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Dynamical Symmetry Breaking in an E_6 GUT Model

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Dynamical symmetry breaking is studied in an E_6 GUT model of a single generation of fermions with strong 4-fermi interactions. The effective potential is analyzed analytically by the help of Michel's conjecture[1] and the result is confirmed numerically. We find that the E_6 symmetry is spontaneously broken either to F_4 or to $Sp(8)$ or G_2 or $SU(3)$, depending on which of the 4-fermi coupling constants G_{27} and G_{351} in the **27**/**351** channels is stronger. The possibilities for obtaining other type of breaking patterns are also discussed.

§ 1. Introduction

Despite remarkable successes of the standard model based on $SU(3)\times SU(2)\times U(1)$, many physicists believe that there exists a more fundamental theory beyond it. The strongest evidence for such theories, usually called grand unified theories (GUT's),[2] are the facts that the quark/lepton charges are quantized and that anomaly is cancelled miraculously between quarks and leptons. In the usual scenario of GUT's, however, the spontaneous symmetry breaking required there is discussed by introducing some elementary Higgs fields just in the same manner as in the standard model. Then quite a large arbitrariness appears in, e.g., which representations and how many we introduce as the Higgs fields. Moreover this introduces too many arbitrary parameters, even more than in the standard model, in the Higgs Yukawa- and self-couplings.

Dynamical symmmetry breaking scenario[3)~8)] is very attractive in this respect. There one supposes that there exist only matter fermion fields belonging to some representation of a gauge group G and the gauge fields of that group. Then the Lagrangian is uniquely determined by the gauge symmetry alone when we require the renormalizability (and if the fermions are all chiral). The usual Higgs fields are supplied as bound states of the fundamental fermions which are formed by the gauge interaction dynamics itself. So, which types of Higgs fields appear is determined dynamically and all the parameters concerning the Higgs fields, which are arbitrary in the usual scenario, becomes in principle calculable.

Even when it is difficult to solve fully the dynamics, the dynamical symmetry breaking scenario can give several constraints on the possible models for the GUT's, e.g., on possible GUT groups and/or matter contents. For instance, as was emphasized by Barbieri and Nanopoulos[9] and Ramond,[10] E_6 is uniquely selected among many GUT groups if we require i) every generation of quarks/lepton fields belongs to a single irreducible representation of the group, ii) the theory is automatically anomaly free and iii) all the (phenomenological) Higgs fields necessitated for causing the symmetry breakings down to $SU(3)_c\times U(1)_{em}$ fall in the representations which can be supplied by the fermion bilinears. Therefore it is very important to investigate

GUT's from the viewpoint whether they are compatible or not with dynamical symmetry breaking.

In this paper we study dynamical symmetry breaking in an E_6 GUT model. The reason why we adopt E_6 is its unique property stated above. In particular the third point implies the possibility that all the Higgs fields necessary for the symmetry breakings can be formed dynamically as fermion bound states. The unified gauge coupling constant of E_6 suggested by the present experimental data, however, seems not large enough to break the E_6 symmetry itself, and so we expect that some strong gauge interaction yet other than the E_6 one exists and gives a primary driving force for the E_6 symmetry breaking. But we still have no definite idea about that gauge interaction beyond E_6. So we assume in this paper that the strong gauge interaction is effectively treated as a Nambu-Jona-Lasinio type 4-fermi interaction.[4] We include all possible E_6-invariant 4-fermi interaction terms that can contribute to the formation of scalar Higgs fields. We, however, restrict ourselves to the model of a single generation of quarks/leptons, with the hope that the Higgs fields are all supplied as bound states of mainly a single generation of fermions. Following the usual procedure we introduce Higgs fields as auxiliary fields. We analyze the effective potential to find the patterns of dynamical symmetry breaking realized in this model, and see whether the desirable symmetry breaking patterns emerge or not.

This paper is organized as follows. In § 2 we present the model which we study in this paper and give the effective potential of the auxiliary Higgs fields. The analysis of the effective potential is performed analytically in § 3. In a special case in which the 4-fermi interaction is present only in E_6 **27** channel, a complete analysis is possible and is given there. Otherwise, however, such a direct analysis becomes almost impossible and we perform a simplified analysis *assuming* that Michel's conjecture concerning the potential minimum holds. The symmetry breaking patterns found in this way are actually confirmed to be correct by the numerical analysis performed in § 4. Section 5 is devoted to a summary and a conclusion. Three appendices are supplemented; by using the spinor representation of $SO(10)$ presented in Appendix A, representations **27** and **351** of E_6 and an invariant tensor with three **27** indices are explicitly constructed in Appendix B; definition of maximal little groups which appears in Michel's conjecture is presented in Appendix C.

§ 2. The model

As explained in the Introduction, we consider the Nambu-Jona-Lasino type model with a single generation of left-handed fermions, $\psi=(\psi_A)$ $(A=1, \cdots, 27)$, belonging to **27** representation of E_6. The Langrangian is given in the most generic form as follows:

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}},$$

$$\mathcal{L}_0=\bar{\psi}i\gamma^\mu(\partial_\mu-igA_\mu)\psi-\frac{1}{4}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}),$$

$$\mathcal{L}_{\text{int}}=+G_{27}|\psi^{\mathrm{T}}C\psi|^2_{27}+G_{351_S}|\psi^{\mathrm{T}}C\psi|^2_{351_S}+G_{351_A}|\psi^{\mathrm{T}}C\psi|^2_{351_A}. \tag{2·1}$$

In this expression C denotes the charge conjugation matrix of the Lorentz spinor so that $\psi^{\mathrm{T}}C\psi$ is the Lorentz scalar, the G_i's denote coupling constants, subscripts such as **27** mean the projection into the denoted irreducible component of E_6 constructed with the fermion bilinear: $\mathbf{27}\times\mathbf{27}=\overline{\mathbf{27}}+\overline{\mathbf{351}}_{\mathrm{S}}+\overline{\mathbf{351}}_{\mathrm{A}}$. The absolute squares are understood to denote E_6-invariant contractions between those irreducible components and their complex conjugates. For the present case of single generation fermions, the fermion bilinear $\psi_A{}^{\mathrm{T}}C\psi_B$ is symmetric with respect to the indices A and B, and so the last anti-symmetric component $(\psi_A{}^{\mathrm{T}}C\psi_B)_{351_{\mathrm{A}}}$ vanishes identically. Henceforth **351** without subscript always denotes $\mathbf{351}_{\mathrm{S}}$.

Now we introduce auxiliary fields $(H^{\dagger}_{27/351})_{AB}$ standing for $-(\psi_A{}^{\mathrm{T}}C\psi_B)_{27/351}$ and rewrite the interaction part $\mathcal{L}_{\mathrm{int}}$ into

$$\begin{aligned}\mathcal{L}_{\mathrm{int}}=&-\{(\psi_A{}^{\mathrm{T}}C\psi_B)_{27}(H_{27})^{AB}+\mathrm{h.c.}\}-M_{27}^2\mathrm{tr}(H_{27}^{\dagger}H_{27})\\&-\{(\psi_A{}^{\mathrm{T}}C\psi_B)_{351}(H_{351})^{AB}+\mathrm{h.c}\}-M_{351}^2\mathrm{tr}(H_{351}^{\dagger}H_{351})\,,\\M_{27/351}^2\equiv&\frac{1}{G_{27/351}}\,.\end{aligned} \tag{2·2}$$

We evaluate only the fermion one-loop diagram for our effective potential. That is formally the leading term in $1/N_g$ expansion if we introduce N_g copies of our single generation of fermions. We neglect the E_6 gauge interaction since it is expected to be weak. Then the 1-loop effective potential of H is given by

$$\begin{aligned}&\phi(H)=\phi_0(H)+\phi_1(H)\,,\\&\phi_0(H)=M_{27}^2\mathrm{tr}(H_{27}^{\dagger}H_{27})+M_{351}^2\mathrm{tr}(H_{351}^{\dagger}H_{351})\,,\\&\phi_1(H)=-4\int^{\Lambda}\frac{d^4p}{i(2\pi)^4}\ln\det(M^{\dagger}M-p^2)\,,\\&M=2(H_{27}+H_{351})\equiv 2H\,.\end{aligned} \tag{2·3}$$

Here $\int^{\Lambda}d^4p$ denotes that the integral over p is defined with an ultraviolet cutoff $|p_{\mathrm{E}}|\leq\Lambda$ after making the Wick rotation to the Euclidean momentum $p\to p_{\mathrm{E}}$. If the coupling constants are large enough this potential has a minimum away from the symmetric point $H=0$ and the E_6 symmetry is dynamically broken. We can determine the direction of the symmetry breaking by searching a minimum of this potential.

