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DYNAMICAL PROPERTIES OF SOME CLASSES OF ENTIRE FUNCTIONS

by A.E. EREMENKO and M. Yu. LYUBICH

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1. Introduction.

The earliest paper devoted to the iteration theory of transcendental entire functions $f : \mathbb{C} \to \mathbb{C}$ was written by Fatou [F3] in 1926. He showed that the first basic facts are very similar in the rational and transcendental cases. However, further development of the subject showed that some dynamical properties of entire functions may be quite different from those

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of polynomials and rational maps [B3], [EL5]. This paper studies some classes of entire functions for which the dynamics are more or less similar to those of polynomials. The simplest examples of such classes are $\lambda \exp z$ and $a\cos z + b$.

Denote by f^m the m-th iterate of an entire function f. All entire functions considered in this paper are supposed to be non-linear. The maximal open set $N(f)$ where the family of iterates is normal in the sense of Montel [Mo] is called the *set of normality* and its complement $J(f) = \mathbb{C} \backslash N(f)$ is called the *Julia set*. $J(f)$ is a perfect completely invariant set (*i.e.* , $f^{-1}J = J$) which is either nowhere dense or coincides with $\mathbb{C}$. The Julia set of a transcendental entire function is unbounded.

A point $\alpha \in \mathbb{C}$ is called *periodic* if $f^p\alpha = \alpha$ for a natural number p which is called a *period*. If p is the minimal period of the point α then $\lambda = (f^p)'(\alpha)$ is said to be the *multiplier* of α. The periodic point α is called *attracting, repelling,* or *neutral* in the cases $|\lambda| < 1$, $|\lambda| > 1$, and $|\lambda| = 1$ respectively. In the last case α is said to be *rational* (resp., *irrational*) if $\lambda = e^{2\pi i\theta}$ with rational θ (resp., irrational θ). The Julia set of an arbitrary entire function coincides with the closure of repelling periodic points. The only known proof of this fact for transcendental functions [B1] is based on a deep theory of Ahlfors [N], Ch. 13.

Consider the class B consisting of all entire functions f such that the set of singular points of the inverse function f^{-1} is bounded (in other words, f is a covering map over $\{z : |z| > R\}$ for large R). Such functions are studied in §2. First, we prove an elementary but useful fact that all connected components of $N(f)$ are simply connected for transcendental $f \in B$ (it is not the case for arbitrary transcendental entire functions [B4]). Then we describe the logarithmic change of variable in a neighborhood of ∞. It is our main tool which permits us to study the dynamics of f near ∞. As the first application of the logarithmic change of variable we prove

THEOREM 1. — *Let $f \in B$ be a transcendental entire function. If $z \in N(f)$ then the orbit $\{f^m z\}_{m=0}^{\infty}$ does not tend to ∞.*

Most of the results of this paper concern a more restricted class of functions. Let S be the set of all entire functions f such that the set of the singular points of the inverse function f^{-1} is finite. In other words, there exists a finite set A such that $f : \mathbb{C}\backslash f^{-1}A \to \mathbb{C}\backslash A$ is a (unramified) covering map. The polynomials, the functions $\lambda \exp z$ and $a\cos z + b$ belong to S. If h and p are polynomials then

$$f(z) = \int^{z} h(\zeta) \exp p(\zeta) d\zeta \in S. \tag{1.1}$$

The function $f(z) = \dfrac{\sin z}{z}$ provides an elementary example outside of S.

The class S investigated systematically by Nevanlinna, Teichmüller, and others plays an important part in the value distribution theory [N], [W]. It was introduced into the iteration theory in works [EL1], [EL2] and [GK].

In §3 we include every $f \in S$ to a finite dimensional complex analytic manifold $M_f \subset S$. In §4 keeping in mind the further applications we prove various analytical results on M_f. The main result is the following : *the periodic points of a function $g \in M_f$ considered as multi-valued functions on M_f have only algebraic singularities* (Theorem 2).

The main property of the manifold M_f is as follows : if g is an entire function topologically conjugated to f then $g \in M_f$. This property allows one to extend Sullivan's theorem on the non-existence of wandering domains [S1] to the class S [EL1], [EL2], [GK].

Let D be a periodic component of $N(f)$, $f^p D \subset D$. If all orbits originating in D tend to a cycle then D is called a *Fatou domain.* If $f^p|D$ is conformally conjugate to an irrational rotation of the unit disk then D is called a *Siegel disk.* We say that an orbit $\{f^m x\}_{m=0}^{\infty}$ is *absorbed* by an invariant set X if $f^m x \in X$ for some m.

Theorem 1 and the absence of wandering domains yield a complete description of the dynamics of $f \in S$ on the set of normality : *every orbit in the set of normality $N(f)$ is absorbed by a cycle $\bigcup_{k=0}^{p-1} f^k D$ of Fatou domains or Siegel disks.*

We conclude §5 with the finiteness theorem for non-repelling cycles.

In [B2] Baker stated the conjecture that *if a transcendental entire function f has a completely invariant component D of $N(f)$ then $D = N(f)$.* This conjecture for $f \in S$ is proven in §6 (Theorem 6).

In §7 we study the problem of the area of the Julia set $J(f)$. The main difference as compared with a rational case is related to the set $I(f) = \{z | f^n z \to \infty,\ n \to \infty\}$. We prove that area $I(f) = 0$ *provided f is of finite order of growth and f^{-1} has a logarithmic singularity* (Example : λe^z).

Let $f \in S$, $M = M_f$. A function $g \in M$ is said to be *structurally stable* if for every function $h \in M$ close to g there exists a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ close to identity conjugating g and h : $\varphi \circ g = h \circ \varphi$. Using the auxiliary results of §4 we prove in §8 that *the set of structurally stable functions is open and dense in M.*

In the Appendix we discuss the exponential family $M_{\exp}$ of functions $z \mapsto a\exp(bz) + c$.

The results of the present paper were announced in [EL1], [EL3], and their proofs in Russian were given in [EL2], [EL4].

Finally, let us refer to the surveys [Bla], [L3], [EL6], [Mi] for a general introduction to holomorphic dynamics.

2. The logarithmic change of variable in the class B.

We begin with a simple proposition concerning arbitrary entire functions [B4], [T]. Denote by $\operatorname{ind}\gamma$ the index of a curve γ with respect to 0.

PROPOSITION 1. — *Let f be a transcendental entire function and D be a multiply connected component of $N(f)$ Then*

(a) *$f^m z \to \infty$ uniformly on compact subsets in D.*

(b) *For every Jordan curve γ non-contractible in D $\operatorname{ind}(f^n\gamma) \neq 0$ for all sufficiently large n.* □

The following consequence of Proposition 1 is a convenient sufficient condition of simply connectedness of all components of $N(f)$.

PROPOSITION 2. — *Let an entire function f be bounded on a curve Γ tending to ∞. Then all components of $N(f)$ are simply connected.*

Proof. — Otherwise let us consider a non-contractible Jordan curve $\gamma \subset D$. It follows from the above proposition that there exists a sequence $z_n \to \infty$ such that $z_n \in \Gamma \cap f^n\gamma$. This contradicts the boundedness of $f|\Gamma$. □

At this point we restrict the class of functions under consideration. To this end we need some definitions concerning singularities of the inverse function f^{-1}.

A point $a \in \mathbb{C}$ is said to be an *asymptotic value* of f if there exists a curve $\Gamma \subset \mathbb{C}$ tending to ∞ such that $f(z) \to a$ as $z \to \infty$ along Γ. If $f'(c) = 0$ then c is called a *critical point* of f and $f(c)$ is called a *critical value*. By a *singular point* of f^{-1} we mean a critical or an asymptotic value [N]. Denote the set of singular points by $\operatorname{sing} f^{-1}$. Note that this set may be non-closed. It is known that for an open set G such that $G \cap \operatorname{sing} f^{-1} = \emptyset$ the map $f : f^{-1}G \to G$ is an unramified covering [N].

Let B be the class of entire functions f such that the set $\operatorname{sing} f^{-1}$ is bounded. Denote $D(z_0, r) = \{z : |z - z_0| < r\}$. Let $f \in B$ be a transcendental function, $\operatorname{sing} f^{-1} \subset D(0, R/2)$, $A = \mathbb{C}\backslash\overline{D(0,R)}$, $G = f^{-1}A$. It is easy to show that each component V of G is a simply connected domain bounded by a single non-closed analytic curve both ends of which tend to ∞, and $f : V \to A$ is a universal covering. We have $|f(z)| = R$ on this curve, and Proposition 2 implies

PROPOSITION 3. — *If $f \in B$ is transcendental then all components of $N(f)$ are simply connected.* □

If R is chosen so large that $|f(0)| < R$, then $0 \notin G$, and $\exp : W \to G$ is a conformal isomorphism for any component W of the set $U = \ln G$. Considering the half-plane $H = \ln A = \{\xi : \operatorname{Re}\xi > \ln R\}$, we have the following commutative diagram :

$$\begin{array}{ccc} U & \xrightarrow{F} & H \\ \exp\downarrow & & \downarrow\exp \\ G & \xrightarrow{f} & A \end{array} \tag{2.1}$$

Here F is a conformal isomorphism of each connected component of U onto H. The existence of F is obvious because $f \circ \exp : W \to A$ is a universal covering for each connected component W of U. We say that F is obtained from f by the logarithmic change of variable in a neighborhood of ∞. A similar change of variable was used by Teichmüller in value distribution theory [W], 4.2.

LEMMA 1. — $|F'(z)| \geq \dfrac{1}{4\pi}(\operatorname{Re} F(z) - \ln R)$.

