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Dynamical instability for non-adiabatic spherical collapse

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Summary. We study the dynamical instability of a spherically symmetric fluid which collapses non-adiabatically undergoing dissipation in the form of a radial heat flow. The Newtonian and post-Newtonian limits are considered and the range for dynamical instability for the ratio of specific heats Γ is given. We show that the Newtonian correction due to dissipation increases the instability of the fluid while the relativistic correction due to dissipation makes the fluid less unstable.

1 Introduction

In a famous paper Chandrasekhar (1964) studied the dynamical instability of a pulsating spherically symmetric distribution of perfect fluid in the framework of Einstein's general relativity. His main result was establishing the range for an unstable system of the ratio of specific heats.

Here, it is our aim to extend the study of dynamical instability to spherical source distributions, in which perturbations lead to non-adiabatic collapse. The system that we are going to analyse consists of an equilibrium configuration plus a perturbation. The equilibrium configuration is built up by a static, spherically symmetric, perfect fluid distribution, satisfying junction conditions such that its exterior spacetime has the Schwarzschild metric. The perturbed configuration is built up by a spherically symmetric distribution of non-adiabatic fluid undergoing dissipation in the form of a radial heat-flow. The perturbed configuration satisfies junction conditions such that its exterior spacetime is described by Vaidya's (1953) radiating metric. Further, we consider the fluid as shear-free, which is the slowest possible collapse, as has been shown by Raychaudhuri (1957).

In the following two sections, we present the field equations and junction conditions for the equilibrium and perturbed configurations. The expressions for the second law of thermo-

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dynamics and for the ratio of specific heats are presented in Section 4 and in Section 5 we obtain the collapse equation after a long calculation. The study of the Newtonian and post-Newtonian limits for the non-dissipative and dissipative perturbations are studied in Sections 6 and 7, where the range of the ratio of specific heats for dynamical instability is obtained. Concluding remarks are presented in the last section.

2 The field equations and the junction conditions

Here we study the field equations for a spherically symmetric distribution of fluid, undergoing dissipation in the form of a radial heat-flow. While the dissipative fluid collapses, it produces unpolarized radiation. Details of the mathematical description of this physical situation can be found in Santos (1985). Here we restrict ourselves to the main results.

The surface of the collapsing sphere is described by a time-like three-space Σ. It divides spacetime into two distinct four-dimensional manifolds ν^- and ν^+, each of class c^4, containing Σ as its boundary.

The interior spacetime ν^- is described by the general spherically symmetric metric with shear-free fluid motion (Nariai 1968; Glass 1979), using comoving coordinates

$$ds_-^2 = -A^2(r,t)\,dt^2 + B^2(r,t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \quad (1)$$

We describe the exterior spacetime ν^+ by Vaidya's (1953) metric, which represents an outgoing radial flux of unpolarized radiation,

$$ds_+^2 = -\left[1 - \frac{2M(v)}{\mathrm{r}}\right] dv^2 - 2\,dv\,d\mathrm{r} + \mathrm{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2)$$

More details about the analysis of Vaidya's metric can be found in de Oliveira, Kolassis & Santos (1988).

For the junction conditions we use the approach given by Israel (1966). Hence we have to demand when approaching Σ in ν^+ or ν^-, that

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma \quad (3)$$

and also that

$$K_{ij}^- - K_{ij}^+ = 0, \quad (4)$$

where K_{ij} is the extrinsic curvature of Σ.

We assume that the source in Einstein's field equations is given in the interior spacetime ν^- by

$$G_{\alpha\beta}^- = KT_{\alpha\beta} = K[(\bar{\mu} + \bar{p})\,\omega_\alpha\omega_\beta + \bar{p}g_{\alpha\beta} + q_\alpha\omega_\beta + q_\beta\omega_\alpha], \quad (5)$$

where $\bar{\mu}$ is the energy density of the fluid, $\bar{p}$ the isotropic pressure, ω_α the four-velocity, and q_α the radial heat flux vector which has to satisfy $q_\alpha\omega^\alpha = 0$. Since we use comoving coordinates, we shall have

$$\omega^\alpha = \frac{1}{A}\,\delta_0^\alpha \text{ and} \quad (6)$$

$$q^\alpha = q\delta_1^\alpha. \quad (7)$$

The non-vanishing components of the field equations (5) with (1), (6) and (7) are:

