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Research Article

Dynamical Analysis of a Viral Infection Model with Delays in Computer Networks

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This paper is devoted to the study of an SIRS computer virus propagation model with two delays and multistate antivirus measures. We demonstrate that the system loses its stability and a Hopf bifurcation occurs when the delay passes through the corresponding critical value by choosing the possible combination of the two delays as the bifurcation parameter. Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by means of the center manifold theorem and the normal form theory. Finally, some numerical simulations are performed to illustrate the obtained results.

1. Introduction

With the rapid development of computer technologies and network applications, the threat of computer viruses to the world would become increasingly serious. It is of vital importance to understand how computer viruses spread over computer network and to control the computer viruses' propagation in computer networks. To this end, many mathematical models have been studied to illustrate the dynamical behavior of computer viruses spreading since Murray [1] suggested that computer viruses share some traits of biological viruses. In [2, 3], Kephart and White used the SIS model to describe the propagation of computer viruses. In [4], Zou et al. investigated how the spread of red worms is affected by the worm characteristics based on the SIR model. In [5, 6], Yuan et al. proposed the SEIR computer virus model and studied the dynamics of the model, respectively. In [7], Mishra and Pandey formulated an SEIRS model for the transmission of worms in computer network through vertical transmission. In addition, there are also some researchers who proposed the computer virus models with vaccination and quarantine strategy [8–10].

In fact, many computer viruses have different kinds of delays when the viruses spread, such as latent period delay [11, 12], immunity period delay [12–15], and the delay due to the period that the anti-virus software needs to clean the viruses [6]. In [12], Feng et al. proposed the following computer virus propagation model with dual delays and multistate antivirus measures based on the classical SIR epidemic model in [16]:

$$\begin{aligned}\frac{dS(t)}{dt} &= pA - \beta S(t-\tau_1) I(t-\tau_1) - (\mu+\gamma) S(t) \\ &\quad + \delta R(t-\tau_2), \\ \frac{dI(t)}{dt} &= \beta S(t-\tau_1) I(t-\tau_1) - (\mu+\alpha) I(t), \\ \frac{dR(t)}{dt} &= (1-p) A + \gamma S(t) + \alpha I(t) - \delta R(t-\tau_2) - \mu R(t),\end{aligned} \tag{1}$$

where $S(t)$, $I(t)$, and $R(t)$ represent the numbers of susceptible, infected, and recovered hosts in computer networks at time t, respectively. A is the number of the hosts which are attached to the computer networks and p is the proportion of the new hosts which are susceptible. μ is the death rate of the hosts. α, β, γ, and δ are the state transition rates between the classes S, I, and R. $\tau_1 \geq 0$ is the latent period of the computer viruses and $\tau_2 \geq 0$ is the temporary immune period of the recovered hosts. For the convenience of analysis, Feng et al. [12] let $\tau_1 = \tau_2$; then, system (1) becomes the following form:

$$\begin{aligned}
\frac{dS(t)}{dt} &= pA - \beta S(t-\tau) I(t-\tau) - (\mu+\gamma) S(t) \\
&\quad + \delta R(t-\tau), \\
\frac{dI(t)}{dt} &= \beta S(t-\tau) I(t-\tau) - (\mu+\alpha) I(t), \\
\frac{dR(t)}{dt} &= (1-p) A + \gamma S(t) + \alpha I(t) - \delta R(t-\tau) - \mu R(t).
\end{aligned} \tag{2}$$

By regarding the time delay τ as the bifurcation parameter, Feng et al. [12] studied the existence and properties of Hopf bifurcation of system (2). As is known, it needs some time to clean the viruses in the infected hosts for the antivirus software. Therefore, it is reasonable to take into account the time delay due to the period that the antivirus software uses to clean the viruses in the infected hosts in system (2). To this end, we consider the following system with two delays:

$$\begin{aligned}
\frac{dS(t)}{dt} &= pA - \beta S(t-\tau_1) I(t-\tau_1) - (\mu+\gamma) S(t) \\
&\quad + \delta R(t-\tau_1), \\
\frac{dI(t)}{dt} &= \beta S(t-\tau_1) I(t-\tau_1) - \mu I(t) - \alpha I(t-\tau_2), \\
\frac{dR(t)}{dt} &= (1-p) A + \gamma S(t) + \alpha I(t-\tau_2) - \delta R(t-\tau_1) \\
&\quad - \mu R(t),
\end{aligned} \tag{3}$$

where $\tau_1 \geq 0$ is the time delay due to the latent period of the computer viruses and the temporary immune period of the recovered hosts. $\tau_2 \geq 0$ is the time delay due to the period that the antivirus software uses to clean the viruses in the infected hosts.

The remaining materials of this paper are organized in this fashion: local stability and existence of local Hopf bifurcation are discussed in Section 2. Properties of the Hopf bifurcation such as the direction and stability are investigated in Section 3. Some numerical simulations are carried out to verify the theoretical results in Section 4 and, finally, this work is summarized in Section 5.

2. Local Stability and Existence of Local Hopf Bifurcation

By direct computation, it can be concluded that if $R_0 = ((\mu+\delta)pA\beta + (1-p)\beta\delta A + (\mu+\alpha)\delta\gamma)/(\mu+\alpha)(\mu+\gamma)(\mu+\delta) > 1$, then system (3) has a unique positive equilibrium $E_*(S_*, I_*, R_*)$, where

$$\begin{aligned}
S_* &= \frac{\mu+\alpha}{\beta}, \\
I_* &= ((\mu+\delta) pA\beta + (1-p)\beta\delta A + (\mu+\alpha)\delta\gamma \\
&\quad - (\mu+\alpha)(\mu+\gamma)(\mu+\delta)) \\
&\quad \times (\beta(\mu+\alpha)(\mu+\delta) - \alpha\beta\delta)^{-1}, \\
R_* &= \frac{(1-p) A + \alpha I_* + \gamma S_*}{\mu+\delta}.
\end{aligned} \tag{4}$$

The characteristic equation of system (3) at E_* is

$$\begin{vmatrix}
\lambda - a_{11} - b_{11}e^{-\lambda\tau_1} & -b_{12}e^{-\lambda\tau_1} & -b_{13}e^{-\lambda\tau_1} \\
-b_{21}e^{-\lambda\tau_1} & \lambda - a_{22} - b_{22}e^{-\lambda\tau_1} - c_{22}e^{-\lambda\tau_2} & 0 \\
-a_{31} & -c_{32}e^{-\lambda\tau_2} & \lambda - a_{33} - b_{33}e^{-\lambda\tau_1}
\end{vmatrix} = 0, \tag{5}$$

from which one can obtain

$$\begin{aligned}
&\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0) e^{-\lambda\tau_1} \\
&\quad + (C_2\lambda^2 + C_1\lambda + C_0) e^{-\lambda\tau_2} + (D_1\lambda + D_0) e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + (E_1\lambda + E_0) e^{-2\lambda\tau_1} + F_0 e^{-\lambda(2\tau_1+\tau_2)} = 0,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
A_0 &= -a_{11}a_{22}a_{33}, \qquad A_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}, \\
A_2 &= -(a_{11} + a_{22} + a_{33}), \\
B_0 &= a_{22}a_{31}b_{13} - a_{11}a_{22}b_{33} - a_{11}a_{33}b_{22} - a_{22}a_{33}b_{11}, \\
B_1 &= b_{11}(a_{22} + a_{33}) + b_{22}(a_{11} + a_{33}) \\
&\quad + b_{33}(a_{11} + a_{22}) - a_{31}b_{13}, \\
B_2 &= -(b_{11} + b_{22} + b_{33}), \qquad C_0 = -a_{11}a_{33}c_{22}, \\
C_1 &= c_{22}(a_{11} + a_{33}), \\
C_2 &= -c_{22}, \qquad D_0 = c_{22}(a_{31}b_{13} - a_{11}b_{33} - a_{33}b_{11}), \\
D_1 &= c_{22}(b_{11} + b_{33}), \\
E_0 &= a_{31}b_{13}b_{22} - b_{33}(a_{11}b_{22} + a_{22}b_{11}), \\
E_1 &= b_{33}(b_{11} + b_{22}), \qquad F_0 = -(b_{11}b_{33}c_{22} + b_{13}b_{21}c_{32}),
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
&a_{11} = -(\mu+\gamma), \qquad a_{22} = -\mu, \qquad a_{31} = \gamma, \qquad a_{33} - \mu, \\
&b_{11} = -\beta I_*, \qquad b_{12} = -\beta S_*, \qquad b_{13} = \delta, \qquad b_{21} = \beta I_*, \\
&b_{22} = \beta S_*, \qquad b_{33} = -\delta, \qquad c_{22} = -\alpha, \qquad c_{32} = \alpha.
\end{aligned} \tag{8}$$

From the expressions of b_{11}, b_{13}, b_{21}, b_{33}, c_{22}, and c_{32}, one can obtain $F_0 = 0$. Therefore, (6) can be transformed into the following form:

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + \left(B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau_1} + \left(C_2\lambda^2 + C_1\lambda + C_0\right)e^{-\lambda\tau_2} + \left(D_1\lambda + D_0\right)e^{-\lambda(\tau_1+\tau_2)} + \left(E_1\lambda + E_0\right)e^{-2\lambda\tau_1} = 0. \tag{9}$$

Case 1 ($\tau_1 = \tau_2 = 0$). When $\tau_1 = \tau_2 = 0$, (9) is equivalent to

$$\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{10}$$

where

$$A_{10} = A_0 + B_0 + C_0 + D_0 + E_0,\quad A_{11} = A_1 + B_1 + C_1 + D_1 + E_1,\quad A_{12} = A_2 + B_2 + C_2. \tag{11}$$

It is easy to get that $A_{12} = 2\mu + \gamma + \delta + \beta I_* > 0$. Therefore, according to the Routh-Hurwitz criterion, we can conclude that if $A_{12}A_{11} > A_{10} > 0$, then the positive equilibrium E_* of system (3) is locally asymptotically stable when $\tau_1 = \tau_2 = 0$.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). When $\tau_1 > 0$ and $\tau_2 = 0$, (9) becomes the following:

$$\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} + \left(B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)e^{-\lambda\tau_1} + \left(E_{21}\lambda + E_{20}\right)e^{-2\lambda\tau_1} = 0, \tag{12}$$

where

$$A_{20} = A_0 + C_0,\quad A_{21} = A_1 + C_1,\quad A_{22} = A_2 + C_2,\quad B_{20} = B_0 + D_0,\quad B_{21} = B_1 + D_1,\quad B_{22} = B_2,\quad E_{20} = E_0,\quad E_{21} = E_1. \tag{13}$$

