TOKYO J. MATH.
VOL. 4, No. 1, 1981

Dynamical System with Continuous States and Relative Free Energy

Makoto MORI and Masasi KOWADA

Tsukuba University and Nagoya Institute of Technology
(Communicated by T. Onoyama)

Introduction

In this article, we are concerned with a dynamical system on the 1-dimensional integer lattice the state space of which is compact (not necessarily discrete). When the state space of the system is not discrete the usual free energy may diverge and hence instead of it we consider the relative free energy which we introduced in [3]. In § 2, we discuss the variational principle in this system with respect to relative free energy and we show that the measure which minimizes relative free energy is obtained as a unique solution of an eigenvalue problem of a transfer operator, if the potential has not so long range (Theorem 2.1).

Moreover we discuss its cluster property and we prove that the equilibrium measure is mixing under the same assumption on the potential as the above (Theorem 3.1), and especially it is weak Bernoulli, if the potential has finite range (Theorem 3.2).

In the former paper [3], we discussed time evolution of a Markov process μ_t of a speed change model on our dynamical system and we proved that the relative free energy of μ_t decreases according to time evolution. Combining these results with Theorem 2.1 in this article, it follows that every initial state converges to the equilibrium state, if the Gibbs measure is unique.

§ 1. Construction of a shift invariant measure.

Let Ω_0 be a compact Hausdorff space with the second countability axiom and let $\mathscr{B}_0$ be its topological Borel field. We suppose that a probability measure ν_0 and a metric d_0 with $d_0(x, y)\leqq 1$, $x, y\in\Omega_0$ are endowed with $(\Omega_0, \mathscr{B}_0)$. Let $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\nu})$ be the 1-sided countable product

Received June 6, 1980
Revised December 10, 1980

of copies of $(\Omega_0, \mathscr{B}_0, \nu_0)$. Let Ω be a shift invariant closed subset of $\tilde{\Omega}$. The shift on Ω is denoted by σ and the n-th coordinate of an element $\omega \in \tilde{\Omega}$ is denoted by ω_n, $n=1, 2, \cdots$.

Moreover we use the following notations;

$$\Omega^n = \{u \in \Omega_0^n;\ \text{there exists an } \omega \in \Omega \text{ such that } u\omega \in \Omega\},$$

where Ω_0^n is the n-fold product of Ω_0, and by $u\omega$, we mean the element of Ω such that

$$(u\omega)_i = \begin{cases} u_i & (1 \leqq i \leqq n) \\ \omega_{i-n} & (i > n), \end{cases}$$

ν^n = the restriction of the n-fold product measure ν_0^n to Ω^n,

$W^n(\omega) = \{u \in \Omega^n;\ u\omega \in \Omega\}$,

$W_r(u, \omega) = \{v \in \Omega^r;\ uv\omega \in \Omega\}$,

$\Omega^{n,r}(\omega) = \{u \in \Omega^n;\ \nu^r\{W_r(u, \omega)\} > 0\}$.

$A \triangle B$ = the symmetric difference of the sets A and B, and d is the metric on Ω defined by

$$d(\omega, \omega') = \sum_{n=1}^{\infty} 2^{-n} d_0(\omega_n, \omega'_n).$$

Furthermore throughout this article, we assume the following three conditions A), B) and C) on Ω.

A). There exists a certain $r>0$ such that for any ω and $n>0$,

$$\nu^n\{\Omega^{n,r}(\omega)\} = \nu^n(\Omega^n).$$

We fix the r hereafter.

B). For any $\varepsilon>0$, there exists $\delta>0$ such that

$$\sup\left\{\frac{\nu^r\{W_r(u, \omega) \triangle W_r(u, \omega')\}}{\nu^r\{W_r(u, \omega')\}};\ u \in \bigcup_n \{\Omega^{n,r}(\omega) \cap \Omega^{n,r}(\omega')\}\right\} < \varepsilon,$$

whenever $d(\omega, \omega') < \delta$.

C). For any $n=1, 2 \cdots$, $\nu^n(\mathscr{O}) > 0$ for any nonempty open subset $\mathscr{O}$ of Ω^n.

The above three assumptions imply the following lemma which we often use hereafter.

LEMMA 1.1.

(1.1) (1) $\nu^n\{W_n(\omega)\} > 0$ *for any* n *and* ω.

