DYNAMICAL COHERENCE OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS OF THE 3-TORUS

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ABSTRACT. We show that partially hyperbolic diffeomorphisms of the 3-torus are dynamically coherent.

1. INTRODUCTION AND FORMULATION OF RESULTS

The goal of this paper is to show that partially hyperbolic diffeomorphisms of the three-torus are *dynamically coherent*, that is that their center, center-stable, and center-unstable distributions are uniquely integrable.

Let M be a smooth, connected, compact 3-dimensional Riemannian manifold without boundary (a concrete choice of Riemannian metric is of no importance for the sequel). A C^1 diffeomorphism $f\colon M \to M$ is said to be *partially hyperbolic* if there are numbers $0 < \lambda < \gamma_1 \leq 1 \leq \gamma_2 < \mu$ and a df-invariant splitting of the tangent bundle

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^c(x), \qquad x \in M$$

into one-dimensional C^0 distributions E^s, E^u and E^c (called the *stable, unstable* and *center* distributions) such that

$$\begin{aligned}
&df(x)E^a(x) = E^a(f(x)) && \text{for} \quad a = s, u, c\\
&\|df(x)v^s\| \leq \lambda\|v^s\| && \text{for} \quad v^s \in E^s(x)\\
&\mu\|v^u\| \leq \|df(x)v^u\| && \text{for} \quad v^u \in E^u(x)\\
&\gamma_1\|v^c\| \leq \|df(x)v^c\| \leq \gamma_2\|v^c\| && \text{for} \quad v^c \in E^c(x).
\end{aligned}$$

Note that there is a subtle difference between the notions of partial hyperbolicity we employ here and in [BI]: we require here that the constants do not depend on a point in M. Both definitions are quite common, and we abuse terminology and use the same term for a slightly different notion in this paper.

The distributions E^s, E^u, and E^c are Hölder continuous but in general are not C^1 even if f is C^2 or better [Ano67]. We refer to the direct

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sums $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ as the center-stable and center-unstable distributions, respectively.

In this paper, by a C^0 *foliation with* C^1 *leaves* we mean a continuous foliation W of M whose leaves $W(x)$, $x \in M$, are C^1 and their tangent spaces $T_xW(x)$ depend continuously on $x \in M$. For such a foliation W, we denote by TW the tangent distribution of W, i.e., the collection of all tangent spaces to the leaves of W. Note that a C^0 foliation with C^1 leaves is not necessarily a C^1 foliation (as defined in terms of C^1 charts).

The stable E^s and unstable E^u distributions are integrable in the sense that there exist C^0 foliations W^s and W^u (called the *stable* and *unstable foliations*, respectively) such that $TW^s = E^s$ and $TW^u = E^u$. Moreover, the exponential contraction and expansion implies the uniqueness of integral manifolds: if a C^1 curve is everywhere tangent to E^s, then it lies in one leaf of W^s, and similarly for W^u.

By analogy with ordinary differential equations, one says that a continuous k-dimensional distribution E on a manifold M is *uniquely integrable* if there is a C^0 foliation W such that every C^1 curve $\sigma\colon \mathbb{R} \to M$ satisfying $\dot{\sigma}(t) \in E(\sigma(t))$ for all t, is contained in $W(\sigma(0))$.

The unique integrability of E^c, E^{cs}, and E^{cu} (which is referred to as *dynamical coherence*) are important assumptions in the theory of stable ergodicity for partially hyperbolic diffeomorphisms (see [PS97], [BPSW01]).

In higher dimensions, the center distribution E^c fails to be integrable even when the distributions are perfectly smooth (see [Wil98] for a counterexample). In general, it is not known whether the central distribution is uniquely integrable even if it is one-dimensional.

From now on, $M = \mathbb{T}^3$ is the 3-torus and f is a partially hyperbolic diffeomorphism of M. The following Theorem is the main result of this paper.

Theorem 1.1. *The distributions E^{cs}, E^{cu}, and E^c are uniquely integrable, i.e., f is dynamically coherent.*

By passing to a finite cover we may assume that E^s, E^u and E^c are oriented. Let W^s and W^u denote the one-dimensional foliations tangent to E^s and E^u (recall that these distributions are uniquely integrable). We use a tilde to denote lifts of objects to the universal cover $\mathbb{R}^3$ of $M = \mathbb{T}^3$.

Definition 1.2. Let W be a one dimensional foliation of $\mathbb{R}^3$. We say that W *has quasi-isometric leaves* if there exists a constant $C > 0$ such that every segment γ of a leaf of W with endpoints p and q satisfies

$$\text{length}(\gamma) \le C \cdot |p - q| + 1.$$

The main result of [Br] asserts that if the lifts of stable and unstable leaves of a partially hyperbolic diffeomorphism are quasi-isometric, then the diffeomorphism is dynamically coherent. We derive Theorem 1.1 from [Br] by showing that the lifts of E^s and E^u to the universal cover $\tilde{M}$ of M have quasi-isometric leaves (see Definition 1.2).