Multiplying $e^{\lambda\tau_1}$ on both sides of (12), it is easy to get

$$B_{22}\lambda^2 + B_{21}\lambda + B_{20} + \left(\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20}\right)e^{\lambda\tau_1} + \left(E_{21}\lambda + E_{20}\right)e^{-\lambda\tau_1} = 0. \tag{14}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be the root of (14). Then,

$$\begin{aligned}&\left(A_{20} + E_{20} + A_{22}\omega_1^2\right)\cos\tau_1\omega_1 - \left(A_{21}\omega_1 - E_{21}\omega_1 - \omega_1^3\right)\sin\tau_1\omega_1 = B_{22}\omega_1^2 - B_{20},\\ &\left(A_{20} - E_{20} + A_{22}\omega_1^2\right)\sin\tau_1\omega_1 + \left(A_{21}\omega_1 + E_{21}\omega_1 - \omega_1^3\right)\cos\tau_1\omega_1 = -B_{21}\omega_1.\end{aligned} \tag{15}$$

Then, one can obtain

$$\cos\tau_1\omega_1 = \frac{p_{24}\omega_1^4 + p_{22}\omega_1^2 + p_{20}}{\omega_1^6 + q_{24}\omega_1^4 + q_{22}\omega_1^2 + q_{20}},\quad \sin\tau_1\omega_1 = \frac{p_{25}\omega_1^5 + p_{23}\omega_1^3 + p_{21}\omega_1}{\omega_1^6 + q_{24}\omega_1^4 + q_{22}\omega_1^2 + q_{20}}, \tag{16}$$

where

$$\begin{aligned}p_{20} &= B_{20}\left(E_{20} - A_{20}\right),\\ p_{21} &= B_{20}\left(A_{21} + E_{21}\right) - B_{21}\left(A_{20} + E_{20}\right),\\ P_{22} &= B_{22}\left(A_{20} - E_{20}\right) - B_{21}\left(A_{21} - E_{21}\right) + A_{22}B_{00},\\ p_{23} &= A_{22}B_{21} - B_{20} - B_{22}\left(A_{21} + E_{21}\right),\\ p_{24} &= B_{21} - A_{22}B_{22},\quad p_{25} = B_{22},\quad q_{20} = A_{20}^2 - E_{20}^2,\\ q_{22} &= A_{21}^2 - E_{21}^2 - 2A_{20}A_{22},\quad q_{24} = A_{22}^2 - 2A_{21}.\end{aligned} \tag{17}$$

Since $\cos^2\tau_1\omega_1 + \sin^2\tau_1\omega_1 = 1$, we have

$$\omega_1^{12} + e_{25}\omega_1^{10} + e_{24}\omega_1^8 + e_{23}\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0, \tag{18}$$

where

$$\begin{aligned}e_{20} &= q_{20}^2 - p_{20}^2,\quad e_{21} = 2q_{20}q_{22} - 2p_{20}p_{22} - p_{21}^2,\\ e_{22} &= q_{22}^2 - p_{22}^2 + 2q_{20}q_{24} - 2p_{20}p_{24} - 2p_{21}p_{23},\\ e_{23} &= 2q_{20} + 2q_{22}q_{24} - 2p_{22}p_{24} - 2p_{21}p_{25} - p_{23}^2,\\ e_{24} &= q_{24}^2 - p_{24}^2 + 2q_{22} - 2p_{23}p_{25},\quad e_{25} = 2q_{24} - p_{25}^2.\end{aligned} \tag{19}$$

Let $\omega_1^2 = v_1$; then, (18) becomes

$$v_1^6 + e_{25}v_1^5 + e_{24}v_1^4 + e_{23}v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{20}$$

In order to give the main results in the present paper, we make the following assumption.

(H_{21}) Equation (20) has at least one positive real root.

If the condition (H_{21}) holds, then there exists a positive root v_{10} of (20) which can make (14) have a pair of purely imaginary roots $\pm i\omega_{10} = \pm i\sqrt{v_{10}}$. For ω_{10}, the corresponding critical value of delay is

$$\tau_{10} = \frac{1}{\omega_{10}}\arccos\frac{p_{24}\omega_{10}^4 + p_{22}\omega_{10}^2 + p_{20}}{\omega_{10}^6 + q_{24}\omega_{10}^4 + q_{22}\omega_{10}^2 + q_{20}}. \tag{21}$$

Differentiating (14) with respect to τ_1, we get

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\left(2B_{22}\lambda + B_{21} + \left(3\lambda^2 + 2A_{22}\lambda + A_{21}\right)e^{\lambda\tau_1} + E_{21}e^{-\lambda\tau_1}\right) \times \left(\left(E_{21}\lambda^2 + E_{20}\lambda\right)e^{-\lambda\tau_1} - \left(\lambda^4 + A_{22}\lambda^3 + A_{21}\lambda^2 + A_{20}\lambda\right)e^{\lambda\tau_1}\right)^{-1} - \frac{\tau_1}{\lambda}. \quad (22)$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{P_{2R}Q_{2R} + P_{2I}Q_{2I}}{Q_{2R}^2 + Q_{2I}^2}, \quad (23)$$

where

$$\begin{aligned}
P_{2R} &= \left(A_{21} + E_{21} - 3\omega_{10}^2\right)\cos\tau_{10}\omega_{10} - 2A_{22}\omega_{10}\sin\tau_{10}\omega_{10} + B_{21},\\
P_{2I} &= \left(A_{21} - E_{21} - 3\omega_{10}^2\right)\sin\tau_{10}\omega_{10} + 2A_{22}\omega_{10}\cos\tau_{10}\omega_{10} + 2B_{22}\omega_{10},\\
Q_{2R} &= \left(A_{21}\omega_{10}^2 - E_{21}\omega_{10}^2 - \omega_{10}^4\right)\cos\tau_{10}\omega_{10} - \left(A_{22}\omega_{10}^3 - A_{20}\omega_{10} - E_{20}\omega_{10}\right)\sin\tau_{10}\omega_{10},\\
Q_{2I} &= \left(A_{21}\omega_{10}^2 + E_{21}\omega_{10}^2 - \omega_{10}^4\right)\sin\tau_{10}\omega_{10} + \left(A_{22}\omega_{10}^3 - A_{20}\omega_{10} + E_{20}\omega_{10}\right)\cos\tau_{10}\omega_{10}.
\end{aligned} \quad (24)$$

It is obvious that if the condition (H_{21}) $P_{2R}Q_{2R}+P_{2I}Q_{2I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau=\tau_{10}} \neq 0$. According to the Hopf bifurcation theorem in [17], the following results hold.

Theorem 1. *If the conditions (H_{21})-(H_{22}) hold, the positive equilibrium $E_*(S_*, I_*, R_*)$ of system (3) is locally asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and system (3) undergoes a Hopf bifurcation at the positive equilibrium $E_*(S_*, I_*, R_*)$ when $\tau_1 = \tau_{10}$.*

Case 3 ($\tau_1 = 0, \tau_2 > 0$). When $\tau_1 = 0$ and $\tau_2 > 0$, (9) becomes

$$\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} + \left(C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right)e^{-\lambda\tau_2} = 0, \quad (25)$$

where

$$\begin{gathered}
A_{30} = A_0 + B_0 + E_0, \qquad A_{31} = A_1 + B_1 + E_1,\\
A_{32} = A_2 + B_2,\\
C_{30} = C_0 + D_0, \qquad C_{31} = C_1 + D_1, \qquad C_{32} = C_2.
\end{gathered} \quad (26)$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be the root of (25). Then,

$$\begin{aligned}
C_{31}\omega_2\sin\tau_2\omega_2 + \left(C_{30} - C_{32}\omega_2^2\right)\cos\tau_2\omega_2 &= A_{32}\omega_2^2 - A_{30},\\
C_{31}\omega_2\cos\tau_2\omega_2 - \left(C_{30} - C_{32}\omega_2^2\right)\sin\tau_2\omega_2 &= \omega_2^3 - A_{31}\omega_2,
\end{aligned} \quad (27)$$

which follows that

$$\omega_2^6 + e_{32}\omega_2^4 + e_{31}\omega_2^2 + e_{30} = 0, \quad (28)$$

with

$$\begin{aligned}
e_{30} &= A_{30}^2 - C_{30}^2,\\
e_{31} &= A_{31}^2 - C_{31}^2 - 2A_{30}A_{32} + 2C_{30}C_{32},\\
e_{32} &= A_{32}^2 - C_{32}^2 - 2A_{31}.
\end{aligned} \quad (29)$$

Let $\omega_2^2 = v_2$; then, (28) becomes

$$v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30} = 0. \quad (30)$$

Let

$$f_2(v_2) = v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30}. \quad (31)$$

Discussion about the roots of (30) is similar to that in [18].

Lemma 2. *(i) If $e_{30} < 0$, then (30) has at least one positive root.*

(ii) If $e_{30} \geq 0$ and $e_{32}^2 - 3e_{31} \leq 0$, then (30) has no positive root.

(iii) If $e_{30} \geq 0$ and $e_{32}^2 - 3e_{31} > 0$, then (30) has positive root if and only if $v_2^* = (-e_{32} + \sqrt{e_{32}^2 - 3e_{31}})/3 > 0$ and $f_2(v_2^*) < 0$.

In what follows, we suppose that the coefficients in (30) satisfy the following condition:

(H_{31}): (a) $e_{30} < 0$ or (b) $e_{30} \geq 0$, $e_{32}^2 - 3e_{31} > 0$, $v_2^* = (-e_{32} + \sqrt{e_{32}^2 - 3e_{31}})/3 > 0$, and $f_2(v_2^*) < 0$.

If the condition (H_{31}) holds, we know that there exists a positive root v_{20} of (30) such that (25) has a pair of purely imaginary roots $\pm i\omega_{20} = \pm i\sqrt{v_{20}}$. For ω_{20}, the corresponding critical value of time delay is

$$\begin{aligned}
\tau_{20} = \frac{1}{\omega_{20}}\arccos\Big(&\Big(\left(C_{31} - A_{32}C_{32}\right)\omega_{20}^4 + \left(A_{30}C_{32} - A_{31}C_{31} + A_{32}C_{30}\right)\omega_{20}^2 - A_{30}C_{30}\Big)\\
&\times \left(C_{32}\omega_{20}^4 + (C_{31}^2 - 2C_{30}C_{32})\omega_{20}^2 + C_{30}^2\right)^{-1}\Big).
\end{aligned} \quad (32)$$

Differentiating two sides of (25) with respect to τ_2, we have

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = -\frac{3\lambda^2 + 2A_{32}\lambda + A_{31}}{\lambda\left(\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30}\right)} + \frac{2C_{32}\lambda + C_{31}}{\lambda\left(C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right)} - \frac{\tau_2}{\lambda}. \quad (33)$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{20}} = \frac{f_2'(v_{2*})}{\left(B_{21}\omega_{10} - B_{23}\omega_{10}^3\right)^2 + \left(B_{24}\omega_{20}^4 - B_{22}\omega_{20}^2 + B_{20}\right)^2}, \quad (34)$$

where $f_2(v_2) = v_2^3 + e_{32}v_2^2 + e_{31}v_2 + e_{30}$ and $v_{2*} = \omega_{20}^2$.

