)\left[\frac{\beta_1(A-(d+\alpha_2)\varepsilon)}{d(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}\right.\\
&\qquad\left.-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})^2(d+\alpha_1+r_1)}-1\right]-\left[\frac{\beta_1(d+\alpha_1)}{d}+b_1(d+\alpha_1+r_1)\right]\langle I_1(t)\rangle\\
&\qquad-\frac{\beta_1\Theta(t)}{d}+\frac{M(t)}{t},
\end{aligned}
\tag{17}
$$

where $M(t) = \int_0^t \sigma_1 S(\tau)\,\mathrm{d}B_1(\tau)$. The inequality (17) can be rewritten as

$$\begin{aligned}\langle I_1(t)\rangle \geq{}& \frac{1}{\Delta}\Bigg[\left(1+a_1\frac{A}{d}\right)(d+\alpha_1+r_1)\Bigg(\frac{\beta_1(A-(d+\alpha_2)\varepsilon)}{d(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}\\ &-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})^2(d+\alpha_1+r_1)}-1\Bigg)-\frac{\beta_1\Theta(t)}{d}+\frac{M(t)}{t}\\ &-\left(1+a_1\frac{A}{d}\right)\frac{(\ln I_1(t)-\ln I_1(0))}{t}-b_1\frac{I_1(t)-I_1(0)}{t}\Bigg]\\ \geq{}&\begin{cases}\frac{1}{\Delta}[(1+a_1\frac{A}{d})(d+\alpha_1+r_1)(\frac{\beta_1(A-(d+\alpha_2)\varepsilon)}{d(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}\\ \quad-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})^2(d+\alpha_1+r_1)}-1)-\frac{\beta_1\Theta(t)}{d}+\frac{M(t)}{t}\\ \quad+(1+a_1\frac{A}{d})\frac{\ln I_1(0)}{t}-b_1\frac{I_1(t)-I_1(0)}{t}], & 0<I_1(t)<1,\\ \frac{1}{\Delta}[a_1(d+\alpha_1+r_1)(\frac{\beta_1(A-(d+\alpha_2)\varepsilon)}{d(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}\\ \quad-\frac{\sigma_1^2(\frac{A}{d})^2}{2(1+a_1\frac{A}{d})^2(d+\alpha_1+r_1)}-1)-\frac{\beta_1\Theta(t)}{d}+\frac{M(t)}{t}\\ \quad-(1+a_1\frac{A}{d})\frac{(\ln I_1(t)-\ln I_1(0))}{t}-b_1\frac{I_1(t)-I_1(0)}{t}], & 1\leq I_1(t),\end{cases}\end{aligned} \tag{18}$$

where $\Delta = \frac{\beta_1(d+\alpha_1)}{d} + b_1(d+\alpha_1+r_1)$.

By Lemma 2.3, we get $\lim_{t\to+\infty}\frac{M(t)}{t} = 0$. According to Lemma 2.2, one can see that $I_1(t) \leq \frac{A}{d}$. Thus, one has $\lim_{t\to+\infty}\frac{I_1(t)}{t} = 0$, $\lim_{t\to+\infty}\frac{\ln I_1(t)}{t} = 0$ as $I_1(t) \geq 1$ and $\lim_{t\to+\infty}\Theta(t) = 0$.

Taking the inferior limit of both sides of (18) yields

$$\begin{aligned}\liminf_{t\to+\infty}\langle I_1(t)\rangle \geq{}& \frac{(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}{\Delta}\Bigg[\frac{\beta_1(A-(d+\alpha_2)\varepsilon)}{(d+a_1A)(d+\alpha_1+r_1)}\\ &-\frac{\sigma_1^2A^2}{2(d+a_1A)^2(d+\alpha_1+r_1)}-1\Bigg]\\ >{}& 0.\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \geq \frac{d(1+a_1\frac{A}{d})(d+\alpha_1+r_1)}{\beta_1(d+\alpha_1)+b_1d(d+\alpha_1+r_1)}\big(\mathcal{R}_1^*-1\big).$$

Similarly, by using arguments as in the part (i), we can establish the results given in part (ii), and we omit it here.

Part (iii). Notice that

$$\langle S(t)\rangle = \frac{A}{d}-\frac{d+\alpha_1}{d}\langle I_1(t)\rangle-\frac{d+\alpha_2}{d}\langle I_2(t)\rangle-\frac{\Theta(t)}{d}. \tag{19}$$

Define

$$V(t) = \ln\big[I_1^{(1+a_1\frac{A}{d})}(t)I_2^{(1+a_2\frac{A}{d})}(t)\big]+\big[b_1I_1(t)+b_2I_2(t)\big].$$

