Dynamical Degrees of Birational transformations of projective surfaces

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Joint work with S. Cantat (Rennes)

Dynamical degree

In this talk, we work with projective algebraic varieties (mostly smooth surfaces), defined over an algebraically closed field **k**.

Definition

Let X be a smooth projective surface, $f \in \mathrm{Bir}(X)$. The *dynamical degree* $\lambda(f) \in \mathbb{R}$ of f is given by

$$\lambda(f) = \lim_{n\to\infty} \| (f^n)_* \|^{1/n} ;$$

where $(f^n)_* \in \mathrm{End}(\mathrm{NS}_{\mathbb{R}}(X))$ is the action induced by $f^n \in \mathrm{Bir}(X)$.

Remark

For each ample divisor D, we have

$$\lambda(f) = \lim_{n\to\infty} (D \cdot (f^n)_* D)^{1/n} ,$$

In particular, if $X = \mathbb{P}^2$, then $\lambda(f) = \lim \deg(f^n)^{1/n}$.

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Properties of the dynamical degree

- The dynamical degree is invariant under conjugation: if $f \in \mathrm{Bir}(X)$ and $g\colon X \dashrightarrow Y$ is birational, then $\lambda(f) = \lambda(gfg^{-1})$.
- When $\mathbf{k} = \mathbb{C}$, $\log(\lambda(f))$ is an upper bound for the topological entropy of $f\colon X(\mathbb{C}) \dashrightarrow X(\mathbb{C})$ and is often equal to it.
- The dynamical degree measures in some sense the complexity of the dynamics of X.
- A feature of our results may be summarized by the following slogan: *Precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f.*

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Pisot and Salem numbers

Let $f \in \mathrm{Bir}(X)$ (X a smooth projective surface).

- Diller-Favre: By blowing-ups $Y \to X$, one can conjugate f to $g \in \mathrm{Bir}(Y)$, such that $(g^n)_* = (g_*)^n$ for each $n \geq 0$ (g is called algebraically stable).
- Hence, $\lambda(f) = \lambda(g)$ is the eigenvalue of a matrix defined over $\mathbb{Z}$. More precisely, one gets:

Theorem (Diller-Favre 2001)

Le f be a birational transformation of a projective surface. If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Salem or a Pisot number.

- A *Pisot number* is an algebraic integer $\lambda \in \,]1, \infty[$ whose other Galois conjugates lie in the open unit disk.
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Example

Integers $d \geq 2$.
Reciprocal quadratic integers > 1 (solutions of $x^2 + 1 = tx$, $t \in \mathbb{Z}$).
The *plastic number*, or *padovan number*, $\lambda_P \simeq 1.324717$, root of $x^3 = x + 1$. *This is the smallest Pisot number.*

The set $\mathrm{Pis} \subset \mathbb{R}$ of Pisot numbers is closed.
The smallest accumulation point is the golden mean $\lambda_G = (1 + \sqrt{5})/2$.
All Pisot numbers between λ_P and λ_G have been listed.

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Example

The *Lehmer number*, $\lambda_L \simeq 1.176280$, unique root > 1 of the irreducible polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Our present knowledge of Salem numbers is weaker than for Pisot numbers.

Conjecturally, the infimum of Sal $\subset \mathbb{R}$ is larger than 1, and should be equal to λ_L.

Every Pisot number is the limit of a sequence of Salem numbers.

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Let $d \in \mathbb{N}$, and $f \in \mathrm{Aut}(\mathbb{A}^2)$ given by $f\colon (x,y) \dashrightarrow (y + x^d, x)$. Then, $\deg(f^n) = d^n$ for each $n \geq 1$, so $\lambda(f) = d$.

Example

Let $f \in \mathrm{Bir}(\mathbb{A}^2)$ be given by $f\colon (x,y) \dashrightarrow (x^a y^b, x^c y^d)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Then, $\lambda(f)$ is the highest modulus of the eigenvalues of A in $\mathbb{C}$. This yields all reciprocal quadratic integers.

Example (Bedford-Kim-McMullen)

There exists a projective surface X, obtained by blowing-up 10 points of $\mathbb{P}^2$, such that $\mathrm{Aut}(X)$ contains an element f with $\lambda(f) = \lambda_L$ (Lehmer number).

The same holds on some K3 surfaces.

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Salem numbers and automorphisms

Theorem (Diller-Favre 2001)

If f is an automorphism of a projective surface, then $\lambda(f)$ is either 1, reciprocical quadratic or a Salem number.

Theorem (McMullen 2007)

If f is an automorphism of a projective surface, $\lambda(f) = 1$ or $\lambda(f) \geq \lambda_L$.

Theorem (B.-Cantat 2013)

If f is a birational transformation of a surface and $\lambda(f)$ is a Salem number, then f is conjugate to an automorphism of a projective surface.