Obviously, if the condition (H_{32}) $f_2'(v_{2*}) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}} \neq 0$. According to the Hopf bifurcation theorem in [17], the following results hold.

Theorem 3. *If the conditions (H_{31})-(H_{32}) hold, the positive equilibrium $E_*(S_*, I_*, R_*)$ of system (3) is locally asymptotically stable for $\tau_2 \in [0, \tau_{20})$ and system (3) undergoes a Hopf bifurcation at $E_*(S_*, I_*, R_*)$ when $\tau_2 = \tau_{20}$.*

Case 4 ($\tau_1 > 0$, $\tau_2 > 0$, $\tau_2 \in (0, \tau_{20})$). We consider (9) with τ_2 in its stable interval and choose τ_1 as a bifurcation parameter. Multiplying by $e^{\lambda\tau_1}$, (9) becomes

$$\begin{aligned} &B_2\lambda^2 + B_1\lambda + B_0 + \left(\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right)e^{\lambda\tau_1} \\ &\quad + \left(E_1\lambda + E_0\right)e^{-\lambda\tau_1} + \left(D_1\lambda + D_0\right)e^{-\lambda\tau_2} \\ &\quad + \left(C_2\lambda^2 + C_1\lambda + C_0\right)e^{\lambda(\tau_1-\tau_2)} = 0. \end{aligned} \quad (35)$$

Let $\lambda = i\omega_{1*}$ $(\omega_{1*} > 0)$ be the root of (35). Then,

$$\begin{aligned} M_{41}\cos\tau_1\omega_{1*} - M_{42}\sin\tau_1\omega_{1*} &= M_{43}, \\ M_{44}\sin\tau_1\omega_{1*} + M_{45}\cos\tau_1\omega_{1*} &= M_{46}, \end{aligned} \quad (36)$$

where

$$\begin{aligned} M_{41} &= A_0 + E_0 - A_2\omega_{1*}^2 + \left(C_0 - C_2\omega_{1*}^2\right)\cos\tau_2\omega_{1*} \\ &\quad + C_1\omega_{1*}\sin\tau_2\omega_{1*}, \\ M_{42} &= A_1\omega_{1*} - E_1\omega_{1*} - \omega_{1*}^3 - \left(C_0 - C_2\omega_{1*}^2\right)\sin\tau_2\omega_{1*} \\ &\quad + C_1\omega_{1*}\cos\tau_2\omega_{1*}, \\ M_{43} &= B_2\omega_{1*}^2 - B_0 - D_1\omega_{1*}\sin\tau_2\omega_{1*} - D_0\cos\tau_2\omega_{1*}, \\ M_{44} &= A_0 - E_0 - A_2\omega_{1*}^2 + \left(C_0 - C_2\omega_{1*}^2\right)\cos\tau_2\omega_{1*} \\ &\quad + C_1\omega_{1*}\sin\tau_2\omega_{1*}, \\ M_{45} &= A_1\omega_{1*} + E_1\omega_{1*} - \omega_{1*}^3 - \left(C_0 - C_2\omega_{1*}^2\right)\sin\tau_2\omega_{1*} \\ &\quad + C_1\omega_{1*}\cos\tau_2\omega_{1*}, \\ M_{46} &= -B_1\omega_{1*} + D_0\sin\tau_2\omega_{1*} - D_1\omega_{1*}\cos\tau_2\omega_{1*}. \end{aligned} \quad (37)$$

Then, we can obtain

$$\begin{aligned} &\cos\tau_1\omega_{1*} \\ &\quad = \frac{h_{40}(\omega_{1*}) + h_{41}(\omega_{1*})\cos\tau_2\omega_{1*} + h_{42}(\omega_{1*})\sin\tau_2\omega_{1*}}{g_{40}(\omega_{1*}) + g_{41}(\omega_{1*})\cos\tau_2\omega_{1*} + g_{42}(\omega_{1*})\sin\tau_2\omega_{1*}}, \\ &\sin\tau_1\omega_{1*} \\ &\quad = \frac{h_{40}'(\omega_{1*}) + h_{41}'(\omega_{1*})\cos\tau_2\omega_{1*} + h_{42}'(\omega_{1*})\sin\tau_2\omega_{1*}}{g_{40}(\omega_{1*}) + g_{41}(\omega_{1*})\cos\tau_2\omega_{1*} + g_{42}(\omega_{1*})\sin\tau_2\omega_{1*}}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} g_{40}(\omega_{1*}) &= \omega_{1*}^6 + \left(A_2^2 + C_2^2 - 2A_1\right)\omega_{1*}^4 \\ &\quad + \left(A_1^2 + C_1^2 - E_1^2 - 2A_0A_2 - 2C_0C_2\right)\omega_{1*}^2 \\ &\quad + A_0^2 + C_0^2 - C_0^2, \\ g_{41}(\omega_{1*}) &= 2\left(A_2C_2 - C_1\right)\omega_{1*}^4 \\ &\quad + 2\left(A_1C_1 - A_0C_2 - A_2C_0\right)\omega_{1*}^2 + 2A_0C_0, \\ g_{42}(\omega_{1*}) &= -2C_2\omega_{1*}^5 + 2\left(A_1C_2 - A_2C_1 - C_0\right)\omega_{1*}^3 \\ &\quad + 2\left(A_0C_1 - A_1C_0\right)\omega_{1*}, \\ h_{40}(\omega_{1*}) &= \left(B_1 + A_2B_2\right)\omega_{1*}^4 \\ &\quad + \left[A_2B_0 - C_1D_1 + C_2D_0 - B_1\left(A_1 - E_1\right)\right. \\ &\qquad \left. + B_2\left(A_0 - E_0\right)\right]\omega_{1*}^2 \\ &\quad + B_0\left(E_0 - A_0\right) - C_0D_0, \\ h_{41}(\omega_{1*}) &= \left(D_1 - B_2C_2\right)\omega_{1*}^4 \\ &\quad + \left[A_2D_0 + B_0C_2 - B_1C_1\right. \\ &\qquad \left. + B_2C_0 + D_1\left(E_1 - A_1\right)\right]\omega_{1*}^2 \\ &\quad + D_0\left(E_0 - A_0\right) - B_0C_0, \\ h_{42}(\omega_{1*}) &= \left(A_2D_1 - B_1C_2 + B_2C_1 - D_0\right)\omega_{1*}^3 \\ &\quad + \left[D_0\left(A_1 - E_1\right) - D_1\left(A_0 - E_0\right)\right. \\ &\qquad \left. + B_1C_0 - B_0C_1\right]\omega_{1*}, \\ h_{40}'(\omega_{1*}) &= B_2\omega_{1*}^5 \\ &\quad + \left[A_2B_1 + C_2D_1 - B_0 - B_2\left(A_1 + E_1\right)\right]\omega_{1*}^3 \\ &\quad + \left[B_0\left(A_1 + E_1\right) - B_1\left(A_0 + E_0\right)\right. \\ &\qquad \left. - C_0D_1 + C_1D_0\right]\omega_{1*}, \\ h_{41}'(\omega_{1*}) &= \left(A_2D_1 + B_1C_2 - B_2C_1 - D_0\right)\omega_{1*}^3 \\ &\quad + \left[D_0\left(A_1 + E_1\right) - D_1\left(A_0 + E_0\right)\right. \\ &\qquad \left. + B_0C_1 - B_1C_0\right]\omega_{1*}, \end{aligned}$$

$$h'_{42}(\omega_{1*}) = -(B_2C_2 + D_1)\omega_{1*}^4 + [B_0C_2 - B_1C_1 - A_2D_0 + B_2C_0 + D_1(A_1 + E_1)]\omega_{1*}^2 + D_0(A_0 + E_0) - B_0C_0. \tag{39}$$

Then, we can get a function with respect to ω_{1*}:

$$\cos^2\tau_1\omega_{1*} + \sin^2\tau_1\omega_{1*} = 1. \tag{40}$$

Next, we suppose that (H_{41}): (40) has at least one positive real root.

If the condition (H_{41}) holds, then there exists a ω_{10}^* such that (35) has a pair of purely imaginary roots $\pm i\omega_{10}^*$. For ω_{10}^*, the corresponding critical value of time delay is

$$\tau_{10}^* = \frac{1}{\omega_{10}^*}\arccos\left((h_{40}(\omega_{10}^*) + h_{41}(\omega_{10}^*)\cos\tau_2\omega_{10}^* + h_{42}(\omega_{10}^*)\sin\tau_2\omega_{10}^*) \times (g_{40}(\omega_{10}^*) + g_{41}(\omega_{10}^*)\cos\tau_2\omega_{10}^* + g_{42}(\omega_{10}^*)\sin\tau_2\omega_{10}^*)^{-1}\right). \tag{41}$$

Taking the derivative with respect to τ_1 in (35), we get

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{g_{41}(\lambda)}{g_{42}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{42}$$

where

$$\begin{aligned} g_{41}(\lambda) &= 2B_2\lambda + B_1 + E_1e^{-\lambda\tau_1} + (3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau_1} \\ &\quad - (\tau_2D_1\lambda - D_1 + \tau_2D_0)e^{-\lambda\tau_2} \\ &\quad - [\tau_2C_2\lambda^2 - (2C_2 - \tau_2C_1)\lambda + C_1 - \tau_2C_0]e^{\lambda(\tau_1-\tau_2)}, \\ g_{42}(\lambda) &= (E_1\lambda^2 + E_0\lambda)e^{-\lambda\tau_1} \\ &\quad - (\lambda^4 + A_2\lambda^3 + A_1\lambda^2 + A_0\lambda)e^{\lambda\tau_1} \\ &\quad - (C_2\lambda^3 + C_1\lambda^2 + C_0\lambda)e^{\lambda(\tau_1-\tau_2)}. \end{aligned} \tag{43}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}^*} = \frac{P_{4R}Q_{4R} + P_{4I}Q_{4I}}{Q_{4R}^2 + Q_{4I}^2}, \tag{44}$$