Proof (see Figure 1). — Let W be a connected component of U. Note that W contains no vertical segments of length 2π because the exponential map is univalent in W. Let $\Phi : H \to W$ be the inverse of F. The disk $D(F(z), \operatorname{Re} F(z) - \ln R)$ is contained in H. Applying the Koebe 1/4-theorem (see [N], Ch. 4, §3) to the function Φ in this disk we obtain

$$\frac{1}{4}|\Phi'(F(z))|\,|\operatorname{Re}F(z)-\ln R|\leq\pi\,,$$

and the lemma follows. □

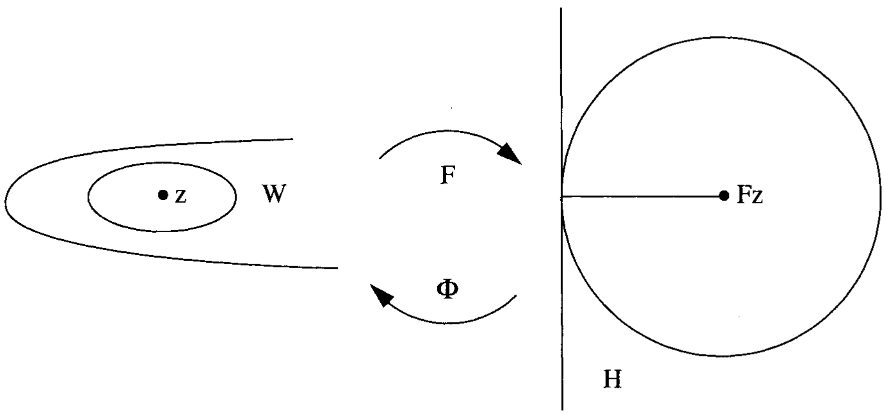

Figure 1

THEOREM 1. — *Let $f\in B$ be a transcendental entire function. If $z\in N(f)$ then the orbit $\{f^m z\}_{m=0}^{\infty}$ does not tend to ∞.*

Proof. — Suppose the orbit $\{z_m\}$ of $z_0\in N(f)$ tends to ∞. Then there exists a disk $B_0=D(z_0,r)$, $r>0$ such that the sequence $\{f^m\}$ tends uniformly to ∞ in B_0. Thus all $B_m=f^m B_0$ except a finite number are contained in G. Further the notations of the diagram (2.1) are used. One may suppose $B_m\subset G$ for all $m\geq 0$. Let C_0 be a component of the set $\ln B_0$, $C_m=F^m C_0$. Then $\exp C_m=B_m$. Consequently $C_m\subset U$ and $\operatorname{Re}F^m$ tends to $+\infty$ uniformly in C_0. Let $\zeta_0\in C_0$, $\zeta_m=F^m\zeta\in C_m$. Denote by d_m the supremum of radii of disks centered at ζ_m and contained in C_m. We have by the Koebe 1/4-theorem that $d_{m+1}\geq\frac{1}{4}d_m|F'(\zeta_m)|$. In view of $\operatorname{Re}F(\zeta_m)\to+\infty$ and Lemma 1, one obtains $|F'(\zeta_m)|\to\infty$. Thus $d_m\to\infty$. This is a contradiction since $C_m\subset U$ and U does not contain vertical segments of length 2π. The theorem is proved. □

Recall that $I(f)=\{z:f^m z\to\infty\}$.

COROLLARY. — *Let $f \in B$ be transcendental. Then $J(f) = \overline{I(f)}$.*

Proof. — It is proved in [E] that $J(f) = \partial I(f)$ for arbitrary entire functions f. By Theorem 1 $I(f) \subset J(f)$ for $f \in B$ and the corollary follows. $\square$

3. Class S and manifolds M_g.

We say that an entire function f belongs to the class S_q if the set $\operatorname{sing} f^{-1}$ contains at most q points. In other words, there exists a set $A = \{a_1, \ldots, a_q\}$ such that $f : \mathbb{C}\backslash f^{-1}(A) \to \mathbb{C}\backslash A$ is a covering map. Set $S = \bigcup_{q=1}^{\infty} S_q$. Some examples of functions of the class S were mentioned in the introduction.

We call entire functions f and g *topologically equivalent* if there exist homeomorphisms $\varphi, \Psi : \mathbb{C} \to \mathbb{C}$ such that

$$\Psi \circ g = f \circ \varphi. \tag{3.1}$$

Fix $g \in S_q$ and denote by $M_g \subset S_q$ the set of all entire functions topologically equivalent to g. The aim of this section is to define on M_g a structure of $(q+2)$-dimensional complex analytic manifold.

Choose β_1 and β_2 such that $g(\beta_i) \notin \operatorname{sing} g^{-1}$. Let $M_g(\beta_1, \beta_2)$ be the set of functions f such that the homeomorphisms φ and Ψ in (3.1) may be chosen in such a way that $\varphi(\beta_i) = \beta_i$. One can easily verify that $M_g = \cup M_g(\beta_1, \beta_2)$. Fix $\beta_1, \beta_2 \notin \operatorname{sing} g^{-1} = \{a_1, \ldots, a_q\}$ and put $a_{q+1} = g(\beta_1)$, $a_{q+2} = g(\beta_2)$.

LEMMA 2. — *Let $\Psi_0 \circ g = f_0 \circ \varphi_0$, $\Psi_1 \circ g = f_1 \circ \varphi_1$, $f_i \in S$, $\varphi_i(b_j) = \beta_j$, $j = 0, 1$. Assume that there exists an isotopy Ψ_t connecting Ψ_0 and Ψ_1 such that $\Psi_t(a_j) = \Psi_j(a_j)$ for $0 \le t \le 1$, $1 \le j \le q+2$. Then $f_1 = f_0$.*

Proof. — By the Covering Homotopy Theorem there exists a continuous family of homeomorphisms h_t such that $h_1 = \varphi_1$ and $\Psi_t \circ g = f_1 \circ h_t$, $0 \le t \le 1$. The functions $t \mapsto h_t(\beta_i)$ are continuous and take a discrete set of values. Hence $h_t(\beta_i) = \beta_i$. Putting $t = 0$ we obtain $f_0 \circ \varphi_0 = \Psi_0 \circ g = f_1 \circ h_0$, thus $f_0 = f_1 \circ (h_0 \circ \varphi_0^{-1})$. The homeomorphism $h_0 \circ \varphi_0^{-1} : \mathbb{C} \to \mathbb{C}$ has two fixed points and is conformal outside a discrete set. Consequently $h_0 \circ \varphi_0^{-1} = \mathrm{id}$ and $f_0 = f_1$. $\square$

Let us define an analytic structure on $M_g(\beta_1, \beta_2)$. To this end consider the space Y of homeomorphisms $\Psi : \mathbb{C} \to \mathbb{C}$ modulo the following equivalence relation : $\Psi_0 \sim \Psi_1$ if there exists an isotopy $\Psi_t : \mathbb{C} \to \mathbb{C}$ such that $\Psi_t(a_j) = \Psi_0(a_j)$, $0 \leq t \leq 1$, $1 \leq j \leq q+2$. The map $Y \to \mathbb{C}^{q+2}$, $\Psi \mapsto (\Psi(a_1), \dots, \Psi(a_q+2))$ being a local homeomorphism defines on Y the structure of a $(q+2)$-dimensional complex analytic manifold. Let us construct a map $\pi : Y \to M_g(\beta_1, \beta_2)$. Observe that every element Ψ of Y can be represented by a quasiconformal homeomorphism. Consider a map $\Psi \circ g$ where Ψ is such a representative. By the Measurable Riemann Theorem [AB] there exists a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(\beta_j) = \beta_j$, $j = 1, 2$ and $\Psi \circ g \circ \varphi^{-1} = f$ is an entire function. Set $\pi(\Psi) = f$. Then π is correctly defined (by Lemma 2). Note that $\operatorname{sing} f^{-1} = \{a_1(f), \dots, a_q(f)\} = \{\Psi(a_1), \dots, \Psi(a_q)\}$.

Clearly π is surjective and locally injective. Consequently π induces a complex analytic structure on $M_g(\beta_1, \beta_2)$. The functions $a_1(f), \dots, a_{q+2}(f)$ are local coordinates on $M_g(\beta_1, \beta_2)$. Finally, the covering $M_g = \cup M_g(\beta_1, \beta_2)$ gives the analytic structure on the whole space M_g.

Note that the topology on M_g is locally equivalent to the topology of uniform convergence on compact subsets of $\mathbb{C}$.

In conclusion let us show that the map

$$M_g \times \mathbb{C} \to \mathbb{C}, \qquad (f, z) \mapsto f(z) \tag{3.2}$$

is analytic. Let $a = (a_1(f), \dots, a_{q+2}(f))$ be the local parameters of $f = f_a$. Then the homeomorphism Ψ_a in (3.1) can be chosen in such a way that $\Psi_a(z)$ analytically depends on a for any $z \in \mathbb{C}$. By the Ahlfors-Bers theorem on the analytic dependence of the solution of the Beltrami equation on parameters [AB] we conclude that φ_a in (3.1) also analytically depends on a. Hence $f_a = \Psi_a \circ g \circ \varphi_a^{-1}$ analytically depends on a. Thus (3.2) is analytic in both variables and we are done.

4. Auxiliary analytic results.

The results of this section will be used only in §8.

In what follows we fix a transcendental function $g \in S$ and denote M_g by M.

Consider periodic points of period p of a function $f \in M$. They are defined by the equation

$$f^p z = z. \tag{4.1}$$

The solution $z = \alpha(f)$ of this equation is a multi-valued analytic function on M. The main result of this section is the following :

THEOREM 2. — *All singularities of the function α on M are algebraic.*

For the proof we need several lemmas.