§ 3. Analysis of the effective potential

3.1. *Case of **27** interaction only*

We first consider the simplest case in which only the **27** part of the 4-fermi interaction is present; namely,

$$G_{27}\neq 0\,,\quad G_{351}=0\,. \tag{3·1}$$

Then clearly the Higgs vacuum expectation value (VEV) can appear only in the **27** component:

$$H_{351}=0\,,\quad H=H_{27}\,. \tag{3·2}$$

As explained in Appendix B, **27** representation of E_6 is decomposed into **1**+**16**+**10** under the $SO(10)$ subgroup and the **27** Higgs field $H=H_{27}$ is expressed in a 'vector' notation as

$$V\equiv\begin{pmatrix} H_0 \\ H_\alpha \\ H_M \end{pmatrix} \begin{matrix} \\ (\alpha=1,2,\cdots,16)\ , \\ (M=1,2,\cdots,10) \end{matrix} \tag{3·3}$$

where the subscripts 0, α and M stand for the $SO(10)$ singlet, **16** spinor and **10** vector representations, respectively. V is embedded into the 27×27 matrix $H=H_{27}$ by the help of the invariant tensor Γ^{ABC} carrying three $\overline{\mathbf{27}}$ indices, which is explicitly given in Appendix B:

$$H^{AB}=\Gamma^{ABC}V_C\,. \tag{3·4}$$

(See Eq. (B·11) for the explicit form of this matrix.)

Any **27** can be E_6 rotated into the following 'standard' reduced-form:

$$V_0=\begin{bmatrix} v_0 \\ \mathbf{0} \\ R+iI \\ m \\ \mathbf{0} \end{bmatrix} \begin{matrix} \}SO(10)\ \text{singlet (real)}\,, \\ \}SO(10)\ \mathbf{16}\,, \\ \}\text{the first component of } SO(10)\ \mathbf{10}\ \text{(complex)}\,, \\ \}\text{the second component of } SO(10)\ \mathbf{10}\ \text{(real)}\,, \\ \}\text{the third to tenth components of } SO(10)\ \mathbf{10}\,. \end{matrix} \tag{3·5}$$

(v_0, R, I, m are all real)

This is seen as follows. First, starting from a generic form (3·3) of V, the spinor component H_α can be rotated away by using the E_6 rotation freedom with the spinor parameter ϵ in (B·2). Next we note that the vector component H_M actually stands for two $SO(10)$ irreducible **10** vectors, the real and imaginary parts. So, using the $SO(10)$ rotation freedom, we can make one of the two **10** vectors, say, the imaginary parts, to have only the first component, and finally, by using the remaining $SO(9)$ rotation freedom, we can make the real part **10** to have only the first two components. Alternatively, one can also convince the validity of the statement as follows: the problem is whether any **27** V can be written in the form gV_0 with $g\in E_6$ by using the standard form V_0 in (3·5). Note as for g in this expression that only the right quotient $E_6/SO(8)$ part is effective since V_0 is invariant under $SO(8)$. So the g part is parameterized by $78-28=50$ parameters, and hence gV_0 spans a $50+4=54$ dimensional space. But it is the same dimensions as the whole **27** complex vector V does.

This standard form (3·5) has four parameters and implies that there exist four E_6-invariants which can be constructed by **27** representation V alone. They can easily be found and are given as follows:

$$X=V^\dagger V\,, \tag{3·6}$$

$$Y\equiv\Gamma^{ABC}V_AV_BV_C \qquad \text{and its complex conjugate} \qquad Y^*\,, \tag{3·7}$$

$$Z\equiv\Gamma^{ABC}V_BV_C(\Gamma^{ADE}V_DV_E)^* . \tag{3·8}$$

These invariants X, Y, Y^* and Z are expressed in terms of the four parameters in (3·5) as

$$\begin{aligned}
&X=v_0^2+R^2+m^2+I^2 ,\\
&Y=v_0(R^2+m^2-I^2+2iRI) , \quad Y^*=v_0(R^2+m^2-I^2-2iRI) ,\\
&Z=\{(R^2+m^2+I^2)^2-4I^2m^2\}+4v_0^2(R^2+m^2+I^2) .
\end{aligned} \tag{3·9}$$

Since the 1-loop effective potential $\phi(H)$ is invariant under E_6, $\phi(H)$ can be expressed in terms of the invariants, X, $Y(Y^*)$ and Z alone. The effective potential (2·3) now reads

$$\begin{aligned}
\phi(H=M/2)&=M_{27}^2\frac{1}{4}\mathrm{tr}(M^\dagger M)-4\int^\Lambda\frac{d^4p}{i(2\pi)^4}\ln\det(M^\dagger M-p^2)\\
&=\frac{1}{4}M_{27}^2\sum_{i=1}^{27}\lambda_i-4\int^\Lambda\frac{d^4p}{i(2\pi)^4}\sum_{i=1}^{27}\ln(\lambda_i-p^2) ,
\end{aligned} \tag{3·10}$$

where the λ_i's are real positive eigenvalues of $M^\dagger M$, given by the roots of the following equations:

$$\lambda_i^3-8X\lambda_i^2+16X^2\lambda_i-64YY^*=0 \quad \text{for } i=1, 2, 3 , \tag{3·11}$$

$$\lambda_i^3-4X\lambda_i^2+4Z\lambda_i-16YY^*=0 \quad \text{for } i=4, \cdots, 27 . \tag{3·12}$$

These equations depend on only three quantities of the four invariants:

$$\begin{aligned}
&X=v_0^2+R^2+m^2+I^2 ,\\
&YY^*=v_0^2\{(R^2+m^2+I^2)^2-4I^2m^2\} ,\\
&Z=\{(R^2+m^2+I^2)^2-4I^2m^2\}+4v_0^2(R^2+m^2+I^2) .
\end{aligned} \tag{3·13}$$

For convenience, we re-parameterize these three quantities as follows:

$$\begin{array}{lcl}
a\equiv v_0^2 , & & X=a+b ,\\
b\equiv R^2+m^2+I^2 & \Longleftrightarrow & YY^*=a(b^2-4c^2) ,\\
c\equiv Im , & & Z=(b^2-4c^2)+4ab .
\end{array} \tag{3·14}$$

The three roots of Eq. (3·11) cannot explicitly be expressed in terms of a, b and c, but the twenty-four roots of Eq. (3·12) are given by

$$\begin{aligned}
&\lambda_i=4a , \quad (i=4, \cdots, 11)\\
&\lambda_i=2b+4c , \quad (i=12, \cdots, 19)\\
&\lambda_i=2b-4c . \quad (i=20, \cdots, 27)
\end{aligned} \tag{3·15}$$

Then, inserting these explicit expressions for the roots λ_i ($i=4, \cdots, 27$) and using an identity

$$(\lambda_1+y)(\lambda_2+y)(\lambda_3+y)=y^3+8Xy^2+16X^2y+64YY^* \tag{3·16}$$

following from Eq. (3·11) for the implicit roots $\lambda_{1,2,3}$, the effective potential (3·10) reduces to

$$\phi(H)=\phi(a,b,c)$$
$$=10M_{27}^2(a+b)-\frac{1}{4\pi^2}\int_0^{\Lambda^2}ydy[\ln(y^3+8Xy^2+16X^2y+64YY^*)$$
$$+8\ln(4a+y)+8\ln(2b+4c+y)+8\ln(2b-4c+y)]\,. \tag{3·17}$$

We now look for the stationary point of the effective potential $\phi(H)=\phi(a,b,c)$. Taking into account that X is independent of c and $\partial(YY^*)/\partial c=-8ac$, the derivative of the potential ϕ with respect to the parameter c is given by

$$\frac{\partial\phi}{\partial c}=-\frac{1}{4\pi^2}\Big[-64\cdot 8ac\,f(\Lambda^2)+8\cdot 4\{g(2b+4c)-g(2b-4c)\}\Big] \tag{3·18}$$

with functions f and g defined by

$$f(x)=\int_0^x dy\frac{y}{y^3+8Xy^2+16X^2y+64YY^*}\,,$$
$$g(x)=\int_0^{\Lambda^2}dy\frac{y}{y+x}=\Lambda^2-x\ln\frac{x+\Lambda^2}{x}\,. \tag{3·19}$$

Note that $f(x)$ is positive for $x>0$ since $f(0)=0$ and

$$f'(x)=\frac{x}{x^3+8Xx^2+16X^2x+64YY^*}>0\,, \tag{3·20}$$

because $X\geq 0$ and $YY^*\geq 0$ by definition (3·14). On the other hand, $g(x)$ is seen to be a monotonically decreasing function of x since

$$g'(x)=-\int_0^{\Lambda^2}\frac{ydy}{(y+x)^2}<0\,. \tag{3·21}$$

Taking into account also that c is bounded ($|c|<|b|/2$) by definition (3·14), we find that

$$\mathrm{sgn}\Big(\frac{\partial\phi}{\partial c}\Big)=\mathrm{sgn}(c)\,. \tag{3·22}$$

This shows that $\phi(a,b,c)$ has an absolute minimum at

$$c=0 \tag{3·23}$$

in the defining region $|c|<|b|/2$.