Therefore, $V(t)$ is bounded. Then we have

$$\begin{aligned}
D^+V(t) &= \sum_{i=1}^{2}\left[\frac{(1+a_i\frac{A}{d})\beta_i S(t)}{1+a_iS(t)+b_iI_i(t)} - \left(1+a_i\frac{A}{d}\right)(d+\alpha_i+r_i)\right.\\
&\quad \left. - \frac{(1+a_i\frac{A}{d})\sigma_i^2S^2(t)}{2(1+a_iS(t)+b_iI_i(t))^2}\right]\mathrm{d}t + \sum_{i=1}^{2}\frac{(1+a_i\frac{A}{d})\sigma_iS(t)\,\mathrm{d}B_i(t)}{1+a_iS(t)+b_iI_i(t)}\\
&\quad + \sum_{i=1}^{2}b_i\left[\frac{\beta_iS(t)I_i(t)}{1+a_iS(t)+b_iI_i(t)} - (d+\alpha_i+r_i)I_i(t)\right]\mathrm{d}t\\
&\quad + \sum_{i=1}^{2}\frac{b_i\sigma_iS(t)I_i(t)\,\mathrm{d}B_i(t)}{1+a_iS(t)+b_iI_i(t)}\\
&\geq \sum_{i=1}^{2}\left[\frac{(1+a_i\frac{A}{d})\beta_i S(t)}{1+a_i\frac{A}{d}+b_iI_i(t)} - \left(1+a_i\frac{A}{d}\right)(d+\alpha_i+r_i)\right.\\
&\quad \left. - \frac{(1+a_i\frac{A}{d})\sigma_i^2S^2(t)}{2(1+a_iS(t)+b_iI_i(t))^2}\right]\mathrm{d}t + \sum_{i=1}^{2}\frac{(1+a_i\frac{A}{d})\sigma_iS(t)\,\mathrm{d}B_i(t)}{1+a_i\frac{A}{d}+b_iI_i(t)}\\
&\quad + \sum_{i=1}^{2}b_i\left[\frac{\beta_iS(t)I_i(t)}{1+a_i\frac{A}{d}+b_iI_i(t)} - (d+\alpha_i+r_i)I_i(t)\right]\mathrm{d}t\\
&\quad + \sum_{i=1}^{2}\frac{b_i\sigma_iS(t)I_i(t)\,\mathrm{d}B_i(t)}{1+a_i\frac{A}{d}+b_iI_i(t)}\\
&\geq \left[(\beta_1+\beta_2)S(t) - \sum_{i=1}^{2}(d+\alpha_i+r_i)\left(1+a_1\frac{A}{d}+b_iI_i(t)\right)\right.\\
&\quad \left. - \sum_{i=1}^{2}\frac{\sigma_i^2(\frac{A}{d})^2}{2(1+a_i\frac{A}{d})}\right]\mathrm{d}t + \sum_{i=1}^{2}\sigma_iS(t)\,\mathrm{d}B_i(t). \qquad (20)
\end{aligned}$$

Integrating from 0 to t and dividing by $t>0$ on both sides of (20) yields

$$\begin{aligned}
\frac{V(t)}{t} - \frac{V(0)}{t} &\geq \left[(\beta_1+\beta_2)\langle S(t)\rangle - \sum_{i=1}^{2}(d+\alpha_i+r_i)\left(1+a_i\frac{A}{d}+b_i\langle I_i(t)\rangle\right)\right.\\
&\quad \left. - \sum_{i=1}^{2}\frac{\sigma_i^2(\frac{A}{d})^2}{2(1+a_i\frac{A}{d})}\right]\mathrm{d}t + \sum_{i=1}^{2}\frac{M_i}{t}\\
&= (\beta_1+\beta_2)\frac{A}{d} - \sum_{i=1}^{2}(d+\alpha_i+r_i)\left(1+a_1\frac{A}{d}\right) - \sum_{i=1}^{2}\frac{\sigma_i^2(\frac{A}{d})^2}{2(1+a_i\frac{A}{d})}\\
&\quad - \sum_{i=1}^{2}\left[\frac{\beta_1+\beta_2}{d}(d+\alpha_i)+b_i(d+\alpha_i+r_i)\right]\langle I_i(t)\rangle\\
&\quad - \frac{\beta_1+\beta_2}{d}\Theta(t) + \sum_{i=1}^{2}\frac{M_i}{t}\\
&\geq \sum_{i=1}^{2}\left(1+a_i\frac{A}{d}\right)(d+\alpha_i+r_i)\left[\frac{\beta_iA}{(d+a_iA)(d+\alpha_i+r_i)}\right.
\end{aligned}$$