Let V be a domain bounded by a simple curve Γ both ends of which tend to ∞, $0 \notin V$. Fix two points b_1 and b_2 in ∂V. Let $z \in V$. Consider the circle $L = \{w : |w| = |z|\}$ and let (b_3, b_4) be the connected component of $V \cap L$ containing z. We say that the point z belongs to a *gulf* if b_1 does not belong to the bounded arc of ∂V between the points b_3 and b_4. The gulfs are relatively closed bounded sets in V. The complement of all gulfs in V is unbounded. If we change b_1 then the notion of gulf will change only in a bounded part of the plane. That is why we shall not emphasize the dependence on the choice of b_1. If $z \in V$ does not belong to a gulf and $|z|$ is sufficiently large then three bounded arcs of ∂V with ends at b_1, b_2, b_3, b_4 and the arc (b_3, b_4) of the circle L form a curvilinear quadrilateral $[b_1, b_2, b_3, b_4]$. If $\gamma \subset V$ is a curve tending to ∞ then there exist points on γ with arbitrarily large moduli which do not belong to any gulf.

The following result is closely related to one due to Ahlfors [A1].

LEMMA 3. — *Let V be any component of the set G from the diagram (2.1), so that $f : V \to A = \mathbb{C}\backslash\overline{D(0,R)}$ is a covering. Fix a branch of $\arg z$ in V. Suppose that a point $z \in V$ does not belong to any gulf. Then*

$$\ln^2 |f(z)| + \arg^2 f(z) \geq C|z| \exp \frac{\arg^2 z}{\ln |z|}$$

for sufficiently large $|z|$. The constant $C > 0$ is independent of z.

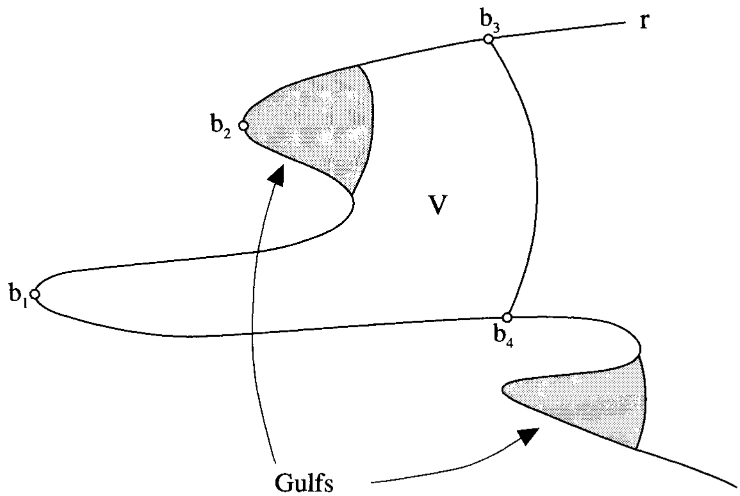

Figure 2

Proof. — Let $\varphi = \ln f : G \to H$, $H_0 = H \backslash D(\ln R, 1)$, $V_0 = (\varphi^{-1} H_0) \cap V$. (We use the notation from the diagram (2.1).) Consider the commutative diagram consisting of conformal homeomorphisms :

$$\text{(4.2)} \qquad \begin{array}{ccc} T & \xrightarrow{\Phi} & E \\ \exp \downarrow & & \downarrow \exp + \ln R \\ V_0 & \xrightarrow{\varphi} & H_0 \end{array}$$

Here T is a half-strip-like domain intersecting all lines $\{\zeta : \operatorname{Re} \zeta = \delta\}$, $\delta > \delta_0$ in a finite union of intervals of total length $\leq 2\pi$; $E = \{s : \operatorname{Re} s > 0, |\operatorname{Im} s| < \pi/2\}$ is a half-strip. Let $z = \mathrm{r}e^{i\theta} \in V$ ($\theta = \arg z$ is the branch of the argument fixed above), $\zeta = \ln z \in T$.

Consider the connected component (d_3, d_4) of the intersection $\{t : \operatorname{Re} t = \ln r\}$ containing ζ. Denote $d_1 = \Phi^{-1}\Big(-i\dfrac{\pi}{2}\Big)$, $d_2 = \Phi^{-1}\Big(i\dfrac{\pi}{2}\Big)$. If z does not belong to a gulf and $|z|$ is sufficiently large, then the curvilinear quadrilateral $\Delta = [d_1, d_2, d_3, d_4]$ is well-defined. It is bounded by three arcs $[d_1, d_2]$, $[d_2, d_3]$ and $[d_1, d_4]$ of the curve ∂T and by the segment $[d_3, d_4]$.

We are going to estimate from below the *extremal length* ℓ of the family of the curves in Δ connecting the sides $[d_1, d_2]$ and $[d_3, d_4]$. (For the definition and the properties of extremal length see [A2], [W].) Consider a metric coinciding with the Euclidean one on the set $\Delta_0 = \Delta \cap \{t : \operatorname{Re} t < \ln r\}$. Let γ be a curve in our family, $\gamma_0 = \gamma \cap \Delta_0$. The horizontal projection of γ_0 has length at least $\ln r + O(1)$, $r \to \infty$. The length of the vertical

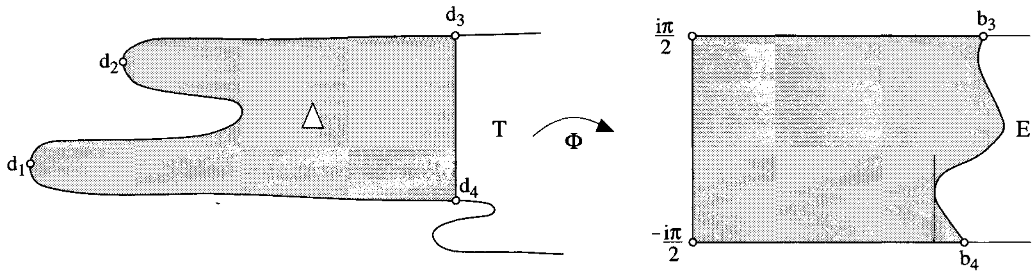

Figure 3

projection is at least $\theta + O(1)$, $r \to \infty$. Thus the length of γ_0 is at least

$$\sqrt{\ln^2 r + \theta^2} + O(1) \quad r \to \infty.$$

The area of Δ_0 does not exceed $2\pi \ln r + O(1)$, $r \to \infty$. Consequently

$$\ell \geq \frac{1}{2\pi}\Big(\ln r + \frac{\theta^2}{\ln r}\Big) + O(1), \qquad r \to \infty. \tag{4.3}$$

Consider the curvilinear quadrilateral $\Phi(\Delta) = \left[-i\frac{\pi}{2}, i\frac{\pi}{2}, b_3, b_4\right]$ where $b_j = \Phi(d_j)$. Observe that three sides of $\Phi(\Delta)$ are line segments and the fourth side is the curve (b_3, b_4). The extremal length of the family of curves in $\Phi(\Delta)$ connecting the side $\left[-i\frac{\pi}{2}, i\frac{\pi}{2}\right]$ with the side (b_3, b_4) is equal to ℓ because the extremal length is a conformal invariant. On the other hand by the well-known estimate due to Ahlfors [A2], p.77 we have

$$\ell \leq \frac{\tau}{\pi} + c_0 \tag{4.4}$$

where $\tau = \inf\{\operatorname{Re} s : s \in (b_3, b_4)\}$, c_0 being an absolute constant. The estimates (4.3), (4.4) imply

$$\tau \geq \frac{1}{2}\Big(\ln r + \frac{\theta^2}{\ln r}\Big) + O(1), \qquad r \to \infty. \tag{4.5}$$

From (4.2) and $\Phi(\zeta) \in (b_3, b_4)$ we obtain

$$\ln|\varphi(z) - \ln R| = \operatorname{Re}\Phi(\zeta) \geq \tau. \tag{4.6}$$

It follows from (4.5), (4.6) that $|\varphi(z)| \geq c\sqrt{r}\exp\frac{\theta^2}{2\ln r}$, where c is independent of z. Lemma 3 is proved since

$$\ln^2|f(z)| + \arg^2 f(z) = |\varphi(z)|^2.$$

$\square$

LEMMA 4. — *Let $\Psi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal homeomorphism, $\Psi(0) = 0$. Let $\arg \Psi(z) - \arg z$ be a uniform branch of the difference of arguments in $\mathbb{C}^*$. Suppose*

$$B^{-1} \leq |\Psi(z_0)| \leq B\,, \qquad |\arg \Psi(z_0) - \arg z_0| \leq B$$

for some $z_0 \in \mathbb{C}^*$. Then for $|z| > |z_0|$ the following estimates hold :

$$C^{-1}|z|^{K_1^{-1}} \leq |\Psi(z)| \leq C|z|^{K_1}\,, \tag{4.7}$$

$$|\arg \Psi(z) - \arg z| \leq K_1 \ln |z| + C. \tag{4.8}$$

Here K_1, C depend on K, z_0, B but do not depend on Ψ and z.