$$G_{00}^- = -\frac{A^2}{B^2}\left[2\,\frac{B''}{B} - \left(\frac{B'}{B}\right)^2 + \frac{4B'}{rB}\right] + 3\left(\frac{\dot{B}}{B}\right)^2 = K\bar{\mu}A^2, \quad (8)$$

$$G_{11}^{-}=\left(\frac{B'}{B}\right)^2+\frac{2}{r}\frac{B'}{B}+\frac{A'B'}{AB}+\frac{2A'}{rA}+\frac{B^2}{A^2}\left[-2\frac{\ddot{B}}{B}-\left(\frac{\dot{B}}{B}\right)^2+2\frac{\dot{A}\dot{B}}{AB}\right]=K\bar{p}B^2, \tag{9}$$

$$G_{22}^{-}=\frac{1}{\sin^2\theta}G_{33}^{-}=r^2\left[\frac{B''}{B}-\left(\frac{B'}{B}\right)^2+\frac{1B'}{rB}+\frac{A''}{A}+\frac{1A'}{rA}\right]+r^2\frac{B^2}{A^2}\left[-2\frac{\ddot{B}}{B}-\left(\frac{\dot{B}}{B}\right)^2+2\frac{\dot{A}\dot{B}}{AB}\right]=K\bar{p}B^2r^2 \tag{10}$$

and

$$G_{01}^{-}=-2\frac{\dot{B}'}{B}+2\frac{B'\dot{B}}{BB}+2\frac{A'\dot{B}}{AB}=-KqB^2A. \tag{11}$$

The dot and the prime stand respectively for differentiation with respect to t and r. The isotropy of pressure allows us to obtain from (9) and (10) an equation relating A and B;

$$\left(\frac{A'}{A}+\frac{B'}{B}\right)'-\left(\frac{A'}{A}+\frac{B'}{B}\right)^2-\frac{1}{r}\left(\frac{A'}{A}+\frac{B'}{B}\right)+2\left(\frac{A'}{A}\right)^2=0, \tag{12}$$

and the Bianchi identifies, $T^{\beta}_{\alpha;\beta}=0$, have two non-identically vanishing components,

$$\dot{\bar{\mu}}+(\bar{\mu}+\bar{p})3\frac{\dot{B}}{B}+q\left(2\frac{A'}{A}+3\frac{B'}{B}+\frac{2}{r}\right)A+q'A=0 \tag{13}$$

and

$$\bar{p}'+(\bar{\mu}+\bar{p})\frac{A'}{A}+5q\frac{\dot{B}B}{A}+\dot{q}\frac{B^2}{A}=0. \tag{14}$$

Using the junction conditions (3) and (4) and the metrics (1) and (2), together with the field equations (8)–(11), we can prove the following relations:

$$(rB)_\Sigma=\text{r}_\Sigma, \tag{15}$$

$$\bar{p}_\Sigma=(qB)_\Sigma \quad \text{and} \tag{16}$$

$$\bar{M}(v)=\left[\frac{r^3\dot{B}^2B}{2A^2}-r^2B'-\frac{r^3B'^2}{2B}\right]_\Sigma. \tag{17}$$

Relation (15) describes the radius of the fluid distribution in both coordinate systems (1) and (2). Relation (16) implies that if we have a spherically symmetric shear-free distribution of a collapsing fluid undergoing dissipation in the form of radial heat-flow, the isotropic pressure on the surface of discontinuity cannot be zero. The pressure will vanish at the boundary only if the fluid stops dissipation, $q_\Sigma=0$, in which case there is also no radiation and the exterior spacetime $\mathcal{V}^+$ reduces to Schwarzschild spacetime. Relation (17) decribes the total energy entrapped inside the surface Σ (Hernandez & Misner 1966; Cahill & Mc Vittie 1970). Then, we are allowed to assume that the total amount of energy contained in a sphere of radius r inside Σ is given by (de Oliveira & Santos 1987):

$$\bar{m}(r,t)=\frac{r^3\dot{B}^2B}{2A^2}-r^2B'-r^2B'-\frac{r^3B'^2}{2B}. \tag{18}$$