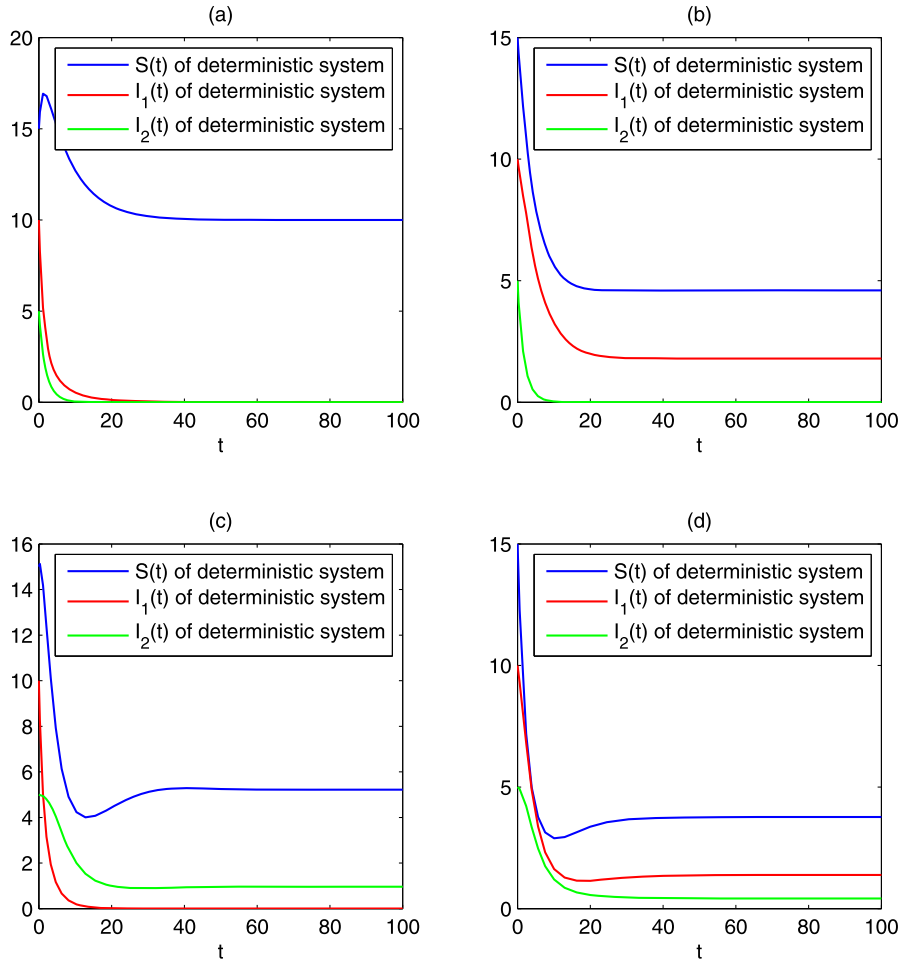

Figure 1 Time evolutions of the deterministic *SIS* system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$. (a) Time series for $S(t)$, $I_1(t)$, $I_2(t)$ with parameters $r_1 = 0.9$, $r_2 = 0.9$, $R_1 = 0.9091$, $R_2 = 0.6696$. **(b)** Time series for $S(t)$, $I_1(t)$, $I_2(t)$ with parameters $r_1 = 0.3$, $r_2 = 0.9$, $R_1 = 1.8182$, $R_2 = 0.6696$. **(c)** Time series for $S(t)$, $I_1(t)$, $I_2(t)$ with parameters $r_1 = 0.9$, $r_2 = 0.3$, $R_1 = 0.9091$, $R_2 = 1.1719$. **(d)** Time series for $S(t)$, $I_1(t)$, $I_2(t)$ with parameters $r_1 = 0.3$, $r_2 = 0.3$, $R_1 = 1.8182$, $R_2 = 1.1719$.

$$
\begin{aligned}
&- \frac{\sigma_i^2 A^2}{2(d + a_i A)^2 (d + \alpha_i + r_i)} - 1 \Bigg] - \Delta_{\max} \big[\big\langle I_1(t) \big\rangle + \big\langle I_2(t) \big\rangle \big] \\
&- \frac{\beta_1 + \beta_2}{d} \Theta(t) + \sum_{i=1}^{2} \frac{M_i}{t}, \qquad (21)
\end{aligned}
$$

where $M_i(t) = \int_0^t \sigma_i S(\tau)\, \mathrm{d}B_i(\tau)$.

The inequality (21) can be rewritten as

$$
\begin{aligned}
\big\langle I_1(t) \big\rangle + \big\langle I_2(t) \big\rangle \geq \frac{1}{\Delta_{\max}} \Bigg[& \sum_{i=1}^{2} \left(1 + a_i \frac{A}{d} \right) (d + \alpha_i + r_i) \left(\frac{\beta_i A}{(d + a_i A)(d + \alpha_i + r_i)} \right. \\
& \left. - \frac{\sigma_i^2 A^2}{2(d + a_i A)^2 (d + \alpha_i + r_i)} - 1 \right) - \frac{V(t)}{t} + \frac{V(0)}{t} \\
& - \frac{\beta_1 + \beta_2}{d} \Theta(t) + \sum_{i=1}^{2} \frac{M_i}{t} \Bigg]. \qquad (22)
\end{aligned}
$$

