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DYNAMICAL FRICTION IN SPHERICAL CLUSTERS

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SUMMARY

The effect of dynamical friction on the density profile of the most massive galaxies in a cluster is calculated both for an isothermal model cluster and for Plummer's model. The resulting profiles show a depletion of massive objects near the centre, but this depletion appears unable to produce a secondary maximum. The application of the models to rich clusters of galaxies is discussed and it is found that the evolution should be at least strong enough to produce central objects the size of cD galaxies. Accurate luminosity profiles for clusters of galaxies are shown to be capable of putting constraints on the mass to light ratios of the member galaxies and to give an indication of the form and distribution of the 'missing mass'.

1. INTRODUCTION

The most massive objects in any self-gravitating system will gradually spiral towards the bottom of the cluster potential well as encounters cause them to lose energy continuously to less massive bodies. This effect may be viewed as resulting from the attempt of massive cluster members to reach equipartition of energy with the other objects in the part of the cluster they traverse. Under certain conditions the cumulative effect of encounters may be modelled by a frictional force resisting the motion of heavy particles (Chandrasekhar 1943). Such an approach to the problem of two-body relaxation in clusters has the advantage that the orbital evolution of a massive body can be computed relatively easily, and it has recently been used to model the formation of galactic nuclei by spiralling globular clusters (Tremaine, Ostriker & Spitzer 1975). Tremaine (1975) has noted that evolution through dynamical friction can cause a depletion of massive objects in the regions near the centre of a cluster rather than the density enhancement that might intuitively be expected. In this paper we calculate the evolution of the distribution of massive objects both for the isothermal model cluster considered by Tremaine *et al.* (1975) and also by Lecar (1975), and in addition for a model of quite different potential and phase space structure. The similarity of the results suggests that the kind of variation of density profile with mass found below may be expected in a wide range of stellar dynamical systems.

The results of this paper show that dynamical evolution in rich clusters of galaxies should cause the formation of condensations at least the size of cD galaxies at the cluster centre, and we calculate below the approximate mass and luminosity to be expected for such objects. The observed brightness of the central regions of clusters of galaxies can be used to put limits on the mass to light ratio of the individual galaxies, and presently available data indicate a value of approximately 26 solar units for the brighter galaxies in the Coma Cluster. Central depletion of the more massive galaxies in clusters appears to be too weak to cause a secondary maximum in the radial density distribution.

2. THEORY

The effect of random encounters on a body moving through a homogeneous and isotropic distribution of lighter bodies was shown by Chandrasekhar (1943) to be closely approximated by a force of dynamical friction,

$$m\frac{d\mathbf{v}}{dt} = -4\pi G^2 m^2 \frac{\mathbf{v}}{|\mathbf{v}|^3} \log_e\left(\frac{d_{\max}}{d_{\min}}\right) \int_0^{|v|} M(v')\,dv'. \tag{1}$$

m and $\mathbf{v}$ here are the mass and velocity of the test body, respectively, $M(v')\,dv'$ is the mass density of background objects with speeds in $(v'+dv')$ and $d_{\max}/d_{\min}$ is ratio of the maximum and minimum impact parameters for which encounters can be considered effective. In a real cluster this frictional force should give an approximate description of the orbital evolution of any object which is appreciably heavier than the bodies which make up the bulk of the mass of the cluster, provided that the mean distance between these background objects is small compared to the curvature of the orbit and that their velocity distribution can be considered isotropic. For a system of point particles $d_{\min} = Gm/\langle v^2\rangle$ (Chandrasekhar 1943) but the appropriate values of the cut-offs are usually difficult to specify. The exact values adopted do not, however, influence the results obtained very strongly. Equation (1) shows that if the dependence of the cut-offs on m can be neglected, the form of the final distribution of a particular population of massive objects depends on the product of their individual mass and the time of evolution.

In the models considered in this paper a spherical distribution of heavy particles is allowed to evolve under the influence of dynamical friction in a fixed spherical background cluster of lighter objects. The contribution of the heavy particles to the cluster potential is neglected, as is the effect of their mutual encounters, and the rates of change of specific energy, E, and specific angular momentum, J, are assumed to be sufficiently slow that they may be calculated by integrating the force (1) along orbits of constant E and J. This approach has already been adopted by Tremaine *et al.* (1975) in the particular case when the background cluster and the massive object distribution are isothermal spheres of the same scale length.