where

$$\begin{aligned} P_{4R} &= B_1 + (D_1 - \tau_2D_0)\cos\tau_2\omega_{10}^* - \tau_2D_1\omega_{10}^*\sin\tau_2\omega_{10}^* \\ &\quad + \Big[A_1 + E_1 - 3(\omega_{10}^*)^2 + (\tau_2C_2(\omega_{10}^*)^2 + \tau_2C_0 - C_1)\cos\tau_2\omega_{10}^* \\ &\quad + (2C_2 - \tau_2C_1)\sin\tau_2\omega_{10}^*\Big]\cos\tau_{10}^*\omega_{10}^* \\ &\quad + \Big[(\tau_2C_2(\omega_{10}^*)^2 + \tau_2C_0 - C_1)\sin\tau_2\omega_{10}^* \\ &\quad - (2C_2 - \tau_2C_1) - 2A_2\omega_{10}^*\Big]\sin\tau_{10}^*\omega_{10}^*, \\ P_{4I} &= 2B_2\omega_{10}^* - (D_1 - \tau_2D_0)\sin\tau_2\omega_{10}^* - \tau_1D_1\omega_{10}^*\cos\tau_2\omega_{10}^* \\ &\quad + \Big[A_1 - E_1 - 3(\omega_{10}^*)^2 + (\tau_2C_2(\omega_{10}^*)^2 + \tau_2C_0 - C_1)\cos\tau_2\omega_{10}^* \\ &\quad + (2C_2 - \tau_2C_1)\sin\tau_2\omega_{10}^*\Big]\sin\tau_{10}^*\omega_{10}^* \\ &\quad + \Big[2A_2\omega_{10}^* - (\tau_2C_2(\omega_{10}^*)^2 + \tau_2C_0 - C_1)\cos\tau_2\omega_{10}^* \\ &\quad + (2C_2 - \tau_2C_1)\sin\tau_2\omega_{10}^*\Big]\cos\tau_{10}^*\omega_{10}^*, \\ Q_{4R} &= \Big[A_0\omega_{10}^* + E_0\omega_{10}^* - A_2(\omega_{10}^*)^3 + C_1(\omega_{10}^*)^2\sin\tau_2\omega_{10}^* \\ &\quad - (C_2(\omega_{10}^*)^3 - C_0\omega_{10}^*)\cos\tau_2\omega_{10}^*\Big]\sin\tau_{10}^*\omega_{10}^* \\ &\quad + \Big[A_1(\omega_{10}^*)^2 - E_1(\omega_{10}^*)^2 - (\omega_{10}^*)^4 + C_1(\omega_{10}^*)^2\cos\tau_2\omega_{10}^* \\ &\quad + (C_2(\omega_{10}^*)^3 - C_0\omega_{10}^*)\sin\tau_2\omega_{10}^*\Big]\cos\tau_{10}^*\omega_{10}^*, \\ Q_{4I} &= \Big[A_2(\omega_{10}^*)^3 - A_0\omega_{10}^* + E_0\omega_{10}^* - C_1(\omega_{10}^*)^2\sin\tau_2\omega_{10}^* \\ &\quad + (C_2(\omega_{10}^*)^3 - C_0\omega_{10}^*)\cos\tau_2\omega_{10}^*\Big]\cos\tau_{10}^*\omega_{10}^* \\ &\quad + \Big[A_1(\omega_{10}^*)^2 + E_1(\omega_{10}^*)^2 - (\omega_{10}^*)^4 + C_1(\omega_{10}^*)^2\cos\tau_2\omega_{10}^* \\ &\quad + (C_2(\omega_{10}^*)^3 - C_0\omega_{10}^*)\sin\tau_2\omega_{10}^*\Big]\sin\tau_{10}^*\omega_{10}^*. \end{aligned} \tag{45}$$

Thus, if the condition (H_{42}) $P_{4R}Q_{4R} + P_{4I}Q_{4I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau_1=\tau_{10}^*} \neq 0$, which implies that the transversality condition is satisfied. According to the Hopf bifurcation theorem in [17], we can conclude the discussions above as follows.

Theorem 4. *If the conditions (H_{41})-(H_{42}) hold and $\tau_2 \in (0, \tau_{20})$, the positive equilibrium $E_*(S_*, I_*, R_*)$ of system (3) is locally asymptotically stable for $\tau_1 \in [0, \tau_{10}^*)$ and system (3) undergoes a Hopf bifurcation at $E_*(S_*, I_*, R_*)$ when $\tau_1 = \tau_{10}^*$.*

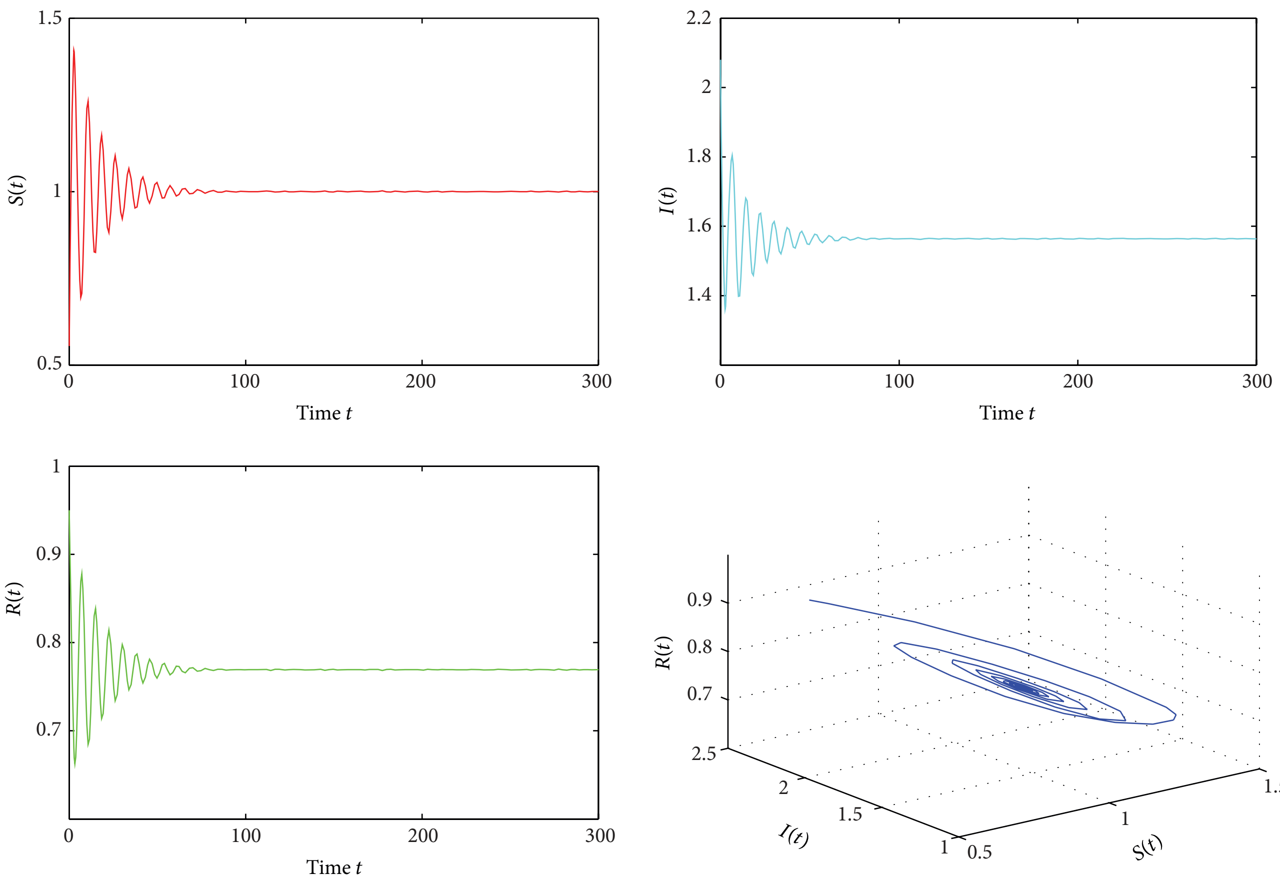

FIGURE 1: E_* is locally asymptotically stable for $\tau_1 = 2.3750 < \tau_{10} = 2.8957$.

Case 5 ($\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$). We consider (9) with τ_1 in its stable interval and τ_2 is considered as a bifurcation parameter.

Let $\lambda = i\omega_{2*}$ $(\omega_{2*} > 0)$ be the root of (9). Then,

$$M_{51} \sin \tau_2\omega_{2*} + M_{52} \cos \tau_2\omega_{2*} = M_{53},$$
$$M_{51} \cos \tau_2\omega_{2*} - M_{52} \sin \tau_2\omega_{2*} = M_{54}, \quad (46)$$

where

$$M_{51} = C_1\omega_{2*} - D_0 \sin \tau_1\omega_{2*} + D_1\omega_{2*} \cos \tau_1\omega_{2*},$$
$$M_{52} = C_0 - C_2\omega_{2*}^2 + D_0 \cos \tau_1\omega_{2*} + D_1\omega_{2*} \sin \tau_1\omega_{2*},$$
$$M_{53} = A_2\omega_{2*}^2 - A_0 - B_1\omega_{2*} \sin \tau_1\omega_{2*} - \left(B_0 - B_2\omega_{2*}^2\right) \cos \tau_1\omega_{2*} - E_1\omega_{2*} \sin 2\tau_1\omega_{2*} - E_0 \cos 2\tau_1\omega_{2*}, \quad (47)$$
$$M_{54} = \omega_{2*}^3 - A_1\omega_{2*} - B_1\omega_{2*} \cos \tau_1\omega_{2*} + \left(B_0 - B_2\omega_{2*}^2\right) \sin \tau_1\omega_{2*} - E_1\omega_{2*} \cos 2\tau_1\omega_{2*} + E_0 \sin 2\tau_1\omega_{2*}.$$

Then, we have

$$g_{50}(\omega_{2*}) + g_{51}(\omega_{2*}) \cos \tau_1\omega_{2*} + g_{52}(\omega_{2*}) \sin \tau_1\omega_{2*} + g_{53}(\omega_{2*}) \cos 2\tau_1\omega_{2*} + g_{54}(\omega_{2*}) \sin 2\tau_1\omega_{2*} = 0, \quad (48)$$

where

$$g_{50}(\omega_{2*}) = \omega_{2*}^6 + \left(A_2^2 + B_2^2 - C_2^2 - 2A_1\right)\omega_{2*}^4 + \left(A_1^2 + B_1^2 - C_1^2 - D_1^2 - E_1^2 - 2A_0A_2 - 2B_0B_2 + 2C_0C_2\right)\omega_{2*}^2 + A_0^2 - C_0^2 - D_0^2 + E_0^2,$$
$$g_{51}(\omega_{2*}) = 2\left(A_2B_2 - B_1\right)\omega_{2*}^4 + 2\left(A_1B_1 - A_0B_2 - A_2B_0 - C_1D_1 + C_2D_0\right)\omega_{2*}^2 + 2\left(A_0B_0 - C_0D_0\right),$$
$$g_{52}(\omega_{2*}) = -2B_2\omega_{2*}^5 + 2\left(A_1B_2 - A_2B_1 + C_2D_1 + B_0\right)\omega_{2*}^3 + 2\left(A_0B_1 - A_1B_0 - C_0D_1 + C_1D_0\right)\omega_{2*},$$