3 Perturbed equations

We assume now that the fluid is in hydrostatic equilibrium, which means that none of the quantities describing the fluid depend on t but only on r. The quantities describing equilibrium are denoted by a subscript zero. We then suppose that the static system having energy density μ_0 and isotropic pressure p_0 is perturbed, undergoing slow dissipative shear-free collapse. We further suppose that the perturbation μ of the energy density and p of the isotropic pressure have the same time dependence and that dissipation is in the form of a radial heat flow. Hence we can consider the metric functions $A(r, t)$ and $B(r, t)$, the energy density $\bar{\mu}(r, t)$ and the isotropic pressure $\bar{p}(r, t)$ given by:

$$A(r, t)=A_0(r)+\varepsilon a(r)T(t), \tag{19}$$

$$B(r, t)=B_0(r)+\varepsilon b(r)T(t), \tag{20}$$

$$\bar{\mu}(r, t)=\mu_0(r)+\varepsilon\mu(r, t) \text{ and} \tag{21}$$

$$\bar{p}(r, t)=p_0(r)+\varepsilon p(r, t); \tag{22}$$

the radial heat flow q being of the order of $\varepsilon(0<\varepsilon\ll 1)$. From (8) and (9) we obtain for the static configuration,

$$K\mu_0=-\frac{1}{B_0^2}\left[2\frac{B_0''}{B_0}-\left(\frac{B_0'}{B_0}\right)^2+\frac{4B_0'}{rB_0}\right], \tag{23}$$

$$Kp_0=\frac{1}{B_0^2}\left[\left(\frac{B_0'}{B_0}\right)^2+\frac{2B_0'}{rB_0}+2\frac{A_0'B_0'}{A_0B_0}+\frac{2A_0'}{rA_0}\right], \tag{24}$$

and from (12), because of the isotropy of p_0,

$$\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)'-\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)^2-\frac{1}{r}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)+2\left(\frac{A_0'}{A_0}\right)^2=0. \tag{25}$$

The perturbed quantities up to the first order in ε are:

$$K\mu=-3K\mu_0\frac{b}{B_0}T+\frac{1}{B_0^3}\left[-\left(\frac{B_0'}{B_0}\right)^2 b+2\left(\frac{B_0'}{B_0}-\frac{2}{r}\right)b'-2b''\right]T, \tag{26}$$

$$Kp=-2Kp_0\frac{b}{B_0}T+\frac{2}{B_0^2}\left[\left(\frac{B_0'}{B_0}+\frac{1}{r}+\frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right]T-2\frac{b}{A_0^2B_0}\ddot{T} \text{ and} \tag{27}$$

$$Kq=\frac{2\varepsilon}{B_0^2}\left(\frac{b}{A_0B_0}\right)'\dot{T}, \tag{28}$$

and due to the isotropy of pressure, equation (12) takes the form,

$$\left[\left(\frac{a}{A_0}\right)'+\left(\frac{b}{B_0}\right)'\right]'-2\left[\left(\frac{a}{A_0}\right)'+\left(\frac{b}{B_0}\right)'\right]\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)-\frac{1}{r}\left[\left(\frac{a}{A_0}\right)'+\left(\frac{b}{B_0}\right)'\right]+4\frac{A_0'}{A_0}\left(\frac{a}{A_0}\right)'=0. \tag{29}$$

From the equations (13) and (14), arising from the Bianchi identities, we have for the static configuration:

$$p_0'+(\mu_0+p_0)\frac{A_0'}{A_0}=0, \tag{30}$$

and for the perturbed configuration

$$\dot{\mu}+3(\mu_0+p_0)\frac{b}{B_0}\dot{T}+q\left(2\frac{A_0'}{A_0}+3\frac{B_0'}{B_0}+\frac{2}{r}\right)A_0+q'A_0=0 \tag{31}$$

and

$$p'+(\mu_0+p_0)\left(\frac{a}{A_0}\right)'T+(\mu+p)\frac{A_0'}{A_0}+\dot{q}\frac{B_0^2}{A_0}=0. \tag{32}$$

By substituting (28) into (31) we can integrate it, and we find that

$$K\mu+3K(\mu_0+p_0)\frac{b}{B_0}T+\left\{\frac{4A_0'}{B_0^2}\left(\frac{b}{A_0B_0}\right)'+\frac{2A_0}{r^2B_0^3}\left[r^2B_0\left(\frac{b}{A_0B_0}\right)'\right]'\right\}T=0. \tag{33}$$