It proves to be convenient to consider the evolution of the distribution function in (E, ϵ) space where ϵ measures the orbital circularity and is given for any particular orbit by

$$\epsilon = J/J_c(E). \tag{2}$$

$J_c(E)$ in equation (2) is the specific angular momentum of the circular orbit of energy E. ϵ varies between zero for linear orbits and one for circular orbits. For a given spherical potential well $U(r)$ containing a particular density distribution $N(E, \epsilon)$ of orbits in (E, ϵ) space, the corresponding space density can be found from

$$\rho(r) = \int f(v)\,d^3v = 4\pi \int_0^\infty \int_0^\infty f(v_t, v_r)\,v_t\,dv_t\,dv_r$$

where $f(v_t, v_r)$ is the six-dimensional phase-space distribution function and the transverse and radial velocities are given by

$$v_t = J/r; \quad v_r = [2(E-U) - J^2/r^2]^{1/2}.$$

Changing the integration variables now shows that

$$\rho(r) = \frac{4\pi}{r}\int_{U(r)}^{U(\infty)}\int_0^{\varepsilon_{\mathrm{m}}(E,\,r)} f(E,\,\epsilon)\,\epsilon J_{\mathrm{c}}(E)(\epsilon_{\mathrm{m}}^2-\epsilon^2)^{-1/2}\,d\epsilon\,dE \tag{3}$$

where

$$\epsilon_{\mathrm{m}}(E,\,r) = r[2(E-U)]^{1/2}/J_{\mathrm{c}}(E). \tag{4}$$

In addition, $N(E,\,\epsilon)$ and $f(E,\,\epsilon)$ are related by

$$N(E,\,\epsilon) = f(E,\,\epsilon)\,Q(E,\,\epsilon) \tag{5}$$

where

$$Q(E,\,\epsilon) = 16\pi^2\epsilon J_{\mathrm{c}}(E)\int_{r_{\mathrm{p}}(E,\,\varepsilon)}^{r_{\mathrm{a}}(E,\,\varepsilon)}(\epsilon_{\mathrm{m}}^2-\epsilon^2)^{-1/2}\,r\,dr. \tag{6}$$

In this last formula $r_{\mathrm{p}}(E,\,\epsilon)$ and $r_{\mathrm{a}}(E,\,\epsilon)$ are the pericentric and apocentric distances of the orbit $(E,\,\epsilon)$ respectively. $Q(E,\,\epsilon)$ is the six-dimensional phase space volume occupied by orbits with energies and circularities in the intervals $(E,\,E+dE)$ and $(\epsilon,\,\epsilon+d\epsilon)$, and it is worth noting that it is, in fact, equal to $16\pi^2\times$(orbital angular momentum)$^2\times$(orbital semi-period)/(orbital circularity).

The final distribution $N_{\mathrm{f}}(E,\,\epsilon)$ which results after a given time from an initial distribution $N_{\mathrm{i}}(E,\,\epsilon)$ of massive particles is calculated in the work here reported by calculating $\dot{E}$ and $\dot{\epsilon}$ on a grid of points in $(E,\,\epsilon)$ space and then integrating back in time to find the initial values of energy and circularity, E_{i} and ϵ_{i}, corresponding to each grid point. From this one has

$$N_{\mathrm{f}}(E,\,\epsilon) = N_{\mathrm{i}}[E_{\mathrm{i}}(E,\,\epsilon),\,\epsilon_{\mathrm{i}}(E,\,\epsilon)]\left|\frac{\partial(E_{\mathrm{i}},\,\epsilon_{\mathrm{i}})}{\partial(E,\,\epsilon)}\right| \tag{7}$$

and the final space density can be found by substituting this expression in equation (3). Some approximation is introduced into the calculations by the finite grid spacing and the finite upper energy of the integration grid, but in practice these do not give rise to any serious error.