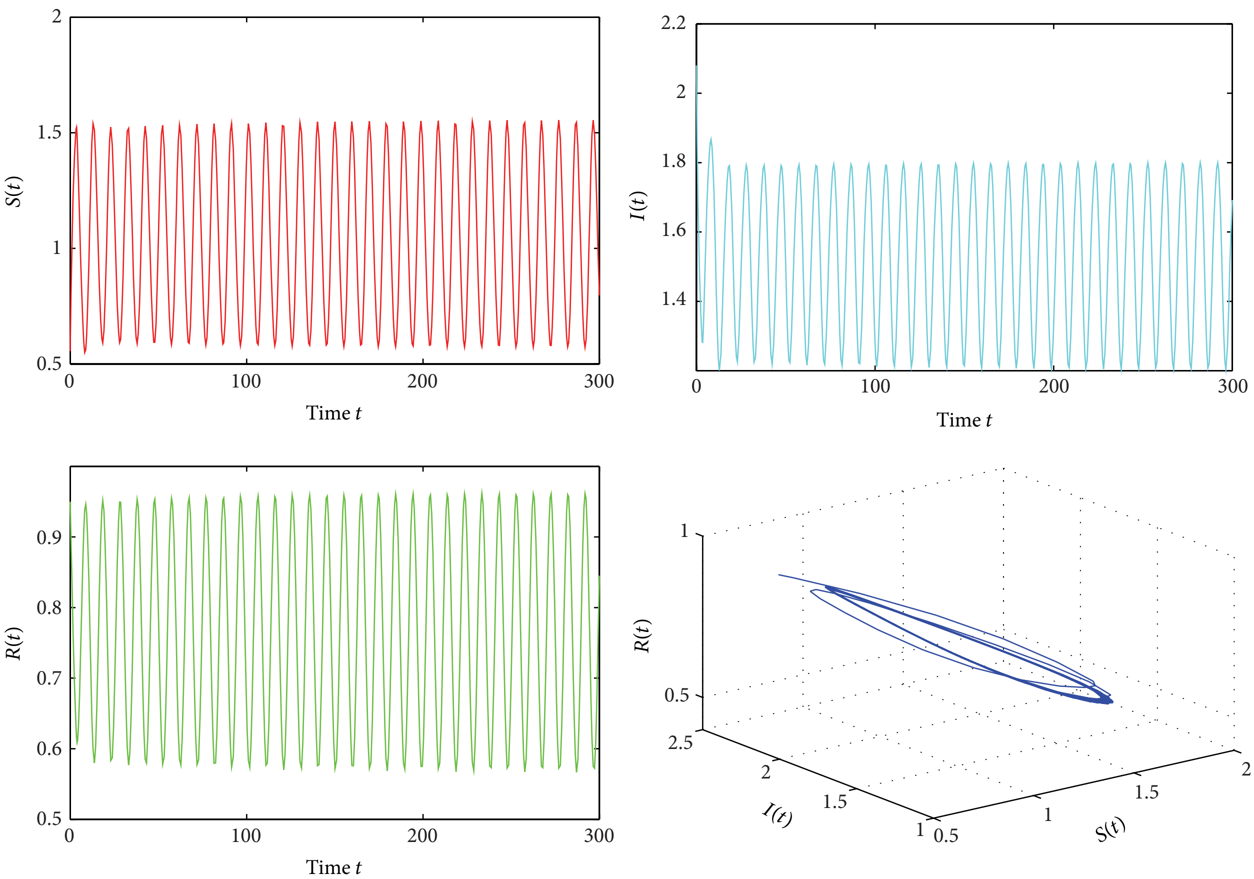

FIGURE 2: E_* is unstable for $\tau_1 = 3.2950 > \tau_{10} = 2.8957$.

$$g_{53}(\omega_{2*}) = -2E_1\omega_{2*}^4 + 2(A_1E_1 - A_2E_0)\omega_{2*}^2 + 2A_0E_0,$$
$$g_{54}(\omega_{2*}) = 2(E_0 - A_2E_1)\omega_{2*}^3 + 2(A_0E_1 - A_1E_0)\omega_{2*}. \tag{49}$$

Similar to Case 4, we suppose that (H_{51}): (48) has at least one positive real root. If the condition (H_{51}) holds, then there exists a ω_{20}^* such that (9) has a pair of purely imaginary roots $\pm i\omega_{20}^*$. For ω_{20}^*, the corresponding critical value of time delay is

$$\tau_{20}^* = \frac{1}{\omega_{20}^*}\arccos\frac{M_{51}\times M_{54} + M_{52}\times M_{53}}{M_{51}^2 + M_{52}^2}\bigg|_{\tau_2=\tau_{20}^*}. \tag{50}$$

Differentiating (9) with respect to τ_2, we have

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{g_{51}(\lambda)}{g_{52}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{51}$$

where

$$\begin{aligned} g_{51}(\lambda) = {} & 3\lambda^2 + 2A_2\lambda + A_1 \\ & + \left[(2B_2 - \tau_1B_1)\lambda - \tau_1B_2\lambda^2 + B_1 - \tau_1B_0\right]e^{-\lambda\tau_1} \\ & + (2C_2\lambda + C_1)e^{-\lambda\tau_2} \\ & + (D_1 - \tau_1D_0 - \tau_1D_1\lambda)e^{-\lambda(\tau_1+\tau_2)} \\ & + (E_1 - 2\tau_1E_0 - 2\tau_1E_1\lambda)e^{-2\lambda\tau_1}, \\ g_{52}(\lambda) = {} & \left(C_2\lambda^3 + C_1\lambda^2 + C_0\lambda\right)e^{-\lambda\tau_2} \\ & + \left(D_1\lambda^2 + D_0\lambda\right)e^{-\lambda(\tau_1+\tau_2)}. \end{aligned} \tag{52}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{20}^*} = \frac{P_{5R}Q_{5R} + P_{5I}Q_{5I}}{Q_{5R}^2 + Q_{5I}^2}. \tag{53}$$

If the condition (H_{52}) $P_{5R}Q_{5R} + P_{5I}Q_{5I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau_2=\tau_{20}^*} \neq 0$. Therefore, according to the Hopf bifurcation theorem in [17], we can conclude the discussions above as follows.

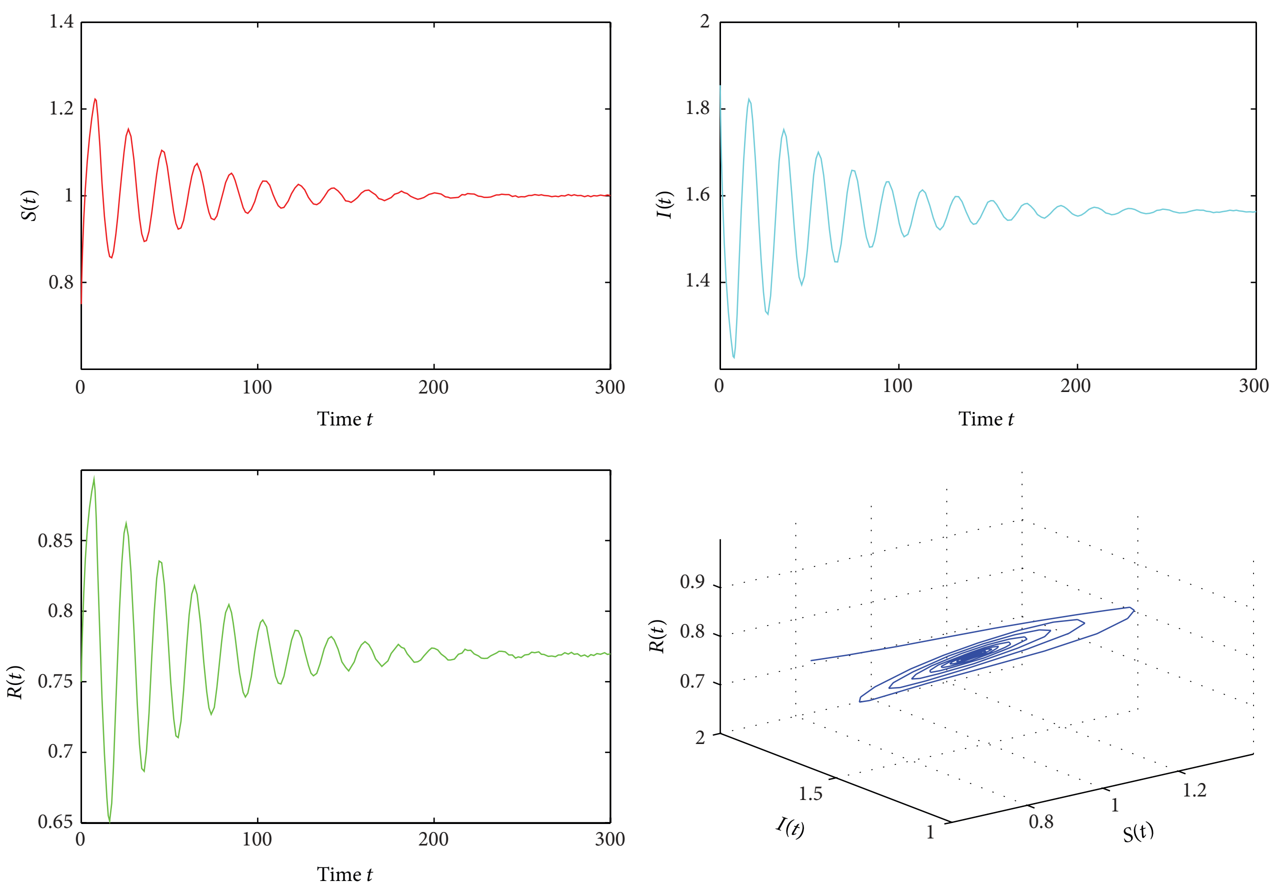

FIGURE 3: E_* is locally asymptotically stable for $\tau_2 = 7.2500 < \tau_{20} = 8.4669$.

Theorem 5. *If the conditions* (H_{51})-(H_{52}) *hold and* $\tau_1 \in (0, \tau_{10})$, *the positive equilibrium* $E_*(S_*, I_*, R_*)$ *of system (3) is locally asymptotically stable for* $\tau_2 \in [0, \tau_{20}^*)$ *and system (3) undergoes a Hopf bifurcation at* $E_*(S_*, I_*, R_*)$ *when* $\tau_2 = \tau_{20}^*$.

3. Direction and Stability of the Hopf Bifurcation

In this section, we determine the properties of the Hopf bifurcation of system (3) with respect to τ_2 for $\tau_1 \in (0, \tau_{10})$. Throughout this section, we assume that $\tau_{1*} < \tau_{20}^*$, where $\tau_1 \in (0, \tau_{10})$.