Proof. — This is a well-known property of quasiconformal homeomorphisms (see for example [LV]). □

LEMMA 5. — *Consider a curve $z = \gamma(t)$, $0 \leq t < 1$ such that $\gamma(t) \to \infty$, $t \to 1$ and a function $f \in S$ such that $f(\gamma(t)) \to \infty$, $t \to 1$. Let $\{h_t : 0 \leq t \leq 1\}$ be a continuous family of K-quasiconformal homeomorphisms satisfying the assumptions of Lemma 4. Then there exists a curve $z = \gamma_1(t)$ such that*

$$f(\gamma_1(t)) = h_t \circ f(\gamma(t))\,, \qquad t_0 < t < 1, \tag{4.9}$$

$$\ln |\gamma_1(t)| = \ln |\gamma(t)| + O(1)\,, \qquad t \to 1, \tag{4.10}$$

$$\arg \gamma_1(t) = \arg \gamma(t) + O(1)\,, \qquad t \to 1. \tag{4.11}$$

Proof. — By Lemma 4, $h_t \circ f(\gamma(t)) \to \infty$, $t \to 1$. There exists $R > 0$ such that

$$f : \mathbb{C}\backslash f^{-1}(D(0,R)) \to \mathbb{C}\backslash D(0,R)$$

is an unramified covering. Consequently we can find a curve γ_1 satisfying (4.9). Let us use the diagram (2.1). We have

$$F(\delta_1(t)) = H_t \circ F(\delta(t))\,, \tag{4.12}$$

where $\delta(t) = \ln \gamma(t)$, $\delta_1(t) = \ln \gamma_1(t)$, $H_t = \ln \circ h_t \circ \exp$. Lemma 4 and (4.12) imply

$$|F(\delta_1(t)) - F(\delta(t))| = O(\operatorname{Re} F(\delta(t)))\,, \qquad t \to 1, \tag{4.13}$$

$$\operatorname{Re} F(\delta_1(t)) \geq K_1^{-1} \operatorname{Re} F(\delta(t)) - \ln C. \tag{4.14}$$

We deduce from Lemma 1 and (4.14) that

$$|\delta_1(t) - \delta_2(t)| \le \frac{\text{const.}}{\operatorname{Re} F(\delta(t))} |F(\delta_1(t)) - F(\delta(t))|.$$

Combining this estimate with (4.13) we obtain (4.10), (4.11). The lemma is proved. □

Proof of Theorem 2. — Consider a germ $z = \alpha(f)$ of the analytic function defined by the equation (4.1) in a neighborhood of $f_0 \in M$. Let f_t, $0 \le t \le 1$ be a curve in M such that the element $\alpha(f)$ can be analytically continued along f_t, $0 \le t < 1$. Two cases are possible :

1 : There exists a sequence $t_n \to 1$ such that $\alpha(f_{t_n})$ tends to a finite limit α_1 as $n \to \infty$. If $(f^p)'(\alpha_1) \neq 1$ then the element $\alpha(f)$ can be continued to the point f_1 by the Implicit Function Theorem. If $(f^p)'(\alpha_1) = 1$ then the function $\alpha(f)$ has an algebraic singularity at $f = f_1$.

2 : $\alpha(t) \equiv \alpha(f_t) \to \infty$ as $t \to 1$. We will show that this is impossible. One has $f_t = \Psi_t \circ f \circ \varphi_t$ where Ψ_t and φ_t are continuous families of K-quasiconformal homeomorphisms. We may suppose without loss of generality that $\varphi_t(0) = 0$, $\Psi_t(0) = 0$, $0 \le t < 1$. Applying Lemmas 4 and 5 repeatedly we find a curve $z = \beta(t)$ such that

$$f_0^p(\beta(t)) = f_t^p(\alpha(t)) = \alpha(t)\,,$$

$$\ln|\alpha(t)| \le C \ln|\beta(t)|\,,$$

$$|\arg\alpha(t) - \arg\beta(t)| \le C\ln|\beta(t)|\,, \qquad t_0 \le t < 1.$$

These estimates imply

$$\ln^2|f_0^p(\beta(t))| + \arg^2 f_0^p(\beta(t)) \le 3C^2\ln^2|\beta(t)| + 2\arg^2\beta(t)$$

which is impossible in view of Lemma 3. The proof is completed. □

Consider now the multiplier $\lambda(f) = (f^p)'(\alpha(f))$ of a periodic point α as a function of $f \in M$.

LEMMA 6. — *All branches of $\lambda(f)$ are non-constant.*

Proof. — Let $f \in M$. Consider the subfamily $f_w = wf \in M$, $w \in \mathbb{C}^*$. It is sufficient to prove that $\lambda(w) = \lambda(wf)$ is non-constant. Denote $\alpha_k(w) = f_w^k(\alpha(w))$, $0 \le k \le p-1$. Then

$$\lambda(w) = w^p \prod_{k=0}^{p-1} f'(\alpha_k(w)). \tag{4.15}$$

Suppose $\lambda(w) \equiv \lambda$.

If $\lambda = 0$ then for some k, $0 \le k \le p-1$ the function $\alpha_k(w)$ is equal identically to a critical point c of the function f. Consequently $f_w^p c = c$. Denote $f_w^k c = g_{c,k}(w)$. We have the recurrent equation

$$g_{c,k+1}(w) = wf(g_{c,k}(w)), \qquad g_{c,0}(w) \equiv c.$$

This implies that the functions $g_{c,k}$ are non-constant for $k \ge 1$. Thus $\lambda \not\equiv 0$.

It follows from Theorem 2 that there exists a curve $w = \gamma(t)$, $0 \le t < 1$ such that $\gamma(t) \ne 0$, $0 \le t < 1$ and $\gamma(t) \to 0$ as $t \to 1$ and the function $\alpha(w)$ can be analytically continued along γ.

The formula (4.15) is valid on γ. Suppose there exists a sequence $w_j \to 0$, $w_j \in \gamma$ such that $|\alpha(w_j)| \le c$. Then

$$\prod_{k=0}^{p-1} |f'(f_{w_j}^k(\alpha(w_j)))| \le c_1$$

and hence $\lambda(w_j) \to 0$ by (4.15). This is a contradiction.

The remaining case to consider is $\alpha(w) \to \infty$ as $w \to 0$ along γ. (We cannot apply Theorem 2 since $f_0 \notin M$.) In such a case we have $\alpha_k(w) \to \infty$ along γ, $1 \le k \le p-1$. Make use of diagram (2.1). We have

$$f'(\zeta) = \frac{f(\zeta)}{\zeta} F'(z), \qquad \zeta = \exp z, \ z \in U,$$

consequently

$$f'(\alpha_k(w)) = F'(z_k(w)) \frac{f(\alpha_k(w))}{\alpha_k(w)}, \qquad z_k(w) = \ln \alpha_k(w).$$

This relation and (4.15) imply

$$\lambda = \prod_{k=0}^{p-1} F'(z_k(w)) \prod_{k=0}^{p-1} \frac{wf(\alpha_k(w))}{\alpha_k(w)} = \prod_{k=0}^{p-1} F'(z_k(w)).$$

The last product tends to ∞ in view of Lemma 1 and $\operatorname{Re} z_k(w) \to +\infty$ as $w \to 0$ along γ. This is a contradiction which proves the lemma. □

Consider an entire function

$$f(z) = \sum_{k=0}^{\infty} d_k z^k$$

and include it in the one-parameter family $f_w(z) = f(wz)$, $w \in \mathbb{C}^*$. Consider a point $z = b$ and the sequence of entire functions

$$\tilde{g}_{b,m}(w) = f_w^m(b), \qquad m = 1, 2, \ldots . \tag{4.16}$$

LEMMA 7. — *If $d_0 \neq b$ and $d_1 \neq 0$ then the functions $\tilde{g}_{b,m}$, $b \in \mathbb{C}$, $m = 1, 2, ...$ are pairwise distinct.*

Proof. — Let

$$\tilde{g}_{b,m}(w) = \sum_{k=0}^{\infty} e_k(b,m) w^k.$$

It is easy to see that

$$e_k(b,m) = \begin{cases} d_1^k b + s_k\,, & k = m \\ d_1^k d_0 + s_k\,, & k < m\,, \end{cases}$$

where s_k are independent of b and m. Consequently if $(b,m) \neq (b',m')$ and $m' \geq m$ then $e_m(b,m) \neq e_m(b',m')$. The lemma is proved. □

Let us consider the following sequence of holomorphic functions on M :

(4.17) $$g_{i,m}(f) = f^m(a_i(f))\,, \qquad 1 \leq i \leq q, \quad m = 1, 2, ...$$

where $\{a_1(f), \dots, a_q(f)\} = \operatorname{sing} f^{-1}$.

LEMMA 8. — *The functions $g_{i,m}$ are pairwise distinct.*

Proof. — Let $f \in M$. Conjugating f by an affine mapping we achieve $f(0) \neq a_i(f)$, $1 \leq i \leq q$; $f'(0) \neq 0$. Then Lemma 7 is applicable to the sequence $\tilde{g}_{a_i,m}(w)$ defined by (4.16). We have $\tilde{g}_{a_i,m}(w) = g_{i,m}(f_w)$ where $f_w(z) = f(wz)$ and Lemma 8 follows from Lemma 7. □

LEMMA 9. — *Let f_t, $0 \leq t \leq 1$ be a curve in M and $\gamma(t)$, $0 \leq t \leq 1$ be a curve in $\mathbb{C}$. Suppose that $\gamma(t) \to \infty$, $f_t(\gamma(t)) \to b \in \mathbb{C}$ as $t \to 1$. Then b is an asymptotic value of the function f_1.*

Proof. — We have $f_t = \Psi_t \circ f_1 \circ \varphi_t$ where Ψ_t and $\varphi_t \to \mathrm{id}$ as $t \to 1$. By Lemma 4 $\varphi_t(\gamma(t)) \to \infty$, $t \to 1$. Furthermore $\lim\limits_{t \to 0} f_1(\varphi_t(\gamma(t))) = b$ and the lemma is proved. □

5. The dynamics of $f \in S$ on the set of normality.

Recall that a domain D is called wandering if $f^m D \cap f^n D = \emptyset$ for $m > n \geq 0$. The first example of an entire function having wandering components of the set of normality was constructed by Baker [B3]. Later on, many other examples having interesting additional properties were constructed [B4], [EL2], [EL5], [H]. On the other hand, rational functions have no wandering components of the set of normality [S1]. It was shown in [EL1], [EL2] and [GK] that this result can be extended to the class S of entire functions (also, Baker [B4] did this for a smaller class of functions). Let us start with a brief discussion of this fact.