(2) *For any* $\varepsilon>0$, *there exists* $\delta>0$ *such that*

$$\nu^n\{W_n(\omega)\Delta W_n(\omega')\}<\varepsilon \text{ for } n\geqq r , \tag{1.2}$$

whenever $d(\omega, \omega')<\delta$.

$$(3) \quad \sup_{\omega,\omega'\in\Omega} \sup \left\{\frac{\nu^r\{W_r(u, \omega)\}}{\nu^r\{W_r(u, \omega')\}};\ u\in\bigcup_n\{\Omega^{n,r}(\omega)\cap\Omega^{n,r}(\omega')\}\right\}<\infty . \tag{1.3}$$

PROOF. (1) Suppose that $\nu^n\{W_n(\omega)\}=0$ for some n and ω. We may assume $n\geqq r$. On the other hand we get by the assumption A)

$$\nu^m\{\Omega^{m,n}(\omega)\}=\nu^m\{u\in\Omega^m;\ \nu^n\{W_n(u, \omega)\}>0\}>0 .$$

This is a contradiction because $W_n(\omega)\supset W_n(u, \omega)$.

(2) This follows easily from the assumption B).

(3) Let $\varepsilon>0$ and let δ be the corresponding positive number in B). Then for any $\omega, \omega'\in\Omega$ we can find $\{\omega^i\in\Omega;\ i=0, 1, \cdots, k\}$ such that

(i) $\omega^0=\omega,\ \omega^k=\omega'$

(ii) $0<d(\omega^i, \omega^{i+1})<\delta$ for $0\leqq i\leqq k-1$,

and it follows from the assumption B) that

$$\frac{\nu^r\{W_r(u, \omega)\}}{\nu^r\{W_r(u, \omega')\}}=\prod_{i=0}^{k-1}\frac{\nu^r\{W_r(u, \omega^i)\}}{\nu^r\{W_r(u, \omega^{i+1})\}}\leqq(1+\varepsilon)^k\leqq(1+\varepsilon)^{1/\delta} .$$

This completes the proof.

Denote

$C(\Omega)=$the set of all continuous functions on Ω,
$C_p(\Omega)=\{f\in C(\Omega);\ f(\omega)$ depends only on the coordinates $\omega_1, \cdots\omega_p\}$

and topologize $C(\Omega)$ and $C_p(\Omega)$ by the norm

$$\|f\|=\sup_\omega|f(\omega)| \text{ for } f\in C(\Omega) . \tag{1.3}$$

Moreover we put

$\mathscr{P}=$the set of all probability measures on Ω

and

$\mathscr{I}=$the set of all shift invariant probability measures on Ω. Let U be a function in $C(\Omega)$ which satisfies the condition D). $\prod_{n=0}^\infty[l]_n<\infty$,

where

$$[l]_n=\sup\{e^{U(\omega')-U(\omega)};\ \omega, \omega'\in\Omega,\ d(\omega, \omega')\leqq 2^{-n}\} .$$

We call U a *potential* and we say U is *of finite range* if $U \in C_p(\Omega)$ for some p.

For given potential U we shall seek for an equilibrium measure μ_U in the sense of the *variational principle*. In the following, appealing to a linear operator L_U on $C(\Omega)$ which is a generalization of transfer matrix, we shall show that μ_U is obtained as a solution of an eigenvalue problem for L_U. An operator $\mathscr{L}_U$ is defined by

$$\mathscr{L}_U f(\omega)=\int_{W_1(\omega)} f(u\omega)e^{U(u\omega)}d\nu_0(u),\ f\in C(\Omega)\ , \tag{1.4}$$

and if no confusion is likely to be, we use the notation $\mathscr{L}$ and l instead of $\mathscr{L}_U$ and $e^{U(\omega)}$ respectively. The dual operator of $\mathscr{L}$ is denoted by $\mathscr{L}^*$; that is $\mathscr{L}^*\mu(f)=\mu(\mathscr{L}f)$ for any finite signed measure μ. An operator L^* on $\mathscr{P}$ is defined by

$$L^*\mu(f)=\mathscr{L}^*\mu(f)/\mathscr{L}^*\mu(1),\ f\in C(\Omega)\ . \tag{1.5}$$

Since $\mathscr{P}$ is weakly compact and convex, Riesz-Schauder's fixed point theorem implies that there exists $\rho=\rho_U\in\mathscr{P}$ such that

$$L^*\rho=\rho\ . \tag{1.6}$$

Define an operator L on $C(\Omega)$ by

$$Lf(\omega)=\alpha^{-1}\mathscr{L}f(\omega)\ \text{for}\ f\in C(\Omega)\ , \tag{1.7}$$

where $\alpha=\alpha_U=\mathscr{L}^*\rho(1)$. Then $\rho(Lf)=L^*\rho(f)=\rho(f)$ holds.

