Research Article

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Dynamical behaviors of a k-order fuzzy difference equation

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Abstract: Difference equations are often used to create discrete mathematical models. In this paper, we mainly study the dynamical behaviors of positive solutions of a nonlinear fuzzy difference equation: $x_{n+1} = \frac{x_n}{A + Bx_{n-k}}$ $(n = 0, 1, 2, \ldots)$, where parameters A, B and initial value $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $k \in \{0, 1, \ldots\}$ are positive fuzzy numbers. We investigate the existence, boundedness, convergence, and asymptotic stability of the positive solutions of the fuzzy difference equation. At last, we give numerical examples to intuitively reflect the global behavior. The conclusion of the global stability of this paper can be applied directly to production practice.

Keywords: fuzzy difference equation, solution, boundedness, convergence

MSC 2020: 39A10, 65K10

1 Introduction

As we all know, the stability theory has always been one of the hotspots in mathematical research, especially the stability theory of time-delay systems [1–3]. Difference equations are usually used to describe discrete-time dynamic systems, which can be widely used to establish mathematical models in many areas, such as computer science, ecology, population dynamics, electrical networks, economics, etc. [4,5]. Due to its wide range of applications, the study of the time-delay difference equation and its dynamic behavior has become an important topic in applied mathematics. In practice, more problems encountered are that the conditional parameters fluctuate or change within a certain range. One of the ways to solve this problem is to fuzz the difference equation based on the uncertainty. Fuzzy set theory is an effective tool for modeling uncertainty and processing vague or subjective information [6,7]. Fuzzy difference equations have gradually attracted attention for their practicability [8–28] and are developing vigorously. Chrysafis et al. [10] studied the fuzzy difference equation of finance, which is an alternative method for studying the time value of money. Deeba and Korvin [11] studied fuzzy difference equation $x_{n+1} = x_n - ABx_{n-1} + C$ about the level of carbon dioxide (CO_2) in the blood.

Let us first review the history of the fuzzy difference equations studied in this article. In [14], Zhang and Liu studied the qualitative behavior of the first-order linear fuzzy difference equation, which is $x_{n+1} = Ax_n + B$ $(n = 0, 1, 2, \ldots)$, where the parameters A, B and initial value are positive fuzzy numbers. In [15], Hatir investigated the existence, oscillatory behavior, and asymptotic behavior of the positive solutions of

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second-order fuzzy difference equation $x_{n+1} = A + \frac{B}{x_{n-1}}$ $(n = 0, 1, 2, \dots)$, where parameters A, B and initial value are positive fuzzy numbers. Subsequently, many researchers used similar methods to study different fuzzy difference equations and came to many conclusions, such as [23–27].

In [16], Papaschinopoulos and Schinas studied the global properties of a high-order nonlinear difference equation system $x_{n+1} = A + \frac{y_n}{x_{n-p}}$, $y_{n+1} = A + \frac{x_n}{y_{n-q}}$ $(n = 0, 1, 2, \dots)$, where p, q are positive integers, parameter A and initial values are positive real numbers. Later, he studied m-order fuzzy difference equation $x_{n+1} = A + \frac{x_n}{x_{n-m}}$ $(n = 0, 1, 2, \dots)$, in which parameter A and initial value are positive fuzzy numbers. In [17], Zhang et al. showed the existence and the asymptotic stability of the positive solutions about first-order fuzzy difference equation $x_{n+1} = \frac{A + x_n}{B + x_n}$ $(n = 0, 1, 2, \dots)$, in which parameter A, B and initial value are positive fuzzy numbers.

Motivated by the discussions above, we study arbitrary k-order nonlinear fuzzy difference equation in this paper as follows,

$$x_{n+1} = \frac{x_n}{A + Bx_{n-k}}, \quad n = 0, 1, 2, \dots, \tag{1}$$

and also the existence, stability, and other dynamic behaviors of the positive solution of equation (1), where parameters A, B and initial value $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$, $k \in \{0, 1, \dots\}$ are positive fuzzy numbers.

For further study of (1), we need some basic definitions and lemmas which are related to fuzzy difference equations, which are obtained from reference resources [14–21]. In this paper, $R = (-\infty, +\infty)$, $R^+ = (0, +\infty)$, F^+ denote the set of all positive fuzzy numbers.

Definition 1.1. [18] A function $A : R \to [0, 1]$ is said to be a fuzzy number if it satisfies the following conditions (i)–(iv):

(i) A is normal, i.e., there exists $x \in R$ such that $A(x) = 1$;

(ii) A is fuzzy convex, i.e., for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1 - t)x_2) \geq \min\{A(x_1), A(x_2)\};$$

(iii) A is upper semicontinuous;

(iv) The support of A, defined as supp$A = \overline{\bigcup_{\alpha \in (0,1]} [A]_\alpha} = \overline{\{x : A(x) > 0\}}$, is compact.

Definition 1.2. [18] For $\alpha \in (0, 1]$ we define the α-cuts of fuzzy number A with $[A]_\alpha = \{x \in R : A(x) \geq \alpha\}$. In particular, for $\alpha = 0$, the support of A is defined as supp$A = [A]_0 = \overline{\{x \in R | A(x) > 0\}}$.

It is clear that $[A]_\alpha$ is a closed interval. Fuzzy number A is positive if min(supp$A$) > 0.

It is obvious that if A is a positive real number, then A is a positive fuzzy number and $[A]_\alpha = [A, A]$, $\alpha \in [0, 1]$. In this case, we say A is a trivial fuzzy number.