Models calculated in this manner should give an approximate description of the evolution of real spherical clusters containing a wide range of masses on time scales for which only a small part of the total mass of the system resides in those objects which evolve significantly. On such times scales the results presented below should be applicable outside a small central region in which a substantial fraction of the mass is made up of the evolved distribution of heavy objects, and where the dominant contributions to the background density are from regions of phase space which experience appreciable changes in energy content as a result of heat input from the evolving masses. This region of inapplicability of the results will always exist but it is reasonably small in many models of interest.

3. RESULTS

For an isothermal (Maxwellian) background cluster the dynamical friction force (1) becomes

$$m\frac{d\mathbf{v}}{dt} = -4\pi G^2m^2\frac{\mathbf{v}}{|\mathbf{v}|^3}\log_{\mathrm{e}}\left(\frac{d_{\max}}{d_{\min}}\right)\rho_{\mathrm{bkgrd}}(r)[\phi(jv)-jv\phi'(jv)] \tag{8}$$

where $j^2 = \frac{3}{2}\langle v^2\rangle$ and ϕ is the error function. The evolution of the space density and of the projected surface density of an initially isothermal distribution of objects of a single mass m in such a cluster is shown in Fig. 1. The unit of length here is the isothermal scale length, the unit of density is arbitrary and the evolution parameter, Z, is given by

$$Z = 17 \cdot 6 \frac{mT}{\sigma \alpha^2} \tag{9}$$

where T is the time of evolution in units of 10^{10} yr, m is in units of $10^{12} M_{\odot}$, σ is the line of sight velocity dispersion in units of 10^3 km s^{-1} and α is the isothermal scale length in units of 50 kpc. (α is defined by the relation $4\pi G \rho_0 \alpha^2 = \sigma^2$ where ρ_0 is the central density of the background isothermal sphere). These parameters are all near unity in rich clusters of galaxies. For the present calculations $d_{\max}$ was taken as max(r, α) where r is the distance from the cluster centre, and $d_{\min}$ was taken as $\alpha/10$. These numbers are again appropriate to rich clusters of galaxies, but the main results are not very sensitive to their value. This particular model is similar to the one used by Tremaine *et al.* (1975) to examine the evolution of the globular cluster population in M31. With their parameters for the bulge of the galaxy $Z = 58 \cdot 8$ for a $10^6 M_{\odot}$ globular cluster, though their calculations show a higher degree of evolution than this implies because they take $d_{\min} = \alpha/77$. It can be shown, however, that the appropriate value of $d_{\min}$ for the globular cluster problem is about one-fifth of the tidal radius, a factor of order 10 larger than this (White 1976).

It can be seen from Fig. 1 that the model predicts a significant depletion of massive objects near the centre for all but the least evolved part of the massive body spectrum. The excess mass at the centre at any stage may be taken as equal