Next consider the derivatives of $\phi(a,b,c)$ with respect to a and b; at $c=0$ they are given respectively by

$$\frac{\partial\phi}{\partial a}=10M_{27}^2-\frac{1}{4\pi^2}\Big[\int_0^{\Lambda^2}ydy\frac{8y^2+32Xy}{y^3+8Xy^2+16X^2y+64YY^*}$$
$$+64\cdot b^2f(\Lambda^2)+8\cdot 4g(4a)\Big]\,, \tag{3·24}$$

$$\frac{\partial\phi}{\partial b}=10M_{27}^2-\frac{1}{4\pi^2}\left[\int_0^{\Lambda^2} ydy\frac{8y^2+32Xy}{y^3+8Xy^2+16X^2y+64YY^*}\right.$$
$$\left.+64\cdot 2ab\,f(\Lambda^2)+8\cdot 4g(2b)\right]. \tag{3·25}$$

Stationarity requirement $\partial\phi/\partial a=0$ and $\partial\phi/\partial b=0$, or only the difference $\partial\phi/\partial a-\partial\phi/\partial b=0$, leads to the following condition:

$$2b(2a-b)f(\Lambda^2)=g(4a)-g(2b)\,. \tag{3·26}$$

Since a, $b>0$ by definition and $f(\Lambda^2)>0$ as mentioned above, the sign of the LHS is $\mathrm{sgn}(2a-b)$ while the sign of the RHS is opposite, $-\mathrm{sgn}(2a-b)$, since $g(x)$ is a monotonically decreasing function. Therefore $\phi(a, b, c)$ can have a minimum only at

$$2a=b\,. \tag{3·27}$$

Under the conditions (3·23) and (3·27), the **27** vector (3·5) now takes the form:

$$V=\begin{pmatrix} v \\ \mathbf{0} \\ \sqrt{2}v \\ \mathbf{0}\end{pmatrix}\begin{array}{l} \}SO(10)\text{ singlet (real)}\,, \\ \}SO(10)\ \mathbf{16}\,, \\ \}\text{the first component of } SO(10)\ \mathbf{10}\text{ (real)}\,, \\ \}\text{the second to tenth components of } SO(10)\ \mathbf{10}\,. \end{array} \tag{3·28}$$

For the VEV of this form, the 27 eigenvalues λ_i of the fermion squared mass matrix $M^\dagger M$, determined by (3·11) and (3·12), now become explicit and show an interesting degeneracy: one $16v^2$ and twenty six $4v^2$'s. Then the potential (3·10) is given by

$$\phi(H)=30M_{27}^2v^2-\frac{1}{4\pi^2}\int_0^{\Lambda^2} ydy\Big[\ln(16v^2+y)+26\ln(4v^2+y)\Big], \tag{3·29}$$

the stationarity of which determines the magnitude of the VEV v:

$$30M_{27}^2=\frac{1}{4\pi^2}\Big(16g(16v^2)+26\cdot 4g(4v^2)\Big). \tag{3·30}$$

The critical coupling $G_{27}^{\rm cr}=1/(M_{27}^{\rm cr})^2$ beyond which non-zero VEV is realized is found by taking $v^2\to 0$:

$$G_{27}^{\rm cr}=\frac{\pi^2}{\Lambda^2}. \tag{3·31}$$

Note that on this vacuum one fermion has mass $4v$ and all the other 26 fermions have a degenerate mass $2v$. This implies that the original fermion multiplet **27** splits into **1**+**26**. This branching pattern indicates that the symmetry breaking realized in this case is

$$E_6 \longrightarrow F_4\,. \tag{3·32}$$

This can also be confirmed by calculating branching pattern of the E_6 gauge boson masses on this vacuum; **78** → **26** (massive)+**52** (massless), where **52** is massless gauge bosons of the unbroken F_4 group.

3.2. *Case of* **351** *interaction only*

Next we consider the case in which only the **351** part of the 4-fermi interaction is present; namely,

$$G_{351}\neq 0\,,\quad G_{27}=0\,,\tag{3·33}$$

in which case the Higgs VEV appears only in the **351** component:

$$H_{27}=0\,,\quad H=H_{351}\,.\tag{3·34}$$

Contrary to the previous case, there are many E_6 invariants and it is almost impossible to perform the same kind of analysis as in the previous subsection.

Now we invoke Michel's conjecture,[1),11)] which claims that the following statement holds when the potential system contains only a real irreducible representation of scalar fields, or a self-conjugate pair of a complex irreducible representations: *The group symmetry can break down only to one of the maximal little groups of the representation considered.* (See Appendix C for the definition of maximal little groups.)

For illustration, let us apply this conjecture to the previous case; there, the fields appearing in the potential are (auxiliary) Higgs fields of **27** and its conjugate $\overline{\mathbf{27}}$. Then the maximal little groups are $SO(10)$ and F_4. We in fact found the symmetry breaking $E_6 \to F_4$ in the above, so that the conjecture was actually correct.

In the present case of **351** Higgs fields, the maximal little groups are $SO(10)$, F_4, $Sp(8)$, G_2, $SU(3)$ and $SU(2)\otimes SU(4)$. We assume that the conjecture holds in this case also. Then what we have to do is to calculate the effective potential for each possibility of the Higgs VEV's in the maximal little group directions and to compare the minimum values of the potentials to see which possibility is actually realized.

Let us first determine the form of the VEV for each case of the maximal little groups. From Table I, we see that the fermion **27** of E_6 is again **27** under $Sp(8)$, G_2 and $SU(3)$ but the latter is a real representation. Taking also into account that E_6 **351** is constructed from $\mathbf{27}\otimes\mathbf{27}$ symmetrically, we see that the trace part of H_{351} is a singlet under those groups. Therefore, on a suitable basis, the VEV takes the following form for the cases of those groups:

$$H_{351}=v\otimes\mathbf{1}_{27}\,,\quad (Sp(8),\, G_2,\, SU(3))\tag{3·35}$$

where $\mathbf{1}_n$ denotes $n\times n$ identity matrix.

Similarly, in case of $SU(2)\otimes SU(4)$, a singlet under this group which we get from

Table I. Decomposition of E_6 **27** under the maximal groups H.

$H\subset E_6$	decomposition under H
F_4	$\mathbf{27}=\mathbf{1}+\mathbf{26}$
$Sp(8)$	$\mathbf{27}=\mathbf{27}$
G_2	$\mathbf{27}=\mathbf{27}$
$SU(3)$	$\mathbf{27}=\mathbf{27}$
$SU(2)\otimes SU(4)$	$\mathbf{27}=(\mathbf{2},\mathbf{6})+(\mathbf{1},\mathbf{15})$

a symmetric product of $\mathbf{27}\times\mathbf{27}$ comes from the component $(\mathbf{1},\mathbf{15})\times(\mathbf{1},\mathbf{15})$ and hence, on a basis,

$$H_{351}=\begin{pmatrix} v\otimes\mathbf{1}_{15} & \\ & 0\otimes\mathbf{1}_{12}\end{pmatrix}. \quad (SU(2)\otimes SU(4)) \tag{3·36}$$

It is a bit more complicated to get the form of VEV in cases of $SO(10)$ and F_4, since, as is seen from Table I, we get two singlets from the symmetric product $\mathbf{27}\times\mathbf{27}$ for each case of $SO(10)$ and F_4. In view of $\mathbf{27}\otimes\mathbf{27}|_{\rm sym}=\overline{\mathbf{27}}+\overline{\mathbf{351}}$, we see that one singlet is in H_{27} and the other is in H_{351}. We can find the form of $SO(10)$ and F_4 singlets in $\mathbf{27}$; from (3·4), (3·5) and (3·28), they are given, respectively, by

$$H_{27}=\begin{pmatrix} 0\otimes\mathbf{1}_{17} & \\ & V\otimes\mathbf{1}_{10}\end{pmatrix}, \quad (SO(10)) \tag{3·37}$$

$$H_{27}=\begin{pmatrix} -2V & \\ & V\otimes\mathbf{1}_{26}\end{pmatrix}. \quad (F_4) \tag{3·38}$$

We see from (3·37) that the $SO(10)$ singlet in H_{27} comes solely from $\mathbf{10}\times\mathbf{10}$. Therefore the $SO(10)$ singlet in H_{351} must be the other singlet made from $\mathbf{1}\times\mathbf{1}$:

$$H_{351}=\begin{pmatrix} v & \\ & 0\otimes\mathbf{1}_{26}\end{pmatrix}. \quad (SO(10)) \tag{3·39}$$

(This can also be seen directly from (B·10) in Appendix B.) In case of F_4, the two F_4 singlets come from $\mathbf{26}\times\mathbf{26}$ and $\mathbf{1}\times\mathbf{1}$, one combination of which is (3·38) contained in H_{27}. Since $\mathbf{351}$ is orthogonal to $\mathbf{27}$, ${\rm tr}H_{27}^\dagger H_{351}=0$, the F_4 singlet in H_{351} should thus have the form:

$$H_{351}=\begin{pmatrix} 13v & \\ & v\otimes\mathbf{1}_{26}\end{pmatrix}. \quad (F_4) \tag{3·40}$$