Let $f \in S_q$. Then f belongs to the $(q+2)$-dimensional complex analytic manifold M_f (see §3). By the definition of M_f it satisfies the following property : if an entire function g is topologically conjugate to f then $g \in M_f$. This remark permits one to repeat word by word the proof by Sullivan. Moreover, the argument for a transcendental function $f \in S$ is even easier than the argument for a rational function due to the fact that all components of $N(f)$ are simply connected (Proposition 3). Thus we have

THEOREM 3. — *Let $f \in S$. Then $N(f)$ has no wandering components.*

This theorem immediately implies that for $f \in S$ each orbit in $N(f)$ is absorbed by a cycle of components of $N(f)$. One may obtain the classification of such cycles by an argument similar to the one used for the proof of the Denjoy-Wolff theorem (see [L3], [S2], [V]). Let f be an arbitrary entire function, D be a periodic component of $N(f)$, $f^p D \subset D$. Then one of the following possibilities holds :

(i) *D is a Fatou domain.* In such a case all orbits originating in D tend to an attracting or to a neutral rational cycle $\{\alpha_k\}_{k=0}^{p-1}$. The cycle of domains $\bigcup_{k=0}^{p-1} f^k D$ is called an *immediate attractive region* of $\{\alpha_k\}$. Each immediate attractive region contains a singular point of f^{-1} (for the proof see [F2], [Bla], [L3], [Mo]).

(ii) *D is a Siegel disk.* Then $f^p|D$ is conformally conjugate to an irrational rotation of the round disk. Hence each cycle of Siegel disks contains a neutral irrational cycle. In addition, the following inclusion holds :

$$\partial D \subset \overline{\bigcup_{k=1}^{\infty} f^k(\operatorname{sing} f^{-1})} \tag{5.1}$$

(see [F2], [L3]).

(iii) *D is a Baker domain.* We call a Baker domain a periodic component D of $N(f)$ such that $f^m z \to \infty$ as $m \to \infty$ for $z \in D$.

It follows from Theorem 1 that a transcendental entire function $f \in S$ cannot have Baker domains. Thus we obtain

THEOREM 4. — *Let $f \in S$. Then every orbit in $N(f)$ is absorbed by a cycle of Fatou domains or by a cycle of Siegel disks.*

For examples of transcendental entire functions having Baker domains see [EL5], [H].

In conclusion we show that the number of Fatou domains and Siegel disks is finite. Denote by n_F the number of the cycles of Fatou domains and by n_I the number of irrational neutral cycles. It is clear that $n_F \leq q$ for $f \in S_q$ because every cycle of Fatou domains contains a singular point of f^{-1}.

THEOREM 5. — *Let $f \in S_q$. Then $n_F + n_I \leq q$.*

Sketch of the proof (Compare [S]). — Suppose first that there is only one irrational neutral periodic point z_0. If $n_F \leq q-1$, there is nothing to prove. Otherwise all singular points of f^{-1} are attracted to attracting and neutral rational cycles. In view of (5.1) z_0 cannot be the center of a Siegel disk. So $z_0 \in J(f)$ (a "*Cremer point*"). But then

$$z_0 \in \overline{\bigcup_{k=1}^{\infty} f^k \operatorname{sing} f^{-1})}$$

(see [L3], §1.14) which gives a contradiction again.

Now assume that there are at least two neutral irrational periodic points. Then one of them, say z_0, has a preimage z_1 which does not belong to the cycle of z_0. One can construct a homeomorphism $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ conformal in $\bar{\mathbb{C}} \backslash D(z_1, \varepsilon)$ and having the following properties :

(i) $h(\infty) = \infty$,

(ii) $n_F(f \circ h) \geq n_F(f) + n_I(f)$,

(iii) z_0 is an attracting periodic point of $f \circ h$ with immediate attractive region V and $f \circ h(D(z_1, \varepsilon)) \subset V$.

Then using the Measurable Riemann Theorem one can find a quasi-conformal homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $f_1 = \varphi^{-1} \circ f \circ h \circ \varphi$ is an entire function. Thus $n_F(f) + n_I(f) \leq n_F(f_1) \leq q$ because $f_1 \in M_f \subset S_q$. $\square$

Remark. — One can deduce from Lemma 6 the weaker estimate $n_F(f) + \frac{1}{2}n_I(f) \leq q$ using the following elementary

LEMMA 10 (See [F2], [Mo]). — *Consider n functions $\lambda_1, \dots, \lambda_n$ analytic and non-constant in a neighborhood of the origin, $|\lambda_j(0)| = 1$, $1 \leq j \leq n$. Then there exists an arbitrarily small t such that at least $n/2$ of the functions satisfy $|\lambda_j(t)| < 1$.* $\square$

6. Completely invariant components of $N(f)$.

In what follows we shall need a more detailed description of singularities of functions f^{-1}, where f is entire. A point $a \in \mathbb{C}$ is called a *logarithmic singularity* of f^{-1} if there exists a disk $V = D(a, r)$ such that $f^{-1}(V)$ contains an unbounded component W such that $f : W \to V \backslash \{a\}$ is a universal covering. For $f \in S$ all asymptotic values are logarithmic singularities. We shall use

GROSS THEOREM [N]. — *Let f be an entire function and g be an element of f^{-1} defined in a neighborhood of $w_0 \in \mathbb{C}$. Then g can be analytically continued along almost all rays $\{w_0 + te^{i\theta} : 0 \leq t < \infty\}$, $\theta \in [-\pi, \pi]$.*

The following result is an extension of Theorem 2 from [B2].

LEMMA 11. — *Assume that a transcendental entire function f has a completely invariant domain D. Then all critical values and logarithmic singularities of f^{-1} are contained in D.*

Proof. — Assume that $a \notin D$ is a critical value or a logarithmic singularity. Let $V = D(a, r)\backslash\{a\}$ with a sufficiently small $r > 0$ and W be a component of $f^{-1}V$ such that $f : W \to V$ is an unramified covering but not a homeomorphism. (If a is a logarithmic singularity then $f|W$ is a universal covering. If a is a critical value then W is double connected and $f|W$ is a covering with finite valency).

Fix two points b_1 and b_2 in W such that $f(b_1) = f(b_2) = b$. Denote by g_i the branches of f^{-1} such that $g_i(b) = b_i$, $i = 1, 2$. Using the Gross theorem we find a segment $[b, c]$, $c \in D$ such that g_i can be analytically continued along $[b, c]$. Let $\gamma_i = g_i([b, c])$. The curves γ_i connect b_i with some c_i, $i = 1, 2$. We have $f(c_1) = f(c_2) = c \in D$. Thus c_1 and c_2 belong to D since D is completely invariant. There exists a simple curve $\gamma_0 \subset D$ which connects c_1 and c_2. We have $f(\gamma_0) \subset D$ since D is invariant. There exists a small r', $0 < r' < r$ such that $D(a, 2r') \cap f(\gamma_0 \cup \gamma_1 \cup \gamma_2) = \emptyset$. Thus the component W_1 of $f^{-1}(D(a, 2r') \backslash \{\alpha\})$ which belongs to W does not intersect $\gamma_0 \cup \gamma_1 \cup \gamma_2$. (When $r' \to 0$, W_1 tends uniformly either to a critical point $z_0 \notin D$ or to infinity.) Choose a point $d \in \partial D(a, r')$ such that the segment $[b, d]$ has the properties : $[b, d] \cap D(a, r') = \emptyset$ and $[b, d] \cap [b, c] = \{b\}$. The elements g_i can be analytically continued along $[b, d]$ because $f : W \to V$ is a covering. We obtain two disjoint simple curves $\beta_i = g_i([b, d])$ which connect the points b_i with points d_i, $f(d_1) = f(d_2) = d$. Then we connect d_1 and d_2 by a simple curve β such that $\beta \cap \beta_i = \{d_i\}$ and $f(\beta)$ is the circle $\partial D(a, r')$.

Denote $\delta_i = \beta_i \cup \gamma_i$, $i = 1, 2$. Then the simple curves δ_1, δ_2 and β have pairwise disjoint interiors and $\beta \cap \gamma_0 = \emptyset$. Let $\gamma_0(t)$, $0 \le t \le 1$ be a parametrization of γ_0, $\gamma_0(0) = c_1$, $\gamma_0(1) = c_2$. There exist t_1 and t_2 in $[0, 1]$ such that $\gamma' = \{\gamma_0(t) : t_1 < t < t_2\} \cap (\delta_1 \cup \delta_2) = \emptyset$, $\gamma_0(t_1) = c'_1 \in \delta_1$ and $\gamma_0(t_2) = c'_2 \in \delta_2$. Denote by δ'_i the part of δ_i from d_i to c'_i. Then $\Gamma = \beta \cup \delta'_1 \cup \delta'_2 \cup \gamma'$ is a Jordan curve. Denote by A the bounded component of its complement. The image $f(\Gamma)$ consists of the following parts :

(i) the circle $\partial D(a, r')$,

(ii) the curve $f(\delta'_1 \cup \delta'_2)$ which is a part of $[b, d] \cup [b, c]$,

(iii) the curve $f(\gamma') \subset f(\gamma_0) \subset D$ which is disjoint from $D(a, 2r')$.

Note that D is simply connected since all unbounded components of $N(f)$ for entire transcendental f are simply connected [B2]. Thus $D(a, 2r')$ lies in an unbounded component of $\mathbb{C} \backslash f(\gamma')$.

Consider the point $\{w\} = \partial D(a, 2r') \cap f(\Gamma) = \partial D(a, 2r') \cap [b, d]$ and a disk $C = D(w, \varepsilon)$. Here $\varepsilon > 0$ is so small that $\varepsilon < r'$ and $C \cap ([b, c] \cup f(\gamma')) = \emptyset$. It follows from (i)-(iii) that the index of $f(\Gamma)$ with respect to all points of $C \backslash [b, d]$ is equal to zero. On the other hand $w \in [b, d] \subset \overline{f(A)}$ and $f(A)$ is an open set. This is a contradiction which proves the lemma. $\square$

Remarks. — 1. Essentially the same proof shows that if an entire function f has a completely invariant domain D then all *direct* transcendental singularities of f^{-1} lie in D. (For the classification of singularities see [N]). The question of whether *indirect* singularities are contained in D remains open.