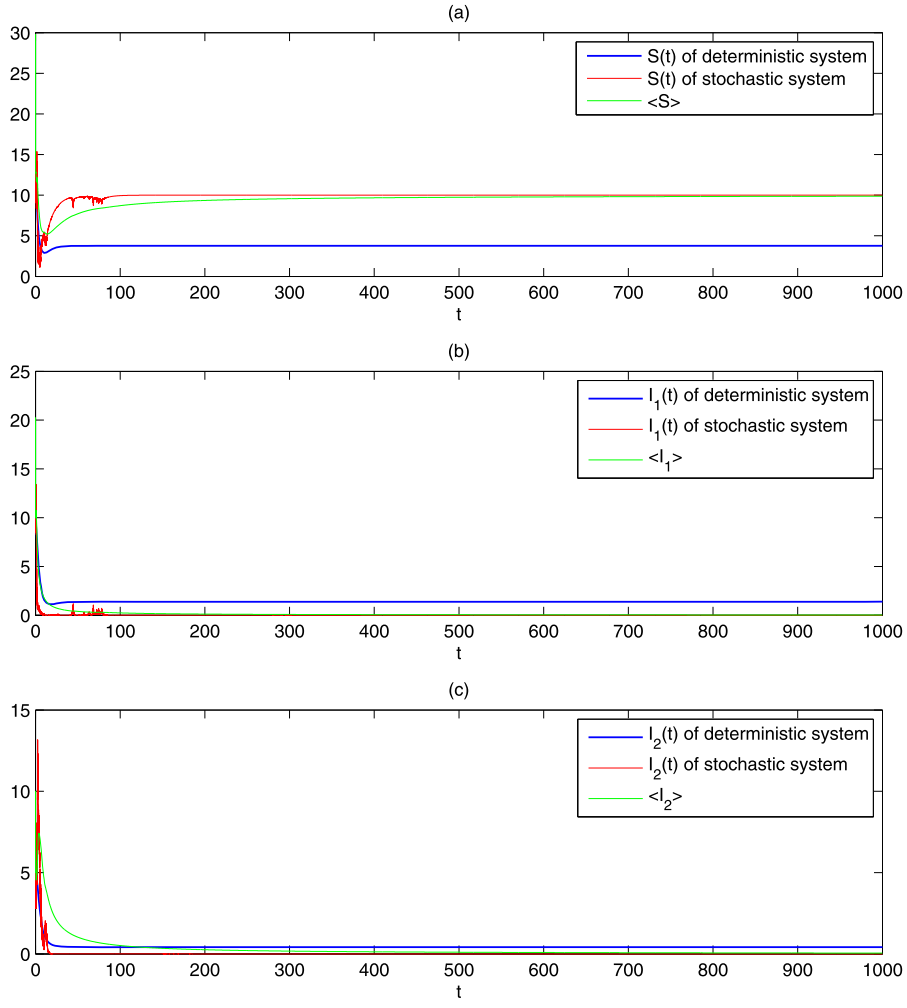

Figure 2 Dynamic behaviour comparisons of $S(t)$, $I_1(t)$, $I_2(t)$ in the deterministic and the stochastic *SIS* system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $r_1 = 0.3$, $r_2 = 0.3$, $\sigma_1 = 1.2$, $\sigma_2 = 1.2$. (a) Time series for $S(t)$ and $\langle S(t)\rangle$. **(b)** Time series for $I_1(t)$ and $\langle I_1(t)\rangle$. **(c)** Time series for $I_2(t)$ and $\langle I_2(t)\rangle$.

By Lemma 2.3, we have $\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0$, for $i = 1, 2$. According to Lemma 2.2, one can see that $\lim_{t\to+\infty} \Theta(t) = 0$ and $\lim_{t\to+\infty} \frac{V(t)}{t} = 0$.

Taking the inferior limit of both sides of (22) yields

$$\liminf_{t\to+\infty} \langle I_1(t) + I_2(t)\rangle \geq \frac{1}{\Delta_{\max}} \sum_{i=1}^{2} \left(1 + a_i \frac{A}{d}\right)(d + \alpha_i + r_i)\left(R_i^* - 1\right) > 0.$$

This completes the proof of Theorem 4.3. □

Remark 4.3 Theorem 4.3 shows that both diseases will prevail if the white noise disturbances are small enough such that $\mathcal{R}_i^* > 1$, conversely, if the white noise disturbances are large enough, then both diseases will become extinct. This implies that the stochastic disturbance may cause epidemic diseases to die out.