The total energy entrapped inside Σ, given by equation (17), and up to a radius r inside Σ, equation (18), are for the static and perturbed configurations:

$$\bar{M}(v)=M_0+\varepsilon M(v) \text{ and} \tag{34}$$

$$\bar{m}(r,t)=m_0(r)+\varepsilon m(r,t), \tag{35}$$

where

$$M_0=\left(-r^2B_0'-\frac{r^3B_0'^2}{2B_0}\right)_\Sigma, \tag{36}$$

$$M(v)=\left[-r^2b'-\frac{r^3B_0'^2}{2B_0}\left(2\frac{b'}{B_0'}-\frac{b}{B_0}\right)\right]_\Sigma T \quad \text{and} \tag{37}$$

$$m_0(r)=-r^2B_0'-\frac{r^3B_0'^2}{2B_0}, \tag{38}$$

$$m(r,t)=\left[-r^2b'-\frac{r^3B_0'^2}{2B_0}\left(2\frac{b'}{B_0'}-\frac{b}{B_0}\right)\right]T. \tag{39}$$

We can rewrite equations (27) and (28) in the following way:

$$Kp=2Kp_0\frac{b}{B_0}T+2\frac{b}{A_0^2B_0}(\alpha T-\ddot{T}) \quad \text{and} \tag{40}$$

$$Kq=4\frac{\varepsilon b}{A_0^2B_0^2}\beta\dot{T}, \tag{41}$$

where

$$\alpha=\frac{A_0^2}{B_0b}\left[\left(\frac{B_0'}{B_0}+\frac{1}{r}+\frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right] \quad \text{and} \tag{42}$$

$$\beta=\frac{A_0^2}{2b}\left(\frac{b}{A_0B_0}\right)'. \tag{43}$$

Considering the junction condition (16) and also the relation $p_{0\Sigma}=0$, we obtain from (40) and (41) a second order linear differential equation for $T(t)$, namely,

$$\alpha_\Sigma T-\ddot{T}=2\beta_\Sigma \dot{T}>0. \tag{44}$$

As we want only real solutions of (44), we have to restrict our perturbations $a(r)$ and $b(r)$ so that $\alpha_\Sigma>0$, and as we are interested in a solution describing collapse, we find

$$T(t)=\exp[(-\beta_\Sigma-\sqrt{\alpha_\Sigma+\beta_\Sigma^2})\,t]. \tag{45}$$

4 The second law of thermodynamics

The second law of thermodynamics can be expressed by the relation,

$$k\mathscr{T}\,dS=\bar{p}\,d\left(\frac{1}{\bar{n}}\right)+d\left(\frac{\bar{\mu}}{\bar{n}}\right), \tag{46}$$

where k is the Boltzmann's constant, $\mathscr{T}$ the temperature, S the entropy and $\bar{n}$ the baryon number of the thermodynamical system. If the perturbed system does not dissipate, then evidently $dS=0$. If the perturbed system dissipates, then $dS=0(\varepsilon^2)$, because entropy production is proportional to $q_\mu q^\mu$ (Israel 1976). With these considerations we can write (46) for both cases, i.e. non-dissipative and dissipative perturbations, in the form,

$$\bar{p}d\left(\frac{1}{\bar{n}}\right)+d\left(\frac{\bar{\mu}}{\bar{n}}\right)=0. \tag{47}$$

We assume that there is an equation of state,

$$\bar{n}=\bar{n}(\bar{p},\bar{\mu}), \tag{48}$$

which after differentiation becomes,

$$d\bar{n}=\frac{\partial\bar{n}}{\partial\bar{\mu}}\,d\bar{\mu}+\frac{\partial\bar{n}}{\partial\bar{p}}\,d\bar{p}, \tag{49}$$

and we define Γ, the ratio of specific heats, as:

$$\Gamma=\left(\bar{p}\,\frac{\partial\bar{n}}{\partial\bar{p}}\right)^{-1}\left[\bar{n}-(\bar{\mu}+\bar{p})\,\frac{\partial\bar{n}}{\partial\bar{\mu}}\right]. \tag{50}$$