2. If $f \in S$ then the use of the Gross Theorem becomes unnecessary.

THEOREM 6. — *Let $f \in S$ be a transcendental entire function having a completely invariant component D of the set $N(f)$. Then $D = N(f)$.*

Proof. — If $D \neq N(f)$ then there exists a periodic component G of the set $N(f)$ different from D. This follows from Theorem 3. This component G cannot be a Fatou domain because $\operatorname{sing} f^{-1} \subset D$. On the other hand it is evident that D is a Fatou domain. Thus the set

$$\bigcup_{n>0} f^n(\operatorname{sing} f^{-1})$$

has only one limit point. Consequently G cannot be a Siegel disk in view of (5.1). The theorem is proved. □

7. The area of the Julia set.

Let $\theta_R(r, f)$ be the linear measure of the set $\{\theta : |f(re^{i\theta})| < R\}$. In this section we consider entire functions satisfying the following property :

$$\liminf_{r\to\infty} \frac{1}{\ln r} \int_1^r \theta_R(t, f) \frac{dt}{t} > 0. \tag{7.1}$$

There exists a simple sufficient condition for (7.1). To state it recall that the order of growth of an entire function f is

$$\rho = \liminf_{r\to\infty} \frac{\ln\ln M(r, f)}{\ln r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$. Observe that all functions of class S mentioned in the introduction have finite order.

PROPOSITION 4. — *If the order of an entire function f is finite and its inverse f^{-1} has a logarithmic singularity $a \in \mathbb{C}$ (see §6) then (7.1) is satisfied.*

This proposition may be proved by the argument used in the proof of the Denjoy-Carleman-Ahlfors Theorem [N], Ch. XI, §4.

It is plausible that for a function $f \in S$ of a finite order the property (7.1) is equivalent to having a (finite) asymptotic value.

Recall that $I(f) = \{z : f^n z \to \infty\}$.

THEOREM 7. — *Let $f \in B$ be a transcendental entire function satisfying (7.1). Then* area $I(f) = 0$. *Moreover, there exists an $M > 0$ such that*

$$\liminf_{n\to\infty} |f^n z| < M \quad \text{a.e. in } \mathbb{C}.$$

Remark. — For any function of the form $f_{a,b}(z) = a\cos z + b \in S_2$ (7.1) fails (it has a finite order but $f_{a,b}^{-1}$ has no (finite) logarithmic singularities). Mc Mullen [McM] obtained a surprising result that area $I(f_{a,b}) > 0$ for arbitrary a, b ($a \neq 0$). So (7.1) is essential in Theorem 7.

We shall use the following classical

KÖBE DISTORTION THEOREM (see [V]). — *Let g be a univalent holomorphic function in the disk $D(z_0, r)$ and $k < 1$. Then*

(i) $|g'(z_0)|\dfrac{kr}{(1+k)^2} \le |g(z) - g(z_0)| \le |g'(z_0)|\dfrac{kr}{(1-k)^2}\,,\ z \in \partial D(z_0, kr)$

(ii) $\left|\dfrac{g'(z_1)}{g'(z_2)}\right| \le T(k)\,;\quad z_1, z_2 \in D(z_0, kr)$.

Proof of Theorem 7. — If the assumption (7.1) holds for some $R > 0$ then it holds for every $R' > R$. Fix $R \ge 1$ so large that in addition to (7.1) we have $\operatorname{sing} f^{-1} \subset D(0, R/2)$, $|f(0)| < R$. We use the notation of diagram (2.1). Let $\varphi(t)$ be the length of the intersection of the set U with the segment $[t, t + 2\pi i]$, $t > 0$. It follows from (7.1) that for some constants $t_0 > 0$ and $\eta > 0$

$$\int_0^t \varphi(t)dt \le t(2\pi - \eta)\,, \qquad t > t_0.$$

Consequently there exist the constants $C_0 > 0$ and $\varepsilon > 0$ such that

$$\frac{\operatorname{area}(D(z, r/4) \cap U)}{\operatorname{area} D(z, r/4)} \le 1 - \varepsilon\,, \qquad r = \operatorname{Re} z > C_0. \tag{7.2}$$

Choose C such that $C > C_0$ and $C > 2\ln R + 32\pi$. Then in view of Lemma 1

$$F'(z) \ge 8 \quad \text{if } \operatorname{Re} f(z) > C. \tag{7.3}$$

Denote by Y the set $\{z : \operatorname{Re} F^m z > C,\ m = 0, 1, 2, \ldots\}$. We shall prove that area $Y = 0$. By the Lebesgue Theorem it is sufficient to prove that the lower density of the set Y at an arbitrary point $z \in Y$ is less than 1.

Let $z_0 \in Y$, $z_n = F^n z_0$, $r_n = \operatorname{Re} z_n$. Denote by $F_m^{-1} : H \to U$ the branch of the inverse function for which $F_m^{-1} z_m = z_{m-1}$. The function F_m^{-1} is univalent in the disk $D(z_m, r_m/2) \subset H$. The image of this disk is contained in U and thus it cannot contain a vertical segment of length 2π. By the 1/4-theorem we have $|(F_m^{-1})'(z_m)| \le 8\pi/r_m$. Applying the Köbe Distortion Theorem (i) one obtains

$$(7.4) \qquad F_m^{-1} D(z_m, r_m/4) \subset D(z_{m-1}, d), \qquad d = 8\pi.$$

Now let $1 \le n \le m-1$. The function F_n^{-1} is univalent in the disk $D(z_n, 2d)$ and $|(F_n^{-1})'(z_n)| < 1/8$ in view of (7.3). Using the Köbe Distortion Theorem (i), we obtain that

$$(7.5) \qquad F_n^{-1} D(z_n, d) \subset D(z_{n-1}, d/2), \qquad 1 \le n \le m-1.$$

It follows from (7.4), (7.5) that

$$(7.6) \qquad B_m = F^{-m} D(z_m, r_m/4) \subset D(z_0, 2^{-m+1} d),$$

where $F^{-m} = F_1^{-1} \circ F_2^{-1} \circ \cdots \circ F_m^{-1}$. Applying the Köbe Distortion Theorem (i) to the function F^{-m} univalent in $D(z_m, r_m/2)$ we see that the oval B_m has bounded distortion, *i.e.* ,

$$(7.7) \qquad D(z_0, t s_m) \subset B_m \subset D(z_0, s_m)$$

where t is independent of m, and s_m is the radius of the smallest disk centered at z_0 containing B_m. It follows from (7.6) that

$$(7.8) \qquad s_m \to 0 \ \text{ as } \ m \to \infty.$$

Applying the Köbe Distortion Theorem (ii) to the function F^{-m} in view of (7.2) we obtain

$$(7.9) \qquad \frac{\operatorname{area}(B_m \cap Y)}{\operatorname{area} B_m} \le 1 - T(1/2)^{-2} \varepsilon.$$

From this and (7.7), (7.8) it follows that the lower density of Y at z_0 is less than 1. Consequently area $Y = 0$. □

THEOREM 8 (*cf.* [DH2], [L1]). — *Let $f \in S$ be a transcendental entire function satisfying (7.1). Assume that the orbit of every singular point of f^{-1} is either absorbed by a cycle or converges to an attracting or to a neutral rational cycle. Then either $J(f) = \mathbb{C}$ or area $J(f) = 0$.*

Remark. — In the latter case all orbits in $N(f)$ converge to attracting or neutral rational cycles in view of Theorem 4. One may show that in such a case there exists a singular point whose orbit is not absorbed by a cycle (see [L3], Theorem 1.4). So if the orbits of all singular points are absorbed by cycles then $J(f) = \mathbb{C}$. Example : $f(z) = 2\pi i e^z$.

Proof. — Observe first that there are no neutral irrational cycles. Indeed, if α is such a cycle then

$$\alpha \subset \overline{\{f^n c\}_{n=0}^{\infty}} \backslash \{f^n c\}_{n=0}^{\infty}$$

for some point $c \in \operatorname{sing} f^{-1}$ ([L3], Prop. 1.11) which contradicts the assumptions.

Further, by Theorem 7

$$\liminf_{m \to \infty} |f^m z| < M \tag{7.10}$$

for almost all $z \in J(f)$. Consider a point $z \in J(f)$ satisfying (7.10), the orbit of which is not absorbed by any cycle. Then it is not attracted by any cycle. It is obvious for repelling cycles and follows from the results due to Fatou for neutral rational cycles (Fatou [F1] proved that a rational neutral cycle may attract only points of $N(f)$).

Let $C_n = \bigcup_{k=1}^{n} f^k(\operatorname{sing} f^{-1})$, $1 \leq n \leq \infty$. Since z is not attracted by any cycle, there exists a sequence $m_j \to \infty$ such that

$$f^{m_j} z \to w\,, \qquad \operatorname{dist}(f^{m_j} z, C_\infty) > 2\delta > 0$$

for some $w \in \mathbb{C}$ and $\delta > 0$. But $C_n = \operatorname{sing}(f^{-n})$. Hence there exist branches f^{-m_j} which map univalently the disks $D(f^{m_j} z, 2\delta)$ onto neighborhoods of z. If $J(f)$ is nowhere dense, we have

$$\inf_{\substack{|\zeta| \leq 2|w| \\ \zeta \in J(f)}} \frac{\operatorname{area}(D(z,\delta) \cap N(f))}{\operatorname{area} D(z,\delta)} \geq \varepsilon > 0.$$

This inequality and the Köbe Distortion Theorem (ii) imply

$$\frac{\operatorname{area}(B_j \cap N(f))}{\operatorname{area} B_j} \geq T\Big(\frac{1}{2}\Big)^{-2} \varepsilon \tag{7.11}$$

where $B_j = f^{-m_j} D(f^{m_j} z, \delta)$. Furthermore $|(f^{-m_j})'| \to 0$ uniformly in $D\Big(w, \dfrac{3}{4}\delta\Big)$ (see [F2] or [L3]) and hence $\operatorname{diam} B_j \to 0$.