In the following we shall construct a solution $h\in C(\Omega)$ of the equation $Lh=h$, which is unique as can be seen later.

LEMMA 1.2. *Suppose $f\in C(\Omega)$ is nonnegative and $f\not\equiv 0$. Then for sufficiently large n we get $L^n f(\omega)>0$ for any $\omega\in\Omega$.*

PROOF. If there exists some n such that $L^n f>0$, then we get

$$L^{n+1}f(\omega)=\alpha^{-1}\int_{W_1(\omega)} L^n f(u\omega)l(u\omega)d\nu_0(u)>0$$

for any $\omega\in\Omega$. On the other hand, there exist an integer $k>0$, an open set $\mathscr{O}\subset\Omega^k$ and $\delta>0$ such that $f(\omega)>\delta$ for any $\omega=(\omega_1,\omega_2,\cdots)\in\Omega$ with $(\omega_1,\omega_2,\cdots,\omega_k)\in\mathscr{O}$, and hence by A) we get

$$\begin{aligned}\mathscr{L}^{k+r}f(\omega)&=\int_{\Omega^{k,r}(\omega)} d\nu^k(u)\int_{W_r(u,\omega)} d\nu^r(v)f(uv\omega)\prod_{i=0}^{k+r-1} l(\sigma^i uv\omega)\\ &\geqq\delta(\inf l)^{k+r}\int_{\Omega^{k,r}(\omega)\cap\mathscr{O}} d\nu^k(u)\nu^r\{W_r(u,\omega)\}>0\ .\end{aligned}$$

COROLLARY 1.1. *$\rho(\mathcal{O})>0$ for any open set $\mathcal{O}\subset\Omega$.*

PROOF. For any $\mathcal{O}$, we can find a closed set $C\subset\mathcal{O}$ and $f\in C(\Omega)$ such that $0\leqq f(\omega)\leqq 1$ and

$$f(\omega)=\begin{cases}1 & \omega\in C\\ 0 & \omega\in\mathcal{O}^c.\end{cases}$$

It follows for sufficiently large n,

$$\rho(\mathcal{O})\geqq\rho(f)=L^{*n}\rho(f)=\rho(L^n f)>0 .$$

COROLLARY 1.2. *If $f\in C(\Omega)$ satisfies $\rho(|f|)=0$, then $f\equiv 0$.*

PROOF. Suppose $f\not\equiv 0$. Then by Lemma 1.2 for sufficiently large n we have $L^n|f|>0$. On the other hand, we have

$$\rho(L^n|f|)=\rho(|f|)=0 .$$

This is a contradiction.

LEMMA 1.3. *If $f\in C_p(\Omega)$ for some p and $f>0$, we get*

$$\frac{L^n f(\omega)}{L^n f(\omega')}\leqq C \tag{1.8}$$

and

$$\|L^n f\|\leqq C\|f\| \tag{1.9}$$

for any $n\geqq p+r$ and any ω, $\omega'\in\Omega$.

PROOF. We get by Lemma 1.1 3) and the assumption A)

$$\begin{aligned}
\frac{L^n f(\omega)}{L^n f(\omega')}&=\frac{\int_{\Omega^{n-r,r}(\omega)} f(uv\omega)d\nu^{n-r}(u)\int_{W_r(u,\omega)} d\nu^r(v)\prod l(\sigma^i uv\omega)}{\int_{\Omega^{n-r,r}(\omega')} f(uv\omega')d\nu^{n-r}(u)\int_{W_r(u,\omega')} d\nu^r(v)\prod l(\sigma^i uv\omega')} \\
&\leqq\left\{\prod_{i=0}^{n-r}[l]_i([l]_0)^r\right\}\sup\left\{\frac{\nu^r\{W_r(u,\omega)\}}{\nu^r\{W_r(u,\omega')\}};\ u\in\Omega^{n-r,r}(\omega)\cap\Omega^{n-r,r}(\omega')\right\} \\
&\leqq\left\{\prod_{i=0}^{\infty}[l]_i([l]_0)^r\right\}\sup_{\omega,\omega'\in\Omega}\sup\left\{\frac{\nu^r\{W_r(u,\omega)\}}{\nu^r\{W_r(u,\omega')\}};\ u\in\bigcup_n\{\Omega^{n,r}(\omega)\cap\Omega^{n,r}(\omega')\}\right\} \\
&<\infty .
\end{aligned}\tag{1.10}$$