Now that we have found the form of VEV's, we can calculate the minimum value of the potential by substituting those matrix into (2·3) for each case and compare their minimum values to find the direction of symmetry breaking. First we define a function

$$F(v)\equiv M_{351}^2 v^2-4\int^\Lambda \frac{d^4p}{i(2\pi)^4}\ln(4v^2-p^2)\,. \tag{3·41}$$

Then the potential value in the directions $Sp(8)$, G_2 and $SU(3)$ is commonly given by

$$\phi_{Sp}=27F(v)\,. \tag{3·42}$$

In the directions $SU(2)\otimes SU(4)$, $SO(10)$ and F_4, it is given respectively by

$$\phi_{SS}=15F(v)\,, \tag{3·43}$$

$$\phi_{SO}=F(v)\,, \tag{3·44}$$

$$\phi_{F_4}=26F(v)+F(13v)\,. \tag{3·45}$$

When symmetry breaking occurs, $F(v)$ has a minimum at a certain point v_0 and takes a negative value there. The potentials (3·42), (3·43) and (3·44) take their minima at the same point v_0 and hence we immediately see for the minimal values

$$\phi_{Sp}<\phi_{SS}<\phi_{SO}\,. \tag{3·46}$$

The minimum of (3·45) is realized at a certain point v_1, which is different from the minimum point v_0 of $F(v)$, so that

$$F(v_0)\leq F(v_1) \quad \text{and} \quad F(v_0)\leq F(13v_1)\,, \tag{3·47}$$

and hence

$$\phi_{Sp}\leq\phi_{F_4}\,. \tag{3·48}$$

We thus find that the symmetry breaking in this pure **351** interaction case is

$$E_6 \quad\longrightarrow\quad Sp(8) \text{ or } G_2 \text{ or } SU(3)\,. \tag{3·49}$$

These three group cases cannot be distinguished in the present approximation in which only the fermion one-loop vacuum energy is counted, since the fermions get quite the same masses for those three breakings. This degeneracy will be lifted if the vacuum energy due to gauge boson loops is taken into account.

3.3. *General case*

Finally in this section we study the general case in which both **27** and **351** 4-fermi interactions are present. Strictly speaking, Michel's conjecture is inapplicable to this general case since there appear two fields of different representations **27** and **351** in the potential. Nevertheless we assume that this conjecture still holds and determine the symmetry breaking pattern in this case also using the same analysis method as in the previous subsection.

Candidate groups are the same as the **351** case: $SO(10)$, F_4, $Sp(8)$, G_2, $SU(3)$ and $SU(2)\otimes SU(4)$. Of these groups $Sp(8)$, G_2, $SU(3)$ and $SU(2)\otimes SU(4)$ have their singlet only in **351** of E_6, and so the VEV's and potentials are the same as in the previous subsection. Therefore

$$\phi_{Sp}<\phi_{SS} \tag{3·50}$$

is always realized.

On the other hand $SO(10)$ and F_4 have their singlets in both **27** and **351** representations of E_6. Their singlets in **351** are contained in the form (3·39) and (3·40) and those in **27** are in the form (3·37) and (3·38), for the $SO(10)$ and F_4 cases, respectively. Thus the general forms of VEV's for these group cases are given respectively by

$$H=\begin{pmatrix} v & & \\ & 0\otimes\mathbf{1}_{16} & \\ & & V\otimes\mathbf{1}_{10}\end{pmatrix}, \quad (SO(10)) \tag{3·51}$$

$$H=\begin{pmatrix} 13v-2V & \\ & (v+V)\otimes \mathbf{1}_{26} \end{pmatrix}. \quad (F_4) \tag{3·52}$$

By using (3·51), the potential corresponding to $SO(10)$ breaking ($\equiv \phi'_{SO}$) is

$$\phi'_{SO}=10(M_{27}^2-M_{351}^2)V^2+10F(V)+F(v) \tag{3·53}$$

and by using (3·52), that of F_4 breaking ($\equiv \phi'_{F_4}$) is

$$\phi'_{F_4}=30(M_{27}^2-M_{351}^2)V^2+F(13v-2V)+26F(v+V). \tag{3·54}$$

Now let us compare ϕ_{S_P}, ϕ'_{SO} and ϕ'_{F_4} at their minimum points. First of all we clearly see the relation

$$\phi_{S_P}=\phi'_{F_4}<\phi'_{SO} \text{ when } M_{27}^2=M_{351}^2. \tag{3·55}$$

In view of this, we study the potential in two cases (a) $M_{27}^2>M_{351}^2$ and (b) $M_{27}^2<M_{351}^2$, separately.

(a) $M_{27}^2>M_{351}^2$

In this case we have from (3·53)

$$\begin{aligned}\phi'_{SO}&=10(M_{27}^2-M_{351}^2)V^2+10F(V)+F(v)\\ &>11F(v_0)>27F(v_0)=\phi_{S_P},\end{aligned} \tag{3·56}$$

and from (3·54)

$$\begin{aligned}\phi'_{F_4}&=30(M_{27}^2-M_{351}^2)V^2+F(13v-2V)+26F(v+V)\\ &>F(13v-2V)+26F(v+V)\\ &>27F(v_0)=\phi_{S_P}.\end{aligned} \tag{3·57}$$

Hence we conclude that the symmetry breaking pattern in this case is given by

$$E_6 \longrightarrow Sp(8) \text{ or } G_2 \text{ or } SU(3). \tag{3·58}$$

(b) $M_{27}^2<M_{351}^2$

Taking into account that ϕ_{S_P} does not depend on M_{27}^2, we first study the derivative of ϕ'_{F_4} with respect to M_{27}^2, with M_{351}^2 kept fixed. The arguments V and v of ϕ'_{F_4} are set equal to the values realizing the stationary point of ϕ'_{F_4} and so they depend on M_{27}^2.

$$\begin{aligned}\frac{\partial\phi'_{F_4}(\text{stationary point})}{\partial M_{27}^2}&=30V^2+\frac{\partial V}{\partial M_{27}^2}\frac{\partial\phi'_{F_4}}{\partial V}+\frac{\partial v}{\partial M_{27}^2}\frac{\partial\phi'_{F_4}}{\partial v}\\ &=30V^2\geq 0.\end{aligned} \tag{3·59}$$

This implies that the minimum value of ϕ'_{F_4} is monotonically increasing as a function of M_{27}^2, and hence together with (3·55) that

$$\phi'_{F_4}<\phi_{S_P} \tag{3·60}$$

in this region $M_{27}^2<M_{351}^2$.

Next we compare ϕ'_{F_4} and ϕ'_{SO}. In the limiting region

$$M_{351}^2 \gg M_{27}^2 \to 0 \quad \text{namely} \quad G_{351} \ll G_{27} \to \infty\,, \tag{3·61}$$

the system is the same as that where there is only **27** 4-fermi interaction and there, as we know, the F_4 vacuum is the lowest one:

$$\phi'_{F_4} < \phi'_{SO}\,. \tag{3·62}$$

On the other hand the relation (3·55) implies that the same inequality holds even in the region $M_{27}^2 \sim M_{351}^2$. This strongly suggests that the inequality (3·62) holds for the whole region $M_{27}^2 < M_{351}^2$. We assume this holds. Then, together with (3·60), we find that the symmetry breaking pattern in this coupling region is

$$E_6 \quad \longrightarrow \quad F_4\,. \tag{3·63}$$

The discussion in this subsection is very incomplete by two reasons. First, this general case is outside the scope of Michel's conjecture. Second, Eq. (3·62) was not proved for the whole region $M_{27}^2 < M_{351}^2$. Nevertheless it suggests a simple symmetry breaking pattern; it is either $E_6 \to F_4$ or $E_6 \to Sp(8)$ or G_2 or $SU(3)$ depending on whether the **27** interaction G_{27} is larger or smaller than the **351** interaction G_{351}, respectively.

§ 4. Numerical analysis

In order to confirm the symmetry breaking pattern suggested by the analysis in the previous section, we numerically search the minimum of the potential (2·3) and calculate a fermion mass spectrum and gauge boson mass spectrum at that point.

4.1. *Algorithm*

We present in this subsection an algorithm for searching the stationary point $H_{\rm st}$: $\partial\phi/\partial H^\dagger|_{H_{st}}=0$. The idea is essentially to apply the Newton method to the derivative $\partial\phi(H)/\partial H^\dagger$ since we want a zero point of this function.