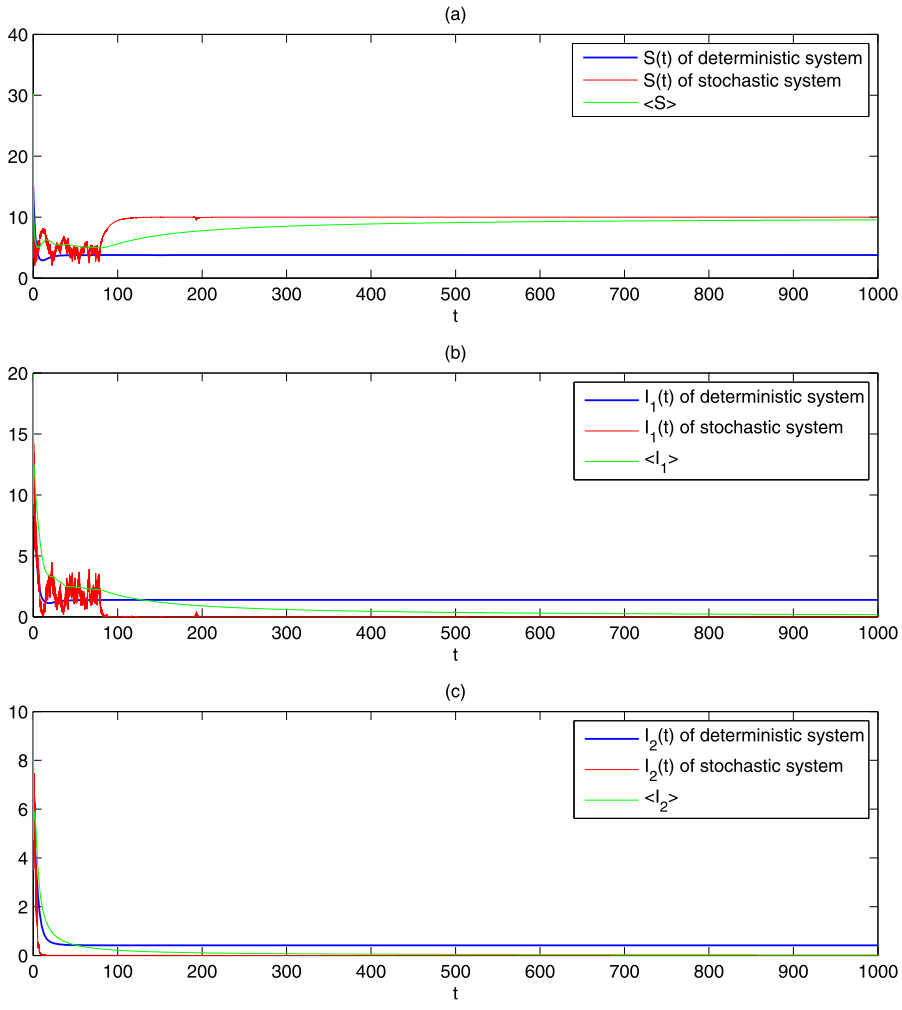

Figure 3 Dynamic behaviour comparisons of $S(t)$, $I_1(t)$, $I_2(t)$ in the deterministic and the stochastic *SIS* system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $r_1 = 0.3$, $r_2 = 0.3$, $\sigma_1 = 1.1$, $\sigma_2 = 1.1$. (a) Time series for $S(t)$ and $\langle S(t)\rangle$. **(b)** Time series for $I_1(t)$ and $\langle I_1(t)\rangle$. **(c)** Time series for $I_2(t)$ and $\langle I_2(t)\rangle$.

5 Conclusion and simulations

This paper proposed two *SIS* epidemic models with Beddington-DeAngelis incidence rate and double epidemic hypothesis from the point of view of deterministic and stochastic aspect. The threshold dynamics of both two systems were investigated and the conditions for extinction and permanence of both epidemic diseases were obtained. From Theorems 4.1 and 4.2, it can be seen that there is a significant difference between the thresholds of the stochastic system and the deterministic system, from which it can be concluded that the conditions for two epidemic diseases to go to extinction in the stochastic system are weaker than those of the deterministic system.

To illustrate the dynamic difference between the deterministic system and the stochastic system, we next carry out some numerical simulations of these cases with respect to different noise disturbance intensity using the Euler Maruyama (EM) method [36, 79].