Denoting the right hand term by C, we get (1.8) and

$$\|L^n f\| \leqq C \inf_{\omega} L^n f(\omega) \leqq C\rho(L^n f) = C\rho(f) \leqq C\|f\| .$$

COROLLARY 1.3. $\{\|L^n\|\}_{n\geqq 1}$ *is bounded.*

LEMMA 1.4. $\{L^n f\}_{n\geqq 0}$ *is equi-continuous for any* $f \in C_\rho(\Omega)$.

PROOF. We may assume $f \geqq 0$ and $f \not\equiv 0$ without loss of generality. Let $n > r+p$. Then we get

$$\begin{aligned}
&|L^n f(\omega) - L^n f(\omega')| \leqq C\|f\| \left| \frac{L^n f(\omega) - L^n f(\omega')}{L^n f(\omega)} \right| \\
&\leqq C\|f\| \Big[\int_{\Omega^{n-r,r}(\omega) \cap \Omega^{n-r,r}(\omega')} d\nu^{n-r}(u) \Big| \int_{W_r(u,\omega)} d\nu^r(v) f(uv\omega) \prod l(\sigma^i uv\omega) \\
&\quad - \int_{W_r(u,\omega')} d\nu^r(v) f(uv\omega') \prod l(\sigma^i uv\omega') \Big| \Big] \\
&\quad \div \Big[\int_{\Omega^{n-r,r}(\omega) \cap \Omega^{n-r,r}(\omega')} d\nu^{n-r}(u) \int_{W_r(u,\omega)} d\nu^r(v) f(uv\omega) \prod l(\sigma^i uv\omega) \Big] \\
&\leqq C\|f\| \max_{u \in \Omega^{n-r,r}(\omega) \cap \Omega^{n-r,r}(\omega')} \frac{\int_{W_r(u,\omega) \cap W_r(u,\omega')} d\nu^r(v) |\prod l(\sigma^i uv\omega) - \prod l(\sigma^i uv\omega')|}{\int_{W_r(u,\omega)} d\nu^r(v) \prod l(\sigma^i uv\omega)} \\
&\quad + C\|f\| \max_u \frac{\int_{W_r(u,\omega) \setminus W_r(u,\omega')} d\nu^r(v) \prod l(\sigma^i uv\omega)}{\int_{W_r(u,\omega)} d\nu^r(v) \prod l(\sigma^i uv\omega)} \\
&\quad + C\|f\| \max_u \frac{\int_{W_r(u,\omega') \setminus W_r(u,\omega)} d\nu^r(v) \prod l(\sigma^i uv\omega')}{\int_{W_r(u,\omega)} d\nu^r(v) \prod l(\sigma^i uv\omega)} \\
&\leqq C\|f\| \max_u \max_{v \in W_r(u,\omega) \cap W_r(u,\omega')} \left| \prod \frac{l(\sigma^i uv\omega')}{l(\sigma^i uv\omega)} - 1 \right| \\
&\quad + C\|f\| \left\{ \prod_{i=0}^{\infty} [l]_i ([l]_0)^r \right\} \max_u \frac{\nu^r\{W_r(u,\omega') \Delta W_r(u,\omega')\}}{\nu^r\{W_r(u,\omega)\}} .
\end{aligned}$$

LEMMA 1.5. *For any* $f \in C(\Omega)$ *such that* $\rho(f) = 0$, *we get*

$$\lim_{n\to\infty} L^n f = 0 . \tag{1.11}$$

Especially, if $f \in C_p(\Omega)$, *then for* $n > r+p$

$$\rho(|L^n f|) \leqq (1 - C^{-1}) \rho(|f|) . \tag{1.12}$$

PROOF. We devide our proof into two cases.

