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Dynamical System on Cantor Set

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1. Introduction.

We will consider Cantor sets generated by piecewise $C^{1+\gamma}$ transformations ($\gamma>0$). In this article, we only consider Markov cases. Non Markov (but piecewise linear) cases will be studied in [6]. A heuristic argument will also appear in that paper.

Let us denote $I=[0, 1]$. We assume that there exists a finite set $\mathscr{A}$ of symbols, and a subinterval $\langle a\rangle\subset I$ corresponds to a symbol $a\in\mathscr{A}$, and

1. $\bigcup_{a\in\mathscr{A}}\langle a\rangle=I$,
2. $\langle a\rangle\cap\langle b\rangle=\varnothing$ if $a\neq b$.

Take a subset $\mathscr{A}_1\subset\mathscr{A}$, and we consider a mapping F from $\bigcup_{a\in\mathscr{A}_1}\langle a\rangle$ to I such that

1. F is monotone on each $\langle a\rangle$ and it can extend to $\overline{\langle a\rangle}$ in $C^{1+\gamma}$ ($\gamma>0$) (piecewise $C^{1+\gamma}$),
2. if $F(\langle a\rangle)\cap\langle b\rangle\neq\varnothing$ for a, $b\in\mathscr{A}_1$, then $\overline{F(\langle a\rangle)}\supset\langle b\rangle$ (Markov),
3. $\xi=\liminf_{n\to\infty}\frac{1}{n}\operatorname{ess\,inf}_{x\in I}\log|F^{n\prime}(x)|>0$ (expanding),
4. for each a, $b\in\mathscr{A}_1$, there exists n such that $\overline{F^n(\langle a\rangle)}\supset\langle b\rangle$ (irreducible),

where we denote the closure of a set J by $\bar{J}$. Note that from the above assumption, we get

$$\log\operatorname*{ess\,inf}_{x\in I}|F^{n\prime}(x)|>0$$

for some $n>0$. Here we denote by F^n the n-th iteration of F:

$$F^n(x)=\begin{cases} x & \text{if } n=0\,,\\ F^{n-1}(F(x)) & \text{if } n\geq 1\,.\end{cases}$$

Thus, hereafter we assume without loss of generality that

$$\xi_0=\log\operatorname*{ess\,inf}_{x\in I}|F'(x)|>0\,.$$

We will consider a set

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$$\mathscr{C}=\{x\in I : F^n(x)\in \bigcup_{a\in\mathscr{A}_1}\langle a\rangle, \forall n\geq 0\},$$

that is, the set of points whose orbits pass only subintervals corresponding to $\mathscr{A}_1$. One of the simplest examples of $\mathscr{C}$ is the usual Cantor set, which is defined by $F(x)=3x$ (mod 1), $\mathscr{A}=\{0, 1, 2\}$, $\langle i\rangle=[i/3, (i+1)/3)$, and $\mathscr{A}_1=\{0, 2\}$, and one of the most important examples is the set of points with digits 1 and 2 in its continued expansion. In the case, $F(x)=1/x$ (mod 1), $\mathscr{A}=\{1, 2, \cdots\}$, $\langle i\rangle=(1/(i+1), 1/i]$ and $\mathscr{A}_1=\{1, 2\}$. The continued fraction expansion has countable symbols, and does not satisfy our assumption. But the transformation on $\bigcup_{a\in\mathscr{A}\setminus\mathscr{A}_1}\langle a\rangle$ is unessential. Moreover, though $F'(1)=-1$, there exists a neighborhood of 1 such that $\{x : F(x)\notin\langle 1\rangle\cup\langle 2\rangle\}$. So, this case $\mathscr{A}_1=\{1, 2\}$ essentially satisfies our assumption (cf. §6).

As usual, we express a point $x\in I$ by a sequence of symbols $a_1^x a_2^x \cdots$ called the expansion of x defined by

$$F^{i-1}(x)\in\langle a_i^x\rangle \qquad i\geq 1 .$$

We call a finite sequence of symbols $w=a_1 \cdots a_m$ a word and define $|w|=m$ (the length of a word w). We define as usual a subinterval $\langle w\rangle$ corresponding to a word w, which is the set of points x such that $F^{i-1}(x)\in\langle a_i\rangle$ for any $1\leq i\leq m$. We call a word w admissible if $\langle w\rangle=\emptyset$. We denote the empty word by ε, and for notational convenience we define $\langle\varepsilon\rangle=I$ (that is, ε is admissible), and $|\varepsilon|=0$. We denote by $\mathscr{W}_m$ the set of admissible words $w=a_1 \cdots a_m$ with $a_i\in\mathscr{A}_1$. Set $\mathscr{W}=\bigcup_{m=0}^{\infty} W_m$. We denote by wx ($w\in\mathscr{W}$, $x\in I$) a point which belongs to $\langle w\rangle$ and $F^{|w|}(wx)=x$ if it exists. Note that $\mathscr{W}$ expresses the set of all the admissible words with symbols only in $\mathscr{A}_1$. Hereafter, we only consider words with symbols in $\mathscr{A}_1$.

We also define for $x\in\langle w\rangle$ ($|w|=m$)

$$F_m^{+\prime}(x)=F_m^{+\prime}(w)=\begin{cases} +\operatorname{ess\,inf}_{y\in\langle w\rangle}|F'(y)| & \text{if } F'(x)>0 ,\\ -\operatorname{ess\,inf}_{y\in\langle w\rangle}|F'(y)| & \text{if } F'(x)<0 ,\end{cases}$$

$$F_m^{-\prime}(x)=F_m^{-\prime}(w)=\begin{cases} +\operatorname{ess\,sup}_{y\in\langle w\rangle}|F'(y)| & \text{if } F'(x)>0 ,\\ -\operatorname{ess\,sup}_{y\in\langle w\rangle}|F'(y)| & \text{if } F'(x)<0 ,\end{cases}$$

$$(F_m^{\tau})^{n\prime}(x)=\prod_{i=0}^{n-1} F_m^{\tau\prime}(F^i(x)) \qquad (\tau\in\{+, -\}) .$$

Note that a transformation F_m^{τ} is only a formal piecewise linear Markov transformation on the symbolic dynamics where F is realized, and it may not be able to express as a map from I into itself.