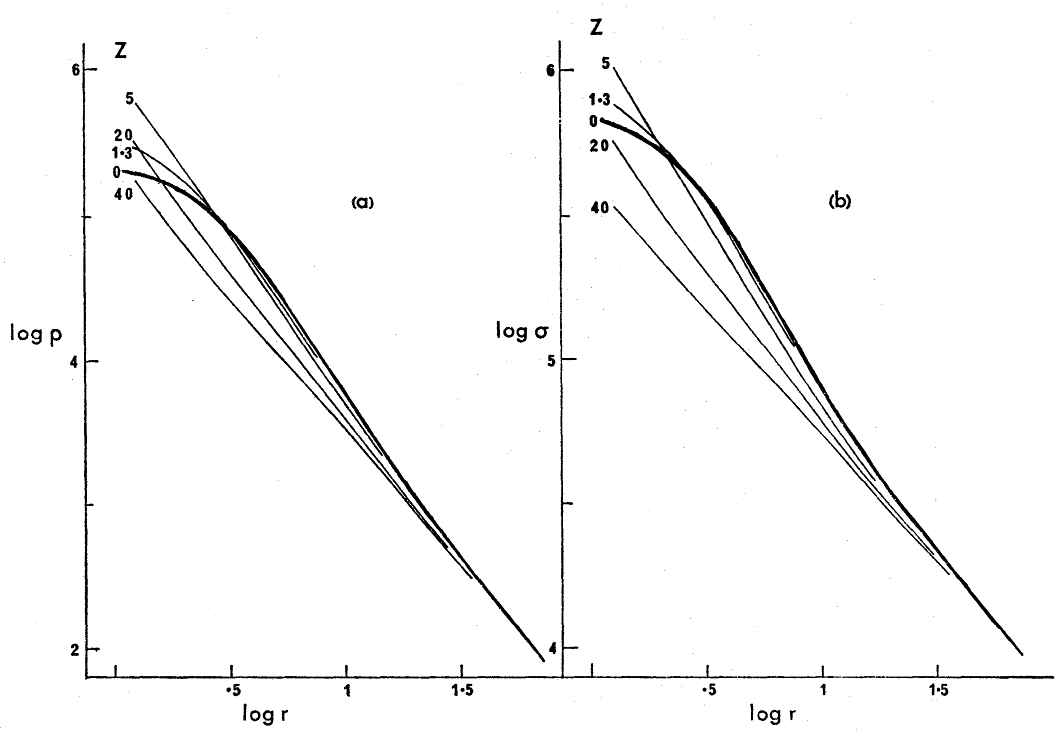

FIG. 1. (*a*) *The space density profiles of the massive object population in an isothermal cluster for various values of the evolution parameter, Z, described in the text* (*equation* (9)). (*b*) *The corresponding projected density profiles. Note the difference in vertical scale between the two diagrams. The unit of radial distance is the scale length, α, in both cases.*

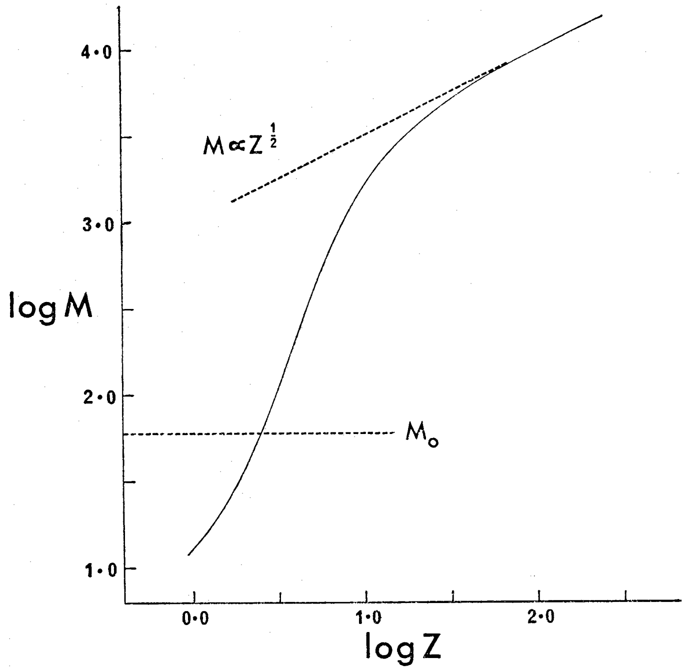

FIG. 2. *The central mass excess in an isothermal model cluster as a function of the evolution parameter, Z. Z is proportional to the time of evolution and to the mass of the objects making up the sub-population under consideration. The mass, M_0, of the subpopulation initially within one scale radius of the centre is shown, as is the asymptotic scaling law $M \propto Z^{1/2}$.*

to the mass of the unevolved distribution contained within the initial energy-circularity surface corresponding to the innermost energy boundary of the final grid ($E/\sigma^2 = 0{\cdot}25$ in the present calculation when the central value of the potential is set equal to zero). The mass of this central excess is plotted against Z in Fig. 2 which also shows the mass initially contained within one scale radius of the centre. It is worth noticing that the value of this mass only approaches the $Z^{1/2}$ scaling law given by Tremaine *et al.* (1975) for rather large values of Z.

The numerical calculations show that the energy loss rates in orbits of a particular energy increase strongly with decreasing ϵ for orbits which remain outside the core and that ϵ changes rather slowly as particles spiral into the centre. Over most of the cluster the evolution is such as to increase ϵ and circularize the orbits, but in the neighbourhood of the centre this effect is actually reversed and circular orbits become unstable. This may, however, be an artificial property of the model since it depends on the form taken for d_{max}/d_{min}.