Note that in general, the stable and unstable foliations need not have quasi-isometric leaves. E.g., the stable and unstable foliations of the time 1 map of the geodesic flow on a compact surface of negative curvature are not quasi-isometric.

Theorem 1.3. *$\widetilde{W}^s$ and $\widetilde{W}^u$ have quasi-isometric leaves.*

By the main theorem of [Br], Theorem 1.3 implies Theorem 1.1.

The rest of the paper is devoted to a proof of Theorem 1.3.

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2. Preliminaries from [BI]

The proof is heavily based on the following constructions and assertions from [BI].

Existence of pre-foliations. A complete surface is a proper C^1 immersion $F : U \to M$, where U is a connected smooth 2-dimensional manifold without boundary, and the induced length metric on U is complete.

We say that a point $a \in U$ is a *lift* of a point $p \in M$ to F if $F(a) = p$. A curve $\tilde{\gamma} : I \to U$ (where I is an interval) is a *lift* of a curve $\gamma : I \to M$ if $\gamma = F \circ \tilde{\gamma}$. Of course, a lift of a curve is uniquely determined by a lift of its starting point.

A *neighborhood* of F is an immersion $\mathcal{F} : U \times \mathbb{R} \to M$ such that $\mathcal{F}(x, 0) = F(x)$ for all $x \in U$. We say that a curve $\gamma : I \to M$ *crosses* F if there is an interval $J \subset I$ such that $\gamma|_J$ can be represented as $\mathcal{F} \circ \tilde{\gamma}$ where $\mathcal{F}$ is a neighborhood of F and $\tilde{\gamma} : J \to U \times \mathbb{R}$ is a curve which intersects both $U \times (0, +\infty)$ and $U \times (-\infty, 0)$.

We say that surfaces F and G *topologically cross* if there is a curve which lies on F and crosses G. It is easy to see that this definition is symmetric with respect to F and G.

A *branching foliation* in M is a collection of complete open surfaces tangent to a continuous 2-dimensional distribution such that no two of the surfaces topologically cross and their images cover M.

Theorem 2.1. *(Theorem 4.1 from* [BI]*) There exist branching foliations tangent to E^{cs} and E^{cu} and invariant under any C^1 diffeomorphism $f : M \to M$ which preserves the oriented distributions E^s, E^c, and E^u.*

In the sequel, W^{cs} and W^{cu} denote branching foliations tangent to E^{cs} and E^{cu}. We refer to leaves of these branching foliations as cs- and cu-leaves.

No transverse contractible cycles. We say that a closed differentiable curve in M is a *transverse contractible cycle* if the curve is transverse to E^{cs} (or E^{cu}) and homotopic to a point.

Lemma 2.1. *(Lemma 2.3 from* [BI]*) There are no transverse contractible cycles.*

Partial hyperblicity of the induced map in first homologies. The following theorem is the main result of [BI].

Theorem 2.2. *(Theorem 1.2 from* [BI]*) The induced map f_* of the first homology group $H_1(M, \mathbb{R})$ is also partially hyperbolic, i.e., it has eigenvalues α_1 and α_2 with $|\alpha_1| > 1$ and $|\alpha_2| < 1$.*

We use these results as well as the notations throughout the proof of Theorem 1.1.

3. Proof of Theorem 1.3

We begin with several lemmas that refine the "no transverse cycles" assertion and provide basic information about cs- and cu-leaves.

Lemma 3.1. *A C^1 curve transverse to $\widetilde{E}^{cs}$ cannot intersect a leaf of $\widetilde{W}^{cs}$ more than once.*

Proof. Suppose that a curve γ transverse to $\widetilde{E}^{cs}$ connects points x and y on the same cs-leaf $S \subset \mathbb{R}^3$. Connect x and y by a C^1 curve γ_1 in S. Since the foliations are oriented, the loop $\gamma \cup \gamma_1$ can be perturbed into a C^1 loop transverse to E^{cs}, i. e. there exists a transverse contractible cycle. This contradicts to Lemma 2.1. □

Lemma 3.2. *The leaves of $\widetilde{W}^{cs}$ are properly embedded C^1-submanifolds of $\mathbb{R}^3$ and have uniformly bounded geometry in the following sense: there is a $\delta > 0$ such that every Euclidean ball of radius δ can be covered by a coordinate neighborhood U such that the intersection of* *every cs-leaf S with U is either empty or close to a planar disc. In the latter case, this intersection is represented in these coordinates as the graph of a function $h_S : \mathbb{R}^2 \to \mathbb{R}$.*