Now considering equations (47), (49) and (50) we obtain an equation relating the Lagrangian variations $d\bar{p}$ and $d\bar{\mu}$,

$$d\bar{p}=\Gamma\,\frac{\bar{p}}{\bar{\mu}+\bar{p}}\,d\bar{\mu}. \tag{51}$$

Here we assume that up to $O(\varepsilon)$, the Lagrangian variations $d\bar{p}$ and $d\bar{\mu}$ can be replaced by the Eulerian variations εp and $\varepsilon\mu$, hence with the help of equations (21) and (22), equation (51) leads to:

$$p=\Gamma\,\frac{p_0}{\mu_0+p_0}\,\mu, \tag{52}$$

and by substituting (33) into (52) we obtain:

$$p=-3\Gamma p_0\frac{b}{B_0}T-\frac{1}{K}\Gamma\frac{p_0}{\mu_0+p_0}\phi bT; \tag{53}$$

where

$$\phi(r)=\left\{\frac{4A_0'}{B_0^2}\left(\frac{b}{A_0B_0}\right)'+\frac{2A_0}{r^2B_0^3}\left[r^2B_0\left(\frac{b}{A_0B_0}\right)'\right]'\right\}\frac{1}{b}. \tag{54}$$

5 The collapse equation

Now with the equations so far obtained we can construct the collapse equation. This is done via equation (32) by substituting into it the following equations: equation (53); equation (27) with (53), which gives,

$$\left(\frac{a}{A_0}\right)'=\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)^{-1}\left[-\frac{3}{2}K\Gamma p_0b-\frac{1}{2}\Gamma\frac{p_0}{\mu_0+p_0}B_0\phi b+Kp_0b+\frac{b}{A_0^2}\frac{\ddot{T}}{T}\right.$$
$$\left.-\frac{1}{B_0}\left(\frac{B_0'}{B_0}+\frac{1}{r}+\frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)'\right]B_0; \tag{55}$$

equation (33) with (54),

$$\mu=-3(\mu_0+p_0)\frac{b}{B_0}T-\frac{1}{K}\phi bT; \tag{56}$$

and equation (41), or

$$\dot{q}\frac{B_0^2}{A_0}=\frac{4}{K}\frac{\varepsilon b}{A_0^3}\beta\dot{T}; \tag{57}$$

then we obtain

$$-\left[3\Gamma p_0\frac{A_0}{B_0}b+\frac{1}{K}\Gamma\frac{p_0}{\mu_0+p_0}A_0\phi b\right]'\frac{1}{A_0}+\frac{\mu_0+p_0}{B_0'/B_0+1/r}\left[-\frac{3}{2}K\Gamma p_0b\right.$$
$$\left.-\frac{1}{2}\Gamma\frac{p_0}{\mu_0+p_0}B_0\phi b+Kp_0b+\frac{b}{A_0^2}\frac{\ddot{T}}{T}-\frac{1}{B_0}\left(\frac{B_0'}{B_0}+\frac{1}{r}+\frac{A_0'}{A_0}\right)\left(\frac{b}{B_0}\right)'\right]B_0$$
$$-\left[3(\mu_0+p_0)\frac{b}{B_0}+\frac{1}{K}\phi b\right]\frac{A_0'}{A_0}+\frac{4}{K}\frac{\beta}{A_0^3}b\frac{\dot{T}}{T}=0. \tag{58}$$

Further, by taking into consideration equation (30),

$$\frac{A_0'}{A_0}=-\frac{p_0'}{\mu_0+p_0}; \tag{59}$$

equation (38) written in the form,

$$\frac{B_0'}{B_0} = -\frac{1}{r} + \frac{1}{r}\sqrt{1 - \frac{2m_0(r)}{rB_0}}; \tag{60}$$

and the solution (45) for $T(t)$,

$$\frac{\ddot{T}}{T} = (\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})^2; \tag{61}$$

and substituting them into (58), we finally obtain the collapse equation,

$$-3(\Gamma p_0)' + 3\Gamma p_0\left(\frac{p_0'}{\mu_0 + p_0} - \frac{1}{r} + \frac{1}{r}\sqrt{1 - \frac{2m_0}{rB_0}} - \frac{b'}{b}\right) - \frac{1}{K}\left(\Gamma\frac{p_0}{\mu_0 + p_0}\right)' B_0\phi$$