Let B_i ($i = 0, 1, \dots, k$, k is a positive integer) be fuzzy numbers such that

$$[B_i]_\alpha = [B_{i,l,\alpha}, B_{i,r,\alpha}], \quad i = 0, 1, \dots, k, \quad \alpha \in (0, 1],$$

and for any $\alpha \in (0, 1]$

$$C_{l,\alpha} = \max\{B_{i,l,\alpha}, i = 0, 1, \dots, k\}, C_{r,\alpha} = \max\{B_{i,r,\alpha}, i = 0, 1, \dots, k\},$$

then by ([19], Theorem 2.1), $(C_{l,\alpha}, C_{r,\alpha})$ determines a fuzzy number C such that

$$[C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}], \alpha \in (0, 1].$$

According to [20] and ([21], Lemma 2.3), we can define

$$C = \max\{B_i, i = 0, 1, \dots, k\}.$$

Definition 1.3. [20] Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, $\alpha \in (0, 1]$. The fuzzy number space norm is defined as follows:

$$\|A\| = \sup_{\alpha \in (0,1]} \max\{|A_{l,\alpha}|, |A_{r,\alpha}|\}.$$

The distance between two arbitrary fuzzy numbers A and B is defined as follows:

$$D(A,B)=\sup_{\alpha\in(0,1]}\max\{|A_{l,\alpha}-B_{l,\alpha}|,|A_{r,\alpha}-B_{r,\alpha}|\}.$$

Definition 1.4. [22] We say that x_n is a positive solution of (1) if x_n is a sequence of positive fuzzy numbers, which satisfies (1).

We say that a sequence of positive fuzzy numbers x_n is persistent (resp. is bounded) if there exists a positive number M (resp., N) such that

$$\text{supp}x_n\subset[M,+\infty),\ \ (\text{resp. supp}x_n\subset(0,N]),\quad n=1,2,\ldots.$$

In addition, we say that x_n is bounded and persists if there exist numbers $M,N\in(0,+\infty)$ such that

$$\text{supp}x_n\subset[M,N],\ \ n=1,2,\ldots.$$

Lemma 1.1. [18] *Let function $f:R^+\times R^+\times R^+\to R^+$ be continuous, A,B,C be fuzzy numbers. Then*

$$[f(A,B,C)]_\alpha=f([A]_\alpha,[B]_\alpha,[C]_\alpha),\quad \alpha\in(0,1].$$

2 Main results

In this section, we will discuss some dynamical characters of the fuzzy equation (1). We know that the existence of the positive solution is important for equation, which is the basis for discussing all of the dynamic characters. So, let us first prove that for fuzzy difference equation (1) there exists unique positive solution for any positive initial value.

Theorem 2.1. *For any positive fuzzy numbers $x_{-k},x_{-k+1},\ldots,x_{-1},x_0$, $k\in\{0,1,2\cdots\}$, fuzzy difference equation (1), there exists a unique positive solution x_n whose initial value is $x_{-k},x_{-k+1},\ldots,x_{-1},x_0$, $k\in\{0,1,2\cdots\}$.*

Proof. For all positive fuzzy numbers $x_{-k},x_{-k+1},\ldots,x_{-1},x_0$, $k\in\{0,1,2\ldots\}$, suppose there exists a fuzzy number sequence that satisfies equation (1) whose initial value is $x_{-k},x_{-k+1},\ldots,x_{-1},x_0$. Consider their α-cuts, $\alpha\in(0,1]$,

$$\begin{cases}[x_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\\ [A]_\alpha=[A_{l,\alpha},A_{r,\alpha}],\quad n=0,1,2\ldots.\\ [B]_\alpha=[B_{l,\alpha},B_{r,\alpha}].\end{cases}\tag{2}$$

Following (1), (2), and Lemma 1.1 in the literature [18], we have

$$\begin{aligned}[x_{n+1}]_\alpha=[L_{n+1,\alpha},R_{n+1,\alpha}]&=\left[\frac{x_n}{A+Bx_{n-k}}\right]_\alpha=\frac{[x_n]_\alpha}{[A]_\alpha+[B]_\alpha\times[x_{n-k}]_\alpha}\\ &=\frac{[L_{n,\alpha},R_{n,\alpha}]}{[A_{l,\alpha},A_{r,\alpha}]+[B_{l,\alpha},B_{r,\alpha}]\times[L_{n-k,\alpha},R_{n-k,\alpha}]}\\ &=\left[\frac{L_{n,\alpha}}{A_{r,\alpha}+B_{r,\alpha}\times R_{n-k,\alpha}},\frac{R_{n,\alpha}}{A_{l,\alpha}+B_{l,\alpha}\times L_{n-k,\alpha}}\right],\end{aligned}$$

where $n=0,1,2,\ldots$, $\alpha\in(0,1]$, so we obtain the related equation system

$$\begin{cases}L_{n+1,\alpha}=\dfrac{L_{n,\alpha}}{A_{r,\alpha}+B_{r,\alpha}\times R_{n-k,\alpha}},\\ R_{n+1,\alpha}=\dfrac{R_{n,\alpha}}{A_{l,\alpha}+B_{l,\alpha}\times L_{n-k,\alpha}}.\end{cases}\tag{3}$$

Obviously, for any initial value $(L_{i,\alpha}, R_{i,\alpha})$, $i = -k, -k+1, \ldots, 0$, $\alpha \in (0,1]$, system (3), there exists a unique positive solution $(L_{n,\alpha}, R_{n,\alpha})$, $\alpha \in (0,1]$. Now we prove that $[L_{n,\alpha}, R_{n,\alpha}]$, $\alpha \in (0,1]$ determines the solution x_n of (1) whose initial value is $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the positive solution of system (3) with initial value $(L_{i,\alpha}, R_{i,\alpha})$, $i = -k, -k+1, \ldots, 0$, such that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0,1], \quad n = 0, 1, 2, \ldots. \tag{4}$$