Proof. Fix a smooth unit vector field V on $\mathbb{T}^3$ almost orthogonal to E^{cs}. Let δ be so small that E^{cs} and V have almost constant directions in every ball of radius 10δ. Let $p \in \mathbb{R}^3$, S_0 be a cs-surface passing through p and B the intrinsic ball of radius 2δ in S_0 centered at p. Introduce coordinates (x, y) in B. Consider a map $\phi : B \times (-2\delta, 2\delta) \to \mathbb{R}^3$ defined by $\phi(q, z) = \gamma_q(z)$ where γ_q is the integral curve of V such that $\gamma_q(0) = q$. This map defines a C^1 coordinate system (x, y, z) in a neighborhood U containing the Euclidean ball of radius δ centered at p. By Lemma 3.1, a curve γ_q, $q \in B$, intersects every cs-surface S at most once, hence $U \cap S$ is a graph of a function defined in an open subset of $B \simeq \mathbb{R}^2$. To finish the proof, cut off the top and bottom of U by the inf and sup of all cs-leafs passing through $\gamma_p(2\delta)$ and $\gamma_p(-2\delta)$ respectively. □

Lemma 3.3. *There is a constant C such that, for every segment J of an unstable leaf, one has $vol(U_1(J)) \geq C \cdot length(J)$ where U_1 denotes the neighborhood of radius 1.*

Proof. Suppose the contrary, then for every $\varepsilon > 0$ there is a segment J of an unstable leaf such that length$(J) > 1$ and the distance between the endpoints of J is less than ε. Perturbing such a segment yields a transverse contractible cycle. This contradicts to Lemma 2.1. □

Proposition 3.4. *If S is a closed embedded cs-surface in $\mathbb{T}^3$, then*

1. *S is homeomorphic to the 2-torus;*
2. *S does not divide $\mathbb{T}^3$;*
3. *A homomorphism $i_* : \pi_1(S) \to \pi_1(\mathbb{T}^3)$ induced by inclusion $i : S \to \mathbb{T}^3$ is injective.*

Proof. The first assertion is trivial. Indeed, S is homeomorphic to the 2-torus since it is orientable and admits a nonzero tangent vector field (e. g., E^s).

Lemma 3.5. *Assertions 2 and 3 are equivalent.*

Proof. Note that S divides $\mathbb{T}^3$ if and only if the induced map $i_* : H_2(S) \to H_2(\mathbb{T}^3)$ is trivial. It remains to prove that $i_* : H_2(S) \to H_2(\mathbb{T}^3)$ is trivial if and only if $i_* : \pi_1(S) \to \pi_1(\mathbb{T}^3)$ is not injective. Since $\mathbb{T}^3$ is aspherical, the homotopy type of i is uniquely determined by i_*. Hence it suffices to prove the above equivalence for linear maps

$i : \mathbb{T}^2 \to \mathbb{T}^3$. It is straightforward to check that degenerate (resp. non-degenerate) linear maps $\mathbb{T}^2 \to \mathbb{T}^3$ induce the zero map (resp. nonzero maps) of the second homologies. $\square$

Reasoning by contradiction, we can now assume the negations of both assertions 2 and 3. There are two cases.

Case 1. $i_* : \pi_1(S) \to \pi_1(\mathbb{T}^3)$ is the zero map. Then S is contractible in $\mathbb{T}^3$. Consider a segment of a stable leaf in S that almost recurs to itself. One can close it up so that the resulting loop is transverse to E^u and hence to E^{cu}. This loop is contractible since S is, therefore we constructed a transverse (to E^{cu}) contractible loop, contrary to Lemma 2.1 (applied to f^{-1}).

Case 2. $i_* : \pi_1(S) \to \pi_1(\mathbb{T}^3)$ is a nonzero map. Then $\operatorname{rank} i_* = 1$. Recall that S divides $\mathbb{T}^3$ into two components U and V. Since the set of cs-leaves is compact in the compact-open topology, we may assume that one of the components (say, U) has the minimal volume among all regions bounded by closed cs-leaves. Then U (and hence S) is f^n-invariant for some n. Replacing f by f^n we may assume that S is f-invariant. We need the following lemma from [BBI].