Let $\tau_2 = \tau_{20}^* + \mu$, $\mu \in R$; then, $\mu = 0$ is the Hopf bifurcation value of system (3). Rescale the time delay $t \rightarrow (t/\tau_2)$. Let $u_1(t) = S(t) - S_*$, let $u_2(t) = I(t) - I_*$, and let $u_3(t) = R(t) - R_*$; then, system (3) can be transformed into an FDE in $C = C([-1, 0], R^3)$:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \quad (54)$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$ and $L_\mu : C \rightarrow R^3$ and $F : R \times C \rightarrow R^3$ are given, respectively, by

$$L_\mu \phi = (\tau_{20}^* + \mu)\left(A'\phi(0) + B'\phi\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) + C'\phi(-1)\right),$$

$$F(\mu, \phi) = (\tau_{20}^* + \mu)\begin{pmatrix} -\beta\phi_1\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\phi_2\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) \\ \beta\phi_1\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\phi_2\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) \\ 0 \end{pmatrix}, \quad (55)$$

with

$$A' = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}, \qquad B' = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix},$$
$$C' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & c_{32} & 0 \end{pmatrix}. \quad (56)$$

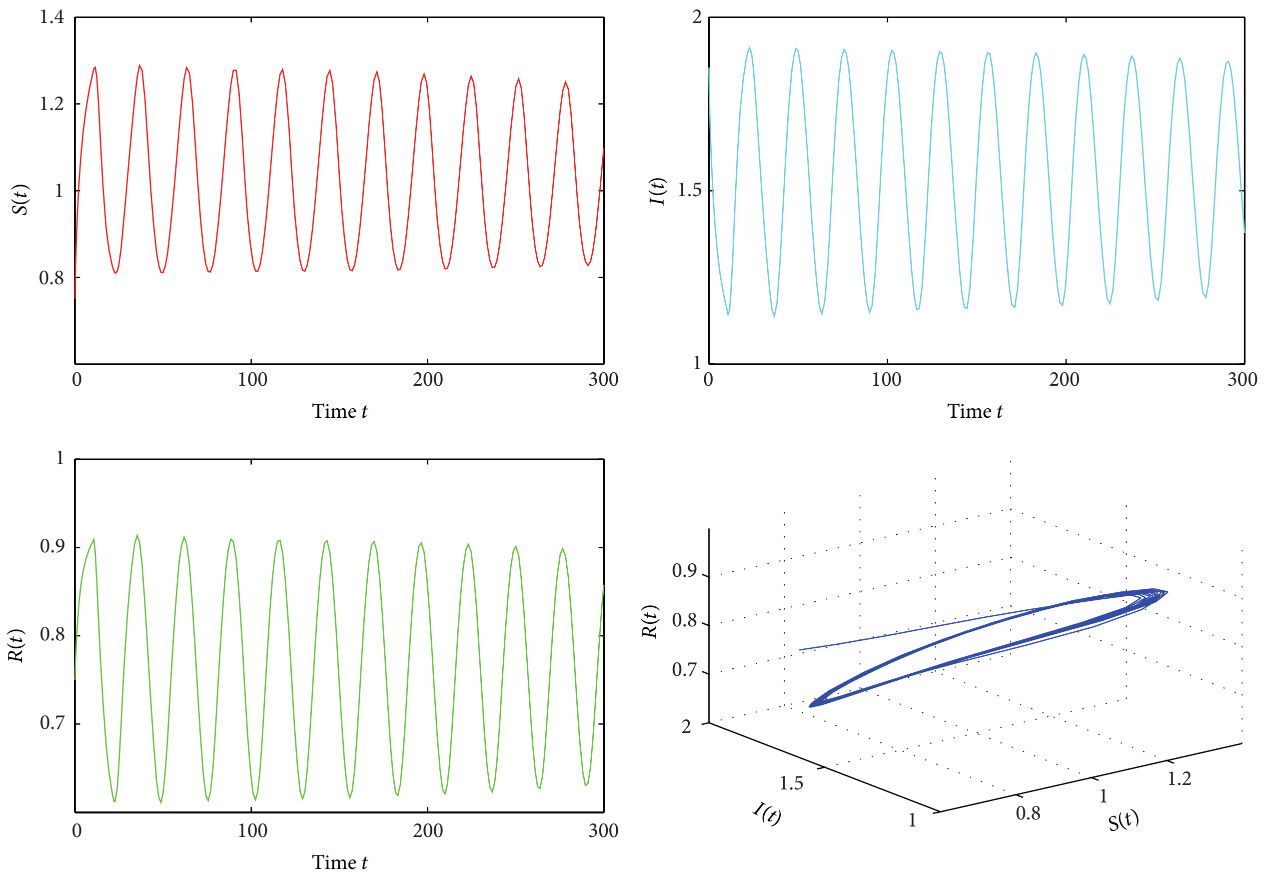

FIGURE 4: E_* is unstable for $\tau_2 = 10.7500 > \tau_{20} = 8.4669$.

By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), \quad \phi \in C\left([-1,0], R^3\right). \tag{57}$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases} (\tau_{20}^* + \mu)\left(A' + B' + C'\right), & \theta = 0, \\ (\tau_{20}^* + \mu)\left(B' + C'\right), & \theta \in \left[-\dfrac{\tau_{1*}}{\tau_{20}^*}, 0\right), \\ (\tau_{20}^* + \mu) C', & \theta \in \left(-1, -\dfrac{\tau_{1*}}{\tau_{20}^*}\right), \\ 0, & \theta = -1. \end{cases} \tag{58}$$

For $\phi \in C([-1,0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), & \theta = 0, \end{cases} \tag{59}$$

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then, system (54) is equivalent to

$$\dot{u}(t) = A(\mu) u_t + R(\mu) u_t, \tag{60}$$

where $u_t = u(t+\theta)$ for $\theta \in [-1,0]$.

For $\varphi \in C([-1,0], (R^3)^*)$, define

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\,\varphi(-s), & s = 0, \end{cases} \tag{61}$$

and the bilinear form

$$\langle \varphi(s), \phi(\theta) \rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \tag{62}$$

where $\eta(\theta) = \eta(\theta,0)$.

Let $q(\theta) = (1, q_2, q_3)^T e^{i\omega_{20}^*\tau_{20}^*\theta}$ and $q^*(s) = V(1, q_2^*, q_3^*)e^{i\omega_{20}^*\tau_{20}^* s}$ be the eigenvectors of A and A^* corresponding to

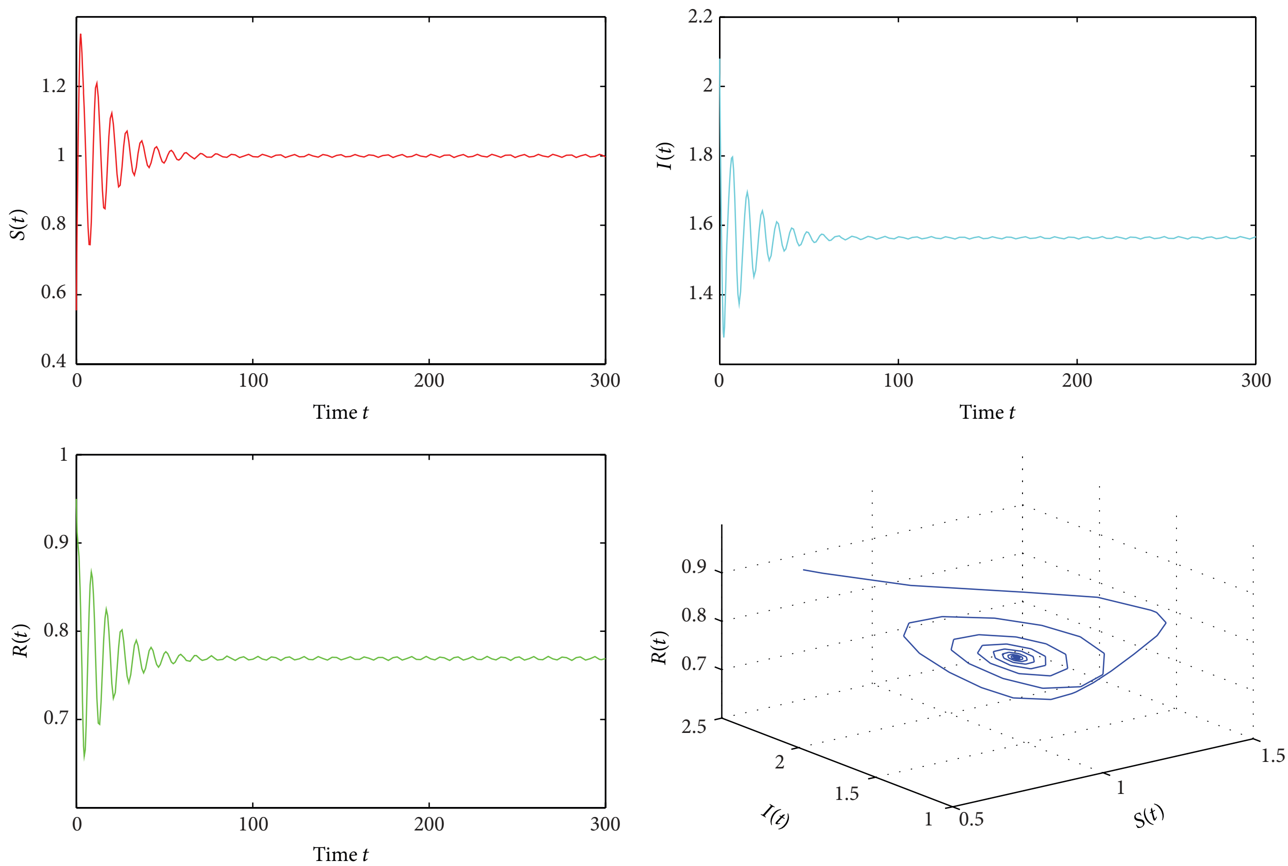

Figure 5: E_* is locally asymptotically stable for $\tau_1 = 2.0300 < \tau_{10}^* = 2.5386$ and $\tau_2 = 1.05$.