According to the literature [23] and $A, B, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $k \in \{0, 1, 2 \cdots\}$ are positive fuzzy numbers, for any $\alpha_1, \alpha_2 \in (0,1]$, $\alpha_1 \le \alpha_2$, we have

$$\begin{cases} 0 < A_{l,\alpha_1} \le A_{l,\alpha_2} \le A_{r,\alpha_2} \le A_{r,\alpha_1}, \\ 0 < B_{l,\alpha_1} \le B_{l,\alpha_2} \le B_{r,\alpha_2} \le B_{r,\alpha_1}, \\ 0 < L_{-k,\alpha_1} \le L_{-k,\alpha_2} \le R_{-k,\alpha_2} \le R_{-k,\alpha_1}, \\ 0 < L_{-k+1,\alpha_1} \le L_{-k+1,\alpha_2} \le R_{-k+1,\alpha_2} \le R_{-k+1,\alpha_1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ 0 < L_{0,\alpha_1} \le L_{0,\alpha_2} \le R_{0,\alpha_2} \le R_{0,\alpha_1}. \end{cases} \tag{5}$$

By induction and (3), (5), we will show

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}, \quad n = 0, 1, 2, \ldots. \tag{6}$$

By (5), when $n = 0$, it is obvious (6) is true. When $n = 1$, since

$$\begin{aligned} L_{1,\alpha_1} &= \frac{L_{0,\alpha_1}}{A_{r,\alpha_1} + B_{r,\alpha_1} \times R_{-k,\alpha_1}} \le \frac{L_{0,\alpha_2}}{A_{r,\alpha_2} + B_{r,\alpha_2} \times R_{-k,\alpha_2}} = L_{1,\alpha_2} \\ &= \frac{L_{0,\alpha_2}}{A_{r,\alpha_2} + B_{r,\alpha_2} \times R_{-k,\alpha_2}} \le \frac{R_{0,\alpha_2}}{A_{l,\alpha_2} + B_{l,\alpha_2} \times L_{-k,\alpha_2}} = R_{1,\alpha_2} \\ &= \frac{R_{0,\alpha_2}}{A_{l,\alpha_2} + B_{l,\alpha_2} \times L_{-k,\alpha_2}} \le \frac{R_{0,\alpha_1}}{A_{l,\alpha_1} + B_{l,\alpha_1} \times L_{-k,\alpha_1}} = R_{1,\alpha_1}, \end{aligned}$$

so $L_{1,\alpha_1} \le L_{1,\alpha_2} \le R_{1,\alpha_2} \le R_{1,\alpha_1}$. Suppose (6) is true when $n \le k$, $k \in \{1, 2, \ldots\}$, following (3) and (5), we have

$$\begin{aligned} L_{k+1,\alpha_1} &= \frac{L_{k,\alpha_1}}{A_{r,\alpha_1} + B_{r,\alpha_1} \times R_{0,\alpha_1}} \le \frac{L_{k,\alpha_2}}{A_{r,\alpha_2} + B_{r,\alpha_2} \times R_{0,\alpha_2}} = L_{k+1,\alpha_2} \\ &\le \frac{R_{k,\alpha_2}}{A_{l,\alpha_2} + B_{l,\alpha_2} \times L_{0,\alpha_2}} = R_{k+1,\alpha_2} \le \frac{R_{k,\alpha_1}}{A_{l,\alpha_1} + B_{l,\alpha_1} \times L_{0,\alpha_1}} = R_{k+1,\alpha_1}, \end{aligned}$$

so $L_{k+1,\alpha_1} \le L_{k+1,\alpha_2} \le R_{k+1,\alpha_2} \le R_{k+1,\alpha_1}$, $k = 1, 2, \ldots$. By induction, (6) holds true.

Following (3), we have

$$\begin{cases} L_{1,\alpha} = \dfrac{L_{0,\alpha}}{A_{r,\alpha} + B_{r,\alpha} \times R_{-k,\alpha}}, \\ R_{1,\alpha} = \dfrac{R_{0,\alpha}}{A_{l,\alpha} + B_{l,\alpha} \times L_{-k,\alpha}}. \end{cases} \tag{7}$$

For $A, B, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $k \in \{0, 1, 2 \ldots\}$ are positive fuzzy numbers and Lemma 2.2 in the literature [24], we know $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}, L_{-k,\alpha}, R_{-k,\alpha}, \ldots, L_{-1,\alpha}, R_{-1,\alpha}, L_{0,\alpha}, R_{0,\alpha}$ are left continuous, by (7) we have $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By induction, we have $L_{n,\alpha}, R_{n,\alpha}$ $(n = 1, 2, \ldots)$ are left continuous.

We assert that the support of x_n, $\mathrm{supp}x_n = \overline{\bigcup_{\alpha \in (0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. We just need to prove that $\bigcup_{\alpha \in (0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded. When $n = 1$, since $A, B, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $k \in \{0, 1, 2 \cdots\}$ are positive fuzzy numbers, there exist constants $M_A, N_A, M_B, N_B, M_i, N_i$, $i = -k, -k+1, \ldots, -1, 0$, such that for all $\alpha \in (0,1]$, we have

$$[A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], [L_{i,\alpha}, R_{i,\alpha}] \subset [M_i, N_i]. \tag{8}$$

Following systems (7) and (8) we obtain $[L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_0}{N_A + N_B \times N_{-k}}, \frac{N_0}{M_A + M_B \times M_{-k}}\right](\alpha \in (0,1])$, so $\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_0}{N_A + N_B \times N_{-k}}, \frac{N_0}{M_A + M_B \times M_{-k}}\right]$. It implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact, and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0, +\infty)$. By induction, we obtain that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact, and $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0, +\infty)$ for every n =1, 2, … . Following this and (6), and because the left continuous of $L_{n,\alpha}, R_{n,\alpha}$, $n = 1, 2, \dots$, $[L_{n,\alpha}, R_{n,\alpha}]$ establishes a sequence of positive fuzzy numbers x_n such that (4) holds.