$$-\frac{1}{K}\Gamma\frac{p_0 B_0}{\mu_0 + p_0}\left[\left(-\frac{p_0'}{\mu_0 + p_0} + \frac{b'}{b}\right)\phi + \phi'\right] + \frac{(\mu_0 + p_0) rB_0^2}{\sqrt{1 - (2m_0/rB_0)}}\left[Kp_0\left(1 - \frac{3}{2}\Gamma\right)\right.$$

$$\left.-\frac{1}{2}\Gamma\frac{p_0}{\mu_0 + p_0}B_0\phi + (\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})^2\frac{1}{A_0^2}\right] + \frac{p_0' r}{\sqrt{1 - (2m_0/rB_0)}}\left(\frac{b'}{b} + \frac{1}{r} - \frac{1}{r}\sqrt{1 - \frac{2m_0}{rB_0}}\right)$$

$$-(\mu_0 + p_0)\left(\frac{b'}{b} + \frac{1}{r} - \frac{1}{r}\sqrt{1 - \frac{2m_0}{rB_0}}\right) + 3p_0' + \frac{1}{K}\frac{p_0'}{\mu_0 + p_0}B_0\phi$$

$$+\frac{4}{K}(\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2})^2\frac{B_0}{A_0^3}\beta = 0. \tag{62}$$

The expression for ϕ can be given in terms of b and its derivatives and quantities which describe the equilibrium configuration.

In the following two sections we study equation (62), first by considering a non-dissipative perturbation, i.e. in the case of vanishing heat-flow, and second a dissipative perturbation, i.e. the case of non-vanishing heat-flow. We also analyse the Newtonian and post-Newtonian limits.

6 Non-dissipative perturbation

Here we analyse the collapse equation (62), assuming that the perturbation does not dissipate, i.e. the heat-flow vanishes. Hence we have from equation (28),

$$kq = \frac{2\varepsilon}{B_0^2}\left(\frac{b}{A_0 B_0}\right)'\dot{T} = 0, \tag{63}$$

which implies that

$$b = A_0 B_0. \tag{64}$$

Then from (43) we find that $\beta = 0$ and the solution of the differential equation (44) reduces to:

$$T(t) = \exp(-\sqrt{\alpha_\Sigma}\, t). \tag{65}$$

Substituting solution (64) into (62) and taking into account that in this case $\phi = 0$, because of (54), we obtain the collapse equation for a perfect fluid,

$$-3(\Gamma p_0)'+6\Gamma p_0\frac{p_0'}{\mu_0+p_0}+\frac{(\mu_0+p_0)rB_0^2}{\sqrt{1-(2m_0/rB_0)}}\left[Kp_0\left(1-\frac{3}{2}\Gamma\right)+\frac{\alpha_\Sigma}{A_0^2}\right]$$

$$-\frac{p_0'^2 r}{(\mu_0+p_0)\sqrt{1-(2m_0/rB_0)}}+4p_0'=0. \qquad (66)$$

In what follows, we discuss (66) in the Newtonian and post-Newtonian limits, assuming the ratio of specific heats is constant throughout the fluid distribution or throughout the region that we want to study.

6.1 NEWTONIAN LIMIT

Here we consider the limit where $A_0=1$, $B_0=1$ and $\mu_0 \gg p_0$, then the collapsing equation (66) becomes:

$$\left(\frac{4}{3}-\Gamma\right)p_0'+\mu_0\frac{\alpha_\Sigma}{3}r=0. \qquad (67)$$

The physical demand that we have to put on the pressure is $p_0'<0$. Hence we obtain from (67) the collapsing condition for the ratio of specific heats in the case of a perfect fluid source,

$$\Gamma<\tfrac{4}{3}. \qquad (68)$$

6.2 POST-NEWTONIAN LIMIT

For this limit we consider $A_0=1-(m_0/r)$, $B_0=1+(m_0/r)$ and relativistic corrections up to order $O(m_0/r)$. Then equation (66) becomes:

$$\left(\frac{4}{3}-\Gamma\right)p_0'+K\mu_0p_0\left(1-\frac{3}{2}\Gamma\right)\frac{r}{3}+\mu_0\frac{\alpha_\Sigma}{3}r=0. \qquad (69)$$

Demanding here too that $p_0'<0$ we find the collapse condition for

$$\Gamma<\frac{4}{3}+\left[\frac{K\mu_0p_0}{|p_0'|}\frac{r}{3}\right]_{\mathrm{MAX}}, \qquad (70)$$

the last term being the maximal value of the expression inside the brackets. Comparing (70) to (68), we can say that relativistic effects increase the unstable range of Γ, which corresponds to the results obtained by Chandrasekhar (1964).