(i) Suppose $f\in C_p(\Omega)$ and $\rho(f)=0$.
Then we have

$$\rho(f^+)=\rho(f^-)\ ,$$

and for $n>r+p$ by (1.9) we get

$$L^n f^\pm(\omega)\geqq C^{-1}\max L^n f^\pm\geqq C^{-1}\rho(L^n f^\pm)=C^{-1}\rho(f^\pm)\ .$$

It follows that

$$\begin{aligned}|L^n f|&=|L^n f^+-L^n f^-|=|L^n f^+-C^{-1}\rho(f^+)+C^{-1}\rho(f^-)-L^n f^-|\\&\leqq L^n f^+-C^{-1}\rho(f^+)+L^n f^--C^{-1}\rho(f^-)\\&=L^n|f|-C^{-1}\rho(|f|)\ .\end{aligned}$$

Hence we get

$$\rho(|L^n f|)\leqq(1-C^{-1})\rho(|f|)\ .$$

From this inequality it follows that

$$\lim_{n\to\infty}\rho(|L^n f|)=0\ . \tag{1.13}$$

Suppose that

$$\overline{\lim_{n\to\infty}}\ \|L^n f\|>0\ .$$

Then there exists a subsequence $\{n_k\}$ and $\delta>0$ such that

$$\|L^{n_k}f\|>\delta \text{ for any } k\ .$$

Since $\{L^{n_k}f\}_{k\geqq 1}$ is relatively compact by Corollary 1.3 and Lemma 1.4, there exists a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ and $g\in C(\Omega)$ such that

$$\lim L^{n_{k_j}}f=g \text{ in } C(\Omega)\ .$$

Hence we get

$$\|g\|=\lim\|L^{n_{k_j}}f\|\geqq\delta\ .$$

On the other hand, by (1.13)

$$\rho(|g|)=\lim\rho(|L^{n_{k_j}}f|)=0\ .$$

This contradicts to that $\|g\|\geqq\delta$ by virtue of Corollary 1.2.

(ii) Suppose $f\in C(\Omega)$ and $\rho(f)=0$, and let $f_p\in C_p(\Omega)$ $(p=1, 2, \cdots)$ be a sequence such that

$$\lim_{p\to\infty}\|f-f_p\|=0 .$$

Then we get

$$\begin{aligned}\|L^n f\| &\leqq \|L^n f-L^n f_p\|+\|L^n f_p\| \\ &\leqq C\|f-f_p\|+\|L^n f_p\|\end{aligned}$$

and hence $\lim_{n\to\infty}\|L^n f\|=0$. This completes the proof.

From the above two lemmas, it follows that $\{L^n 1\}_{n\geq 1}$ is relatively compact and so there exists a subsequence $\{L^{n_k}1\}$ and $h=h_U\in C(\Omega)$ such that

$$h=\lim_{k\to\infty}L^{n_k}1 \text{ in } C(\Omega) . \tag{1.14}$$

Note that $h\in C_{p-1}(\Omega)$ if $U\in C_p(\Omega)$.

LEMMA 1.6. *For the above h, it holds that*

$$(1)\quad \rho(h)=1 , \tag{1.15}$$

$$(2)\quad Lh=h . \tag{1.16}$$

PROOF. (1) $\rho(h)=\lim_{k\to\infty}\rho(L^{n_k}1)=\rho(1)=1$.

$$\begin{aligned}(2)\quad \|Lh-h\| &\leqq \|Lh-L^{n_k+1}1\|+\|L^{n_k+1}1-L^{n_k}1\|+\|L^{n_k}1-h\| \\ &\leqq (C+1)\|L^{n_k}1-h\|+\|L^{n_k}(L1-1)\| .\end{aligned} \tag{1.17}$$

Since $\rho(L1-1)=0$, by Lemma 1.5 the right term in (1.17) converges to 0 as $k\to\infty$. Thus we get $Lh=h$.

LEMMA 1.7. *For any $f\in C(\Omega)$ and $\mu\in\mathscr{P}$, it holds that*

$$(1)\quad \lim_{n\to\infty}\|L^n f-\rho(f)h\|=0 , \tag{1.18}$$

$$(2)\quad \lim_{n\to\infty}\left|\left(\frac{1}{\alpha}\mathscr{L}^*\right)^n\mu(f)-\mu(h)\rho(f)\right|=0 . \tag{1.19}$$