Models were also calculated using modified forms for the initial heavy object distribution but retaining the isothermal background cluster. Introducing anisotropy of the velocity distribution or a difference in temperature between the massive bodies and the background did not cause any marked difference in the pattern of evolution. In no case could a strong enough depletion of objects be achieved to give even an approach to a secondary maximum in the radial density distribution such as has recently been found by some workers (Oemler 1974; Bahcall 1971; Austin & Peach 1974). To see if such a secondary maximum might be generated by using a more favourable cluster model, the calculations were repeated for Plummer's Model (a polytrope of index 5) both the background cluster and the heavy particle population being taken as having the same initial distribution. Plummer's Model is a finite mass cluster in which the space density falls as r^{-5} at large radii, and which has a finite depth potential well in which $N(E, \epsilon)$ decreases as $(U_\infty - E)$ for E near U_∞, the value of the potential at infinity. This may be compared with the isothermal sphere which has infinite mass, density proportional to r^{-2} at large radii, an infinite potential well and $N(E, \epsilon)$ increasing as $\exp(E/2\sigma^2)$ for large E. The frictional force in Plummer's Model is

$$m\frac{d\mathbf{v}}{dt} = -4\pi G^2 m^2 \frac{\mathbf{v}}{|\mathbf{v}|^3} \log_e\left(\frac{d_{\max}}{d_{\min}}\right) \rho_{\text{bkgrd}}\,\Phi(v, r)$$

in which

$$\Phi(v, r) = \frac{\int_0^v [-U(r) - v'^2/2]^{7/2}\, 4\pi v'^2\, dv'}{\int_0^{[-2U(r)]^{1/2}} [-U(r) - v'^2/2]^{7/2}\, 4\pi v'^2\, dv'}$$

$$= \frac{1}{\pi}\left(X + \sin X - \tfrac{2}{7}\sin 2X - \tfrac{13}{42}\sin 3X - \tfrac{3}{28}\sin 4X - \tfrac{1}{70}\sin 5X\right)$$

where

$$X = 2\sin^{-1}[v(-2U(r)^{-1/2}] \tag{10}$$

and it can be shown that under the assumptions of this paper the heavy particle distribution must develop a secondary maximum as its evolution proceeds. Plummer's Model and the isothermal sphere have gross properties which are likely to bracket those of real clusters except in regions where the structure is primarily determined by the anisotropy of the distribution function. These regions might conceivably embrace the whole cluster in some cases.

Fig. 3 shows the evolution of the space density in Plummer's model, the evolution parameter, Z, in this diagram being given by

$$Z = 51{\cdot}8\frac{mT}{M^{1/2}a^{3/2}} \tag{11}$$

where M is the total mass of the cluster, a its scale length, m and T are as before and the units are the same as for equation (9). The ratio $d_{\max}/d_{\min}$ was again taken as $10\max(1, r/a)$. It may be seen from Fig. 3 that even in this case it is singularly difficult to produce a maximum in the space density profile, and in fact such a maximum appears only after almost all the mass has fallen into the cluster centre. The tardy appearance of the secondary bump is in part caused by the fact that particle orbits are on the whole rather eccentric ($\bar{\epsilon} \simeq \frac{2}{3}$ for all centrally-condensed clusters with an isotropic velocity distribution). Not only do particles of a particular energy contribute to the density over a large range of radii, but also the energy loss rates decrease much less strongly with r in eccentric orbits than in circular orbits. This is only a subsidiary reason, however, since the secondary maximum appears little earlier in the rather artificial case when the massive particles are taken to be in predominantly circular orbits. It can easily be shown that it is impossible to construct a model cluster which has both a secondary maximum and an isotropic velocity distribution, and the existence of the secondary maximum found by Oemler (1974), Bahcall (1971) and others implies a considerable excess of orbits with pericentres or apocentres at a particular distance. Dynamical friction appears unable to produce such an excess.