First of all we note:

1) $\phi(H)$ is a function of 378×2 variables as H is a 27×27 symmetric and complex matrix.
2) $\partial\phi(H)/\partial H^\dagger \equiv V(H)$ is a gradient of $\phi(H)$ in the 756 dimension space, which can be written down in a closed matrix form:

$$\begin{aligned} V(H) &\equiv \frac{\partial\phi}{\partial H^\dagger}(H) \\ &= (M_{27}^2 - M_{351}^2)H_{27} + M_{351}^2 H \\ &\quad -\frac{1}{\pi^2}H\Big[\Lambda^2 - 4H^\dagger H\Big(\mathrm{Ln}(4H^\dagger H+\Lambda^2) - \mathrm{Ln}(4H^\dagger H)\Big)\Big]. \end{aligned} \tag{4·1}$$

3) On the contrary we have no such a simple analytic expression for the second derivative of $\phi(H)$.

We now outline how the iteration method goes for searching the minimum. (We assume in the following for simplicity that $\phi(H)$ is *concave* in the considered region.)

i) We take randomly a starting point $H \equiv H_0$, and calculate the gradient $V(H_0) = (\partial\phi/\partial H^\dagger)(H_0)$ there.

ii) To find the next point which is nearer to the stationary point, we consider the potential function $\phi(H)$ in a cross section in the gradient direction; namely, we consider the following function of one real parameter t:

$$f(t) \equiv \phi(H_0 + Vt)\,. \tag{4·2}$$

If we find a zero of the first derivative function

$$g(t) \equiv \frac{df(t)}{dt} = \mathrm{tr}\left[V^\dagger \frac{\partial\phi}{\partial H^\dagger}(H_0 + Vt)\right] \tag{4·3}$$

at $t = t_0$, then $H = H_0 + Vt_0$ will be the lowest point of $\phi(H)$ in this cross section.

iii) Starting from $t = 0$, the Newton method applied to this function $g(t)$ gives at the first iteration step

$$t_1 = -\frac{g(0)}{g'(0)}\,, \tag{4·4}$$

as a nearer point to the zero t_0 of $g(t)$. We do not continue this Newton's iteration any further since even if t_0 is found more exactly the point $H = H_0 + Vt_0$ is merely the lowest point of $\phi(H)$ inside this cross section. So we adopt $H_1 = H_0 + Vt_1$ as a nearer point to a true stationary point of $\phi(H)$.

iv) If t_1 is already small enough we consider H_0 is a stationary point. Otherwise we take $H_1 \equiv H_0 + Vt_1$ as H_0 in the step i) and repeat the procedure.

What we get by this iteration procedure is, logically speaking, not a minimum point but a stationary point. But, in practice in this calculation, we actually obtained a minimum although it may not be a global minimum.

4.2. *Result*

We have run the above procedure for searching the stationary point for the potential with various sets of parameters, M_{351}^2/M_{27}^2 and Λ^2/M_{27}^2; more explicitly, we have swept the region $0 \le M_{351}^2/M_{27}^2 \le 10^3$ and $40 \le \Lambda^2/M_{27}^2 \le 10^3$. (Note that $\Lambda^2/M_{27}^2 = \pi^2$ is the critical value for the symmetry breaking in the pure **27** interaction case.) We have stored in total about 10^4 data of the stationary points H_{st} for the potentials with the parameters in this region, in particular, in the region $10^{-3} \le M_{351}^2/M_{27}^2 \le 300$ and $\Lambda^2/M_{27}^2 = 40$, 100 in detail.

Using the obtained stationary point data H_{st}, we have calculated fermion masses and gauge boson masses on those vacua. Fermion masses are calculated as eigenvalues of squared mass matrix $H_{\mathrm{st}}^\dagger H_{\mathrm{st}}$ and those of gauge bosons are as eigenvalues of the squared mass matrix $G = (G_{ab}) \equiv (\mathrm{tr}[T_a H_{\mathrm{st}}^\dagger + H_{\mathrm{st}}^\dagger T_a^{\mathrm{T}}])(\mathrm{tr}[T_b^* H_{\mathrm{st}} + H_{\mathrm{st}} T_b^\dagger])$ where the T_a's are E_6 generators in the **27** representation, whose explicit form is given in Appendix B. We can judge the symmetry breaking pattern from those mass spectra for each case.

The result of our numerical calculation is summarized as follows.

1) When $M_{27} \le M_{351}$, namely, **27** 4-fermi interaction is dominant, we found in every

case of our search the following. 26 fermions have a degenerate mass and the rest one fermion has another mass. On the other hand, 52 gauge bosons are massless and the rest 26 have a degenerate non-zero mass. All these clearly imply that the symmetry breaking pattern in this coupling region is

$$E_6 \rightarrow F_4 . \tag{4·5}$$

This completely agrees with the result obtained in the previous section, despite that the latter was based on a bit non-rigorous arguments.

2) When $M_{27} \geq M_{351}$, namely, **351** 4-fermi interaction is dominant, we found in every case the following. All of the 27 fermions have a degenerate mass while the gauge bosons become all massive but not degenerate at all. The degenerate fermion spectrum implies that the symmetry breaking pattern in this case is

$$E_6 \longrightarrow Sp(8) \text{ or } G_2 \text{ or } SU(3) , \tag{4·6}$$

agreeing again with the result of the previous analysis. But it seems strange why all the gauge bosons are massive and non-degenerate. If there remains some symmetry, the corresponding gauge bosons should remain massless and the spectrum should show some multiplet structure. The reason why this strange thing happens is in the particular nature in this breaking: namely, in this case, the three different vacua with symmetries $Sp(8)$, G_2 and $SU(3)$ are degenerate. They place at different points in the potential but realize the same stationary value. Then, if there is a path connecting these three points through which the potential is flat (or almost flat within the calculation error), all the points on the path realize the same stationary values but have no symmetries at all. Nevertheless, the fermion mass degeneracy is still realized since the present effective potential counts only the fermion vacuum energy and the degeneracy of the potential value along the path means the fermion mass degeneracy. All the stationary points we found are such points on the path. This is our interpretation, but we confirmed this by examining the potential values realized by our stationary points. They all coincided with $\phi_{Sp}=27f(v_0)$ which we obtained analytically in the previous section by using Michel's conjecture.

Table II. Mass spectra found numerically for the cases $M_{27} \leq M_{351}$ and $M_{27} \geq M_{351}$.

	$M_{27} \leq M_{351}$	$M_{27} \geq M_{351}$
fermion mass	**1**+**26**	**27**
gauge bosson mass	**52**(massless)+**26**(massive)	**1**(massive)×78

§ 5. Summary and conclusion

We have analyzed an E_6 GUT model of a single generation of fermions with strong 4-fermi interactions. The E_6 symmetry is found to be broken spontaneously either to F_4 or to $Sp(8)$ or G_2 or $SU(3)$ depending on which of the 4-fermi coupling constants G_{27} and G_{351} in the **27**/**351** channels is stronger than the other.

In these symmetry breakings, the fermions turn to belong to *real* representations of the residual symmetry and all of them acquire non-vanishing masses. Since these masses are necessarily of the order of the GUT symmetry breaking ($\sim 10^{16-17}$ GeV), the present model as it stands, unfortunately, turns out to be unrealistic as a GUT model. The quarks and leptons belong to a chiral representation of the standard gauge group and should remain massless at the GUT scale.

We can easily understand the reason why all the fermions get non-vanishing masses in the present model. As mentioned before, our effective potential counts only the fermion one-loop vacuum energy. But the fermion vacuum energy essentially comes from the energy of Dirac's negative energy sea and hence is negative. So, the more massive the fermions become, the more the vacuum energy is lowered. Therefore the desirable symmetry breaking patterns, such as down to $SU(3)\times SU(2)\times U(1)$ under which the fermions are chiral and remain massless, are necessarily disfavorable energetically.

This indicates that the vacuum energy coming from *bosons* should play a central role in order for the present model to produce desirable symmetry breaking patterns. Indeed, Harvey[12] once considered the E_6 symmetry breaking in a Coleman-Weinberg like spontaneous symmmetry breaking scenario and found that E_6 is broken down to $SO(10)$. There the main part of the potential in fact came from the gauge boson loop contribution.

Alternatively, there may be another possibility if we change the fermion content of the model. For instance,[13] we can regard the three generations of quarks/leptons as merely survivals from GUT world where $n+3$ generations and n anti-generations of fermions exist. Then, when the dynamical GUT symmetry breaking occurs, a variety of mixing can generally occur among those fermions, and n generations of fermions as a net number can acquire $O(M_{\mathrm{GUT}})$ masses leaving the usual quarks and leptons massless. Since there are fermions which acquire the masses in this case, there is a possibility that small contributions of the gauge boson loop may be sufficient to realize such desirable breaking down to chiral type symmetry. This type of scenario is very interesting also from the viewpoint of the origin of Cabibbo-Kobayashi-Maskawa mixing as well as of the stability of proton.