PROOF. Since $\rho(f-\rho(f)h)=0$, by Lemma 1.5 we have

$$(1)\quad \lim_{n\to\infty}\|L^n f-\rho(f)h\|=\lim_{n\to\infty}\|L^n f-\rho(f)L^n h\|=\lim_{n\to\infty}\|L^n(f-\rho(f)h)\|=0 ,$$

$$\begin{aligned}(2)\quad \lim_{n\to\infty}\left|\left(\frac{1}{\alpha}\mathscr{L}^*\right)^n\mu(f)-\mu(h)\rho(f)\right| &= \lim_{n\to\infty}|\mu(L^n f)-\mu(h)\rho(f)| \\ &= \lim_{n\to\infty}\left|\mu(L^n f-\rho(f)h)\right|=0 .\end{aligned}$$

THEOREM 1.1. *There exists a unique triplet* $\{\alpha_U, h_U, \rho_U\}$ *which satisfies both equations*

$$(1.20) \qquad (1) \quad \mathscr{L}_U^* \rho_U = \alpha_U \rho_U, \ \rho_U \in \mathscr{P}$$

$$(1.21) \qquad (2) \quad \mathscr{L}_U h_U = \alpha_U h_U, \ \rho_U(h_U) = 1 .$$

PROOF. We have already shown the existence of $\{\alpha_U, h_U, \rho_U\}$. Hence we need only to show the uniqueness.

Suppose that there exist $\{\alpha_U, h_U, \rho_U\}$ and $\{\alpha', h', \rho'\}$ which satisfy (1.20) and (1.21). Then by Lemma 1.7 we get

$$\rho'(h_U)\rho_U = \lim_{n\to\infty} \left(\frac{1}{\alpha_U}\mathscr{L}_U^*\right)^n \rho' = \rho' \lim_{n\to\infty} \left(\frac{\alpha'}{\alpha_U}\right)^n .$$

Thus we get $\alpha_U = \alpha'$, and therefore $\rho_U = \rho'$. Hence we get by Lemma 1.5, together with (1.21)

$$\|h_U - h'\| = \lim_{n\to\infty} \|L^n(h_U - h')\| = 0 .$$

Note that it can be easily shown that

$$(1.22) \qquad \lim_{n\to\infty} \frac{1}{n} \log \mathscr{L}_U^n 1 = \log \alpha_U .$$

We now consider the measure $\mu_U \in \mathscr{P}$ defined by $\mu_U(f) = \rho_U(h_U f)$.

THEOREM 1.2. μ_U *is shift invariant.*

PROOF. Define $f \circ \sigma$ by

$$(1.23) \qquad f \circ \sigma(\omega) = f(\sigma\omega) .$$

Then for $f \in C(\Omega)$, we get

$$(1.24) \qquad \begin{aligned} \mu_U(f \circ \sigma) &= \rho_U(h_U(f \circ \sigma)) \\ &= \rho_U(L_U(h_U(f \circ \sigma))) . \end{aligned}$$

On the other hand, we get

$$(1.25) \qquad \begin{aligned} L_U(h_U(f \circ \sigma))(\omega) &= \alpha_U^{-1} \mathscr{L}_U(h_U(f \circ \sigma))(\omega) \\ &= \alpha_U^{-1} \int_{W_1(\omega)} h_U(u\omega) f \circ \sigma(u\omega) l(u\omega) d\nu(u) \\ &= \alpha_U^{-1} f(\omega) \mathscr{L}_U h_U(\omega) \\ &= f(\omega) h_U(\omega) . \end{aligned}$$

Hence

$$\rho_U(L_U(h_U(f\circ\sigma)))=\rho_U(fh_U)=\mu_U(f)\ .$$

This completes the proof.

LEMMA 1.8. *Any one of* $\{\alpha_U, h_U, \rho_U\}$ *is continuous in* U.

PROOF. Let $\{U_n\}$ be a sequence of potentials which converges to a potential U and choose a subsequence $\{U_{n_k}\}$ for which the following both limits exist

$$\lim_{k\to\infty}\rho_{U_{n_k}}=\rho \tag{1.26}$$

and

$$\lim_{k\to\infty}\alpha_{U_{n_k}}=\alpha\ . \tag{1.27}$$

Here the existence of a subsequence $\{\alpha_{U_{n_k}}\}$ is clear from that $|\alpha|\leqq\mathrm{Max}_\omega\, e^{U(\omega)}$ and from the compactness of $\mathscr{P}$. Then we get