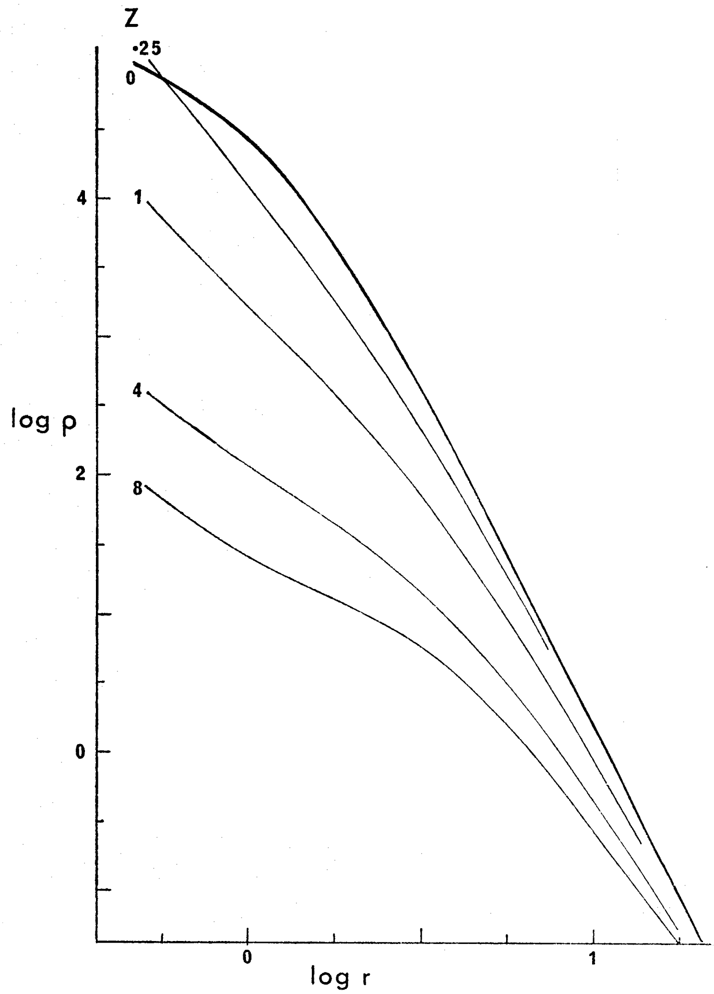

FIG. 3. *The evolution of the space density profile of the massive object population in a polytropic cluster of index* 5. *The evolution parameter, Z, is given by equation* (11).

Turning to the question of the matter which falls to the centre, the values of the central mass enhancement displayed in Fig. 2 can be combined with an assumed luminosity function and mass to light ratio to estimate the mass and luminosity at any time of the central object in an isothermal cluster model. Using Schecter's (1975) luminosity function for cluster galaxies,

$$n(L)\,dL = N\left(\frac{L}{L^*}\right)^{-5/4} e^{-L/L^*}\frac{dL}{L^*} \tag{12}$$

where

$$L^* \simeq 4\times 10^{10}\,L_\odot \quad \text{for} \quad H_0 = 50\,\text{km s}^{-1}\,\text{Mpc}^{-1}$$

and determining N from Oemler's value (1974) for the mean surface brightness at 14′·7 from the centre of the Coma Cluster, the luminosity of the central mass excess in an isothermal model of this cluster has been calculated and is displayed as a function of Z^* in Fig. 4. Z^* is the evolution parameter corresponding to galaxies of the characteristic brightness L^* and is given by