The result is false if $\lambda(f) = 1$ or if $\lambda(f)$ is quadratic: *there are examples conjugate to automorphisms and examples which are not.*

Corollary (Gap property)

If f is a birational transformation of a surface, $\lambda(f) = 1$ or $\lambda(f) \geq \lambda_L$.

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Applications of the gap property

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This has some applications.

Corollary

If $f \in \mathrm{Bir}(\mathbb{P}^2)$ is such that $\lambda(f) > 1$, its centraliser $C(f)$ satisfies that

$$C(f)/\langle f \rangle$$

is a finite group.

Corollary

Two elements $f, g \in \mathrm{Bir}(\mathbb{P}^2)$ with $\lambda(f), \lambda(g) > 1$ are conjugate iff they are conjugate by an element of degree $\leq (2 \max\{\deg f, \deg g\})^{57}$.

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Non-rational surfaces

If X is a surface, we write $\Lambda(X) = \{\lambda(f) \mid f \in \mathrm{Bir}(X)\}$.

Theorem

Let X be a projective surface that is not rational. Then

1. $\Lambda(X) = \{1\}$ *if X is not birationally equivalent to an abelian surface, a K3 surface, or an Enriques surface;*
2. $\Lambda(X) \setminus \{1\}$ *is made of quadratic integers and of Salem numbers of degree at most 6 (resp. 22, resp. 10) if X is an abelian surface (resp. a K3 surface, resp. an Enriques surface).*
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(*When* $\mathrm{char}(\mathbf{k}) = 0$, *the bounds are* $4, 20, 10$ *instead of* $6, 22, 12$.)

The interesting case is then to study $\Lambda(\mathbb{P}^2)$, which is not discrete (every Pisot number in $\Lambda(\mathbb{P}^2)$ is in fact an accumulation point).

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Lower semi-continuity

Fixing the degree d, the set $\mathrm{Bir}_d(\mathbb{P}^2)$ is an algebraic variety.

Theorem (J. Xie 2011)

The dynamical degree

$$\lambda\colon \mathrm{Bir}_d(\mathbb{P}^2) \to [1, +\infty[$$

is lower semi-continous for the Zariski topology.

The above results says that the sets $\{f \in \mathrm{Bir}_d(\mathbb{P}^2) \mid \lambda(f) \leq C\}$ are closed, for each $d \in \mathbb{N}$, $C \in \mathbb{R}$.
The topology of an algebraic variety being noetherian, we find that

$$\{\lambda(f) \mid f \in \mathrm{Bir}_d(\mathbb{P}^2)\}$$

is a well-ordered subset of $\mathbb{R}$ (i.e. satisfying the DCC condition).

Question

Is this true for $\Lambda(\mathbb{P}^2) = \{\lambda(f) \mid f \in \mathrm{Bir}(\mathbb{P}^2)\}$?

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Minimal degree in a conjugacy class

Definition

Let $f \in \mathrm{Bir}(\mathbb{P}^2)$. We define

$$\mathrm{mcdeg}(f) = \min_{g \in \mathrm{Bir}(\mathbb{P}^2)} \deg(g \circ f \circ g^{-1})$$

Remark

$1 \leq \lambda(f) \leq \mathrm{mcdeg}(f) \leq \deg(f)$.

Theorem

Let $f \in \mathrm{Bir}(\mathbb{P}^2)$.

1. *If* $\lambda(f) \geq 10^6$ *then* $\mathrm{mcdeg}(f) \leq 4700\,\lambda(f)^5$.
2. *If* $\lambda(f) > 1$, *then* $\mathrm{mcdeg}(f) \leq e^{18}\lambda(f)^{345}$.

On the other hand, there are sequences of elements $f_n \in \mathrm{Bir}(\mathbb{P}^2)$ such that $\mathrm{mcdeg}(f_n)$ goes to $+\infty$ with n while $\lambda_1(f_n) = 1$ for all n.

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The descending chain condition

Theorem

The dynamical spectrum $\Lambda(\mathbb{P}^2) \subset \mathbb{R}$ is well ordered, and it is closed if the ground field **k** *is uncountable.*

Corollary

Let Λ be the set of all dynamical degrees of birational transformations of projective surfaces, defined over any field. Then,

(1) *Λ is a well ordered subset of $\mathbb{R}_{\geq 1}$;*

(2) *if λ is an element of Λ, there is a real number $\epsilon > 0$ such that $]\lambda, \lambda + \epsilon]$ does not intersect Λ;*

(3) *there is a non-empty interval $]\lambda_G, \lambda_G + \epsilon]$, on the right of the golden mean, that contains infinitely many Pisot and Salem numbers but does not contain any dynamical degree.*

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