$+i\omega_{20}^*\tau_{20}^*$ and $-i\omega_{20}^*\tau_{20}^*$, respectively. By a direction computation, we get

$$\begin{aligned} q_2 &= \frac{b_{21}e^{-i\omega_{20}^*\tau_{1*}}}{i\omega_{20}^* - a_{22} - b_{22}e^{-i\omega_{20}^*\tau_{1*}} - c_{22}e^{-i\omega_{20}^*\tau_{20}^*}}, \\ q_3 &= \frac{i\omega_{20}^* - a_{11} - b_{11}e^{-i\omega_{20}^*\tau_{1*}} - b_{12}e^{-i\omega_{20}^*\tau_{1*}}q_2}{b_{13}e^{-i\omega_{20}^*\tau_{1*}}}, \\ q_2^* &= -\frac{b_{12}e^{-i\omega_{20}^*\tau_{1*}} + c_{32}e^{-i\omega_{20}^*\tau_{20}^*}q_3^*}{i\omega_{20}^* + a_{22} + b_{22}e^{-i\omega_{20}^*\tau_{1*}} + c_{22}e^{-i\omega_{20}^*\tau_{20}^*}}, \\ q_3^* &= -\frac{b_{13}e^{-i\omega_{20}^*\tau_{1*}}}{i\omega_{20}^* + a_{33} + b_{33}e^{-i\omega_{20}^*\tau_{1*}}}. \end{aligned} \tag{63}$$

From (62), we obtain

$$\begin{aligned} \overline{V} = \Big[& 1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + \tau_{20}^* e^{-i\omega_{20}^*\tau_{20}^*} q_2 \left(c_{22}\overline{q}_2^* + c_{32}\overline{q}_3^*\right) \\ & + \tau_{1*} e^{-i\omega_{20}^*\tau_{1*}} \left(b_{11} + b_{12}q_2 + b_{13}q_3\right. \\ & \left. + \overline{q}_2^*(b_{21} + b_{22}q_2) + b_{33}q_3\overline{q}_3^*\right) \Big]^{-1}. \end{aligned} \tag{64}$$

Then, one can see that $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

Next, we can obtain the coefficients determining the properties of the Hopf bifurcation by the algorithms introduced in [17] and using a computation process similar to that in [19, 20]:

$$\begin{aligned} g_{20} &= 2\beta\tau_{20}^*\overline{V}\left(\overline{q}_2^* - 1\right) q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right), \\ g_{11} &= \beta\tau_{20}^*\overline{V}\left(\overline{q}_2^* - 1\right)\left[q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\overline{q}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\right. \\ &\quad \left. + \overline{q}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\right], \\ g_{02} &= 2\beta\tau_{20}^*\overline{V}\left(\overline{q}_2^* - 1\right)\overline{q}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\overline{q}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right), \\ g_{21} &= 2\beta\tau_{20}^*\overline{V}\left(\overline{q}_2^* - 1\right)\left[W_{11}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\right. \\ &\quad + \frac{1}{2}W_{20}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\overline{q}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) \end{aligned}$$

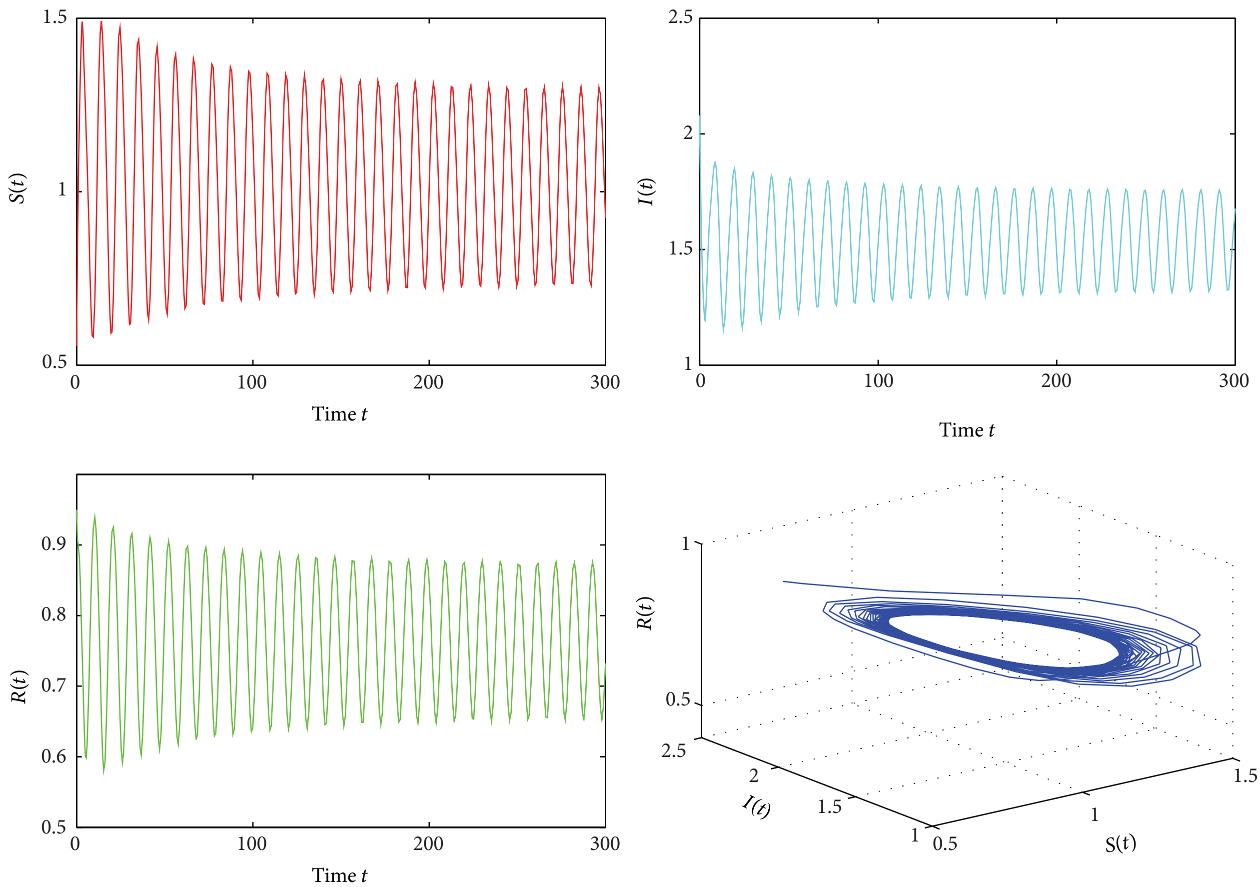

Figure 6: E_* is unstable for $\tau_1 = 2.9732 > \tau_{10}^* = 2.5386$ and $\tau_2 = 1.05$.

$$+ W_{11}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) + \frac{1}{2} W_{20}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) \bar{q}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\Bigg], \quad (65)$$

with

$$W_{20}(\theta) = \frac{i g_{20} q(0)}{\omega_{20}^* \tau_{20}^*} e^{i\omega_{20}^* \tau_{20}^* \theta} + \frac{i \bar{g}_{02} \bar{q}(0)}{3\omega_{20}^* \tau_{20}^*} e^{-i\omega_{20}^* \tau_{20}^* \theta} + E_{20} e^{2i\omega_{20}^* \tau_{20}^* \theta},$$

$$W_{11}(\theta) = -\frac{i g_{11} q(0)}{\omega_{20}^* \tau_{20}^*} e^{i\omega_{20}^* \tau_{20}^* \theta} + \frac{i \bar{g}_{11} \bar{q}(0)}{\omega_{20}^* \tau_{20}^*} e^{-i\omega_{20}^* \tau_{20}^* \theta} + E_{11}, \quad (66)$$

where E_{20} and E_{11} can be calculated by the following two equations:

$$\begin{pmatrix} a_{11}' & -b_{12} e^{-2i\omega_{20}^* \tau_{1*}} & -b_{13} e^{-2i\omega_{20}^* \tau_{1*}} \\ -b_{21} e^{-2i\omega_{20}^* \tau_{1*}} & a_{22}' & 0 \\ -a_{31} & -c_{32} e^{-2i\omega_{20}^* \tau_{20}^*} & a_{33}' \end{pmatrix} E_{20} = 2\begin{pmatrix} E_{20}^{(1)} \\ E_{20}^{(2)} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} a_{11} + b_{11} & b_{12} & b_{13} \\ b_{21} & a_{22} + b_{22} + c_{22} & 0 \\ a_{31} & c_{32} & a_{33} + b_{33} \end{pmatrix} E_{11} = -\begin{pmatrix} E_{11}^{(1)} \\ E_{11}^{(2)} \\ 0 \end{pmatrix} \quad (67)$$

with

$$a_{11}' = 2i\omega_{20}^* - a_{11} - b_{11} e^{-2i\omega_{20}^* \tau_{1*}},$$

$$a_{22}' = 2i\omega_{20}^* - a_{22} - b_{22} e^{-2i\omega_{20}^* \tau_{1*}} - c_{22} e^{-2i\omega_{20}^* \tau_{20}^*},$$

$$a_{33}' = 2i\omega_{20}^* - a_{33} - b_{33} e^{-2i\omega_{20}^* \tau_{1*}},$$

$$E_{20}^{(1)} = -\beta q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right),$$

$$E_{20}^{(2)} = \beta q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right),$$

$$E_{11}^{(1)} = -\beta\left[q^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) \bar{q}^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) + \bar{q}^{(1)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right) q^{(2)}\left(-\frac{\tau_{1*}}{\tau_{20}^*}\right)\right],$$

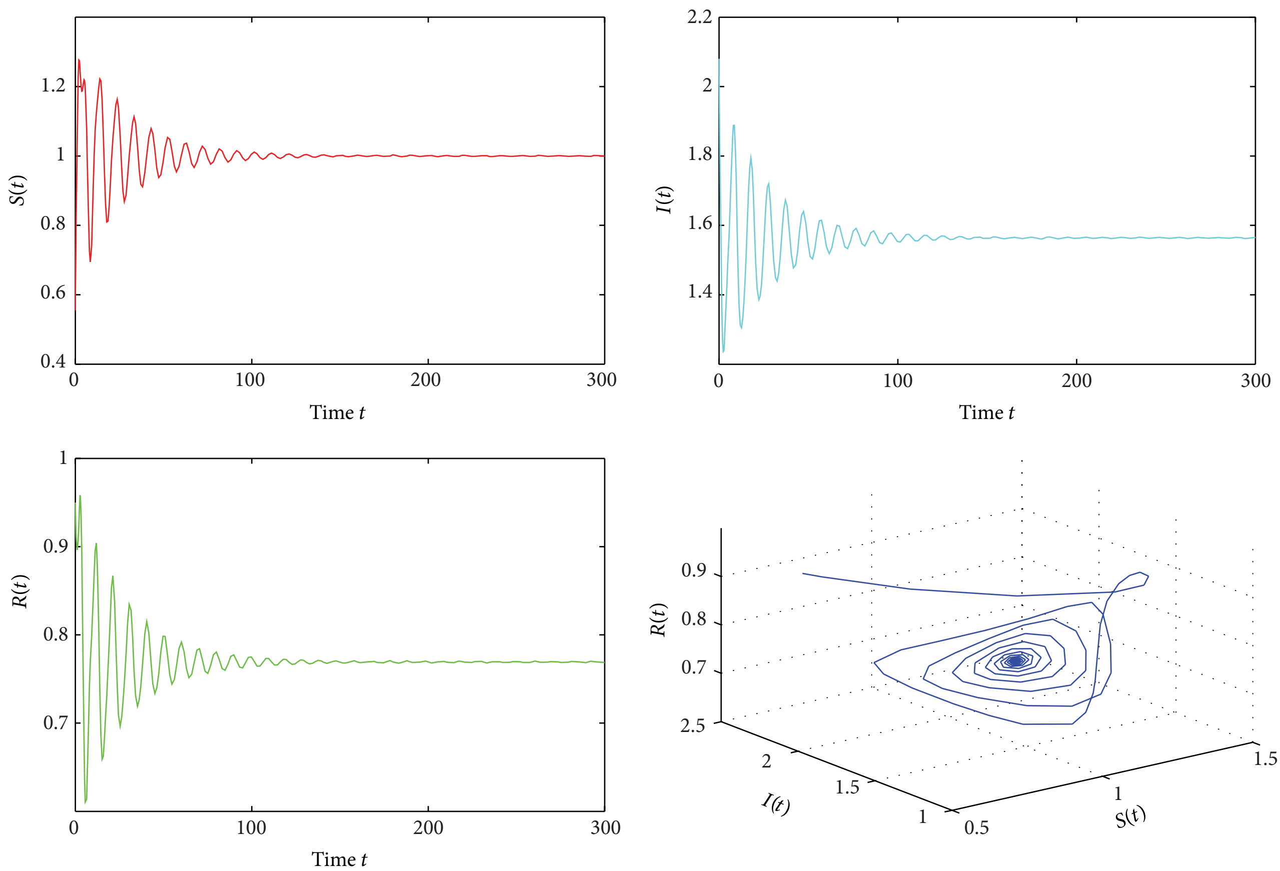

FIGURE 7: E_* is locally asymptotically stable for $\tau_2 = 2.3700 < \tau_{20}^* = 2.9170$ and $\tau_1 = 1.65$.