Lemma 3.6. *(*[BBI]*, Proposition 2.1). If S is an f-invariant 2-torus, then the induced action f_* of f on $\pi_1(S) = \mathbb{Z}^2$ is hyperbolic, that is, it has eigenvalues $|\alpha_1| > 1$ and $|\alpha_2| < 1$.*

On the other hand, the rank-1 subgroup $\ker i_* \subset \pi_1(S)$ is f_*-invariant. Hence the induced action of f on $\pi_1(S)$ has an eigenvalue $\alpha = \pm 1$, a contradiction. $\square$

Proposition 3.7. *There is a cs-leaf $S = S_{cs}$, a plane $P_{cs} \subset \mathbb{R}^3$ and $R > 0$ such that S is contained in the R-neighborhood of P_{cs} and separates $\mathbb{R}^3$ into two components each of which is contained in the R-neighborhood of a half-space bounded by P_{cs}.*

Proof. By Lemma 3.2, every cs-leaf S is properly embedded, hence it separates $\mathbb{R}^3$ into two open components which we denote by S_+ and S_-, where the positive direction of E^u points inward S_+.

Lemma 3.8. *Let S, S' be two cs-leaves. Then*

1. *One of the following possibilities holds:*

$$S'_+ \subset S_+, \qquad S_+ \subset S'_+, \qquad S_+ \cap S'_+ = \emptyset, \qquad S_- \cap S'_- = \emptyset.$$

2. *In the latter two cases the surfaces S and S' are disjoint.*
3. *In the last case $S_+ \cup S'_+ = \mathbb{R}^3$.*

Proof. 1. Since cs-leaves have no topological crossings, S' is contained in one of the closures $\overline{S}_+$ and $\overline{S}_-$. Suppose that $S' \subset \overline{S}_+$. Then S_-

is entirely contained in one of the components S'_+ or S'_-, and this corresponds to the first and last possibilities. Analogously the remaining two cases correspond to the case $S' \subset \overline{S}_+$.

2. Arguing by contradiction, assume that S and S' are not disjoint. Then the contradiction is obtained by inspecting the orientation of E^u at a point where S and S' touch each other.

3. Follows from (2). $\square$

Fix a cs-leaf S. Observe that $\widetilde{W}^{cs}$ is a branching foliation invariant under translations by integer vectors and hence for every $\mathbf{k} \in \mathbb{Z}^3$ the surface $S + \mathbf{k}$ is again a cs-leaf. Then Lemma 3.8 applies to S and $S' = S + \mathbf{k}$. Since translations by integer vectors preserve the orientation of E^u, we have $(S + \mathbf{k})_+ = S_+ + \mathbf{k}$. Define

$$\begin{aligned} \Gamma_+ &= \{\mathbf{k} \in \mathbb{Z}^3 : S_+ + \mathbf{k} \subset S_+\}, \\ \Gamma_- &= \{\mathbf{k} \in \mathbb{Z}^3 : S_+ + \mathbf{k} \supset S_+\}, \\ \Gamma &= \Gamma_+ \cup \Gamma_-. \end{aligned}$$

If $\mathbf{k} \in \mathbb{Z}^3 \setminus \Gamma$, only the last two possibilities from (1) from Lemma 3.8 can occur, and hence by (2) from Lemma 3.8 the surfaces S and $S + \mathbf{k}$ are disjoint.

Lemma 3.9. *Γ is a subgroup of $\mathbb{Z}^3$.*

Proof. Obviously Γ_+ and Γ_- are semi-groups and $\Gamma_- = -\Gamma_+$. It remains to prove that $\mathbf{k}_1 - \mathbf{k}_2 \in \Gamma$ for $\mathbf{k}_1, \mathbf{k}_2 \in \Gamma_+$. Since both sets $S_+ + \mathbf{k}_1$ and $S_+ + \mathbf{k}_2$ are contained in S_+, they cannot cover $\mathbb{R}^3$, and by Lemma 3.8 this leaves only three possibilities: one of the surfaces is contained in the other or they are disjoint. Consider the set $S_+ + \mathbf{k}_1 + \mathbf{k}_2$. It is contained in $S_+ + \mathbf{k}_2$ since $S_+ + \mathbf{k}_1 \subset S_+$. Similarly, the same set is contained in $S_+ + \mathbf{k}_1$. Therefore $S_+ + \mathbf{k}_1$ and $S_+ + \mathbf{k}_2$ have nonempty intersection, hence one of them is contained in the other. Assume for definiteness that $S_+ + \mathbf{k}_1 \subset S_+ + \mathbf{k}_2$. Then $S_+ + (\mathbf{k}_1 - \mathbf{k}_2) \subset S_+$, hence $\mathbf{k}_1 - \mathbf{k}_2 \in \Gamma_+$. $\square$

Lemma 3.10. *Proposition 3.7 is true if there is a point $x_0 \in \mathbb{R}^3$ such that the lattice $x_0 + \mathbb{Z}^3$ is contained in at least one of the sets $\overline{S}_+$ or $\overline{S}_-$.*

Proof. Suppose that $x_0 + \mathbb{Z}^3 \subset \overline{S}_+$. Consider the set

$$\widetilde{A} = \bigcap_{\mathbf{k} \in \mathbb{Z}^3} (\overline{S}_+ + \mathbf{k}).$$

It is nonempty (since it contains x_0), closed and consists of entire cs-leaves.