Next, we show that x_n is the solution of (1) with any initial value $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$, $k \in \{0, 1, 2 \cdots\}$. Since for all $\alpha \in (0, 1]$,

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{L_{n,\alpha}}{A_{r,\alpha} + B_{r,\alpha} \times R_{n-k,\alpha}}, \frac{R_{n,\alpha}}{A_{l,\alpha} + B_{l,\alpha} \times L_{n-k,\alpha}}\right] = \left[\frac{x_n}{A + Bx_{n-k}}\right]_\alpha.$$

So, x_n is the solution of (1) with any initial value $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$, $k \in \{0, 1, 2 \cdots\}$. That is, the positive solution of fuzzy difference equation (1) exists.

Next, we prove that the positive solution of fuzzy difference equation (1) is unique by contradiction. Suppose there exists another solution $\overline{x}_n$ of (1) with initial value $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$, $k \in \{0, 1, 2 \cdots\}$. Similar to the above proof, we can easily prove that

$$[\overline{x}_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{L_{n,\alpha}}{A_{r,\alpha} + B_{r,\alpha} \times R_{n-k,\alpha}}, \frac{R_{n,\alpha}}{A_{l,\alpha} + B_{l,\alpha} \times L_{l-k,\alpha}}\right] = \left[\frac{\overline{x}_n}{A + B\overline{x}_{n-k}}\right]_\alpha,$$

which implies $[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]$ $(n = 0, 1, 2, \dots)$ for any $\alpha \in (0, 1]$. Then, by (4) we have $[\overline{x}_n]_\alpha = [x_n]_\alpha$ $(n = 0, 1, 2, \dots)$ for any $\alpha \in (0, 1]$. So $\overline{x}_n = x_n$, $n = 0, 1, 2, \dots$, which is contradictory. So the positive solution of fuzzy difference equation (1) is unique.

In summary, for fuzzy difference equation (1), there exists a unique positive solution x_n, for every positive initial value $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0$, $k \in \{0, 1, 2, \dots\}$. □

We have proved that the fuzzy difference equation (1) has a unique positive solution, and then we will study the dynamic behavior of the solution. First, we discuss the boundedness of positive solutions of equation (1), which needs to rely on related equation system from (3).

Lemma 2.1. *Consider the system of difference equations*

$$\begin{cases} y_{n+1} = \dfrac{y_n}{a + bz_{n-k}}, \\ z_{n+1} = \dfrac{z_n}{c + dy_{n-k}}, \end{cases} \quad n = 0, 1, 2, \dots, \tag{9}$$

where the initial value y_i, z_i $(i = -k, -k + 1, \dots, -1, 0)$ *and parameters* a, b, c, d *are positive real numbers. If parameters* $a \geq 1$ *and* $c \geq 1$, *then for all* $n \geq k$, y_n, z_n *are bounded and*

$$\begin{cases} \dfrac{1}{(a + bz_0)^{n-k}} y_k \leq y_n \leq \dfrac{1}{a^n} y_0, \\ \dfrac{1}{(c + dy_0)^{n-k}} z_k \leq z_n \leq \dfrac{1}{c^n} z_0, \end{cases} \quad n = 0, 1, 2, \dots.$$

Proof. Let (y_n, z_n) be a positive solution of the difference equation system (9). From (9), the following are concluded by recursion,

$$\begin{cases} y_{n+1} = \dfrac{y_n}{a + bz_{n-k}} \leq \dfrac{1}{a} y_n \leq \dfrac{1}{a^2} y_{n-1} \leq \cdots \leq \dfrac{1}{a^{n+1}} y_0, \\ z_{n+1} = \dfrac{z_n}{c + dy_{n-k}} \leq \dfrac{1}{c} z_n \leq \dfrac{1}{c^2} z_{n-1} \leq \cdots \leq \dfrac{1}{c^{n+1}} z_0, \end{cases} \quad n = 0, 1, 2, \dots,$$

$$\begin{cases} y_{n+1} = \dfrac{y_n}{a + bz_{n-k}} \geq \dfrac{y_n}{a + bz_0} \geq \dfrac{1}{(a + bz_0)^2} y_{n-1} \geq \cdots \geq \dfrac{1}{(a + bz_0)^{n+1-k}} y_k, \\ z_{n+1} = \dfrac{z_n}{c + dy_{n-k}} \geq \dfrac{z_n}{c + dy_0} \geq \dfrac{1}{(c + dy_0)^2} z_{n-1} \geq \cdots \geq \dfrac{1}{(c + dy_0)^{n+1-k}} z_k, \end{cases} \quad n \geq k.$$

It is easy to deduce that y_n, z_n are bounded and

$$\begin{cases} \dfrac{1}{(a + bz_0)^{n-k}} y_k \leq y_n \leq \dfrac{1}{a^n} y_0, \\ \dfrac{1}{(c + dy_0)^{n-k}} z_k \leq z_n \leq \dfrac{1}{c^n} z_0, \end{cases} \quad n \geq k.$$

□

Theorem 2.2. *Consider fuzzy difference equation* (1), *for all* $\alpha \in (0, 1]$, *if*

$$A_{l,\alpha} > 1, \tag{10}$$

then every positive solution x_n *of fuzzy difference equation* (1) *is bounded.*