As in [4] and [5], we will define generating functions for $g\in L^{\infty}$ and $0\leq\alpha\leq 1$

$$s_{g,\alpha}^{w}(z : F)=\sum_{n=0}^{\infty} z^n \sum_{u\in\mathscr{W}_n}\int_{ux\in\langle w\rangle}|F^{n\prime}(ux)|^{-\alpha}g(x)d\bar{x} \tag{1}$$

$$s^w_{g,\alpha}(z : F^\tau_m) = \sum_{n=0}^{\infty} z^n \sum_{u \in \mathscr{W}_n} \int_{ux \in \langle w \rangle} |(F^\tau_m)^{n\prime}(ux)|^{-\alpha} g(x) d\bar{x} \tag{2}$$

for a word $w \in \mathscr{W}$, where $d\bar{x}$ denotes the integral by the Lebesgue measure restricted to $\bigcup_{a \in \mathscr{A}_1} \langle a \rangle$. We also define α^τ_m the maximum α for which $z=1$ is the minimum singularity in modulus of $s^w_{g,\alpha}(z : F^\tau_m)$ for some $g \in L^\infty$ and some word w. We denote by α_0 the corresponding value for F. From the definition, if the right hand term of (1) converges, it is easy to see that

$$|s^w_{g,\alpha}(z : F)| \leq s^w_{|g|,\alpha}(|z| : F)\,.$$

Therefore, since the coefficients of the right hand term of (1) is positive for $g \geq 0$, the minimal singularity of $s^w_{g,\alpha}(z : F)$ is nonnegative. Moreover, for a non-negative valued function g, positive z and $\alpha \geq 0$, if the right hand terms of (1) and (2) converge, we get

$$s^w_{g,\alpha}(z : F) \leq s^\varepsilon_{g,\alpha}(z : F)\,,$$

$$s^w_{g,\alpha}(z : F^-_1) \leq s^w_{g,\alpha}(z : F^-_2) \leq \cdots \leq s^w_{g,\alpha}(z : F) \leq \cdots \leq s^w_{g,\alpha}(z : F^+_2) \leq s^w_{g,\alpha}(z : F^+_1)\,.$$

Therefore we get

$$1 \geq \alpha^+_1 \geq \alpha^+_2 \geq \cdots \geq \alpha_0 \geq \cdots \geq \alpha^-_2 \geq \alpha^-_1 \geq 0\,.$$

REMARK. Let ν_α be a Hausdorff measure with exponent α, that is,

$$\nu_\alpha(J) = \liminf_{\delta \downarrow 0 \{J_i\}} (Lebes(J_i))^\alpha\,,$$

where infimum is taken over all coverings of $\mathscr{C}$ with countable subintervals $\{J_i\}$ with Lebesgue measure $Lebes(J_i)$ less than δ. Then as a formal expression, we can define

$$\begin{aligned}
\bar{s}^w_{g,\alpha}(z : F) &= \sum_{n=0}^{\infty} z^n \int 1_{\langle w \rangle}(x) g(F^n(x)) d\nu_\alpha \\
&= \sum_{n=0}^{\infty} z^n \int_I \sum_{y \in \mathscr{C} : F^n(y) = x} 1_{\langle w \rangle}(y) |F^{n\prime}(y)|^{-\alpha} g(x) d\nu_\alpha \\
&= \sum_{n=0}^{\infty} z^n \sum_{u \in \mathscr{W}_n} \int_{ux \in \langle w \rangle} |F^{n\prime}(ux)|^{-\alpha} g(x) d\nu_\alpha \\
&= \int [(I - zP_\alpha)^{-1} 1_{\langle w \rangle}](x) g(x) d\nu_\alpha\,.
\end{aligned}$$

where P_α is the Perron-Frobenius operator associated with F with respect to ν_α. Therefore, these formal generating functions $\bar{s}_{g,\alpha}$ will express the ergodic properties of the dynamical system. But they are only formal expression up to this point, thus we slightly modify them, and define $s^w_{g,\alpha}(z : F)$.

LEMMA 1. *For fixed* $0 \leq \alpha \leq 1$,

$$\lim_{m\to\infty} \operatorname*{ess\,sup}_{x\in I} \| F_m^{\tau\prime}(x)|^{-\alpha} - |F'(x)|^{-\alpha}| = 0 .$$

PROOF. Note first $|X-Y|^\alpha \geq |X^\alpha - Y^\alpha|$ for X, $Y>0$. Therefore

$$\begin{aligned} \| F_m^{\tau\prime}(x)|^{-\alpha} - |F'(x)|^{-\alpha}| &= \| F_m^{\tau\prime}(x)|^{\alpha} - |F'(x)|^{\alpha}|/(|F_m^{\tau\prime}(x)|^{\alpha}|F'(x)|^{\alpha}) \\ &\leq |F_m^{\tau\prime}(x) - F'(x)|^{\alpha}/(|F_m^{\tau\prime}(x)|^{\alpha}|F'(x)|^{\alpha}) . \end{aligned}$$

Because F is piecewise $C^{1+\gamma}$ and expanding, the lemma is proved.

LEMMA 2. *Set* $0\leq\alpha\leq 1$. *For any* $\varepsilon>0$, *we get for sufficiently large* m *independent of* n

$$|(F_m^\tau)^{n\prime}(x)|^{-\alpha} \begin{cases} \leq (1+\varepsilon)^n |F^{n\prime}(x)|^{-\alpha} , \\ \geq (1-\varepsilon)^n |F^{n\prime}(x)|^{-\alpha} , \end{cases}$$

for any x.