Let $p \in \mathbb{R}^3$. Choose a local coordinate system (x, y, z) in a neighborhood of p as in Lemma 3.2. That is, for every cs-leaf $S + \mathbf{k}$ intersecting this neighborhood, its intersection with the neighborhood is a graph $z = h_{\mathbf{k}}(x, y)$ of a function $h_{\mathbf{k}} : \mathbb{R}^2 \to \mathbb{R}$. Then the intersection of $S_+ + \mathbf{k}$ with the coordinate neighborhood is the epigraph of $h_{\mathbf{k}}$. The intersection of the epigraphs is the epigraph of a function $h = \sup_{\mathbf{k}} h_{\mathbf{k}}$.

Therefore $\widetilde{A}$ is a 3-dimensional submanifold with boundary in $\mathbb{R}^3$ and moreover its boundary is a limit (in the compact-open topology) of cs-leaves. Since our branching foliation is complete, the boundary of $\widetilde{A}$ is a union of cs-leaves. Since $\widetilde{A}$ is invariant under integer translations, it projects down to a submanifold A of $\mathbb{T}^3$ bounded by closed cs-leaves. Let T be a boundary component of A. Since T admits a nonzero tangent vector field, it is homeomorphic to the 2-torus. By Proposition 3.4, the map $i_* : \pi_1(T) \to \pi_1(\mathbb{T}^3)$ induced by the inclusion $i : T \to \mathbb{T}^3$ is injective. Hence i is homotopic to a non-degenerate linear map from $T \simeq \mathbb{T}^2$ to $\mathbb{T}^3$. Then any lift of T to $\mathbb{R}^3$ stays within a bounded distance from a plane and can be taken as a desired cs-leaf C_{cs}. □

Now we can assume that the assumption of Lemma 3.10 does not hold, that is, for every $x \in \mathbb{R}^3$ the lattice $x + \mathbb{Z}^3$ intersects both S_+ and S_-. This means that

$$\bigcup_{\mathbf{k} \in \mathbb{Z}^3} (S_+ + \mathbf{k}) = \bigcup_{\mathbf{k} \in \mathbb{Z}^3} (S_- + \mathbf{k}) = \mathbb{R}^3. \tag{1}$$

Lemma 3.11. *$\Gamma = \mathbb{Z}^3$.*

Proof. Suppose the contrary and let $\mathbf{k}_0 \in \mathbb{Z}^3 \setminus \Gamma$. For definiteness assume that $S_+ \cap (S_+ + \mathbf{k}_0) = \emptyset$. Then for every $\mathbf{k} \in \mathbb{Z}^3 \setminus \Gamma$ one has $S_+ \cap (S_+ + \mathbf{k}) = \emptyset$. Indeed, suppose that there exists $\mathbf{k} \in \mathbb{Z}^3 \setminus \Gamma$ such that $S_- \cap (S_- + \mathbf{k}) = \emptyset$. Then $S_+ + \mathbf{k} \supset S_- \supset S_+ + \mathbf{k}_0$. Pick a point $x \in S_+ \cap (S_+ + \mathbf{k})$. Then $x + \mathbf{k}_0 \in S_+ + \mathbf{k}_0 \subset S_+ + \mathbf{k}$ and $x + \mathbf{k}_0 \in S_+ + \mathbf{k} + \mathbf{k}_0$. Thus $S_+ + \mathbf{k}$ and $S_+ + \mathbf{k} + \mathbf{k}_0$ have nonempty intersection, then so do S_+ and $S_+ + \mathbf{k}_0$, a contradiction.

It follows that for every pair $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^3$ the sets $S_+ + \mathbf{k}_1$ and $S_+ + \mathbf{k}_2$ are either disjoint (if $\mathbf{k}_1 - \mathbf{k}_2 \notin \Gamma$) or nested (if $\mathbf{k}_1 - \mathbf{k}_2 \in \Gamma$). Consider the set $U := \bigcup_{\mathbf{k} \in \Gamma} (S_+ + \mathbf{k})$. Then for every pair $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^3$ the sets $U + \mathbf{k}_1$ and $U + \mathbf{k}_2$ are either disjoint (if $\mathbf{k}_1 - \mathbf{k}_2 \notin \Gamma$) or coincide (if $\mathbf{k}_1 - \mathbf{k}_2 \in \Gamma$). Since these sets are open and $\mathbb{R}^3$ is connected, they cannot cover $\mathbb{R}^3$, contrary to (1). □

Let $\Gamma_0 = \Gamma_+ \cap \Gamma_-$. Obviously Γ_0 is a subgroup of $\mathbb{Z}^3$ and S is invariant under Γ_0. If $\operatorname{rank}(\Gamma_0) = 3$ then S projects down to a closed cs-leaf in

$\mathbb{T}^3$ and the proposition follows similarly to the proof of Lemma 3.10. From now on we assume that $\text{rank}(\Gamma_0) < 3$.