7 Dissipative perturbation

To introduce dissipation, $q\neq 0$, in the collapsing perturbation we can assume for $b(r)$, because of the solution (64), the form

$$b(r)=[1+\xi f(r)]A_0B_0. \qquad (71)$$

Then equation (28) with (71) can be written as:

$$Kq=\frac{2}{B_0^2}\xi f'\varepsilon\dot{T}. \qquad (72)$$

The heat-flow has to satisfy $q>0$ and the fluid is collapsing, $\dot{T}<0$, hence we need $f(r)$, satisfying

$$f'<0. \tag{73}$$

Substituting (71) into the collapsing equation (62) and considering (59) and (60) we obtain:

$$\begin{aligned}
&-3(\Gamma p_0)'+3\Gamma p_0\left(\frac{2p_0'}{\mu_0+p_0}-\frac{\xi f'}{1+\xi f}\right)-\frac{1}{K}\left(\Gamma\frac{p_0}{\mu_0+p_0}\right)'B_0\phi\\
&+\frac{1}{K}\Gamma\frac{p_0B_0}{\mu_0+p_0}\left[\left(\frac{2p_0'}{\mu_0+p_0}+\frac{1}{r}-\frac{1}{r}\sqrt{1-\frac{2m_0}{rB_0}}-\frac{\xi f'}{1+\xi f}\right)\phi-\phi'\right]\\
&+\frac{(\mu_0+p_0)rB_0^2}{\sqrt{1-(2m_0/rB_0)}}\left[Kp_0\left(1-\frac{3}{2}\Gamma\right)-\frac{1}{2}\Gamma\frac{p_0}{\mu_0+p_0}B_0\phi+(\beta_\Sigma+\sqrt{\alpha_\Sigma+\beta_\Sigma^2})^2\frac{1}{A_0^2}\right]\\
&+\frac{p_0'r}{\sqrt{1-(2m_0/rB_0)}}\left(-\frac{p_0'}{\mu_0+p_0}+\frac{\xi f'}{1-\xi f}\right)-(\mu_0+p_0)\frac{\xi f'}{1+\xi f}+4p_0'\\
&+\frac{1}{K}\frac{p_0'}{\mu_0+p_0}B_0\phi+\frac{4}{K}(\beta_\Sigma+\sqrt{\alpha_\Sigma+\beta_\Sigma^2})^2\frac{B_0}{A_0^3}\beta=0.
\end{aligned} \tag{74}$$

In what follows we calculate the Newtonian and post-Newtonian limits of (74) by assuming the ratio of the specific heats Γ is constant throughout the distribution, or at least in the region that we want to study, and ξ is of the order $O(M_0/r_\Sigma)$. Hence it is clear that ϕ given by (54) is of the order $O(M_0/r_\Sigma)$ and β given by (43) becomes:

$$\beta=\frac{A_0}{2B_0}\frac{\xi f'}{1+\xi f}. \tag{75}$$

7.1 NEWTONIAN LIMIT

For this limit, $A_0=1$, $B_0=1$ and $\mu_0 \gg p_0$, equation (74) becomes:

$$\left(\frac{4}{3}-\Gamma\right)p_0'+\frac{2}{3K}\alpha_\Sigma\xi f'+\frac{\alpha_\Sigma}{3}\mu_0 r=0. \tag{76}$$

Now considering $p_0'<0$ and $f'<0$, the collapsing condition for Γ in this case becomes:

$$\Gamma<\frac{4}{3}+\left[\frac{2}{3K}\frac{\alpha_\Sigma\xi|f'|}{|p_0'|}\right]_{\text{MAX}}. \tag{77}$$

It is clear from (77) that dissipation increases the unstable range of Γ.