PROOF. By the chain rule,

$$\begin{aligned} |(F_m^\tau)^{n\prime}(x)|^{-\alpha} &= \prod_{i=0}^{n-1} |F_m^{\tau\prime}(F^i(x))|^{-\alpha} \\ &\leq \prod_{i=0}^{n-1} |F'(F^i(x))|^{-\alpha} \\ &\quad \times (1 + \| F_m^{\tau\prime}(F^i(x))|^{-\alpha} - |F'(F^i(x))|^{-\alpha}| |F'(F^i(x))|^{\alpha}) . \end{aligned}$$

Therefore by Lemma 1, for any $\varepsilon>0$

$$|(F_m^\tau)^{n\prime}(x)|^{-\alpha} \leq (1+\varepsilon)^n |F^{n\prime}(x)|^{-\alpha}$$

holds for any sufficiently large m. In a similar way, we can prove the other inequality.

COROLLARY 1. *For a fixed* $0\leq\alpha\leq 1$, *there exists* $\beta(\alpha)>0$ *such that for* $|z|<\beta(\alpha)$

$$\lim_{m\to\infty} s_{g,\alpha}^w(z : F_m^\tau) = s_{g,\alpha}^w(z : F)$$

for any word $w\in\mathscr{W}$.

PROOF. Note first

$$\begin{aligned} |s_{g,\alpha}^w(z : F)| &\leq \sum_{n=0}^{\infty} |z|^n \sum_{u\in\mathscr{W}_n} \int_{ux\in\langle w\rangle} |F^{n\prime}(ux)|^{-\alpha} |g(x)| d\tilde{x} \\ &\leq \|g\|_\infty \sum_{n=0}^{\infty} (\#\mathscr{A}_1 |z| \operatorname*{ess\,sup}_{x\in I} |F'(x)|^{-\alpha})^n . \end{aligned}$$

Now put

$$\beta(\alpha)=\frac{1}{2}(\sharp\mathscr{A}_1 \operatorname*{ess\,sup}_{x\in I}|F'(x)|^{-\alpha})^{-1}.$$

Then from Lemma 2, $s_{g,\alpha}(z:F_m^\tau)$ and $s_{g,\alpha}(z:F)$ are uniformly bounded in $|z|<\beta(\alpha)$ for sufficiently large m. Therefore the proof follows.

2. α-Fredholm matrix and α-zeta function.

Now we will construct an α-Fredholm matrix $\Phi_\alpha(z:F_m^\tau)$. For $k\geq m$, set $\mathscr{W}_k\times\mathscr{W}_k$ matrix

$$\Phi_{\alpha,k}(z:F_m^\tau)_{u,v}=\begin{cases} z|F_m^\tau(u)'|^{-\alpha} & \text{if } \overline{F(\langle u\rangle)}\supset\langle v\rangle\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We denote $\Phi_{\alpha,m}(z:F_m^\tau)$ by $\Phi_\alpha(z:F_m^\tau)$. Then we get for $u\in\mathscr{W}_k$

$$\begin{aligned} s_{g,\alpha}^u(z:F_m^\tau)&=\sum_{n=0}^{\infty} z^n\sum_{v\in\mathscr{W}_n}\int_{vx\in\langle u\rangle}|(F_m^\tau)^{n\prime}(vx)|^{-\alpha}g(x)d\bar{x} \\ &=\int_{\langle u\rangle}g(x)d\bar{x}+z\sum_{v\in\mathscr{W}_k:\overline{F(\langle u\rangle)}\supset\langle v\rangle}|F_m^{\tau\prime}(u)|^{-\alpha}s_{g,\alpha}^v(z:F_m^\tau)\,. \end{aligned}$$

Therefore we can construct a renewal equation of the form

$$(s_{g,\alpha}^u(z:F_m^\tau))_{u\in\mathscr{W}_k}=\left(\int_{\langle u\rangle}g(x)d\bar{x}\right)_{u\in\mathscr{W}_k}+\Phi_{\alpha,k}(z:F_m^\tau)(s_{g,\alpha}^v(z:F_m^\tau))_{v\in\mathscr{W}_k}\,.$$

Now we define α-zeta function by

$$\zeta_\alpha(z:F)=\exp\left[\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{y\in\mathscr{C}:y=F^n(y)}|F^{n\prime}(y)|^{-\alpha}\right]$$

$$\zeta_\alpha(z:F_m^\tau)=\exp\left[\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{y\in\mathscr{C}:y=F^n(y)}|(F_m^\tau)^{n\prime}(y)|^{-\alpha}\right].$$

LEMMA 3. *for any* $k\geq m$

$$\det(I-\Phi_{\alpha,k}(z:F_m^\tau))=\zeta_\alpha(z:F_m^\tau)^{-1}\,,$$

especially

$$\det(I-\Phi_\alpha(z:F_m^\tau))=\zeta_\alpha(z:F_m^\tau)^{-1}\,.$$

Note that $\Phi_{\alpha,k}(z)$ is essentially a structure matrix of the dynamical system. Namely, the trace of $\Phi_{\alpha,k}(z)^n$ corresponds to periodic orbits with period n. Thus, from the fact that, for a matrix A, $\det A=\exp[\operatorname{tr}\log A]$, we get the proof using the Taylor expansion of $\log(1-z)$, where $\operatorname{tr}A$ means the trace of a matrix A (cf. [4]).

LEMMA 4. *For* $|z|<\beta(\alpha)$,

$$\lim_{m\to\infty} \zeta_\alpha(z : F_m^{\pm}) = \zeta_\alpha(z : F) .$$

The proof easily follows from Lemma 2 and the similar discussion in Corollary 1.