Proof. Let x_n be a positive solution of (1) and satisfy (4). Because $A_{l,\alpha} > 1$, so $A_{r,\alpha} > 1$. From (3) and Lemma 2.1, for any $\alpha \in (0, 1]$ we have

$$\begin{cases} \dfrac{1}{(A_{r,\alpha} + B_{r,\alpha} \times R_{0,\alpha})^{n-k}} L_{k,\alpha} \leq L_{n,\alpha} \leq \dfrac{1}{A_{r,\alpha}^n} L_{0,\alpha}, \\ \dfrac{1}{(A_{l,\alpha} + B_{l,\alpha} \times L_{0,\alpha})^{n-k}} R_{k,\alpha} \leq R_{n,\alpha} \leq \dfrac{1}{A_{l,\alpha}^n} R_{0,\alpha}. \end{cases} \quad n = 0, 1, 2, \ldots.$$

So, $R_{n,\alpha} \leq \frac{1}{A_{l,\alpha}^n} R_{0,\alpha}$ and

$$L_{n,\alpha} > 0. \tag{11}$$

We have $R_{n,\alpha} \leq \frac{1}{A_{l,\alpha}^n} R_{0,\alpha} \leq R_{0,\alpha}$, which is because $A_{l,\alpha} > 1$. Then since x_n is a positive fuzzy number, there exists a constant $J > 0$, such that for all $\alpha \in (0, 1]$, we have

$$R_{0,\alpha} \leq J. \tag{12}$$

From (11) and (12), we obtain that $[L_{n,\alpha}, R_{n,\alpha}] \subset (0, J]$ for all $\alpha \in (0, 1]$, so $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset (0, J]$, which implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0, J]$. Thus, the positive solution x_n of equation (1) is bounded. □

Next, we consider the existence of equilibrium solution of (1). It is obvious that $\bar{x} = 0$ is an equilibrium solution of (1).

Let $\bar{y}, \bar{z}$ be real numbers and satisfy equation systems $\bar{y} = \frac{\bar{y}}{a + b\bar{z}}, \bar{z} = \frac{\bar{z}}{c + d\bar{y}}$. Solving it we have $\bar{y} = \frac{1-c}{d}, \bar{z} = \frac{1-a}{b}$. It is easy to get the following lemma.

Lemma 2.2. *Consider the difference equation system* (9). *Then* (9) *has positive equilibrium point* $(\bar{y}, \bar{z})$ *and* $\bar{y} = \frac{1-c}{d}, \bar{z} = \frac{1-a}{b}$ *when* $a < 1, c < 1$.

Theorem 2.3. *For all* $\alpha \in (0, 1]$, *there is no positive equilibrium solution of fuzzy difference equation* (1).

Proof. Assume there exists a positive fuzzy number x, which satisfies $x = \frac{x}{A + Bx}$, $[x]_\alpha = [L_\alpha, R_\alpha]$, and $L_\alpha, R_\alpha \geq 0$, where $\alpha \in (0, 1]$. So we derive a related equation system as follows:

$$L_\alpha = \frac{L_\alpha}{A_{r,\alpha} + B_{r,\alpha} \times R_\alpha}, \quad R_\alpha = \frac{R_\alpha}{A_{l,\alpha} + B_{l,\alpha} \times L_\alpha}.$$

Suppose $L_\alpha \neq 0, R_\alpha \neq 0$, following Lemma 2.2 we have $L_\alpha = \frac{1-A_{l,\alpha}}{B_{l,\alpha}}, R_\alpha = \frac{1-A_{r,\alpha}}{B_{r,\alpha}}$, that is, $L_\alpha > R_\alpha$. But $[x]_\alpha = [L_\alpha, R_\alpha]$ means $L_\alpha \leq R_\alpha$, which is a contradiction. So it can only be $L_\alpha = R_\alpha = 0$. □

Therefore, there is no positive equilibrium solution of fuzzy difference equation (1), but have equilibrium solution $\bar{x} = 0$.

Next, we discuss the positive solution's convergence and asymptotic stability of fuzzy difference equation (1).

Lemma 2.3. *Consider the difference equation system* (9). *Assume that* $a, c \in (1, +\infty)$, *then the solution of system* (9) *is convergent and* $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} z_n = 0$.

Proof. From Lemma 2.1, we know

$$\begin{cases} \dfrac{1}{(a+bz_0)^{n-k}} y_k \le y_n \le \dfrac{1}{a^n} y_0, \\ \dfrac{1}{(c+dy_0)^{n-k}} z_k \le z_n \le \dfrac{1}{c^n} z_0, \end{cases} \quad n = 0, 1, 2, \ldots.$$

If $a, c \in (1, +\infty)$, then

$$\lim_{n\to\infty} \frac{1}{(a+bz_0)^{n-k}} y_k = 0, \quad \lim_{n\to\infty} \frac{1}{a^n} y_0 = 0, \quad \lim_{n\to\infty} \frac{1}{(c+dy_0)^{n-k}} z_k = 0, \quad \lim_{n\to\infty} \frac{1}{c^n} z_0 = 0.$$

It is obvious that the solution of system (9) (y_n, z_n) is convergent and $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} z_n = 0$. □

Theorem 2.4. *Consider fuzzy difference equation* (1). *Assume* $A_{l,\alpha} > 1$, *then every positive solution of* (1) *converges to equilibrium* $\bar{x} = 0$ *when* $n \to \infty$.