$$\begin{aligned}\left|\int fd\mathscr{L}_U^*\rho-\alpha\int fd\rho\right|&\leqq\left|\int fd\mathscr{L}_U^*\rho-\int fd\mathscr{L}_U^*\rho_{U_{n_k}}\right|\\&+\left|\int fd\mathscr{L}_U^*\rho_{U_{n_k}}-\int fd\mathscr{L}_{U_{n_k}}^*\rho_{U_{n_k}}\right|+\left|\int fd\mathscr{L}_{U_{n_k}}^*\rho_{U_{n_k}}-\alpha_{U_{n_k}}\int fd\rho_{U_{n_k}}\right|\\&+\alpha_{U_{n_k}}\left|\int fd\rho_{U_{n_k}}-\int fd\rho\right|+\left|\alpha_{U_{n_k}}fd\rho-\alpha\int fd\rho\right|\ .\end{aligned}$$

Hence it follows from the continuity of $\mathscr{L}_U^*$ in U that

$$\int fd\mathscr{L}_U^*\rho=\int\alpha fd\rho\ .$$

Therefore by the uniqueness of $\{\alpha_U, h_U, \rho_U\}$ we get

$$\rho=\rho_U\quad\text{and}\quad\alpha=\alpha_U\ .$$

This completes the proof.

§ 2. Variational principle.

We now define the relative free energy for shift invariant measures (for more precise definition, refer to [3]).

Any measurable finite partition ξ_0 of Ω_0 induces a measurable finite partition ξ of Ω each atom of which has the form $\{\omega\in\Omega;\ \omega_1\in I\}$, $I\in\xi_0$ and we say that ξ is induced by ξ_0. The family of all measurable finite partitions of Ω_0 is denoted by Λ and we identify the partition ξ of Ω_0 and the partition of Ω which is induced by ξ.

DEFINITION 2.1. For $\mu \in \mathscr{P}$ and any finite measurable partition ξ of Ω, we put

$$H^{\nu}(\mu, \xi) = \sum_{I \in \xi} \mu(I) \log \frac{\mu(I)}{\nu(I)} \tag{2.1}$$

$$h^{\nu}(\mu, \xi) = \overline{\lim_{n\to\infty}} \frac{1}{n+1} H^{\nu}\left(\mu, \bigvee_{j=0}^{n} \sigma^{-j}\xi\right). \tag{2.2}$$

In (2.1), we take

$$0 \log 0 = 0$$

$$p \log \frac{p}{0} = \begin{cases} 0 & \text{if } p=0 \\ +\infty & \text{if } p>0, \end{cases} \tag{2.3}$$

if they appear.

DEFINITION 2.2. For $\mu \in \mathscr{I}$, let

$$h^{\nu}(\mu) = \sup_{\xi \in \Lambda} h^{\nu}(\mu, \xi). \tag{2.4}$$

We call $h^{\nu}(\mu)$ the *relative entropy of* μ *with respect to* ν.

DEFINITION 2.3. Let

$$f_U^{\nu}(\mu) = \int U(\omega) d\mu(\omega) + h^{\nu}(\mu). \tag{2.5}$$

We call $f_U^{\nu}(\mu)$ the *relative free energy of* μ *with respect to* ν. We call a measure μ_0 an *equilibrium state* if $f_U^{\nu}(\mu_0) = \inf f_U^{\nu}(\mu)$.

It is known that if a measure μ has the density $q^{(n)}$ of the projection of μ to Ω^n with respect to ν^n for $n=1, 2, \cdots$, then $h^{\nu}(\mu)$ is given by

$$h^{\nu}(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log q^{(n)} d\mu \tag{2.6}$$

(refer to [4]). Put

$$\phi(U) = \inf_{\mu \in \mathscr{I}} f_U^{\nu}(\mu) \text{ and } \mathscr{E}(U) = \{\mu \in \mathscr{I};\ f_U^{\nu}(\mu) = \phi(U)\}.$$

LEMMA 2.1. *For any* $\mu \in \mathscr{I}$, *the inequality*

$$f_U^{\nu}(\mu) \leqq -\log \alpha_U \tag{2.7}$$

holds and the equality is attained by $\mu=\mu_U$; *that is,* $\phi(U)=-\log\alpha_U$.