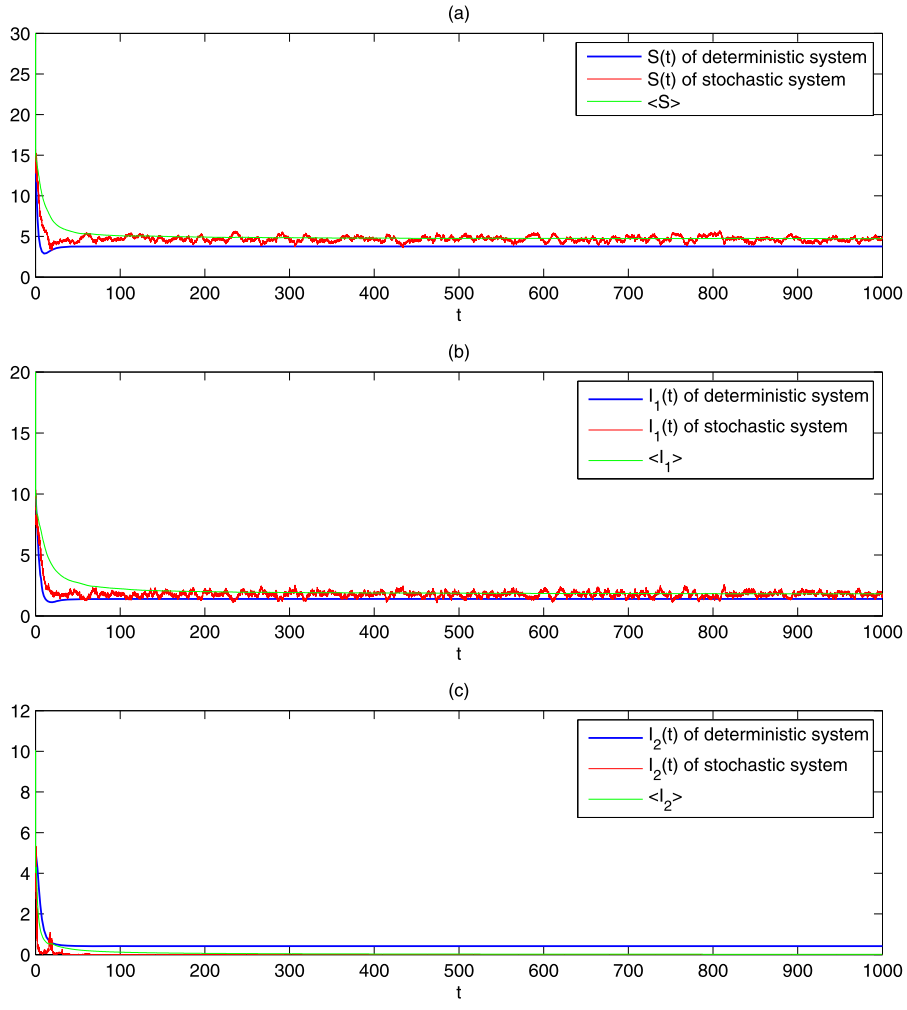

Figure 4 Dynamic behaviour comparisons of $S(t)$, $I_1(t)$, $I_2(t)$ in the deterministic and the stochastic *SIS* system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $r_1 = 0.3$, $r_2 = 0.3$, $\sigma_1 = 0.2$, $\sigma_2 = 1.1$. (a) Time series for $S(t)$ and $\langle S(t)\rangle$. **(b)** Time series for $I_1(t)$ and $\langle I_1(t)\rangle$. **(c)** Time series for $I_2(t)$ and $\langle I_2(t)\rangle$.

Choose the parameters in system (3) and system (4) as follows:

$$A = 1, \quad d = 0.1, \quad \beta_1 = 1.2,$$
$$\beta_2 = 1.5, \quad a_1 = 1, \quad a_2 = 1.5,$$
$$b_1 = 2, \quad b_2 = 1, \quad \alpha_1 = 0.2,$$
$$\alpha_2 = 0.4, \quad r_1 = r_2 = 0.9.$$

A simple computation shows that $R_1 = 0.9091$, $R_2 = 0.6696$, then system (3) has a stable infection-free equilibrium $E_0(10, 0, 0)$, which implies that the two diseases of system (3) will die out ultimately (see Figure 1(a)). If we change $r_1 = 0.9$ to $r_1 = 0.3$, in this case, by simple calculation it can be found that $R_1 = 1.8182$, $R_2 = 0.6696$, the infection-free equilibrium $E_1(4.6, 1.8, 0)$ of system (3) is stable, which implies that the disease I_2 of system