$$E_{11}^{(2)} = \beta \left[q^{(1)} \left(-\frac{\tau_{1*}}{\tau_{20}^*} \right) \bar{q}^{(2)} \left(-\frac{\tau_{1*}}{\tau_{20}^*} \right) + \bar{q}^{(1)} \left(-\frac{\tau_{1*}}{\tau_{20}^*} \right) q^{(2)} \left(-\frac{\tau_{1*}}{\tau_{20}^*} \right) \right]. \quad (68)$$

Then, we can get the following coefficients:

$$C_1(0) = \frac{i}{2\omega_{20}^* \tau_{20}^*} \left(g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{20}^*)\}}, \quad (69)$$
$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{20}^*)\}}{\omega_{20}^* \tau_{20}^*}.$$

By the discussion above, we have the following results about the properties of the Hopf bifurcation.

Theorem 6. *For system (3), the direction of the Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical); the stability of bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable); the period of the bifurcating periodic solutions is determined by the sign of T_2: if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Simulations and Discussion

In this section, in order to support our theoretical results, we will show the interesting dynamical behaviors of system (3) by a special case of system (3). Let $p = 0.9$, $A = 1$, let $\alpha = 0.3$, $\beta = 0.6$, $\gamma = 0.2$, $\delta = 0.7$, and $\mu = 0.3$ and we consider the following system:

$$\frac{dS(t)}{dt} = 0.9 - 0.6S(t-\tau_1) I(t-\tau_1) - 0.5S(t) + 0.7R(t-\tau_1),$$
$$\frac{dI(t)}{dt} = 0.6S(t-\tau_1) I(t-\tau_1) - 0.3I(t) - 0.3I(t-\tau_2),$$
$$\frac{dR(t)}{dt} = 0.1 + 0.2S(t) + 0.3I(t-\tau_2) - 0.7R(t-\tau_1) - 0.3R(t), \quad (70)$$

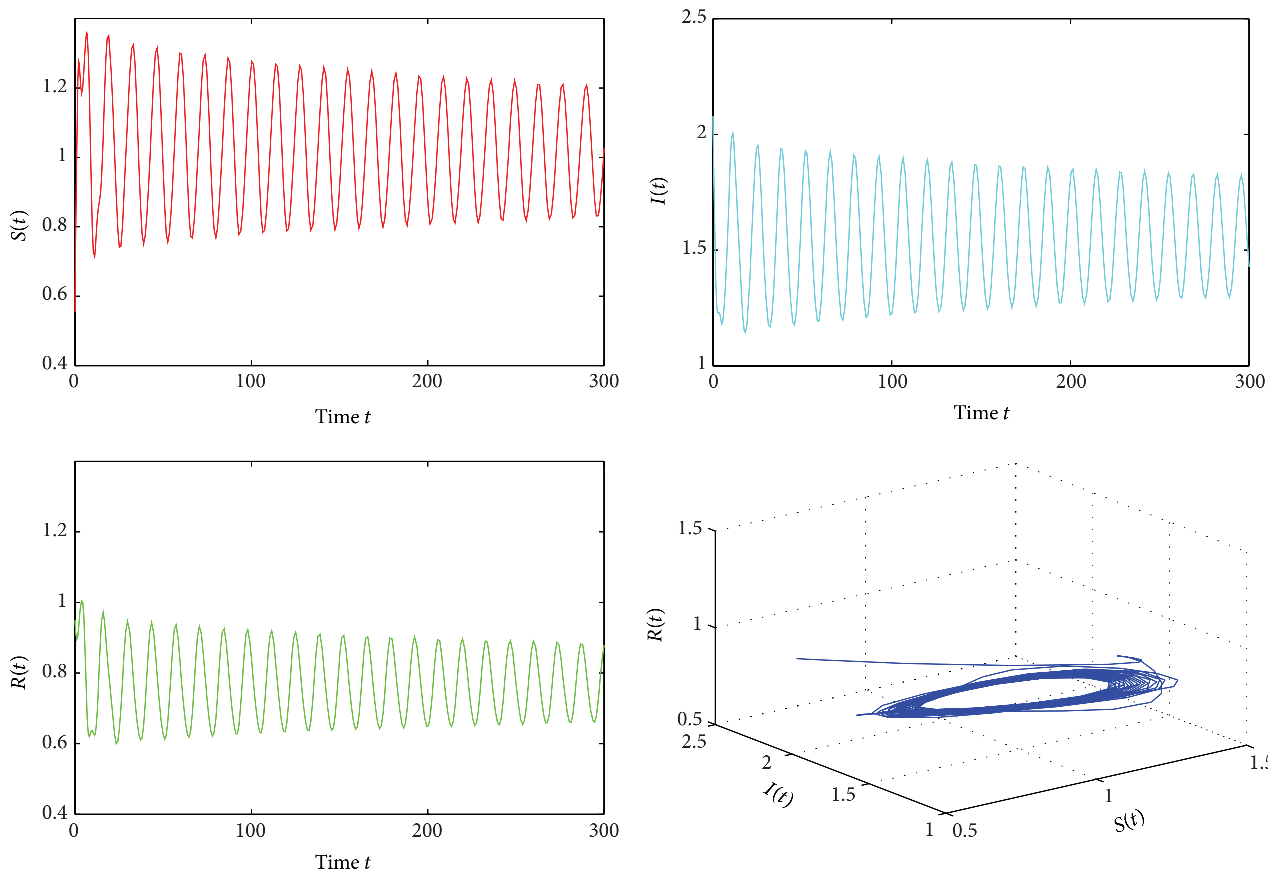

FIGURE 8: E_* is unstable for $\tau_2 = 4.3500 > \tau_{20}^* = 2.9170$ and $\tau_1 = 1.65$.

from which one can get $R_0 = 2.2200$ and the unique positive equilibrium $E_*(1.0000, 1.5641, 0.7692)$. By computing, we obtain $A_{10} = 0.3661$, $A_{11} = 1.8616$, and $A_{12} = 2.4385$. Obviously, $A_{12}A_{11} > A_{10} > 0$.

For $\tau_1 > 0$, $\tau_2 = 0$. Equation (18) has a unique positive root $\omega_{10} = 0.5833$ and one can obtain $\tau_{10} = 2.8957$ from (21). Further, the characteristic equation (14) has a pair of purely imaginary roots $\pm i\omega_{10}$. The computer simulations in Figures 1 and 2 show that $E_*(1.0000, 1.5641, 0.7692)$ is asymptotically stable when $\tau_1 < \tau_{10} = 2.8957$ and when τ_1 passes through the critical value $\tau_{10} = 2.8957$, $E_*(1.0000, 1.5641, 0.7692)$ loses its stability and a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcate from $E_*(1.0000, 1.5641, 0.7692)$. Similarly, we obtain $\omega_{20} = 3.2514$ and $\tau_{20} = 8.4669$. The corresponding trajectories graphs and phase graphs are shown in Figures 3 and 4.

Let $\tau_2 = 1.05 \in (0, \tau_{20})$ and choose τ_1 as a bifurcation parameter. Then, we have $\omega_{10}^* = 1.0514$ and $\tau_{10}^* = 2.5386$. The computer simulations in Figures 5 and 6 show that $E_*(1.0000, 1.5641, 0.7692)$ is asymptotically stable when $\tau_1 < \tau_{10}^* = 2.5386$ and $E_*(1.0000, 1.5641, 0.7692)$ loses its stability and a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcate from $E_*(1.0000, 1.5641, 0.7692)$. Similarly, by some complex computations, we have $\omega_{20}^* = 0.8545$ and $\tau_{20}^* = 2.9170$ when $\tau_1 = 1.65 \in (0, \tau_{10})$ and choose τ_2 as a bifurcation parameter. The corresponding trajectories graphs and phase graphs are shown in Figures 7 and 8. Furthermore, we can compute and obtain $\lambda'(\tau_{20}^*) = 2.3606 + 1.7054i$ and $C_1(0) = -17.9318 + 26.0921i$. It follows from (69) that $\mu_2 = 7.5963 > 0$, $\beta_2 = -35.8636 < 0$, and $T_2 = -15.3981 < 0$. According to Theorem 6, we can conclude that the Hopf bifurcation of system (70) is supercritical, the bifurcating periodic solutions are stable, and the period of the periodic solutions decreases.

In addition, it can be seen from the expression of the positive equilibrium of system (3) that the more the hosts are attached to the computer networks, the more the hosts in networks will be infected. Therefore, the managers of the real networks should properly control the number of the new hosts attached to networks. According to the numerical simulations, we also find that the onset of the Hopf bifurcation can be delayed by the values of the parameters A and p in system (3), which can be controlled by the managers of the real networks. Therefore, the managers of the real networks should properly control the number of the hosts attached to the networks and properly strengthen the immunization of the new hosts in order to control the onset of the Hopf bifurcation, so as to make the propagation of computer viruses predicted and controlled easily.

5. Conclusion

In this paper, an SIRS computer virus propagation model with two delays and multistate antivirus measures is investigated. By choosing the possible combination of the two delays as the bifurcation parameter and analyzing the distribution of the roots of the associated characteristic equation, sufficient conditions for the local stability of the positive equilibrium and existence of local Hopf bifurcation are obtained. Furthermore, the properties of the Hopf bifurcation are determined by using the method in [17].

Compared to the model considered in [12], we consider not only the delay due to the latent period of computer virus and the delay due to the temporary immune period of the recovered hosts, but also the delay due to the period that the antivirus software uses to clean the viruses in the infected hosts. All the possible delays are incorporated into the model and the model considered in this paper is more general. Our analysis shows that the new delay we incorporate into the model can also change the stability of the positive equilibrium of the model and numerical simulations show that our results obtained in the present paper improve some of the existing results on this system that are obtained in [12].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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