Proof. Since

$$L_{n+1,\alpha} = \frac{L_n}{A_{r,\alpha} + B_{r,\alpha} \times R_{n-k}}, \qquad R_{n+1,\alpha} = \frac{R_n}{A_{l,\alpha} + B_{l,\alpha} \times L_{n-k}},$$

following Lemma 2.3 we have $\lim_{n\to\infty} L_{n,\alpha} = 0$, $\lim_{n\to\infty} R_{n,\alpha} = 0$, and according to the distance between two arbitrary fuzzy numbers (Definition 1.3 ([19])), we obtain

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} D(x_n, 0) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \max\{|L_{n,\alpha} - 0|, |R_{n,\alpha} - 0|\} = 0.$$

So $\lim_{n\to\infty} x_n = \bar{x} = 0$. That is every positive solution of equation (1) converges to equilibrium $\bar{x} = 0$ when $n \to \infty$. □

Lemma 2.4. *For the equilibrium point* $(0, 0)$ *of equation system* (9), *if* $a, c \in (1, +\infty)$, *then the equilibrium point* $(0, 0)$ *is locally asymptotically stable.*

Proof. The linearized equation of equation system (9) about the equilibrium point $(0, 0)$ is

$$\varphi_{n+1} = A\varphi_n, \tag{13}$$

where $\varphi_n = (y_n, y_{n-1}, \ldots, y_{n-k}, z_n, z_{n-1}, \ldots, z_{n-k})^T$, and

$$A = \begin{pmatrix} 1/a & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1/c & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The characteristic equation with equation (13) is $P(\lambda) = (-\lambda)^{k-2}\left(\frac{1}{a} - \lambda\right)\left(\frac{1}{c} - \lambda\right)$, and this shows that all eigenvalues are $|\lambda| < 1$. Thus, the equilibrium (0,0) is locally asymptotically stable. □

From Lemmas 2.3 and 2.4, we have that the equilibrium point (0, 0) of equation (9) is global asymptotically stable. Combined with Theorem 4, we can easily obtain the following theorem:

Theorem 2.5. *Consider fuzzy difference equation* (1). *If* $A_{l,\alpha} > 1$, *then the equilibrium point* $\bar{x} = 0$ *is global asymptotically stable.*

3 Numerical example

In this section, we give numerical examples on the main conclusions of Section 2.
Model. Take $k = 2$ in fuzzy difference equation (1). Considering the following fuzzy difference equation:

$$x_{n+1} = \frac{x_n}{A + Bx_{n-2}}, \quad n = 3, 4, 5 \ldots, \tag{14}$$

in which parameters A, B and initial value x_1, x_2, x_3 are positive fuzzy numbers. According to Theorems 2.1–2.3, we know the following conclusions.

For any positive solution x_n of (14), $[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]$, a difference equation system with a parameter α is obtained

$$L_{n+1,\alpha} = \frac{L_{n,\alpha}}{A_{r,\alpha} + B_{r,\alpha} \times R_{n-2,\alpha}}, \qquad R_{n+1,\alpha} = \frac{R_{n,\alpha}}{A_{l,\alpha} + B_{l,\alpha} \times L_{n-2,\alpha}}, \quad \alpha \in (0, 1], \tag{15}$$

and we know the unique equilibrium solution is $\bar{x} = 0$.

Example 1. Take A, B and initial value x_1, x_2, x_3 as follows:

$$A(t) = \begin{cases} 3t - 7, & \frac{7}{3} \le t \le \frac{8}{3}, \\ -3t + 9, & \frac{8}{3} \le t \le 3, \end{cases} \quad B(t) = \begin{cases} 2t - 5, & \frac{5}{2} \le t \le 3, \\ -2t + 7, & 3 \le t \le \frac{7}{2}, \end{cases}$$

$$x_1(t) = \begin{cases} t - 4, & 4 \le t \le 5, \\ -\frac{1}{2}t + \frac{7}{2}, & 5 \le t \le 7, \end{cases} \quad x_2(t) = \begin{cases} \frac{1}{2}t - 4, & 8 \le t \le 10, \\ -\frac{1}{2}t + 6, & 10 \le t \le 12, \end{cases}$$

$$x_3(t) = \begin{cases} \frac{1}{3}t - 2, & 6 \le t \le 9, \\ -\frac{1}{3}t + 4, & 9 \le t \le 12. \end{cases}$$

From function $A(x), B(x), x_1(x), x_2(x), x_3(x)$, for all $\alpha \in (0, 1]$, we have

$$[A]_\alpha = \left[\frac{7}{3} + \frac{\alpha}{3}, 3 - \frac{\alpha}{3}\right], \quad [B]_\alpha = \left[\frac{5}{2} + \frac{\alpha}{2}, \frac{7}{2} - \frac{\alpha}{2}\right],$$
$$[x_1]_\alpha = [4 + \alpha, 7 - 2\alpha], \quad [x_2]_\alpha = [8 + 2\alpha, 12 - 2\alpha], \quad [x_3]_\alpha = [6 + 3\alpha, 12 - 3\alpha].$$

So the support of them are

$$\overline{\bigcup_{\alpha \in (0,1]} [A]_\alpha} = \left[\frac{7}{3}, 3\right], \quad \overline{\bigcup_{\alpha \in (0,1]} [B]_\alpha} = \left[\frac{5}{2}, \frac{7}{2}\right],$$

$$\overline{\bigcup_{\alpha \in (0,1]} [x_0]_\alpha} = [4, 7], \quad \overline{\bigcup_{\alpha \in (0,1]} [x_{-1}]_\alpha} = [8, 12], \quad \overline{\bigcup_{\alpha \in (0,1]} [x_{-2}]_\alpha} = [6, 12].$$

It is clear that $A_{l,\alpha} > 1$ and (14) satisfies Theorems 2.1–2.5. So, every positive solution of (14) is bounded and persists, and converges to equilibrium $\bar{x} = 0$ when $n \to +\infty$. Moreover, the unique equilibrium $\bar{x} = 0$ is global asymptotically stable (Figures 1–3).