Using the Köbe Distortion Theorem once more, we see that the B_j are ovals with uniformly bounded ratio of axes. This and (7.11) imply that the lower density of $J(f)$ at z is less than one. By the Lebesgue Theorem $\operatorname{area} J(f) = 0$. □

8. The structural stability.

Let W be a simply connected manifold, $f_0 \in W$.

DEFINITION. — *A holomorphic motion of a set $A \subset \mathbb{C}$ over W (originating at f_0) is a map $\varphi : W \times A \to \mathbb{C}$ satisfying the following conditions :*

a) *The map $f \mapsto \varphi(f, a)$ is analytic in f for every $a \in A$;*

b) *The map $\varphi_f : a \mapsto \varphi(f, a)$ is injective for every $f \in W$;*

c) $\varphi_{f_0} = \mathrm{id}$.

λ-LEMMA. — a) *A holomorphic motion φ of a set A may be extended to a holomorphic motion of the closure $\bar{A}$ [L2], [MSS];*

b) *The map $\varphi_f : \bar{A} \to \mathbb{C}$ is quasiconformal for any $f \in W$ [MSS].* □

Remark. — The quasiconformality of a map defined in a non-open set is understood in the sense of I.N. Pesin (see [BRo]).

Let us consider a manifold M defined in §3. An entire function $f_0 \in M$ is said to be *J-stable* (in M) if for all $f \in M$ sufficiently close to f_0 the transformations $f_0|J(f_0)$ and $f|J(f)$ are topologically conjugate and the conjugating homeomorphism $\varphi_f : J(f_0) \to J(f)$ depends continuously on f (the space of maps $J(f_0) \to \mathbb{C}$ is endowed with the topology of uniform convergence on compact sets).

Let us consider the multi-valued analytic function $\alpha_p : M \to \mathbb{C}$ satisfying the equation $f^p(\alpha) = \alpha$. By Theorem 2 this function has only algebraic singularities. Denote by N_p the set of these singularities (this is a subset of M). Put $N = \overline{\bigcup_{p=1}^{\infty} N_p}$, $\Sigma = M \backslash N$. The following result is an analog of the theorem obtained in [L2], [MSS] for rational maps.

THEOREM 9. — *All functions $f \in \Sigma$ are J-stable. The set Σ is open and dense in M.*

Proof. — Let $\lambda_p(f) = (f^p)'(\alpha_p(f))$. ($\lambda_p(f)$ is the multiplier of $\alpha_p(f)$ or some power of it). It follows from the Implicit Function Theorem that if $f \in N_p$ then $\tilde{\lambda}_p(f) = 1$ for a branch of λ_p (thus f has a neutral rational cycle).

Let $f_0 \in \Sigma$. Consider a simply connected neighborhood $U \subset \Sigma$ of f_0. Then all branches $\alpha_{p,i}$ of α_p are single-valued in U. Furthermore if $\alpha_{p,i}(f) = \alpha_{q,j}(f)$ for some $f \in U$ then $\alpha_{p,i} \equiv \alpha_{q,j}$. For otherwise f is a singular point of α_{pq}. The family of functions $\alpha_{p,i}$ defines the holomorphic motion of the set of periodic points $\operatorname{Per} f_0$ over U. Namely $\varphi_f : \alpha_{p,i}(f_0) \mapsto \alpha_{p,i}(f)$. By the λ-Lemma this motion may be extended to $\overline{\operatorname{Per} f_0}$. This extension conjugates $f_0|\overline{\operatorname{Per} f_0}$ to $f|\operatorname{Per} f$. But the Julia set $J(f) \subset \overline{\operatorname{Per} f}$ is distinguished from $\overline{\operatorname{Per} f}$ by the purely topological property : $J(f)$ consists of non-isolated points in $\overline{\operatorname{Per} f}$. Hence φ_f maps $J(f_0)$ onto $J(f)$, and J-stability is proved.

Let us show that Σ is dense in M. Denote by $s(f)$ the number of attracting cycles of f. Let $f_0 \in N$ and $\varepsilon > 0$. Then there exists $\tilde{f} \in N_p$ such that $\operatorname{dist}(f_0, \tilde{f}) < \varepsilon$. We have $\lambda_{p,i}(\tilde{f}) = 1$ for a suitable branch of λ_p, and $\lambda_{p,i} \not\equiv 1$ by Lemma 6. Consequently there exists $f_1 \in M$ such that $|\lambda(f_1)| < 1$ and $\operatorname{dist}(\tilde{f}, f_1) < \varepsilon$. Since attracting cycles are stable under perturbation, $s(f_1) > s(f_0)$ for sufficiently small ε. If $f_1 \in N$, the process can be repeated, and the number of attracting cycles increases. By Theorem 5 the process breaks off no later than at the q-th step. As a result we obtain a function $f \in \Sigma$ close to f_0. The theorem is proved. □

Remark. — One may show that the set of J-stable functions coincides with Σ and give some other characterizations of Σ (see [L2]).

Recall that an entire function $f_0 \in M$ is called structurally stable (in M) if for every $f \in M$ close enough to f_0 the transformations $f_0 : \mathbb{C} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ are topologically conjugate, and the conjugating homeomorphism depends continuously on f.

THEOREM 10. — *The set of structurally stable endomorphisms is open and dense in M. The conjugating homeomorphisms can be chosen to be quasiconformal.*

Proof (Compare [MSS]). — Let $f_0 \in \Sigma$ be a J-stable function. Then f_0 has no neutral rational cycles (see the definition of Σ). Hence f_0 has no neutral cycles at all. Otherwise f_0 can be perturbed so that an irrational neutral cycle turns into a rational one (apply Lemma 6). By Theorem 4 all orbits in $N(f_0)$ tend to attracting cycles. To simplify the notation we assume that there is a unique attracting fixed point $\alpha(f_0)$ which attracts all points of $N(f_0)$.

Let $\varphi_f : J(f_0) \to J(f)$ be a homeomorphism conjugating f_0 to a close function $f \in M$. The problem is to extend φ_f to the attracting region of $\alpha(f_0)$. Let $\alpha(f)$ be the attracting fixed point of f obtained by a perturbation of $\alpha(f_0)$. The singular points $a_1(f), \dots, a_q(f)$ can be enumerated so that they depend continuously on f (recall that $a_j(f)$ are local parameters on M). Suppose that the first r singular points of f_0^{-1} lie in the attracting region of $\alpha(f_0)$ while the others lie in the Julia set $J(f_0)$. It follows from the J-stability of f_0 that the same properties hold for any close function f. Let all the above-mentioned properties be valid in a neighborhood W_0 of f_0.

Consider the set $\Lambda \subset W_0$ such that for some $m, \ell \geq 0$, $i, j \in [1, q]$

$$f^m(a_i(f)) = f^\ell(a_j(f)). \tag{8.1}$$

Let us show that Λ is closed and nowhere dense in W_0. Denote by Z the set of $f \in W_0$ for which the multiplier $\lambda(f)$ of the fixed point $\alpha(f)$ vanishes. By Lemma 6, Z is a proper analytic subset of W_0. Therefore, it is sufficient to show that Λ is closed and nowhere dense in a neighborhood W_1 of $f_1 \in W_0 \backslash Z$.

Let $\overline{W}_1 \subset W_0 \backslash Z$. Then there is an $\varepsilon > 0$ such that any function $f \in W_1$ univalently maps the disk $D(\alpha(f), \varepsilon)$ into itself. On the other hand, there is such a number k that

$$|f^m a_j(f) - \alpha(f)| < \varepsilon \quad \text{for } m \geq k\,, \quad f \in W_1\,, \quad 1 \leq j \leq r.$$

Consequently, if $f \in W_1 \cap \Lambda$ then f satisfies some equality (8.1) with $\ell = k$.

Consider now the set X of $f \in W_1$ such that $f^k(a_j(f)) = \alpha(f)$ for some j. By Lemma 8, X is a proper analytic subset of W_1. Hence it is sufficient to show that Λ is closed and nowhere dense in a neighborhood W_2 satisfying $\overline{W}_2 \subset W_1 \backslash X$. But

$$\inf\{|f^k a_i(f) - \alpha(f)| : f \in W_2\,, \quad 1 \leq i \leq r\} > 0$$

while $f^m a_i(f) \to \alpha(f)$, $m \to \infty$ uniformly in W_2. Therefore the equations (8.1) for $\ell = k$ and large m have no solutions in W_2. Thus there exists a N such that

$$\Lambda \cap W_2 = \bigcup_{m \leq N} (\Lambda_{k,m} \cap W_2)\,, \tag{8.2}$$

where $\Lambda_{k,m} = \{f \in W_0 : f^m a_i(f) = f^k a_j(f) \text{ for some } i, j \in [1, r]\}$. By Lemma 8 each $\Lambda_{k,m}$ is a proper analytic subset of W_0. Thus $\Lambda \cap W_2$ is also a proper analytic subset of W_2.

Now we show that every endomorphism $f \in W_0 \backslash \Lambda$ is structurally stable. If $f \in W_0 \backslash \Lambda$ then the multiplier $\lambda(f)$ is not zero. Denote by

$K_f : z \mapsto z + \beta(f)z^2 + \cdots$ the normalized König function for f (see [V]). It is univalent in a neighborhood V_f of $\alpha(f)$ and satisfies the Schröder equation $K_f(fz) = \lambda(f)K_f(z)$. One may easily verify that $K_f(z)$ is analytic in both variables. Diminish the neighborhood $\bigcup_{f \in W_0} V_f$ (without changing the notation) so that $K_f(V_f) = D(0, \varepsilon)$ and the orbits $\{f^m a_j(f)\}_{m=0}^{\infty}$ are disjoint with ∂V_f. Let $d_j(f)$ be the first point of $\{f^m a_j(f)\}_{m=0}^{\infty}$ that falls into V_f, $1 \le j \le r$. Then $d_i(f) \ne d_j(f)$ for all $i \ne j$ and $f \in W_0 \backslash \Lambda$. Set $b_j(f) = K_f(d_j(f))$.