$$Z^* = 17{\cdot}6\,\frac{\kappa L^* T}{\sigma\alpha^2}. \tag{13}$$

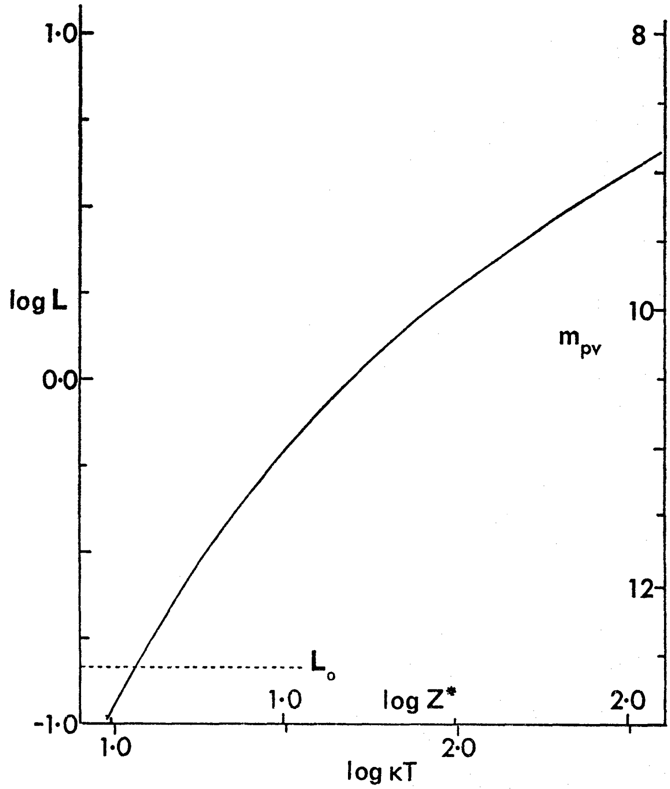

FIG. 4. *The brightness of the central mass excess in an isothermal sphere model for the Coma Cluster as a function of the degree of cluster evolution. The vertical scales give the apparent magnitude of the central condensation and the logarithm of its luminosity in units of* $10^{12}\ L_{\odot}$ *for an assumed Hubble constant* $H_0 = 50$ *km* s^{-1} *Mpc*$^{-1}$. *The horizontal scales give* $\log_{10}(Z^*)$ *and* $\log_{10}(\kappa T)$ *where the cluster evolution parameter,* Z^*, *the mass to light ratio,* κ, *and the evolution time,* T, *are all as defined in the text. The initial projected brightness of the model within one scale radius of the centre,* L_0, *is also shown for comparison.*

In equation (13) L^* is in units of $10^{12}\ L_{\odot}$, κ is the mass to light ratio of the massive galaxies in solar units and the other parameters are the same as in equation (9). Taking $\alpha = 2'$ and $\sigma = 861$ km s^{-1} following Rood *et al.* (1972) we get $Z^* = 0{\cdot}32\ \kappa T$. Note that Z^* does not depend upon the adopted value of H_0. Fig. 4 shows that the effect of dynamical evolution is strong, and that for reasonable values of κ the brightness of the central excess is greater than the initial brightness of the centre. The projected brightness within one scale length of the centre of the unevolved model is also shown in Fig. 4 for comparison.

Rood *et al.* (1972) estimate the total brightness of the innermost 9′ of the Coma Cluster as $m_{\text{pv}} = 9{\cdot}66$. The initial brightness expected in this region on the above model is $m_{\text{pv}} = 9{\cdot}95$, and if the change in the projected luminosity profile outside the immediate centre is neglected the difference between these two implies a magnitude for the central excess of galactic light of about $m_{\text{pv}} = 11{\cdot}24$. Comparison with Fig. 4 shows that this figure implies a mass to light ratio of about 26 for the brighter galaxies in Coma if T is assumed to be 1, however, it is possible that the magnitude estimate of Rood *et al.* (1972) neglects a significant amount of galactic material despite a 40 per cent upward correction which they adopt to allow for faint galaxies and the outer envelopes of bright galaxies. A further half magnitude correction to the value of the central brightness given by Rood *et al.* would give rise to an estimate, $m_{\text{pv}} = 9{\cdot}87$, for the magnitude of the central excess, and an implied mass to light ratio from Fig. 4 of 89. Such a large correction is hard to rule out on the available data, and all that can presently be said is that values of κT as high as 200 seem to be incompatible with the model for the evolution of the Coma Cluster presented here. More suitable photometry could put quite strong limits on the possible values of κT.

4. DISCUSSION

The models explored in this paper will approximately describe the effects of two-body relaxation on the massive end of a cluster mass spectrum in a variety of astronomical situations. They have already been used by Tremaine, Ostriker & Spitzer (1975) and by Tremaine (1975) to discuss the formation of galactic nuclei from globular clusters, and curves such as those of Fig. 1 can be combined with any assumed initial mass distribution of clusters to give an expected final distribution.