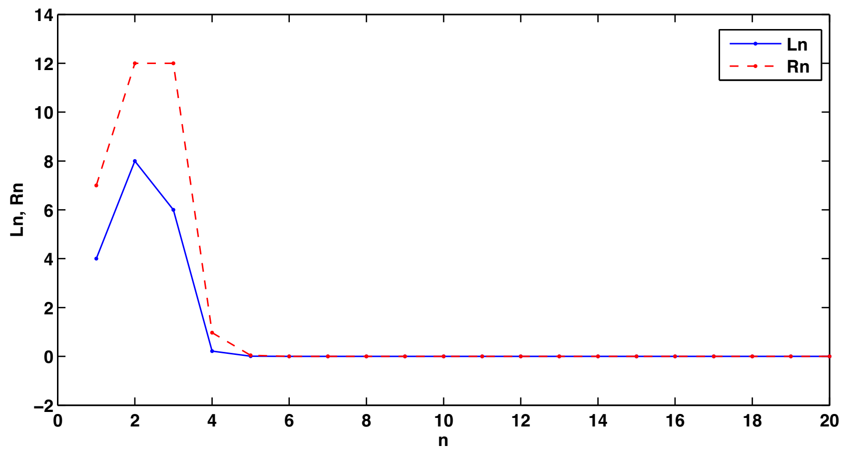

Figure 1: In Example 1, the solution of system (15) when $\alpha = 0$.

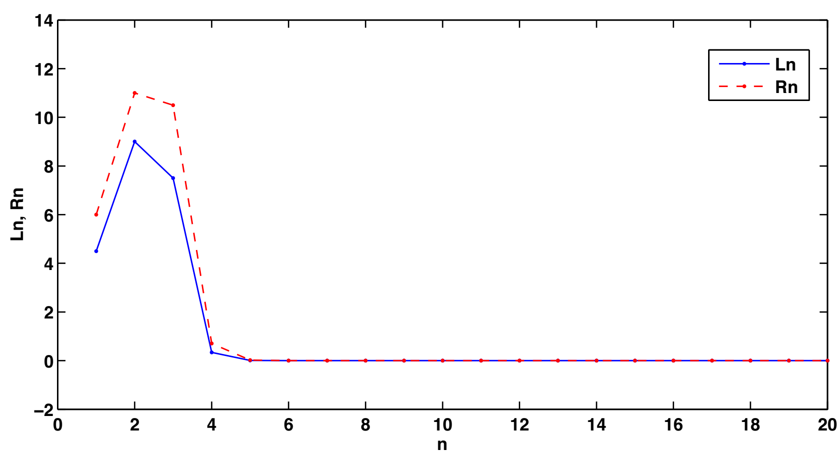

Figure 2: In Example 1, the solution of system (15) when $\alpha = 0.5$.

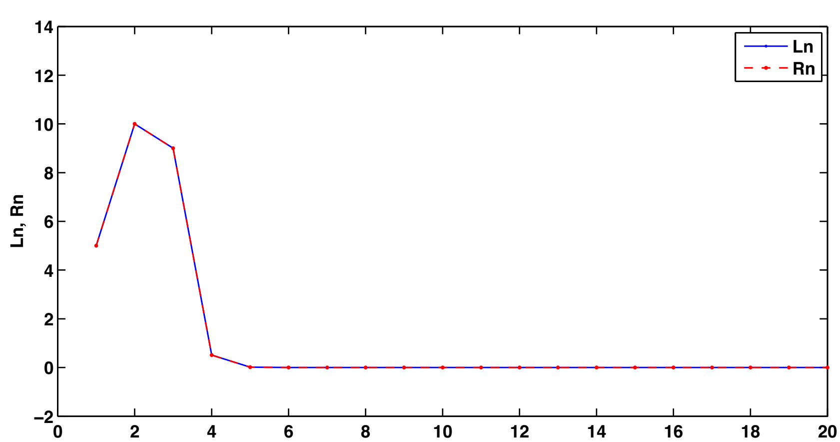

Figure 3: In Example 1, the solution of system (15) when $\alpha = 1$.

Remark 1. There exists positive fuzzy number A with $A_{l,\alpha} < 1$, such that the equilibrium point $\bar{x} = 0$ of (1) is not global asymptotically stable (see the following Examples 2–3).

Example 2. Let us still consider equation (14), whose parameter B and initial value x_1, x_2, x_3 are the same as those in Example 1, but parameter A changes to

$$A(t) = \begin{cases} 8t - 3, & \frac{3}{8} \le t \le \frac{1}{2}, \\ -8t + 5, & \frac{1}{2} \le t \le \frac{5}{8}. \end{cases} \tag{16}$$

So, we have $[A]_\alpha = \left[\frac{3}{8} + \frac{\alpha}{8}, \frac{5}{8} - \frac{\alpha}{8}\right]$ and $\overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha} = \left[\frac{3}{8}, \frac{5}{8}\right]$. It is clear that $A_{l,\alpha} < 1$, $A_{R,\alpha} < 1$, so Example 2 does not satisfy the conditions of Theorems 2.2 and 2.4. Let us see the behaviors of the solution through the following graph (Figure 4).