Lemma 3.12. *Γ_+ and Γ_- are half-lattices. That is, there is a plane $P \subset \mathbb{R}^3$ (containing the origin) such that each set Γ_+ and Γ_- is contained in a half-space bounded by P.*

Proof. Let A_+ and A_- be the convex hulls of Γ_+ and Γ_-, respectively. If the interiors of A_+ and A_- are disjoint, they are separated by a plane since they are nonempty open convex sets. This plane is a desired P.

It remains to consider the case when the intersection $A_+ \cap A_-$ has a nonempty interior. Pick three linearly independent vectors $y_1, y_2, y_3 \in A_+ \cap A_-$ with rational coordinates. Each y_i is a positive rational combination of points from Γ_+ and at the same time a positive rational combination of points from Γ_-. Hence a positive multiple of y_i belongs to Γ_0 for $i = 1, 2, 3$. Then $\text{rank}(\Gamma_0) = 3$, contrary to our assumption. □

For $x \in \mathbb{R}^3$ let $\mathcal{O}_+(x) = (x + \mathbb{Z}^3) \cap \overline{S}_+$ and $\mathcal{O}_-(x) = (x + \mathbb{Z}^3) \cap \overline{S}_-$. We have $\mathcal{O}_+(x) \neq \emptyset$ and $\mathcal{O}_-(x) \neq \emptyset$ (otherwise apply Lemma 3.10). Observe that $\mathcal{O}_+(x) + \Gamma_+ \subset \mathcal{O}_+(x)$ and $\mathcal{O}_-(x) + \Gamma_- \subset \mathcal{O}_-(x)$. Since Γ_+ and Γ_- are half-lattices separated by a plane P (cf. Lemma 3.12), it follows that the sets $\mathcal{O}_+(x)$ and $\mathcal{O}_-(x)$ are non-strictly separated by a plane P_x parallel to P. Choose a $\delta > 0$ as in Lemma 3.2 and let $\{x_i\}$, $i = 1, \dots, N$, be a $(\delta/2)$-net in the fundamental domain of Γ. Let $P_i = P_{x_i}$ and let H_i^+ and H_i^- denote the half-spaces of P_i containing $\mathcal{O}_+(x_i)$ and $\mathcal{O}_-(x_i)$ respectively. Then S_+ is contained in the δ-neighborhood of the half-space $\bigcup_k H_i^+$ and S_- is contained in the δ-neighborhood of the half-space $\bigcup_k H_i^-$. The intersection of these half-spaces is a slice between two parallel planes and S lies in this slice. This finishes the proof of Proposition 3.7. □

Corollary 3.13. *There is $R_1 > 0$ such that every cs-leaf lies in the R_1-neighborhood of a plane parallel to P_{cs}.*

Proof. Let $\mathbf{k} \in \mathbb{Z}^3$ be such that $dist(k, P_{cs}) > 2R$. The surfaces $P_{cs} + n\mathbf{k}$, $n \in \mathbb{Z}$, split $\mathbb{R}^3$ into components each of which lies in a slice of width $dist(k, P) + 2R$ between two planes parallel to P_{cs}. □

Swapping "stable" and "unstable" we obtain a plane P_{cu} such that

(a) every cu-leaf lies within a uniformly bounded distance from a plane parallel to P_{cu}, and

(b) there exists a cu-leaf S_{cu} separating $\mathbb{R}^3$ into two regions each of which contains a half-space.

We may assume that the planes P_{cs} and P_{cu} contain the origin of $\mathbb{R}^3$.

Fix a lift $\widetilde{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ of f. The branching foliations $\widetilde{W}^{cs}$ and $\widetilde{W}^{cu}$ are invariant under $\widetilde{f}$. Let $f_*\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the induced linear transformation of the first homology group. Note that $|\widetilde{f}(x) - f_*(x)| \le \mathit{const}$ for all $x \in \mathbb{R}^3$.