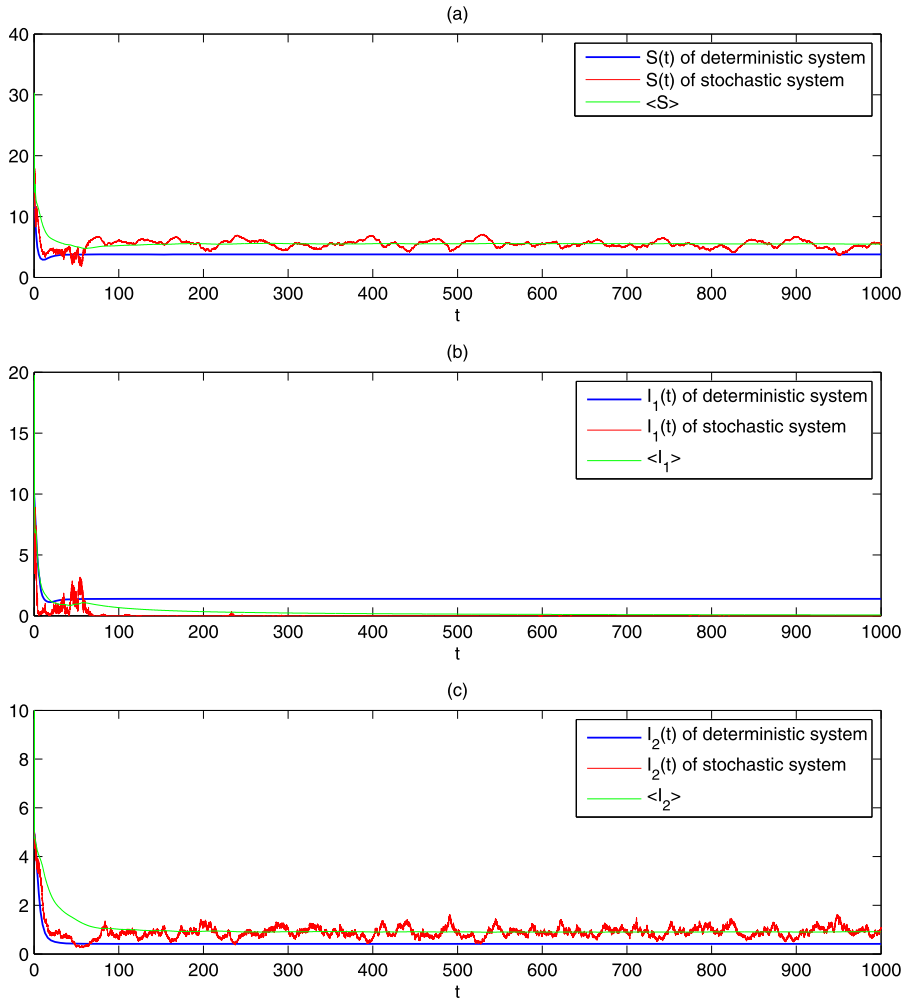

Figure 5 Dynamic behaviour comparisons of $S(t)$, $I_1(t)$, $I_2(t)$ in the deterministic and the stochastic SIS system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $r_1 = 0.3$, $r_2 = 0.3$, $\sigma_1 = 1.1$, $\sigma_2 = 0.2$. (a) Time series for $S(t)$ and $\langle S(t)\rangle$. **(b)** Time series for $I_1(t)$ and $\langle I_1(t)\rangle$. **(c)** Time series for $I_2(t)$ and $\langle I_2(t)\rangle$.

(3) will die out ultimately and the disease I_1 of system (3) will be persistent ultimately (see Figure 1(b)). If we update $r_2 = 0.9$ to $r_2 = 0.3$, in this case, $R_1 = 0.9091$, $R_2 = 1.1719$, the infection-free equilibrium $E_1(5.2174, 0, 0.9565)$ of system (3) is stable, which implies the disease I_1 of system (3) will die out ultimately and the disease I_2 of system (3) will be persistent ultimately (see Figure 1(c)). If we change $r_1 = 0.9$, $r_2 = 0.9$ to $r_1 = 0.3$, $r_2 = 0.3$, respectively, in this case, $R_1 = 1.8182$, $R_2 = 1.1719$, then (3) has a stable infection equilibrium $E^*(3.7714, 1.3857, 0.4143)$, which implies that two diseases of model (3) will be persistent ultimately (see Figure 1(d)).

Next, we consider the effect of stochastic white noise based on the persistent system. Let us choose both σ_1 and σ_2 as 1.2, in this case, σ_1 and σ_2 satisfy $\sigma_i > \frac{\beta_i}{\sqrt{2(d+\alpha_i+r_i)}}$, $i = 1, 2$. By Theorem 4.1, two diseases of system (4) will die out ultimately under a large white noise disturbance (see Figure 2(b) and Figure 2(c)). If we reduce both intensities of noise σ_1, σ_2 to 1.1, in this case, by simple calculation it can be found that $R_1^* = 0.9848$, $R_2^* = 0.8765$,