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Appendix A
—— *SO(10) γ-Matrices* ——

For any $SO(2n)$, the γ-matrices ${}^{2n}\Gamma_M$ $(M=1, 2, \cdots, 2n)$ satisfying ${}^{2n}\Gamma_M{}^{2n}\Gamma_N+{}^{2n}\Gamma_N{}^{2n}\Gamma_M=2\delta_{MN}$ take the form

$$ {}^{2n}\Gamma_M = \begin{pmatrix} 0 & ({}^{2n}\sigma_M)_{\alpha\beta} \\ ({}^{2n}\sigma_M^\dagger)^{\alpha\beta} & 0 \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} \xi_\beta \\ \eta^\beta \end{pmatrix}, \tag{A·1} $$

where ξ_α and η^α are $2^{(n/2)-1}$-component Weyl spinors with chiralities ${}^{2n}\Gamma_{2n+1}=+1$ and -1, respectively. The superscript on the left shoulder indicates the dimension $2n$ of $SO(2n)$. The σ's are the γ-matrices on the Weyl spinor basis.

Totally anti-symmetric multi-indexed γ-matrices ${}^{2n}\Gamma_{M_1M_2\cdots M_k}$ are defined by

$$ {}^{2n}\Gamma_{M_1M_2\cdots M_k} = \frac{1}{k!}\Big({}^{2n}\Gamma_{M_1}{}^{2n}\Gamma_{M_2}\cdots{}^{2n}\Gamma_{M_k} + (\text{anti-symmetrization})\Big), $$

$$ \equiv \begin{cases} \begin{pmatrix} {}^{2n}\sigma_{M_1M_2\cdots M_k} & 0 \\ 0 & {}^{2n}\bar{\sigma}_{M_1M_2\cdots M_k} \end{pmatrix} & \text{for } k=\text{even}, \\ \begin{pmatrix} 0 & {}^{2n}\sigma_{M_1M_2\cdots M_k} \\ {}^{2n}\bar{\sigma}_{M_1M_2\cdots M_k} & 0 \end{pmatrix} & \text{for } k=\text{odd}, \end{cases} $$

$$ {}^{2n}\sigma_{M_1M_2\cdots M_k} = \frac{1}{k!}\Big({}^{2n}\sigma_{M_1}{}^{2n}\sigma_{M_2}^\dagger{}^{2n}\sigma_{M_3}{}^{2n}\sigma_{M_4}^\dagger\cdots{}^{2n}\sigma_{M_k}^{(\dagger)} + (\text{anti-symmetrization})\Big), $$

$$ {}^{2n}\bar{\sigma}_{M_1M_2\cdots M_k} = \frac{1}{k!}\Big({}^{2n}\sigma_{M_1}^\dagger{}^{2n}\sigma_{M_2}{}^{2n}\sigma_{M_3}^\dagger{}^{2n}\sigma_{M_4}\cdots{}^{2n}\sigma_{M_k}^{(\dagger)} + (\text{anti-symmetrization})\Big), \tag{A·2} $$

The $SO(2n)$ generators T_{MN} satisfying $[T_{MN}, T_{KL}] = -(\delta_{NK}T_{ML} + (\text{anti-symmetrization}))$ are expressed in this spinor representation by the matrix

$$ {}^{2n}\Sigma_{MN} \equiv \frac{1}{2i}\,{}^{2n}\Gamma_{MN} = \frac{1}{2i}\begin{pmatrix} {}^{2n}\sigma_{MN} & 0 \\ 0 & {}^{2n}\bar{\sigma}_{MN} \end{pmatrix}. \tag{A·3} $$

The charge conjugation matrix ${}^{2n}C$ exists such that[14]

$$ {}^{2n}C\,{}^{2n}\Gamma_M\,{}^{2n}C^{-1} = \eta\,{}^{2n}\Gamma_M{}^{\mathrm{T}}, $$

$$ {}^{2n}C^\dagger\,{}^{2n}C = 1, \quad {}^{2n}C^{\mathrm{T}} = \epsilon\,{}^{2n}C \quad \text{with } \epsilon = \cos\frac{n\pi}{2} + \eta\sin\frac{n\pi}{2} \tag{A·4} $$

for either choice of $\eta=\pm 1$, where the superscript T denotes transposed. Henceforth we always choose $\eta=+1$ for convenience.

A.1. $SO(6)$

We first construct $SO(6)$ γ-matrices on the Weyl spinor basis: it is convenient to take the 4×4 ${}^6\sigma_m$ matrices as

$$ {}^6\sigma_m = ({}^6\sigma_{i=1,2,3}, {}^6\sigma_{i+3=4,5,6}), $$

$$ \begin{cases} ({}^6\sigma_i)_{\alpha\beta} = \varepsilon_{i4\alpha\beta} + \delta^{i4}_{\alpha\beta}, \\ ({}^6\sigma_{i+3})_{\alpha\beta} = i(\varepsilon_{i4\alpha\beta} - \delta^{i4}_{\alpha\beta}), \end{cases} \tag{A·5} $$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is rank-4 totally anti-symmetric tensor and $\delta^{\gamma\delta}_{\alpha\beta}$ is multi-index anti-symmetric Kronecker's delta defined by $\delta^{\gamma\delta}_{\alpha\beta} \equiv \delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta$. The index i here, running over 1, 2, 3, will correspond to the color index of $SU(3)\subset SO(6)$.

These ${}^6\sigma_m(m=1, 2, \cdots, 6)$ possess the following properties:

$$
\begin{aligned}
&{}^6\sigma_m=-{}^6\sigma_m^{\mathrm{T}}\,, \quad \text{(anti-symmetric)}\\
&({}^6\sigma_m)_{\alpha\beta}=-\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}({}^6\sigma_m^\dagger)^{\gamma\delta}\,, \quad \text{(anti-selfduality)}\\
&\frac{1}{2}({}^6\sigma_m)_{\alpha\beta}({}^6\sigma_m^\dagger)^{\gamma\delta}=-\delta_{\alpha\beta}^{\gamma\delta} \quad \leftrightarrow \quad \frac{1}{4}\mathrm{tr}({}^6\sigma_m{}^6\sigma_n^\dagger)=\delta_{mn}\,,\\
&\frac{1}{2}({}^6\sigma_m)_{\alpha\beta}({}^6\sigma_m)_{\gamma\delta}=\varepsilon_{\alpha\beta\gamma\delta}\,.
\end{aligned}
\tag{A·6}
$$

An $SO(6)$-vector V_m is equivalent to a rank-2 antisymmetric tensor $V_{[\alpha\beta]}$ of $SU(4)$; they are related with each other via

$$
V_{[\alpha\beta]}=\frac{1}{\sqrt{2}}({}^6\sigma_m)_{\alpha\beta}V_m \qquad \leftrightarrow \qquad V_m=\frac{1}{2}\left(\frac{1}{\sqrt{2}}{}^6\sigma_m^\dagger\right)^{\alpha\beta}V_{[\alpha\beta]}\,. \tag{A·7}
$$

Decomposition of the $SO(6)$ vector V_m into $\mathbf{3}+\mathbf{3}^*$ under the color group $SU(3)\subset SU(4)\simeq SO(6)$ is given by

$$
\begin{aligned}
&\mathbf{3}:\quad V_{[i4]}=\frac{1}{\sqrt{2}}(V_i-iV_{i+3})\,,\\
&\mathbf{3}^*:\quad \frac{1}{2}\varepsilon^{ijk}V_{[jk]}=\frac{1}{\sqrt{2}}(V_i+iV_{i+3})\,.
\end{aligned}
\tag{A·8}
$$

The $SO(6)$ generators are given by the general expression (A·3), which defines ${}^6\sigma_{mn}$ and ${}^6\bar{\sigma}_{mn}$. Then the 15 matrices ${}^6\sigma_{mn}$ ($m, n=1, \cdots, 6$) together with a unit matrix span a complete set of 4×4 matrices and satisfy the following completeness relation:

$$
\frac{1}{4}(1)_\alpha{}^\gamma(1)_\beta{}^\delta+\frac{1}{2}({}^6\sigma_{mn})_\alpha{}^\gamma({}^6\sigma_{mn})_\beta{}^\delta=\delta_\alpha^\delta\delta_\beta^\gamma\,. \tag{A·9}
$$

The charge conjugation matrix ${}^6C$ defined generally in (A·4) is now given by

$$
{}^6C=\begin{pmatrix} 0 & -\mathbf{1}_4 \\ \mathbf{1}_4 & 0 \end{pmatrix}=-{}^6C^{\mathrm{T}} \tag{A·10}
$$

with $\mathbf{1}_m$ denoting $m\times m$ unit matrix.