LEMMA 5.

$$\lim_{m\to\infty} \alpha_m^+ = \lim_{m\to\infty} \alpha_m^- = \alpha_0 .$$

PROOF. For any $\varepsilon>0$, take m sufficiently large such that

$$\frac{|(F_m^+)'(x)|}{|(F_m^-)'(x)|} \geq \frac{1-\varepsilon}{1+\varepsilon} .$$

From Lemma 3, for any $\alpha>\alpha_m^-$, $\zeta_\alpha(z : F_m^-)$ has no singularity in the unit disk. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y\in\mathscr{C}:y=F^n(y)} |(F_m^-)^{n\prime}(y)|^{-\alpha} < \infty .$$

On the other hand, for any $\alpha'>\alpha$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y\in\mathscr{C}:y=F^n(y)} |(F_m^-)^{n\prime}(y)|^{-\alpha} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n\alpha} \sum_{y\in\mathscr{C}:y=F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n\alpha} \sum_{y\in\mathscr{C}:y=F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha'} \prod_{i=0}^{n-1} |(F_m^+)'(F^i(y))|^{\alpha'-\alpha} . \qquad (3)
\end{aligned}$$

Now we choose any $\alpha'>\alpha$ such that

$$\inf_x |F'(x)|^{\alpha'-\alpha} \left|\frac{1-\varepsilon}{1+\varepsilon}\right|^{\alpha} > 1 .$$

Namely,

$$\alpha' > \alpha \frac{1+\log[(1+\varepsilon)/(1-\varepsilon)]}{\log \inf_x |F'(x)|} . \qquad (4)$$

Note here the right hand term of (4) tends to α as ε tends to 0. Thus

$$\text{the right hand side of (3)} \geq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{y\in\mathscr{C}:y=F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha'} .$$

This shows that $\zeta_{\alpha'}(z : F_m^+)$ is analytic in $|z|\leq 1$. Thus $\det(I-\Phi_{\alpha'}(z : F_m^+))$ has no zero in $|z|\leq 1$, that is, $\alpha'>\alpha_m^+$. This proves the lemma. Indeed, for example, if there exists $\delta>0$

such that $\alpha_0 > \lim_{m\to\infty} \alpha_m^- + \delta$, then we can take $\varepsilon > 0$ sufficiently small such that there exists $\alpha' < \alpha_0$ which satisfies (4). This is the contradiction, because $\alpha_m^+ < \alpha' < \alpha_0 \le \alpha_m^+$. Therefore $\alpha_0 = \lim_{m\to\infty} \alpha_m^-$. In a similar way, we can show $\alpha_0 = \lim_{m\to\infty} \alpha_m^+$.

3. Construction of a transformation G.

First note that

LEMMA 6. *Let $e_{m,k}^\tau = (e_{m,k}^\tau(w))_{w\in\mathscr{W}_k}$ be an eigenvector of $\Phi_{\alpha,k}(1 : F_m^\tau)$ associated with maximal eigenvalue. Set $e_m^\tau(v) = \sum_{\langle u\rangle \subset \langle v\rangle} e_{m,k}^\tau(u)$ for $v \in \mathscr{W}_m$. Then $e_m^\tau = (e_m^\tau(v))_{v\in\mathscr{W}_m}$ is the eigenvector of $\Phi_\alpha(1 : F_m^\tau)$ with the eigenvalue λ.*

PROOF. Let $\lambda > 0$ be the maximal eigenvalue, and $(e_{m,k}^\tau(u))$ be an eigenvector associated with it. Since $\Phi_{\alpha_{m,k}^\tau}(1 : F_m^\tau)$ is a nonnegative irreducible matrix and 1 is its maximal eigenvalue, we can take $e_{m,k}^\tau(w) > 0$ for any $w \in \mathscr{W}_k$. For $v \in \mathscr{W}_m$, the v-component of $\Phi_\alpha(z : F_m^\tau)(e_m^\tau(v))$ $(v \in \mathscr{W}_m)$ equals

$$
\begin{aligned}
&= \sum_{w\in\mathscr{W}_m} \Phi_\alpha(1 : F_m^\tau)_{v,w} e_m^\tau(w) \\
&= \sum_{w\in\mathscr{W}_m : \overline{F(\langle v\rangle)} \supset \langle w\rangle} |F_m^{\tau\prime}(v)|^{-\alpha} e_m^\tau(w) \\
&= \sum_{w\in\mathscr{W}_k : \overline{F(\langle v\rangle)} \supset \langle w\rangle} |F_m^{\tau\prime}(v)|^{-\alpha} e_{m,k}^\tau(w) \\
&= \sum_{v'\in\mathscr{W}_k : \langle v'\rangle \subset \langle v\rangle} \Phi_{\alpha,k}(1 : F_m^\tau)_{v',w} e_{m,k}^\tau(w) \\
&= \lambda \sum_{v'\in\mathscr{W}_k : \langle v'\rangle \subset \langle v\rangle} e_{m,k}^\tau(v') = \lambda e_m^\tau(v) .
\end{aligned}
$$

Let $e_m^\tau = (e_m^\tau(w))_{w\in\mathscr{W}_m}$ be an eigenvector associated with the eigenvalue 1 of $\Phi_{\alpha_m^\tau}(1 : F_m^\tau)$ such that $\sum_{w\in\mathscr{W}_m} e_m^\tau(w) = 1$. We can construct a piecewise linear Markov mapping $G_m^\tau : [0, 1] \to [0, 1]$ as follows:

Define a natural order on $\mathscr{A}_1$, that is, $a < b$ if and only if $x \in \langle a\rangle$ and $y \in \langle b\rangle$ satisfy $x < y$. We also introduce a natural order on $\mathscr{W}_m$. For $u = a_1 \cdots a_m$ and $v = b_1 \cdots b_m$ with $a_1 \cdots a_i = b_1 \cdots b_i$, and $a_{i+1} < b_{i+1}$. Then $u < v$ if $F^{i\prime}(y) > 0$ for $y \in \langle u\rangle$, and $u > v$ otherwise. Arrange all the words in $\mathscr{W}_m$ in this order

$$w_{m,1} < w_{m,2} < \cdots < w_{m,\#\mathscr{W}_m} .$$

Take

$$c_{m,i}^\tau = \begin{cases} 0 & \text{for } \ i = 0 , \\ \sum_{j=1}^i e_m^\tau(w_{m,j}) & \text{for } \ 0 < i \le \#\mathscr{W}_m . \end{cases}$$