If the mass binding the great clusters of galaxies were distributed amongst the observed galaxies in proportion to their luminosity, these galaxies would have a mass to light ratio of about 200. Such a high figure would seem to imply stronger evolution than is actually observed. (The present model is not, of course, really applicable to this case, but the result itself is still likely to hold, and in any case is not new: *cf.* Rood (1965).) If the ' missing mass ' is present as gas then there will still be an effective frictional drag acting on galaxies, but its magnitude will no longer be given by equation (1). If, however, the missing mass is present in the form of intergalactic stars, small black holes, faint galaxies or discrete dense gas clouds, then the results presented above are consistent with the presently inferred masses of cD galaxies and with the existence of multiple nuclei in cD galaxies such as NGC 6166, and they explain the apparent lack of any strong segregation by luminosity in galaxy counts in clusters. Contrary to what has sometimes been assumed, the curves of Fig. 1 show that two-body relaxation will result in a depletion in the number of massive galaxies towards the cluster centre, at least as long as the evolving population can be assumed to fall into one supermassive object at the centre without grossly distorting the potential and background structure. For the sort of cluster parameters assumed and derived in the last section the background cluster will only be strongly influenced by the evolution within 2 to 4 scale lengths of the centre.

The luminosities found on this model for the central objects in clusters are higher than those normally quoted for cD galaxies but this could well be due to an underestimate of their extent which would be very large on the present hypothesis (following Gallagher & Ostriker 1972) that they are formed from the remnants of the tidal breakup of other galaxies. The present model does not, of course, take account of the possibility of binary galaxy formation as has occurred in Coma, though it seems unlikely that this will affect either the radial density distribution or the total mass reaching the central regions. It may also have some difficulty in accounting for the lack of many severely distorted galaxies near the centres of clusters and for the high metallicities inferred for cD galaxies by Faber (1973).

Though the models of this paper are probably not applicable to star clusters where binary evolution seems to be an important dynamical effect, the curves of Fig. 1 indicate that the difference in central concentration between the red giant and horizontal branch star distributions found by Woolf (1964) should not necessarily be treated as inexplicable in terms of normally assumed masses and lifetimes, but might rather be expected to occur as a result of normal relaxation processes provided that most of the cluster mass is in less massive stars and that there is some sink at the centre (such as a massive black hole, perhaps) which can soak up those stars which reach it.

In future work we hope to test the applicability of the models of this paper by direct comparison with N-body calculations. A photometric program is also being planned to obtain accurate photometry of the central regions of clusters which, together with more detailed comparison with galaxy counts and radial velocity measures in clusters, should enable restrictions to be placed on the range of possible cluster models. This in turn may allow the placing of quite strong limits on the mass to light ratio of cluster galaxies and shed some light on the form, distribution and total amount of missing mass.

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REFERENCES

Austin, T. B. & Peach, J. V., 1974. *Mon. Not. R. astr. Soc.*, **167**, 437.
Bahcall, N. A., 1971. *Astr. J.*, **76**, 995.
Chandrasekhar, S., 1943. *Astrophys. J.*, **97**, 255.
Faber, S. M., 1973. *Astrophys. J.*, **179**, 731.
Gallagher, J. S. & Ostriker, J. P., 1972. *Astr. J.*, **77**, 288.
Lecar, M., 1975. *IAU Symp.*, *No.* 69, in press.
Oemler, A., 1974. *PhD thesis* for California Institute of Technology.
Schecter, P., 1975. *Astrophys. J.*, in press.
Rood, H. J., 1965. *PhD thesis*, University of Michigan.
Rood, H. J., Page, T. L., Kintner, E. C. & King, I. R., 1972. *Astrophys. J.*, **175**, 627.
Tremaine, S. D., 1975. *Astrophys. J.*, in press.
Tremaine, S. D., Ostriker, J. P. & Spitzer, L., 1975. *Astrophys. J.*, **196**, 407.
White, S. D. M., 1976. *Mon. Not. R. astr. Soc.*, **174**, in press.
Woolf, N. J., 1964. *Astrophys. J.*, **139**, 1081.