It is easy to construct a holomorphic motion $g_f : D(0, \varepsilon) \to D(0, \varepsilon)$ over some neighborhood $\Omega \subset W$ such that

(i) g_f conjugates $z \mapsto \lambda(f_0)z$ to $z \mapsto \lambda(f)z$

(ii) $g_f : b_i(f_0) \mapsto b_i(f)$, $1 \le i \le r$.

Let $\varphi_f = K_f^{-1} \circ g_f \circ K_f$. Then $\varphi_f : V_{f_0} \to V_f$ is a holomorphic motion over Ω conjugating $f_0|V_{f_0}$ to $f|V_f$ and such that

$$\varphi_f : d_i(f_0) \mapsto d_i(f). \tag{8.3}$$

We will extend φ_f to the whole attracting region of $\alpha(f_0)$.

Let $z \in f_0^{-k} V_{f_0}$ and $f_0^k z \notin \operatorname{sing} f_0^{-k}$. Consider the functional equation

$$f^k(\Psi_z(f)) = \varphi_f(f_0^k z), \quad \Psi_z(f_0) = z. \tag{8.4}$$

By the Implicit Function Theorem it has an analytic solution $\zeta = \Psi_z(f)$ in a neighborhood of f_0. Let us show that Ψ_z may be analytically extended to the whole domain Ω (assuming without loss of generality that Ω is simply connected).

Let $\{f_t\}_{0 \le t \le 1}$ be a path in Ω such that Ψ_z is analytically continued along the path $\{f_t\}_{0 \le t < 1}$. If f_1 is an algebraic singularity of Ψ_z then $\Psi_z(f_1)$ is a critical point of f_1^k. Hence $f_1^k(\Psi_z(f_1)) = f_1^m a_j(f_1)$ for some $j \in [1, r]$ and $m \in [0, k-1]$. By (8.4)

$$\varphi_{f_1}(f_0^k z) = f_1^m a_j(f_1) = f_1^s d_j(f_1)$$

for some $s \in [0, m]$. Now (8.3) implies

$$f_0^k z = f_0^s d_j(f_0) \in \operatorname{sing} f_0^{-k}$$

which contradicts the assumption.

Assume now that $\Psi_z(f_t) \to \infty$ as $t \to 1$. By Lemma 9 $\varphi_{f_1}(f_0^k z)$ is an asymptotic value of f_1^k, i.e. $\varphi_{f_1}(f_0^k z) = f_1^m a_j(f_1)$ for some $m \in [0, k-1]$, and we obtain a contradiction through the same argument as we just used above.

Thus, φ_f may be extended to the set $\bigcup_{k=0}^{\infty} f_0^{-k} V_{f_0}$ punctured in the inverse images of $a_j(f_0)$ of all orders. Since the closure of this set is $\mathbb{C}$, the application of the λ-Lemma completes the proof. $\square$

Remarks. — 1. As in [L2], [MSS] Theorems 9 and 10 may be proved for any analytic subfamily $\mathcal{M} \subset M$.

2. Let W be a connected component of the set of structurally stable functions in M modulo the action of the affine group by conjugations. Then W can be represented as $T(f)/\mathrm{Mod}(f)$ where $T(f)$ is the Teichmüller space and $\mathrm{Mod}(f)$ is the modular group associated with f (Sullivan [S2]).

We say that an entire function $f \in S$ satisfies Axiom A if the orbits of all singular points of f^{-1} tend to attracting cycles.

PROPOSITION 5. — *A function $f \in S$ satisfying Axiom A is J-stable (in the family M_f).*

Proof. — It is easy to see that all functions $g \in M_f$ close to f also satisfy Axiom A and hence have no neutral cycles. Thus $f \in \Sigma$. $\square$

The converse statement is one of the central problems of holomorphic dynamics. For rational maps it is known as Fatou's conjecture (see [F2], p.73).

9. Appendix : The exponential family.

In conclusion let us discuss the family $M_{\exp}$ of entire functions $z \mapsto \lambda \exp wz + a$ equivalent to $\exp z$ (in the sense of §3), attracting a good deal of interest during the last decade [BR], [D], [DGH], [EL1-4], [L4], [M], [McM], [R]. Factorizing $M_{\exp}$ modulo the action of the affine group by conjugations we obtain the reduced family $\widetilde{M}_{\exp} = \{\exp wz : w \in \mathbb{C}^*\}$. We would like to consider the family $\{f_a : z \mapsto \exp z + a\}$. The natural projection of this family onto the reduced family is $w = \exp a$. The following theorem was independently proved in [BR] (except the results concerning the area of $J(f)$, which were independently proved in [McM]) :

THEOREM 11. — *Let $f_a : z \mapsto \exp z + a$. Then one of the following possibilities holds :*

(i) *The function* f_a *has a unique attracting cycle* $\{\alpha_k\}_{k=0}^{p-1}$. *The set of normality* $N(f_a)$ *coincides with the attractive region of this cycle. The area of* $J(f_a)$ *is equal to zero. The singular point* a *belongs to the immediate attractive region of* $\{\alpha_k\}$ *but its orbit is not absorbed by this cycle. The function* f_a *has no neutral cycles.*

(ii) *The function* f_a *has a unique neutral rational cycle* $\{\alpha_k\}_{k=0}^{p-1}$. *The other properties of* f_a *are the same as in case (i).*

(iii) *The function* f_a *has a cycle of Siegel disks.*

(iv) *The Julia set* $J(f_a)$ *coincides with the entire plane* $\mathbb{C}$.

The theorem follows immediately from the results of §5 and Theorem 8. For real a cases (i), (ii), and (iv) hold for $a < -1$, $a = -1$ and $a > -1$ respectively. The fact that $J(f_a) = \mathbb{C}$ for $a = 0$ was proved for the first time by Misiurewicz [M]. The Hausdorff dimension of $J(f_a)$ in all cases is equal to 2 [McM].

Let $\Sigma \subset \mathbb{C}$ be as in §8 the set of a for which the function f_a is J-stable. In view of Theorem 11, Σ consists of two parts : $\Sigma = \Sigma_1 \cup \Sigma_2$. Here Σ_1 is the set of a for which f_a has an attracting cycle, Σ_2 is the interior of the set of a for which $J(f_a) = \mathbb{C}$. If $a \in \Sigma_1$ then by Theorem 11 the orbit $\{f_a^n a\}_{n=0}^{\infty}$ is not absorbed by the cycle. Hence f_a is structurally stable (see the description of structurally stable functions in the proof of Theorem 10). Thus in the exponential family J-stability implies structural stability. The analogue of the "Fatou conjecture" stated in §8 is the following :

CONJECTURE. — $\Sigma_2 = \emptyset$. *If* $J(f_a) = \mathbb{C}$ *then the function* $z \mapsto \exp z + a$ *is not structurally stable.*

It is known that f_0 is not structurally stable [D]. It also follows from the result of [L4] stating that $z \mapsto \exp z$ has no ergodic components of positive measure.

Denote by W_p the subset of Σ_1 in which the minimal period of the attracting cycle $\alpha(a)$ of f_a is equal to p. Let $W_{p,n}$ be the connected components of W_p and $\lambda_{p,n}(a)$ be the multiplier of $\alpha(a)$. One can easily prove

PROPOSITION 6. — *The domains* $W_{p,n}$ *are simply connected and unbounded.*

One may describe explicitly the sets W_1 and W_2. W_1 is the domain lying on the left of the cycloid $a = i\theta - e^{i\theta}$, $-\infty < \theta < \infty$. W_2 has the unique component $W_{2,n}$ in each strip $\Pi_{2,n} = \{a : 2\pi in < \operatorname{Im} a < 2\pi i(n+1)\}$, $n = 0, \pm 1, \dots$. The boundary of $W_{2,n}$ is a curve $a = i(\theta + u) - e^{i(\theta - u)}$ where $u = u(\theta)$ satisfies the equation $(\sin u)/u = -e^{i\theta}$ and $\operatorname{Im} u \geq 0$, $u(\pi(2n+1)) = 0$. The curve $\partial W_{2,n}$ is tangent to the cycloid ∂W_1 at the point $a_n = 1 + i\pi(2n+1)$.

There are infinitely many other components $W_{p,n}$ touching the cycloid ∂W_1 at the dense set of points (for which the multiplier is rational). Infinitely many new components touch each of these components and so on. The situation is quite similar to that which occurs for the quadratic family $z^2 + c$.

We conclude the section by stating an analogue of the Douady-Hubbard Theorem on the Multiplier [DH1] :

THEOREM 12. — *The multiplier* $\lambda_{p,n} : W_{p,n} \to D^* = \{z : 0 < |z| < 1\}$ *is the universal covering map.*

Sketch of the proof. — Following Sullivan [S2] (see also [L3], proof of Theorem 2.8) one may construct the following commutative diagram

$$\begin{tikzcd} W_{p,n} \arrow[dr, "\lambda_{p,n}"'] & & T(S_a) \approx \{z : \operatorname{Im} z > 0\} \arrow[ll, "\Psi"'] \arrow[dl, "\pi"] \\ & D^* & \end{tikzcd}$$

Here $a \in W_{p,n}$, S_a is the Riemann surface associated with f_a (a torus), $T(S_a)$ is the corresponding Teichmüller space (the half-plane), Ψ is the projection modulo the action of modular group $\operatorname{Mod}(f_a)$ on $T(S_a)$, π is the projection modulo the action of the cyclic group $\Gamma = \{z \mapsto z + n\}_{n \in \mathbf{Z}}$ generated by the Dehn twist map of the torus. So, π is a covering map and hence $\lambda_{p,n}$ is also a covering map. Since $W_{p,n}$ is simply connected, $\lambda_{p,n}$ is the universal covering map. □

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