Set

$$\langle w_{m,i}\rangle=\begin{cases}[c^{\tau}_{m,i-1}, c^{\tau}_{m,i}) & \text{for } 1\le i\le \#\mathscr{W}_m-1\,,\\ [c^{\tau}_{m,\#\mathscr{W}_m-1}, 1] & \text{for } i=\#\mathscr{W}_m\,,\end{cases}$$

$$d^{\tau}_{m,i}=\inf\left\{\bigcup_{b\in\mathscr{A}_1:a_2\cdots a_mb\in\mathscr{W}_m}\langle a_2\cdots a_mb\rangle\right\},$$

$$d^{\tau}_{m,i}{}'=\sup\left\{\bigcup_{b\in\mathscr{A}_1:a_2\cdots a_mb\in\mathscr{W}_m}\langle a_2\cdots a_mb\rangle\right\},$$

where $w_{m,i}=a_1a_2\cdots a_m$. Then for $x\in\langle w_{m,i}\rangle$, define

$$G^{\tau}_m(x)=\frac{d^{\tau}_{m,i}{}'-d^{\tau}_{m,i}}{c^{\tau}_{m,i}-c^{\tau}_{m,i-1}}(x-c^{\tau}_{m,i-1})+d^{\tau}_{m,i}\,.$$

Note that $|G^{\tau}_m{}'(x)|=(|F^{\tau}_m{}'(y)|)^{\alpha^{\tau}_m}$ and $G^{\tau}_m{}'>0$ if and only if $F^{\tau}_m{}'(y)>0$, where the expansion of $y\in I$ by F equals that of $x\in[0, 1]$ by G^{τ}_m. This shows that $\Phi_{\alpha^{\tau}_m}(1:F^{\tau}_m)$ is the Fredholm matrix of G^{τ}_m (cf. [4]). Also note that G^{τ}_m is expanding and $([0, 1], G^{\tau}_m)$ has the same symbolic dynamics with $(\mathscr{C}, F)$.

Set for a word $w\in\mathscr{W}$

$$s^{w}_g(z:G^{\tau}_m)=\sum_{n=0}^{\infty}z^n\int_0^1 1_{\langle w\rangle}(x)g((G^{\tau}_m)^n(x))dx\,,$$

where $\langle w\rangle\subset[0, 1]$ is the subinterval associated with a word w which is induced by G^{τ}_m. Then we can get a renewal equation of the form

$$(s^{w}_g(z:G^{\tau}_m))_{w\in\mathscr{W}_m}=\left(\int_{\langle w\rangle}g(x)dx\right)_{w\in\mathscr{W}_m}+\Phi_{\alpha^{\tau}_m}(z:F^{\tau}_m)(s^{w}_g(z:G^{\tau}_m))_{w\in\mathscr{W}_m}\,.$$

Now we will construct a transformation G: $[0, 1]\to[0, 1]$ which corresponds to F. For $m>k$, set $c^{\tau}_{m,k,0}=0$ and for $0<l\le\#W_k$

$$c^{\tau}_{m,k,l}=\sum_{i=1}^{l}\sum_{u\in\mathscr{W}_m:\langle u\rangle\subset\langle w_{k,i}\rangle}e^{\tau}_m(u)\,,$$

that is, $[c^{\tau}_{m,k,l-1}, c^{\tau}_{m,k,l})$ corresponds to a word $w_{k,l}$ with respect to the mapping G^{τ}_m. Take a subsequence which we also express by $\{m\}$ such that the sequence $c^{\tau}_{m,1,1}$ converges to a point which we express by $\bar{c}^{\tau}_{1,1}$. Next we choose again a subsequence $\{m\}$ of the above subsequence to converge $c^{\tau}_{m,1,2}$ to some point $\bar{c}^{\tau}_{1,2}$, and so on, and we can define $\bar{c}^{\tau}_{k,l}$ for all k and $0\le l\le\#W_k$. Then $[\bar{c}^{\tau}_{k,l-1}, \bar{c}^{\tau}_{k,l})$ is a new subinterval corresponding to a word $w_{k,l}\in\mathscr{W}_K$. Using these subintervals, we can also define $\bar{d}^{\tau}_{k,l}$ and $\bar{d}^{\tau}_{k,l}{}'$. Then we can define mappings $\bar{G}^{\tau}_k$ as before. From the construction, $\bar{G}^{\tau}_k$ maps a subinterval corresponding to a word $w=a_1\cdots a_k$ to the union of the subintervals corresponding to $a_2\cdots a_ka$ $(a\in\mathscr{A}_1)$. Now we will fix a word $w_{k,l}=a_1\cdots a_k\in\mathscr{W}_k$, and we assume that its image

$\langle a_2 \cdots a_k \rangle$ corresponds to a set of words $w_{k,n}$ $(l_1 \le n \le l_2)$. Then we can choose a subsequence $\{m\}$ such that $c^{\tau}_{m,k,n}$ converges to $\bar{c}^{\tau}_{k,n}$ for $n = l-1, l, l_1-1, l_2$. Then

$$\bar{G}^{\tau\prime}_k(w) = \frac{\bar{c}^{\tau}_{k,l_2} - \bar{c}^{\tau}_{k,l-1}}{\bar{c}^{\tau}_{k,l} - \bar{c}^{\tau}_{k,l-1}} = \lim_{m\to\infty} \frac{c^{\tau}_{m,k,l_2} - c^{\tau}_{m,k,l_1-1}}{c^{\tau}_{m,k,l} - c^{\tau}_{m,k,l-1}},$$

$$\inf_{x\in\langle w\rangle} |F'(x)|^{\alpha^{\tau}_m} \le \left| \frac{c^{\tau}_{m,k,l_2} - c^{\tau}_{m,k,l_1-1}}{c^{\tau}_{m,k,l} - c^{\tau}_{m,k,l-1}} \right| \le \sup_{x\in\langle w\rangle} |F'(x)|^{\alpha^{\tau}_m}.$$