A.2. *SO(4)*

$SO(4)$ γ-matrices ${}^4\sigma_\mu$ ($\mu=7, 8, 9, 0$) on the Weyl spinor basis are 2×2 matrices which we take as follows:

$$
{}^4\sigma_\mu=(-i\sigma_1,\ -i\sigma_2,\ -i\sigma_3,\ \mathbf{1}_2) \tag{A·11}
$$

with $\sigma_{1,2,3}$ being the Pauli matrices. Then $SO(4)$ generators ${}^4\Sigma_{\mu\nu}$ defined by (A·3) split into 3+3 generators of $SU(2)_{\mathrm{L}}\times SU(2)_{\mathrm{R}}\simeq SO(4)$:

$$\Sigma_{\mathrm{L}i}=\frac{1}{2}\left(\frac{1}{2}\varepsilon_{ijk}{}^{4}\Sigma_{j+6,k+6}+{}^{4}\Sigma_{0,i+6}\right)=\begin{pmatrix}\frac{1}{2}\sigma_k & \\ & 0\end{pmatrix},$$

$$\Sigma_{\mathrm{R}i}=\frac{1}{2}\left(\frac{1}{2}\varepsilon_{ijk}{}^{4}\Sigma_{j+6,k+6}-{}^{4}\Sigma_{0,i+6}\right)=\begin{pmatrix}0 & \\ & \frac{1}{2}\sigma_k\end{pmatrix}. \tag{A·12}$$

The charge conjugation matrix ${}^4C$ is given by

$${}^4C=\begin{pmatrix}i\sigma_2 & 0\\ 0 & i\sigma_2\end{pmatrix}=-{}^4C^{\mathrm{T}}. \tag{A·13}$$

A.3. *SO(10)*

$SO(10)$ γ-matrices ${}^{10}\Gamma_M$ $(M=1, 2, \cdots, 9, 0)$ are constructed by a tensor product of the $SO(6)$ and $SO(4)$ γ-matrices as follows:

$${}^{10}\Gamma_M=\begin{cases}{}^6\Gamma_m\otimes{}^4\Gamma_5 \text{ for } M=m=1, 2, \cdots, 6\\ \mathbf{1}_4\otimes{}^4\Gamma_\mu \text{ for } M=\mu=7, 8, 9, 0\end{cases} \quad \text{with } {}^4\Gamma_5=\begin{pmatrix}\mathbf{1}_2 & 0\\ 0 & -\mathbf{1}_2\end{pmatrix}. \tag{A·14}$$

Then the γ-matrices in the Weyl basis, σ_M, for which we omit the superscript 10 implying $SO(10)$ for notational simplicity, are 8×8 matrices taking the following form:

$$\sigma_{M=m}=\begin{pmatrix}0 & {}^6\sigma_m\otimes\mathbf{1}_2\\ -{}^6\sigma_m^\dagger\otimes\mathbf{1}_2 & 0\end{pmatrix}, \quad \sigma_{M=\mu}=\begin{pmatrix}\mathbf{1}_4\otimes{}^4\sigma_\mu & 0\\ 0 & \mathbf{1}_4\otimes{}^4\sigma_\mu^\dagger\end{pmatrix}. \tag{A·15}$$

The charge conjugation matrix ${}^{10}C$ takes the form

$${}^{10}C={}^6C\otimes{}^4C=\begin{pmatrix}0 & C\\ C & 0\end{pmatrix}, \tag{A·16}$$

where C is 8×8 matrix given by

$$C=\begin{pmatrix}0 & -\mathbf{1}_4\otimes i\sigma_2\\ \mathbf{1}_4\otimes i\sigma_2 & 0\end{pmatrix}=C^{\mathrm{T}}=C^{-1}=C^\dagger. \tag{A·17}$$

From ${}^{10}C^{\mathrm{T}}={}^{10}C$, it follows that the matrices ${}^{10}C\,{}^{10}\Gamma_{\mathrm{M}}$ are symmetric, and so are $C\sigma_M^{(\dagger)}$ and $\sigma_M^{(\dagger)}C$. Similarly we see that $C\bar{\sigma}_{M_1M_2\cdots M_5}$ and $\sigma_{M_1M_2\cdots M_5}C$ are symmetric. Since they are selfdual,

$$C\bar{\sigma}_{M_1\cdots M_5}=\frac{1}{5!}i\varepsilon_{M_1\cdots M_5N_1\cdots N_5}C\bar{\sigma}_{N_1\cdots N_5}, \tag{A·18}$$

they give ${}_{10}C_5/2=126$ symmetric matrices, and hence, together with the ten $C\sigma_M^\dagger$ matrices, span a complete set in the space of 16×16 symmetric matrices; the completeness relation reads

$$2^{-4}\left[(\sigma_M C)_{\alpha\beta}(C\sigma_M^\dagger)^{\gamma\delta}+\frac{1}{2\cdot 5!}(\sigma_{M_1\cdots M_5}C)_{\alpha\beta}(C\bar{\sigma}_{M_1\cdots M_5})^{\gamma\delta}\right]=\frac{1}{2}(\delta_\alpha^\delta\delta_\beta^\gamma+\delta_\alpha^\gamma\delta_\beta^\delta)\,, \tag{A·19}$$

because of the normalization condition

$$2^{-4}\mathrm{tr}(C\sigma_M^\dagger\sigma_N C)=\delta_{MN}\,,$$
$$2^{-4}\mathrm{tr}(C\bar{\sigma}_{M_1\cdots M_5}\sigma_{N_1\cdots N_5}C)=\delta_{M_1\cdots M_5}^{N_1\cdots N_5}+i\varepsilon_{M_1\cdots M_5 N_1\cdots N_5}\,. \tag{A·20}$$

In the same way ${}_{10}C_3=120$ matrices $C\bar{\sigma}_{M_1M_2M_3}$ (or, $\sigma_{M_1M_2M_3}C$) turn out to give a complete set of 16×16 anti-symmetric matrices so that

$$2^{-4}\frac{1}{3!}(\sigma_{M_1M_2M_3}C)_{\alpha\beta}(C\bar{\sigma}_{M_1M_2M_3})^{\gamma\delta}=\frac{1}{2}(\delta_\alpha^\delta\delta_\beta^\gamma-\delta_\alpha^\gamma\delta_\beta^\delta)\,. \tag{A·21}$$

Appendix B
—— *Some Representations of E_6* ——

The E_6 algebra is most easily expressed by referring to its maximal subgroup $SO(10)\times U(1)$. The generators are given by 16 $SO(10)$ Weyl-spinor generators E_α ($\alpha=1,\cdots,16$) and their complex conjugates $\bar{E}^\alpha=(E_\alpha)^\dagger$ in addition to the 45 $SO(10)$ generators T_{MN} and one $U(1)$ generator T. The algebra is given by[15]

$$[T_{MN},\ T_{KL}]=-i(\delta_{NK}T_{ML}+\delta_{ML}T_{NK}-\delta_{MK}T_{NL}-\delta_{NL}T_{MK})\,,$$

$$[T_{MN},\begin{pmatrix}E_\alpha\\ \bar{E}^\alpha\end{pmatrix}]=-\begin{pmatrix}(\sigma_{MN})_\alpha{}^\beta & 0\\ 0 & (-\sigma_{MN}^*)^\alpha{}_\beta\end{pmatrix}\begin{pmatrix}E_\beta\\ \bar{E}^\beta\end{pmatrix},$$

$$[T,\begin{pmatrix}E_\alpha\\ \bar{E}^\alpha\end{pmatrix}]=\frac{\sqrt{3}}{2}\begin{pmatrix}E_\alpha\\ -\bar{E}^\alpha\end{pmatrix},$$

$$[E_\alpha,\bar{E}^\beta]=-\frac{1}{2}(\sigma_{MN})_\alpha{}^\beta T_{MN}+\frac{\sqrt{3}}{2}\delta_\alpha{}^\beta T\,. \tag{B·1}$$

The simplest representation of E_6 is **27** which is decomposed into $\mathbf{1}_4+\mathbf{16}_1+\mathbf{10}_{-2}$ under the maximal subgroup $SO(10)\times U(1)$. (The suffices denote the value of $U(1)$ charge $2\sqrt{3}T$.) So the **27** representation can be denoted as $\psi_A\equiv(\psi_0,\psi_\alpha,\psi_M)$ with α and M being $SO(10)$ (Weyl-)spinor and vector indices, respectively. The E_6 generators act on this representation as[15]

$$\left(\theta T+\frac{1}{2}\theta_{KL}T_{KL}+\bar{\epsilon}^\gamma E_\gamma+\bar{E}^\gamma\epsilon_\gamma\right)\begin{pmatrix}\psi_0\\ \psi_\alpha\\ \psi_M\end{pmatrix}$$

$$=\begin{bmatrix}\frac{2}{\sqrt{3}}\theta & \bar{\epsilon}^\beta & 0\\ \epsilon_\alpha & \frac{1}{2}\theta_{KL}(\sigma_{KL})_\alpha{}^\beta+\frac{1}{2\sqrt{3}}\theta\delta_\alpha{}^\beta & -\frac{1}{\sqrt{2}}(\bar{\epsilon}\sigma_N C)_\alpha\\ 0 & -\frac{1}{\sqrt{2}}(C\sigma_M^\dagger\epsilon)^\beta & -i\theta_{MN}-\frac{1}{\sqrt{3}}\theta\delta_{MN}\end{bmatrix}\begin{bmatrix}\psi_0\\ \psi_\beta\\ \psi_N\end{bmatrix}\,. \tag{B·2}$$