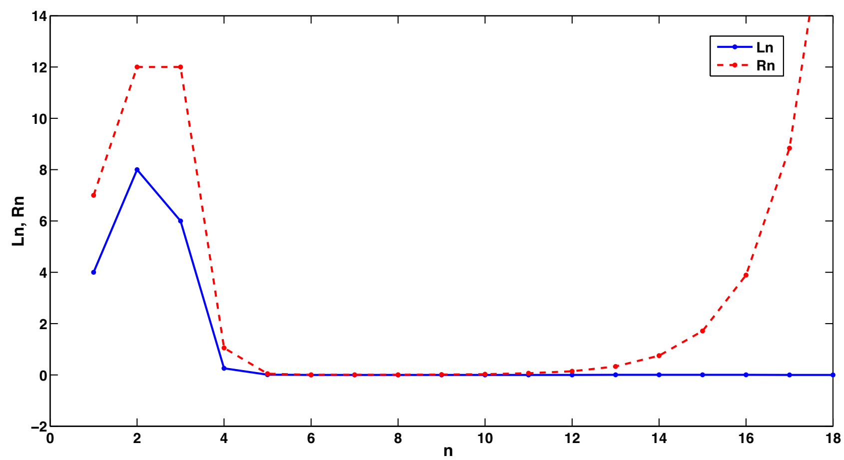

Figure 4: In Example 2, the solution of system (15) when $a = 0.5$.

In Figure 4, it is obvious that L_n tends to 0 and R_n tends to positive infinity when n is sufficiently large. So, the solutions of Example 2 do not converge and the equilibrium point $\bar{x} = 0$ of (1) is not global asymptotically stable.

Example 3. Let us still consider equation (14), where the initial values x_1, x_2, x_3 are the same as those in Example 1, but parameters A, B change to

$$A(t) = \begin{cases} 5t - 3, & \frac{3}{5} \le t \le \frac{4}{5}, \\ 1, & \frac{4}{5} \le t \le 2, \end{cases} \quad B(t) = \begin{cases} 5t - 1, & \frac{1}{5} \le t \le \frac{2}{5}, \\ \frac{9}{5} - 2t, & \frac{2}{5} \le t \le \frac{9}{10}. \end{cases} \tag{17}$$

So we have $[A]_\alpha = \left[0.6 + \frac{\alpha}{5}, 2\right]$, $\overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha} = [0.6, 2]$, $[B]_\alpha = \left[0.2 + \frac{\alpha}{5}, 0.9 - \frac{\alpha}{2}\right]$, and $\overline{\bigcup_{\alpha\in(0,1]}[B]_\alpha} = [0.2, 0.9]$. It is clear that $A_{l,\alpha} < 1$, $A_{R,\alpha} > 1$, Example 3 does not satisfy the conditions of Theorems 2.2 and 2.4. Let us see the behaviors of the solution through the following graph (Figure 5).

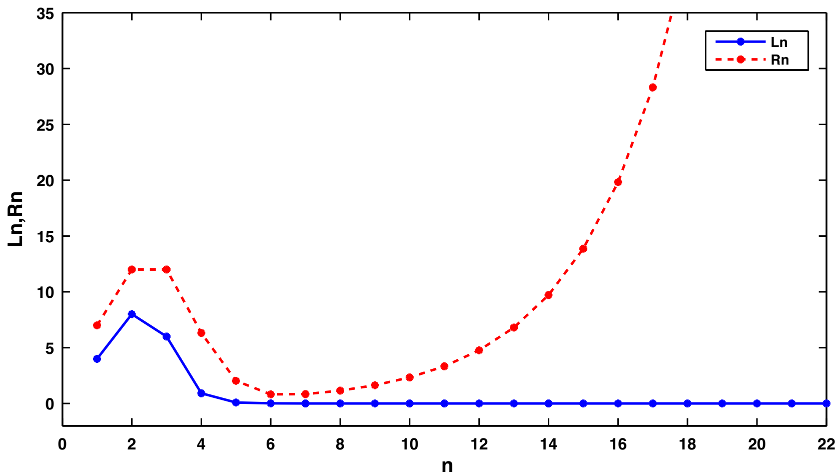

Figure 5: In Example 3, the solution of system (15) when $a = 0.5$.

In Figure 5, it is obvious that L_n tends to 0 and R_n tends to positive infinity when n is sufficiently large. So, the solutions of Example 3 do not converge, the equilibrium point $\bar{x} = 0$ of (1) is not global asymptotically stable.

In [17], Zhang studied a first-order fuzzy difference equation $x_{n+1} = \frac{A + x_n}{B + x_n}$. Later, he studied second-order exponential-type fuzzy difference equation $x_{n+1} = \frac{A + Be^{-x_n}}{C + x_{n-1}}$ in [27]. Similarly, we have also studied k-order exponential-type fuzzy difference equation which is related to equation (1):

Remark 2. We amend equation (1) to be exponential-type fuzzy difference equation

$$x_{n+1} = \frac{x_n e^{-x_{n-m}}}{A + Bx_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where $m \in \{0, 1, \ldots k\}$, parameters A, B and initial value $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $k \in \{0, 1, \ldots\}$ are positive fuzzy numbers. We can obtain the same conclusion as equation (1) on the existence, boundedness, convergence, and global asymptotic stability of the solution.

The proof method and process are similar to the conclusion proof of equation (1).

Remark 3. The existence and uniqueness of the positive solution is important for equation, which is the basis for discussing all of the dynamic characters of solutions. The commonly used method of proof is the definition method, and the proof of this article also uses the same method.

4 Conclusion

In this paper, we investigate the dynamical behaviors of positive solution of the fuzzy difference equation (1).

First, we have proved that for the fuzzy difference equation (1) there exists a unique positive solution. Then, using the relationship between the fuzzy difference equation and the relative real number difference equation system, we have proved that if $A_{l,\alpha} > 1$ then every positive solution of (1) is bounded and converges to equilibrium $x = 0$ when $n \to \infty$. Moreover, we obtain that the equilibrium $x = 0$ is global asymptotically stable. At last, we give numerical examples about equation (1) where $k = 2$ draw trajectories of the solutions to reflect the global behaviors of fuzzy difference equation intuitively.

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