Therefore, since $\lim_{m\to\infty} \alpha^{\tau}_m = \alpha_0$, this implies

$$\inf_{x\in\langle w\rangle} |F'(x)|^{\alpha_0} \le |\bar{G}^{\tau\prime}_k(w)| \le \sup_{x\in\langle w\rangle} |F'(x)|^{\alpha_0}.$$

For each word w, the endpoints of the subintervals corresponding to w converge. Therefore there exists a limit $G^{\tau\prime}(x) = \lim_{k\to\infty} \bar{G}^{\tau\prime}_k(x)$, if x does not coincide with endpoints of any word. Namely, there exists $G'(x)$ except countably many points (that is, there exists G' in L^1 sense). It is also easy to see even for an endpoint of a word whose expansion equals $s = a_1 a_2 \cdots$, there exists a limit $\lim_{m\to\infty} G^{\tau\prime}_m(s)$. However, it may not coincide with the limit corresponding to another sequence of symbols which express the same endpoint. Hereafter, we fix G^+ or G^- and denote it by G. From the construction, G and F on $\mathscr{C}$ has the same symbolic dynamics. We denote the 1 to 1, onto mapping by ϕ: $[0, 1] \to \mathscr{C}$ for which $x \in [0, 1]$ and $\phi(x) \in \mathscr{C}$ has the same expansion. Then

$$G'(x) = \begin{cases} +|F'(\phi(x))|^{\alpha_0} & \text{if } F'(\phi(x)) > 0, \\ -|F'(\phi(x))|^{\alpha_0} & \text{if } F'(\phi(x)) < 0. \end{cases}$$

To emphasize the notation, we denote by $\langle w \rangle_F$ and $\langle w \rangle_G$ subintervals associated with a word w which correspond to F on I and G on $[0, 1]$, respectively.

LEMMA 7. *Assume that $\alpha_0 > 0$. Then there exists a constant $\gamma' > 0$ such that $1/G'(x)$ is of universally bound γ'-variation, where a function f is of universally bounded p-variation if*

$$\mathrm{var}_p(f) = \sup_{0 \le x_0 < \cdots < x_n \le 1} \left(\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p \right)^{1/p} < \infty.$$

PROOF. For $x < y$ which do not coincide with endpoints of any word,

$$\begin{aligned} |1/G'(x) - 1/G'(y)| &= ||F'(\phi(x))|^{-\alpha_0} - |F'(\phi(y))|^{-\alpha_0}| \\ &\le ||F'(\phi(x))|^{-1} - |F'(\phi(y))|^{-1}|^{\alpha_0} \\ &\le ||F'(\phi(x))| - |F'(\phi(y))||^{\alpha_0} e^{-2\alpha_0\xi_0}. \end{aligned}$$

Since F' is Hölder continuous with Hölder exponent γ, it is also of universally bounded $1/\gamma$ variation (cf. [3]), therefore $1/G'$ is of universally $\gamma' = \gamma/\alpha_0$ bounded variation. This proves the lemma.

4. Hausdorff dimension.

First we will mention easy lemmas.

LEMMA 8. *For* $0<\alpha<1$, *and* $x_i>0$ $(1\leq i\leq k)$, *we get*

$$k^{\alpha-1}\left(\sum_{i=1}^{k} x_i^\alpha\right)\leq\left(\sum_{i=1}^{k} x_i\right)^\alpha\leq\sum_{i=1}^{k} x_i^\alpha .$$

LEMMA 9. *There exists a constant* $K>1$ *such that*

$$\frac{1}{K}<\frac{Lebes(\langle wa\rangle)}{Lebes(\langle wb\rangle)}<K,$$

for any word w *and any symbols* $a, b\in\mathscr{A}$ *such that* wa, wb *are admissible.*

The proofs of both lemmas are trivial.

For a probability measure μ on I, we define another Hausdorff dimension $\dim_\mu$ as follows. Let for $\delta>0$

$$\mu_\alpha(C,\delta)=\inf\sum_i \mu(\langle v_i\rangle)^\alpha ,$$

where infimum is taken over all covering by words $\{v_i\}$ such that $\mu(\langle v_i\rangle)<\delta$. Then as usual we can define the Hausdorff dimension $\dim_\mu(\mathscr{C})$ of $\mathscr{C}$ with respect to the probability measure μ as a critical point whether $\lim_{\delta\to 0}\mu_\alpha(\mathscr{C},\delta)$ converges or diverges.

We will use the following theorem.

THEOREM 1 (Billingsley [1]). *For probability measures* μ_1, μ_2 *such that*

$$\mathscr{C}\subset\left\{x:\lim_{n\to\infty}\frac{\log\mu_1(\langle a^x[1,n]\rangle)}{\log\mu_2(\langle a^x[1,n]\rangle)}=\alpha\right\}$$

for some $0\leq\alpha\leq\infty$,

$$\dim_{\mu_2}(\mathscr{C})=\alpha\dim_{\mu_1}(\mathscr{C}),$$

where $a^x[1,n]$ *is a word with length* n *such that* $\langle a^x[1,n]\rangle\ni x$.

We will fix $\tau\in\{+,-\}$ and express F_m^τ and G_m^τ simply by F_m and G_m.

LEMMA 10. *Let* $\varepsilon>0$ *be any constant. Then for sufficiently large* m *and any word* $w\in\mathscr{W}$ $(|w|>m)$

$$Lebes(\langle w\rangle_F)\begin{cases}\leq(1+\varepsilon)^{|w|}K^{2m}|F_m^{|w|\prime}(w)|^{-1},\\ \geq(1-\varepsilon)^{|w|}K^{-2m}|F_m^{|w|\prime}(w)|^{-1},\end{cases}$$

$$Lebes(\langle w\rangle_G)\begin{cases}\leq(1+\varepsilon)^{|w|}K^{2m}|G_m^{|w|\prime}(w)|^{-1},\\ \geq(1-\varepsilon)^{|w|}K^{-2m}|G_m^{|w|\prime}(w)|^{-1},\end{cases}$$

where

$$K=\max\{\sup_{x\in I}|F'(x)|, \sup_{x\in I}|F'(x)|^{-1}\}.$$

PROOF. The proofs of above four inequalities are almost the same, so we will only show the first one. Note first

$$Lebes(\langle w\rangle_F)=\int_{F^{|w|}(\langle w\rangle)}|F^{|w|\prime}(wx)|^{-1}dx.$$