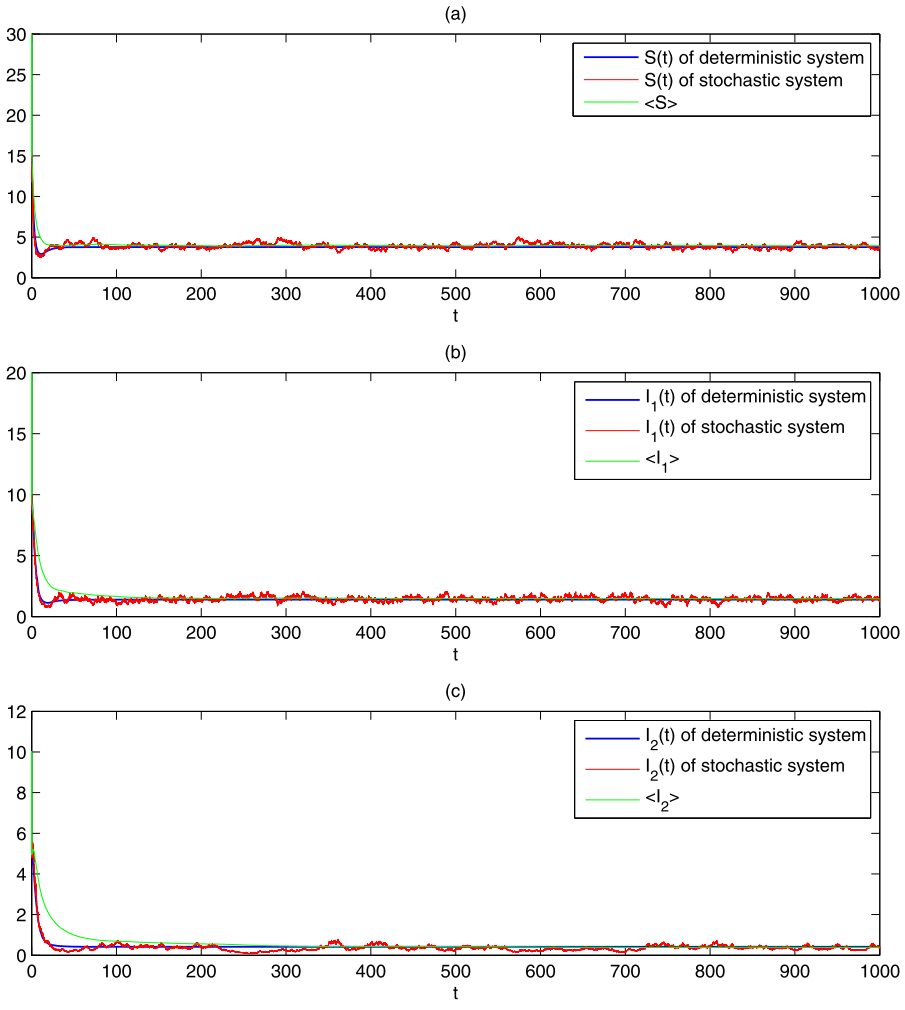

Figure 6 Dynamic behaviour comparisons of $S(t)$, $I_1(t)$, $I_2(t)$ in the deterministic and the stochastic SIS system with parameters $A = 1$, $d = 0.1$, $\beta_1 = 1.2$, $\beta_2 = 1.5$, $a_1 = 1$, $a_2 = 1.5$, $b_1 = 2$, $b_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $r_1 = 0.3$, $r_2 = 0.3$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$. (a) Time series for $S(t)$ and $\langle S(t)\rangle$. **(b)** Time series for $I_1(t)$ and $\langle I_1(t)\rangle$. **(c)** Time series for $I_2(t)$ and $\langle I_2(t)\rangle$.

σ_1 and σ_2 satisfy $\sigma_i \leq \sqrt{\frac{\beta_i(Aa_i+d)}{A}}$, where $i = 1, 2$. Then from Theorem 4.2, two diseases of system (4) will die out ultimately (see Figure 3(b) and Figure 3(c)).

If we reduce the intensity of noise σ_1 to 0.2 and keep the other system parameters the same as that in Figure 3, by computation, we have $R_1^* = 1.7906$, $R_2^* = 0.8765$, and σ_2 satisfy $\sigma_2 \leq \sqrt{\frac{\beta_2(Aa_2+d)}{A}}$. Then from Theorem 4.3, the disease $I_2(t)$ of system (4) will die out ultimately (see Figure 4(c)) and the disease $I_1(t)$ of system (3) will be persistent ultimately (see Figure 4(b)). On the contrary, if we keep the intensity of noise of $\sigma_1 = 1.1$ and reduce the intensity of noise σ_2 to 0.2, we can conclude that the disease $I_1(t)$ of system (4) will die out ultimately (see Figure 5(b)) and the disease $I_2(t)$ of system (3) will be persistent ultimately (see Figure 5(c)). Finally, if we respectively reduce the intensities of noise σ_1 and σ_2 at the same time, from 1.1 to 0.2, by computation, we have $R_1^* = 1.7906$, $R_2^* = 1.1612$, then from Theorem 4.3, the two diseases of system (4) will be persistent ultimately (see Figure 6(b) and Figure 6(c)).

Acknowledgements
This work is supported by Shandong Provincial Natural Science Foundation (No. ZR2015AQ001), National Natural Science Foundation of China (No. 11371230), Project for Higher Educational Science and Technology Program of Shandong Province (No. J13LI05), Research Funds for Joint Innovative Center for Safe and Effective Mining Technology and Equipment of Coal Resources by Shandong Province and SDUST Research Fund (2014TDJH102).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors worked together to produce the results and read and approved the final manuscript.

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Publisher's Note

Received: 16 April 2017 Accepted: 19 July 2017 Published online: 04 August 2017

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