Lemma 3.14. *The planes P_{cs} and P_{cu} are invariant under f_*.*

Proof. Note that $\widetilde{f}(S_{cs})$ lies within bounded distance from $f_*(P_{cs})$. Hence if $f_*(P_{cs}) \ne P_{cs}$, the surfaces S_{cs} and $\widetilde{f}(S_{cs})$ would have a topological crossing, contradiction. □

Proposition 3.15. $P_{cs} \ne P_{cu}$.

Proof. Let α_i, $i = 1,2,3$, be the eigenvalues of f_*, $|\alpha_1| \ge |\alpha_2| \ge |\alpha_3|$. Recall that $|\alpha_1| > 1$ and $|\alpha_3| < 1$.

We assume that $|\alpha_2| \le 1$ (if $|\alpha_2| > 1$, just use f^{-1} instead of f). Then α_1 is a single real eigenvalue. Passing to f^2 if necessary, we may assume that $\alpha_1 > 0$, then $\alpha_1 > 1$. We decompose $\mathbb{R}^3 = E_1 \oplus E_{2,3}$ where E_1 is the eigenline of f^* corresponding to α_1 and $E_{2,3}$ is the invariant plane corresponding to α_2 and α_3. We are going to show that $P_{cs} = E_{2,3}$ and $P_{cu} \supset E_1$.

Let $\mathrm{Pr}_1 : \mathbb{R}^3 \to E_1$ and $\mathrm{Pr}_{2,3} : \mathbb{R}^3 \to E_{23}$ be the projections defined by this decomposition. Since $\alpha_1 > 1$, $|\alpha_3| < 1$ and $|\alpha_2| \le 1$, there is a norm $\|\cdot\|$ on $\mathbb{R}^3$ such that

$$\|\mathrm{Pr}_1(f_*(x) - f_*(y))\| = \alpha_1 \cdot \|\mathrm{Pr}_1(x-y)\|,$$
$$\|\mathrm{Pr}_{2,3}(f_*(x))\| \le \|\mathrm{Pr}_{2,3}(x)\|.$$

for all $x \in \mathbb{R}^3$. Since $\|\widetilde{f} - f_*\| \le \mathit{const}$, these inequalities imply that

$$\|\mathrm{Pr}_1(\widetilde{f}^n(x))\| \le \alpha_1^n \cdot \|\mathrm{Pr}_1(x)\| + C_0 \tag{2}$$

$$\|\mathrm{Pr}_1(\widetilde{f}^n(x) - \tilde{f}^n(y))\| \ge \alpha_1^n \cdot (\|\mathrm{Pr}_1(x-y)\| - C_0), \tag{3}$$

$$\|\mathrm{Pr}_{2,3}(\widetilde{f}^n(x))\| \le \|x\| + nC_0. \tag{4}$$

for all $n \ge 1$ and some constant C_0.

Let J be a unit-length interval of an unstable leaf. Then (2) and (4) imply that

$$\mathrm{diam}\ \mathrm{Pr}_1(\widetilde{f}^n(J)) \le \mathit{const} \cdot \alpha_1^n,$$
$$\mathrm{diam}\ \mathrm{Pr}_{2,3}(\widetilde{f}^n(J)) \le \mathit{const} \cdot n.$$

Hence $\widetilde{f}^n(J)$ is contained in a tube of volume $\mathit{const} \cdot n^2 \alpha_1^n$. On the other hand, $\mathrm{length}(\widetilde{f}^n(J)) \ge \mu^n$ where μ is the constant from the definition of partial hyperbolicity. Then Lemma 3.3 implies that $\alpha_1 \ge \mu$.

Suppose that $P_{cs} \neq E_{2,3}$. Then there exist $x, y \in S_{cs}$ such that $\|\mathrm{Pr}_1(x-y)\| > C_0$. Then by (3) we have

$$\|\widetilde{f}^n(x) - \widetilde{f}^n(y)\| \geq \mathit{const} \cdot \alpha_1^n.$$

On the other hand, x and y are connected by a curve γ tangent to $\widetilde{E}^{cs}$, hence

$$\|\tilde{f}^n(x) - \tilde{f}^n(y)\| \leq \mathit{const} \cdot \mathrm{length}(\tilde{f}^n(\gamma)) \leq \mathit{const} \cdot \gamma_2^n \ll \mu^n \leq \alpha_1^n$$

where γ_2 and μ are from the definition of partial hyperbolicity. This contradiction shows that $P_{cs} = E_{2,3}$.