Therefore for any m and for any word w with $|w|>m$

$$\begin{aligned}Lebes(\langle w\rangle_F)&=\int_{F^{|w|}(\langle w\rangle)}\prod_{n=1}^{|w|}|F'(a_n\cdots a_{|w|}x)|^{-1}dx\\&\le K^m\int\prod_{n=1}^{|w|-m}|F'(a_n\cdots a_{|w|}x)|^{-1}dx,\end{aligned}$$

where $w=a_1\cdots a_{|w|}$. Thus by Lemma 2, we take sufficiently large m, and get

$$Lebes(\langle w\rangle_F)\le(1+\varepsilon)^{|w|-m}K^{2m}|F_m^{|w|\prime}(w)|^{-1}.$$

This proves the lemma.

Now take μ_1 as Lebesgue measure on I, and μ_2 the measure induced by ϕ from the Lebesgue measure on [0, 1] where G acts. Then for $x\in\mathscr{C}$,

$$\frac{\log\mu_1(\langle a^x[1,n]\rangle)}{\log\mu_2(\langle a^x[1,n]\rangle)}=\frac{\log(Lebes(\langle a^x[1,n]\rangle_F))}{\log(Lebes(\langle a^x[1,n]\rangle_G))}$$

tends to $1/\alpha_0$ as $n\to\infty$. Because for example, from Lemma 10

$$\frac{\log(Lebes(\langle a^x[1,n]\rangle_F))}{\log(Lebes(\langle a^x[1,n]\rangle_G))}\le\frac{n\log(1+\varepsilon)+2m\log K-\log|F_m^{n\prime}(x)|}{n\log(1-\varepsilon)-2m\log K-\alpha_m^{\tau}\log|F_m^{n\prime}(x)|},$$

where we take $G_m=G_m^{\tau}$. Taking $n\to\infty$, and as we can take ε arbitrarily small, we get the left hand term is less than or equal to $1/\alpha_0$. We can get the opposite inequality in a same way. Therefore, by Billingsley's theorem, we get

$$\dim_{\mu_1}(\mathscr{C})=\alpha_0\dim_{\mu_2}(\mathscr{C})=\alpha_0\dim_{Lebes}([0,1])=\alpha_0.$$

THEOREM 2. *The Hausdorff dimension of* $\mathscr{C}$ *equals* α_0.

PROOF. Since $\dim_{\mu_1}(\mathscr{C})$ is greater than or equal to the Hausdorff dimension of $\mathscr{C}$, we only need to show the opposite inequality. Let $\{J_i\}$ be a covering by intervals such that $\sum(Lebes(J_i))^{\alpha}<M<\infty$. For each J_i, let

$$n_i=\min\{n:|w|=n, \langle w\rangle\subset J_i\}.$$

If J_i intersects with $\langle u_1\rangle, \cdots, \langle u_k\rangle$ with $|u_j|=n_i-1$, we divide J_i into k intervals $J_i \cap u_j$ $(1 \le j \le k)$. Note that $k \le \#\mathscr{A}$. We denote new covering by intervals also by $\{J_i\}$. Therefore we can assume that J_i is contained in some $\langle u\rangle$ with $|u|=n_i-1$, and that contains at least one $\langle v\rangle$ with $|v|=n_i$. Then by Lemma 8 and the assumption,

$$\sum_i (Lebes(J_i))^\alpha \le \#\mathscr{A}^{1-\alpha} M < \#\mathscr{A} \cdot M .$$

We take all the words $w_{i,1}, \cdots, w_{i,i_1}$ contained in J_i with length n_i. Then take all the words $w_{i,i_1+1}, \cdots, w_{i,i_2}$ contained in $J_i \setminus \bigcup_{j=1}^{i_1} \langle w_{i,j}\rangle$ with length n_i+1, and continue this procedure. Then we get a sequence of words $\{w_{i,j}\}$. Note that, from Lemma 9, the length of any word w with length n_i which intersect with J_i (not only words $w_{i,1}, \cdots, w_{i,i_1}$) is less than or equal to $KLebes(\langle w_{i1}\rangle) \le KLebes(J_i)$. Also noticing $Lebes(\langle wa\rangle) \le e^{-\xi_0} Lebes(\langle w\rangle)$ for any word w and $a \in \mathscr{A}_1$, we get

$$\begin{aligned}\sum_i \sum_j (Lebes(\langle w_{i,j}\rangle))^\alpha &\le 2\#\mathscr{A} K^\alpha \sum_i (Lebes(J_i))^\alpha/(1-e^{-\xi_0\alpha}) \\ &\le 2(\#\mathscr{A})^2 K^\alpha M/(1-e^{-\xi_0\alpha}) .\end{aligned}$$

Now take any α which is greater than the Hausdorff dimension of $\mathscr{C}$. Then for any $\varepsilon>0$ there exists a covering by intervals $\{J_i\}$ such that $\sum_i (Lebes(J_i))^\alpha < \varepsilon$. Then we can choose a covering by words $\{\langle w_{ij}\rangle\}$ such that

$$\sum_i \sum_j (Lebes(\langle w_{ij}\rangle))^\alpha < 2K^\alpha (\#\mathscr{A})^2 \varepsilon/(1-e^{-\xi_0\alpha}) .$$

This proves α_0 smaller than or equal to the Hausdorff dimension of $\mathscr{C}$. This proves the theorem.

5. Invariant measures.

We have proved in Lemma 7 that $1/G'$ is of universally bounded γ'-variation, that is, G satisfies the assumptions of Theorem 3.5 in [3]. Hence, there exists an invariant probability measure μ_G which is absolutely continuous with respect to the Lebesgue measure, and the dynamical system $([0, 1], \mu_G, G)$ is weakly mixing. We will denote by μ_F the induced measure of μ_G to $\mathscr{C}$ by $\phi: [0, 1] \to \mathscr{C}$ such that $x \in [0, 1]$ and $\phi(x) \in \mathscr{C}$ has same expansion by G and F, respectively.