7.2 POST-NEWTONIAN LIMIT

Here again we limit relativistic corrections up to the order $O(m_0/r)$. We therefore shall have $A_0=1-(m_0/r)$ and $B_0=1+(m_0/r)$, and equation (74) becomes:

$$\left(\frac{4}{3}-\Gamma\right)p_0'+\frac{K}{3}\left(1-\frac{3}{3}\Gamma\right)\mu_0p_0r-\frac{1}{3}\mu_0\xi f'+\frac{2}{3K}\alpha_\Sigma\xi f'+\frac{1}{3}\alpha_\Sigma\mu_0 r=0. \tag{78}$$

Considering again $p_0' < 0$ and $f' < 0$, we find that the collapsing condition for the ratio of the specific heats becomes:

$$\Gamma < \frac{4}{3} + \left[\frac{2}{3K} \alpha_\Sigma \frac{\xi |f'|}{|p_0'|} + \frac{K}{3} \frac{\mu_0 p_0}{|p_0'|} r - \frac{1}{3} \frac{\mu_0}{|p_0'|} \xi |f'| \right]_{\text{MAX}}. \tag{79}$$

Equation (79) shows that the unstable range of Γ is increased by the Newtonian term due to dissipation, and also increased by the relativistic correction due to the static background fluid configuration, but is diminished by the relativistic correction due to dissipation.

8 Conclusion

The most interesting result emerging from our calculations concerns the collapse condition for the ratio of the specific heats which includes the relativistic correction due to the dissipation. Since the last term in (79) is obviously negative, it follows that thermal effects (relativistic), while decreasing the unstable range of Γ, tend to make the sphere less unstable.

Although we are aware that the above result has been obtained in the post-Newtonian approximation, there are nevertheless two facts which suggest that the same result will hold in the exact relativistic discussion, namely:

(i) In the case of non-dissipative perturbations, the effect of relativistic corrections is the same in the post-Newtonian limit as in the exact relativistic discussion (Bondi 1964).

(ii) In recent calculations on the effect of thermal conduction on gravitational collapse (Herrera, Jiménez & Esculpi 1987), it was found that for a specific model thermal conduction tends to reduce the ratio of specific heats required for an equilibrium (static) configuration. Hence, at least for one particular model, it seems that relativistic corrections due to dissipation (all orders in m_0/r included), make the system less unstable.

On the basis of the results of the preceding section and the above comments, we can say that there is strong theoretical evidence suggesting that relativistic corrections due to dissipation, do in fact decrease the unstable range of Γ. It is also worth noticing that the unstable range of Γ will be modified differently for different parts of the sphere (it depends on r). This suggests the possibility of a 'fragmentation' of the sphere as a consequence of variations in the stiffness along the radial direction produced by thermal conduction.

As an example, we consider a pre-supernova of about 15 $M_\odot$ mass with a central core of 1.9 $M_\odot$ mass, density 10^6 g cm^{-3} and luminosity 10^{39} erg s^{-1} in the neighbourhood of its radius 5×10^8 cm. The equation of state for the core can be written (Hoyle & Fowler 1960) $p_0 = \mathscr{K}(\mu_0/\mu_e)^{4/3}$, where $\mathscr{K} = 1.243 \times 10^{15}$ and the mean molecular weight per electron $\mu_e = 2$. We consider that the gradient of the pressure can be obtained from the equation of hydrostatic equilibrium, $|p_0'| = [\mu_0 m_0(r)]/r^2$. We assume also that $\alpha_\Sigma^{1/2}$ is of the order of the frequency of oscillation of the pre-supernova, or $\alpha_\Sigma^{-1/2} = r(\mu_0/p_0)^{1/2}$. With these values we obtain for the terms in (79),

$$\frac{K}{3} \frac{\mu_0 p_0}{|p_0'|} r = 1.387 \times 10^{-5},$$

$$\frac{2}{3K} \alpha_\Sigma \frac{\xi |f'|}{|p_0'|} = 7.1 \times 10^{-13} \quad \text{and}$$

$$\frac{1}{3} \frac{\mu_0}{|p_0'|} \xi |f'| = 1.105 \times 10^{-11}.$$

Finally, we would like to conclude with the following comment. Although the Eckart and the Landau–Lifshitz approaches to include thermal conduction in general relativity have been widely used in the past, it is known that those theories present difficulties concerning causality. Israel (1976) and Israel & Stewart (1979) succeeded in restoring causality to the relativistic thermodynamics, by introducing second order terms proportional to the heat-flow. Hence the result in the post-Newtonian approximation is independent of the pathologies of the relativistic theory of dissipative fluids.

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