To check that this representation for the E_6 generators really satisfies the algebra (B·1), we need the following identities for the $SO(10)$ γ-matrices:

$$C\bar{\sigma}_{MN}C=-\sigma^{\mathrm{T}}_{AB}\,,\quad C\bar{\sigma}_{M_1M_2M_3M_4}C=\sigma^{\mathrm{T}}_{M_1M_2M_3M_4}\,,$$

$$\sigma_{MN}\sigma_K-\sigma_K\bar{\sigma}_{MN}=i(\delta_{MK}\sigma_N-\delta_{NK}\sigma_M)\,,$$

$$\frac{1}{2}(C\sigma^\dagger_M)^{\beta\gamma}(\sigma_M C)_{\delta\alpha}-\delta^\gamma_\alpha\delta^\beta_\delta=\frac{1}{4}\delta^\beta_\alpha\delta^\gamma_\delta-\frac{1}{2}(\sigma_{MN})_\alpha{}^\beta(\sigma_{MN})_\delta{}^\gamma\,. \tag{B·3}$$

The last identity follows from the Fierz transformation of the LHS:

$$\delta^\gamma_\alpha\delta^\beta_\delta=2^{-4}\Big[(1)_\alpha{}^\beta(1)_\delta{}^\gamma-\frac{1}{2}(2i\sigma_{MN})_\alpha{}^\beta(2i\sigma_{MN})_\delta{}^\gamma+\frac{1}{4!}(\sigma_{M_1\cdots M_4})_\alpha{}^\beta(\sigma_{M_1\cdots M_4})_\delta{}^\gamma\Big]\,,$$

$$(C\sigma^\dagger_M)^{\beta\gamma}(\sigma_M C)_{\delta\alpha}=2^{-4}\Big[(C1C)^\beta{}_\alpha(\sigma_K 1\sigma^\dagger_K)_\delta{}^\gamma-\frac{1}{2}(C2i\bar{\sigma}_{MN}C)^\beta{}_\alpha(\sigma_K 2i\bar{\sigma}_{MN}\sigma^\dagger_K)_\delta{}^\gamma$$
$$+\frac{1}{4!}(C\bar{\sigma}_{M_1\cdots M_4}C)^\beta{}_\alpha(\sigma_K\bar{\sigma}_{M_1\cdots M_4}\sigma^\dagger_K)_\delta{}^\gamma\Big]\,,$$

$$\sigma_K\bar{\sigma}_{MN}\sigma^\dagger_K=6\sigma_{MN}\,,\quad \sigma_K\bar{\sigma}_{M_1\cdots M_4}\sigma^\dagger_K=2\sigma_{M_1\cdots M_4}\,. \tag{B·4}$$

Tensor product of two **27** representations gives

$$\mathbf{27}\times\mathbf{27}=\overline{\mathbf{27}}_{\mathrm{S}}+\overline{\mathbf{351}}_{\mathrm{S}}+\overline{\mathbf{351}}'_{\mathrm{A}}\,. \tag{B·5}$$

This implies that there is an invariant tensor Γ^{ABC} which gives $\overline{\mathbf{27}}$ from $\mathbf{27}\times\mathbf{27}$:

$$\bar{\Psi}^A=\Gamma^{ABC}\phi_B\phi_C\,. \tag{B·6}$$

This Γ^{ABC} is found to be given by

$$\Gamma^{ABC}\text{: totally symmetric}\quad\begin{cases}\Gamma^{0MN}=\delta_{MN}\,,\\ \Gamma^{M\alpha\beta}=\dfrac{1}{\sqrt{2}}(C\sigma^\dagger_M)^{\alpha\beta}\,,\\ \text{otherwise }0\,,\end{cases} \tag{B·7}$$

or equivalently, in terms of the components of $\bar{\Psi}^A$,

$$\bar{\Psi}^0=\phi_M\phi_M\,,$$

$$\bar{\Psi}^M=\frac{1}{\sqrt{2}}\phi^{\mathrm{T}}C\sigma^\dagger_M\phi+2\phi_0\phi_M\,,$$

$$\bar{\Psi}^\alpha=\sqrt{2}\phi_M(C\sigma^\dagger_M\phi)^\alpha\,. \tag{B·8}$$

To check that this $\bar{\Psi}$ transforms correctly as $\overline{\mathbf{27}}$, we need an identity:

$$(\epsilon^{\mathrm{T}}C\sigma^\dagger_M\phi)(\phi^{\mathrm{T}}C\sigma^\dagger_M\eta)=-\frac{1}{2}(\epsilon^{\mathrm{T}}C\sigma_M\eta)(\phi^{\mathrm{T}}C\sigma^\dagger_M\phi)\,, \tag{B·9}$$

which follows from Fierzing $\sigma^\dagger_M\phi$ and η and using $\sigma^\dagger_M\sigma_K\sigma^\dagger_M=-8\sigma^\dagger_K$ and $\sigma^\dagger_M\sigma_{K_1\cdots K_5}\sigma^\dagger_M=0$.

The **351** can be represented by a symmetric tensor Φ with two $\overline{\mathbf{27}}$ indices A and B:

$$\Phi^{AB}=\begin{bmatrix} \Phi^0(\mathbf{1}) & \frac{1}{\sqrt{2}}\Phi^{\beta}(\overline{\mathbf{16}}) & \frac{1}{\sqrt{2}}\Phi^{N}(\mathbf{10}) \\ \frac{1}{\sqrt{2}}\Phi^{\alpha}(\overline{\mathbf{16}}) & \Phi^{\alpha\beta}(\overline{\mathbf{126}}) & \frac{1}{\sqrt{2}}\Phi^{\alpha N}(\mathbf{144}) \\ \frac{1}{\sqrt{2}}\Phi^{M}(\mathbf{10}) & \frac{1}{\sqrt{2}}\Phi^{M\beta}(\mathbf{144}) & \Phi^{MN}(\mathbf{54}) \end{bmatrix}, \tag{B$\cdot$10}$$

where the argument in each entry denotes the dimension under $SO(10)$. The previous **27** representation ϕ^A can also be imbedded into a symmetric matrix using the invariant symmetric tensor Γ^{ABC}:

$$\Gamma^{ABC}\phi_C=\begin{bmatrix} 0 & 0 & \phi_N \\ 0 & \frac{1}{\sqrt{2}}\phi_K(C\sigma_K^\dagger)^{\alpha\beta} & \frac{1}{\sqrt{2}}(C\sigma_N^\dagger\phi)^{\alpha} \\ \phi_M & \frac{1}{\sqrt{2}}(C\sigma_M^\dagger\phi)^{\beta} & \delta_{MN}\phi_0 \end{bmatrix}. \tag{B$\cdot$11}$$

Since symmetric tensor product of two $\overline{\mathbf{27}}$ is either **27** or **351**, the **351** matrix Φ^{AB} can be characterized as a general symmetric matrix which contains no **27** components of the form (B$\cdot$11): therefore, the component Φ^{MN} should be traceless, $\Phi^{MM}=0$; $\Phi^{\alpha\beta}$ should contain no $SO(10)$ vector components, $(\sigma_M C)_{\alpha\beta}\Phi^{\alpha\beta}=0$; $\Phi^{M\beta}$ should be γ-traceless, $(\sigma_M C)_{\alpha\beta}\Phi^{M\beta}=0$. But, as a matter of course, these conditions are nothing but the requirements that those entries be irreducible representations under $SO(10)$ as indicated in the arguments in (B$\cdot$10).

Appendix C
—— *Maximal Little Group* ——

A little group of a representation vector ϕ of a group G is defined by

$$H_\phi\equiv\left\{g\,|\,g\phi=\phi,\,g\in G\right\}, \tag{C$\cdot$1}$$

This little group depends not only on the representation but also on the vector ϕ itself.

Consider a single irreducible representation R or a self-conjugate pair $R+R^*$ of a complex irredicible representation R. For this representation R, many little groups appear as the vector ϕ varies in the representation R with the length $|\phi|(\neq 0)$ kept fixed. A little group H is called *maximal* if there is no ϕ with little group H_ϕ satisfying $G\supset H_\phi\supset H$.

Some examples of E_6 maximal little groups are given in the following table.

Table III. Maximal little groups for the representations $R(+R^*)$.

R	Maximal little groups
78	$SU(6)\times U(1)$, $SO(10)\times U(1)$, $SU(5)\times SU(2)\times U(1)$, $[SU(3)]^2\times SU(2)\times U(1)$
27	$SO(10)$, F_4
351	$SO(10)$, F_4, $Sp(8)$, G_2, $SU(3)$, $SU(4)\times SU(2)$

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