LEMMA 11. *The measure μ_2, the induced measure on $\mathscr{C}$ from the Lebesgue measure on $[0, 1]$ by ϕ, is α_0-conformal measure. Here, we call a measure μ α_0-conformal if*

$$\mu(F(A)) = \int_A |F'(x)|^{\alpha_0} d\mu$$

holds for any $\mathscr{A} \subset \langle a\rangle$ $(a \in \mathscr{A}_1)$.

PROOF. Since the Lebesgue measure on [0, 1] is 1-conformal measure with respect to G and $|G'(x)|=|F'(y)|^{\alpha_0}$ for $y=\phi(x)$, it is easy to prove the lemma.

Thus combining the results, we get:

THEOREM 3. *The measure μ_F is an invariant probability measure absolutely continuous with respect to the α_0-conformal measure μ_2, and the dynamical system $(\mathscr{C}, \mu_F, F)$ is weakly mixing.*

Now we will study the relations between the conformal measure and the Hausdorff measure.

DEFINITION. (1) The Cantor set $\mathscr{C}$ has Darboux property if $F(\langle a\rangle\cap\mathscr{C})=F(\langle a\rangle)\cap\mathscr{C}$.

(2) A transformation F satisfies the Misiurewicz condition if the set

$$\bigcup_{i=1}^{N}\left\{\lim_{x\uparrow c_i} F^j(x) : j\geq 1\right\} \cup \bigcup_{i=0}^{N-1}\left\{\lim_{x\downarrow c_i} F^j(x) : j\geq 1\right\}$$

has empty intersection with

$$\bigcup_{i=0}^{N-1}(c_i, c_i+\varepsilon) \cup \bigcup_{i=1}^{N}(c_i-\varepsilon, c_i)$$

for some $\varepsilon>0$, where $\{c_i\}_{i=0}^{N}$ is the set of endpoints of $\langle a\rangle$ $(a\in\mathscr{A}_1)$.

It is easy to see that our $\mathscr{C}$ has Darboux property and F satisfies Misiurewicz condition. Then from Theorem 6 of [2], there exists a constant $c\neq 0$ such that $\nu_{\alpha_0}=c\mu_2$. Thus, summarizing the results, we get:

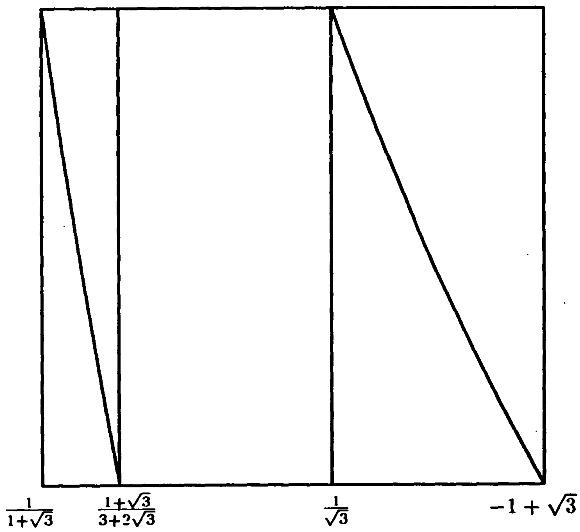
THEOREM 4. *The Hausdorff measure ν_{α_0} is non-zero finite, μ_F is an invariant probability measure absolutely continuous with respect to ν_{α_0}, and the dynamical system $(\mathscr{C}, \mu_F, F)$ is weakly mixing.*

6. Example.

We will calculate the Hausdorff dimension of the set which has only symbols 1 and 2 in continued fraction expansion. We can restrict this map to the interval $[1/(1+\sqrt{3}), -1+\sqrt{3}]$ into itself as in the figure. Namely, all the points which have symbols only 1 and 2 in continued fraction expansion are contained in this interval.

We denote the set $[1/(1+\sqrt{3}), (1+\sqrt{3})/(3+2\sqrt{3})]$, $((1+\sqrt{3})/(3+2\sqrt{3}), 1/\sqrt{3})$ and $[1/\sqrt{3}, -1+\sqrt{3}]$ by symbols $\langle 0\rangle$, $\langle 1\rangle$ and $\langle 2\rangle$, respectively. Denote $\mathscr{A}=\{0, 1, 2\}$, $\mathscr{A}_1=\{0, 2\}$ and $I=[1/(1+\sqrt{3}), -1+\sqrt{3})$,

$$F(x)=\begin{cases} 1/x-2 & \text{if } x\in\langle 0\rangle\,, \\ 1/x-1 & \text{if } x\in\langle 2\rangle\,. \end{cases}$$



Then the Cantor set which we want to calculate can be expressed by

$$\mathscr{C} = \{x \in I : F^n(x) \notin \langle 1 \rangle\} .$$

We can approximate this map by formal piecewise linear transformations. First approximation is formal piecewise linear transformations F_1^+, F_1^- which are linear on each $\langle 0 \rangle$ and $\langle 2 \rangle$, and the second approximation is formal piecewise linear transformations F_2^+, F_2^- which are linear on $\langle 00 \rangle$, $\langle 02 \rangle$, $\langle 20 \rangle$ and $\langle 22 \rangle$ and so on. The Hausdorff dimension α_0 which we calculate by computer satisfies the following:

approximation	minimum	maximum
1	0.4599714039	0.6429535391
2	0.5066200906	0.5573891372
3	0.5239108226	0.5395066173
4	0.52889587337780991 7692	0.53355218785466499004 1
5	0.530600797037892992251	0.532010237643201833751

The program to get these values is very simple, but to get i-th approximation, we need to calculate determinants of 2^i dimensional matrices. Thus, it is not so easy to calculate more precise value. However, we can imagine that α_0 is not so far from the mean value of the fifth approximation $0.531305517\cdots$.

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