Suppose that $E_1 \not\subset P_{cu}$. Then $P_{cu} = E_{2,3}$ since P_{cu} is f_*-invariant (by Lemma 3.14) and $E_{2,3}$ is the only f_*-invariant plane not containing E_1. Let J be an interval of an unstable leaf. Then for every $n \geq 0$, $\widetilde{f}^n(J)$ lies within a uniformly bounded distance from a plane parallel to $E_{2,3}$ Then (4) implies that $\widetilde{f}^n(J)$ lies within a uniformly bounded distance from a two-dimensional ball of radius $\mathit{const} \cdot n$. On the other hand, the length of $\widetilde{f}^n(J)$ grows exponentially in n. This contradicts Lemma 3.3. Thus $E_1 \subset P_{cu}$

Since $P_{cs} = E_{2,3}$ and $E_1 \subset P_{cu}$, it follows that $P_{cs} \neq P_{cu}$. □

Denote $L_c = P_{cs} \cap P_{cu}$. Proposition 3.15 implies that L_c is an eigenline of f_*. Let α_c denote the corresponding eigenvalue of f_*.

Lemma 3.16. $|\alpha_c| \leq \max\{1, \gamma_2\}$ *where* γ_2 *is the maximum expansion in the central distribution.*

Proof. Consider surfaces S_{cs} and S_{cu} from Proposition 3.7. Their intersection is a union of curves tangent to $\widetilde{E}_c$. Recall that each of S_{cs} and S_{cu} splits $\mathbb{R}^3$ into components containing half-spaces (bounded by planes parallel to P_{cs} and P_{cu}). This implies that the intersection $S_{cs} \cap S_{cu}$ contains an unbounded component U. The images $\widetilde{f}^n(U)$, $n \geq 0$, stay within a uniformly bounded distance from lines parallel $\widetilde{f}^n(x) + L_c$. Let $x, y \in U$ be sufficiently far away from each other. Then an argument similar to the proof of Proposition 3.15 shows that

$$\|\widetilde{f}^n(x) - \widetilde{f}^n(y)\| \geq \mathit{const} \cdot |\alpha_c|^n$$

provided that $|\alpha_c| > 1$. On the other hand, for the segment J of U connecting x and y we have

$$\mathrm{length}(\widetilde{f}^n(J)) \leq \mathit{const} \cdot \gamma_2^n.$$

Thus $|\alpha_c| \leq \gamma_2$ if $|\alpha_c| > 1$. □

Proof of Theorem 1.3. Let us prove the theorem for unstable leaves (for stable ones replace f by f^{-1}). Fix a vector $\mathbf{k} \in \mathbb{R}^3$ such that $dist(\mathbf{k}, P_{cs}) > 3R$ where R is from Proposition 3.7. The surfaces $S_{cs} + i\mathbf{k}$, $i \in \mathbb{Z}$, divide $\mathbb{R}^3$ into regions each of which is contained in a slice of width $5R$ between two planes parallel to P_{cs} and contains a similar slice of width R.

Lemma 3.17. *There is $L_0 > 0$ such that every interval of length L_0 of $\widetilde{W}^u$ intersects at least one of the surfaces $S_{cs} + i\mathbf{k}$, $i \in \mathbb{Z}$.*

Proof. Suppose the contrary, then for every $n \geq 0$ there exist a unit-length interval J_n of an unstable leaf such that $\widetilde{f}^n(J_n)$ does not intersect any of these surfaces and therefore is contained in the $5R$-neighborhood of a plane parallel to P_{cs}. By Corollary 3.13 (with swapped "stable" and "unstable") the unstable leaves lie within uniformly bounded distance from planes parallel to P_{cu}. Hence the curves $\widetilde{f}^n(J_n)$ lie within uniformly bounded distance from lines parallel to P_c. Then an iteration argument similar to those in Proposition 3.15 and Lemma 3.16 shows that

$$diam(\widetilde{f}^n(J_n)) \leq const \cdot |\alpha_c|^n \leq const \cdot \gamma_2^n.$$

Hence the 1-neighborhood $U_1(\widetilde{f}^n(J_n))$ of $\widetilde{f}^n(J_n)$ lies within uniformly bounded distance from a segment of length $const \cdot \gamma_2^n$, and therefore the volume of $U_1(\widetilde{f}^n(J_n))$ is at most $const \cdot \gamma_2^n$. On the other hand, the length of $\widetilde{f}^n(J_n)$ is at least $\mu^n \gg \gamma_2^n$. This contradicts Lemma 3.3. $\square$

Since every unstable leaf intersects a cs-leaf at most once (by Lemma 3.1), the above lemma implies that every unstable leaf of length nL_0 intersects at least n of the surfaces $S_{cs} + i\mathbf{k}$, $i \in \mathbb{Z}$ and therefore the distance between its endpoints is at least $(n-1)R$. Thus the foliation $\widetilde{W}^u$ has quasi-isometric leaves.

This completes the proof of Theorem 1.3. As we have already mentioned, Theorem 1.3 implies Theorem 1.1.

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