PROOF. For $f\in C(\Omega)$ we obtain

$$\begin{aligned}\mu_U(f)&=\alpha_U^{-n}\rho_U(\mathscr{L}_U^n(fh_U))\\&=\alpha_U^{-n}\int_{\Omega^n}f(u)d\nu^n(u)\int_\Omega h_U(u\omega)\exp\left\{-\sum_{i=0}^{n-1}U(\sigma^i u\omega)\right\}d\rho_U(\omega)\end{aligned}$$

and hence we can see

$$p^{(n)}(u)=\int\alpha_U^{-n}h_U(u\omega)e^{-\sum_{i=0}^{n-1}U(\sigma^i u\omega)}d\rho_U(\omega) \tag{2.8}$$

is the density of the projection of μ_U to Ω^n with respect to ν^n. We get

$$\begin{aligned}&\lim_{n\to\infty}p^{(n)}(\omega_1,\omega_2,\cdots)/p^{(n-1)}(\omega_2,\omega_3,\cdots)\\&\qquad=\frac{e^{-U(\omega)}h_U(\omega)}{\alpha_U h_U(\sigma\omega)}\text{ (uniformly in }\omega)\end{aligned} \tag{2.9}$$

by the condition D). Hence it follows that

$$\begin{aligned}h^\nu(\mu_U)&=\lim_{n\to\infty}\frac{1}{n}\int\log p^{(n)}d\mu_U=\lim_{n\to\infty}\int\log\frac{p^{(n)}(u)}{p^{(n-1)}(\sigma u)}d\mu_U\\&=-\log\alpha_U-\mu_U(U)\ .\end{aligned}$$

This implies

$$f_U^\nu(\mu_U)=-\log\alpha_U\ .$$

Next we shall show $f_U^\nu(\mu)\geqq-\log\alpha_U$ for any $\mu\in\mathscr{I}$. If the projection of μ to Ω^n is not absolutely continuous with respect to ν^n for some n, then $f_U^\nu(\mu)=+\infty$ and hence the assertion is trivial. Let $p^{(n)}$ and $q^{(n)}$ be densities of $\mu_U^{(n)}$ and $\mu^{(n)}$ with respect to ν^n, respectively. By (2.9), we get

$$U(\omega)=-\log\alpha_U-\lim_{n\to\infty}\log\frac{p^{(n)}(\omega)}{p^{(n-1)}(\sigma\omega)}+\log h_U(\omega)-\log h_U(\sigma\omega)\ . \tag{2.10}$$

Combining (2.10) and the Jensen's inequality, we get

$$f_U^\nu(\mu)=-\log\alpha_U+\overline{\lim_{n\to\infty}}\frac{1}{n}\int\log\frac{q^{(n)}(\omega)}{p^{(n)}(\omega)}d\mu\geqq-\log\alpha_U\ .$$

THEOREM 2.1. $\mathscr{E}(U)=\{\mu_U\}$.

PROOF. By Lemma 2.1, it holds clearly that $\mu_U\in\mathscr{E}(U)$. Since

$$\phi(U+tV)=\inf_{\mu\in\mathscr{I}} f^{\nu}_{U+tV}(\mu)=\inf_{\mu\in\mathscr{I}}\{\mu(U)+h^{\nu}_{\mu}(\mu)+t\mu(V)\}$$
$$\leqq f^{\nu}_{U}(\mu)+t\mu(V)$$

for any $\mu\in\mathscr{I}$, we get

$$\overline{\lim_{t\downarrow 0}}\frac{1}{t}\{\phi(U+tV)-\phi(U)\}\leqq \inf_{\mu\in\mathscr{E}(U)}\mu(V)\ . \tag{2.11}$$

Replacing V by $-V$ in (2.11), we get

$$\underline{\lim_{t\uparrow 0}}\frac{1}{t}\{\phi(U+tV)-\phi(U)\}\geqq \sup_{\mu\in\mathscr{E}(U)}\mu(V)\ . \tag{2.12}$$

On the other hand, since $\mathscr{L}_{U+tV}h_U$ is differentiable at $t=0$ and

$$\frac{1}{t}\{\alpha_{U+tV}-\alpha_U\}=\frac{1}{t}\{\rho_{U+tV}(\mathscr{L}_{U+tV}h_U-\mathscr{L}_U h_U)/\rho_{U+tV}(h_U)$$
$$=\rho_{U+tV}\Big(\frac{1}{t}(\mathscr{L}_{U+tV}-\mathscr{L}_U)h_U\Big)\Big/\rho_{U+tV}(h_U)\ ,$$

$\phi(U+tV)=\log\alpha_{U+tV}$ is also differentiable at $t=0$, and hence

$$\sup_{\mu\in\mathscr{E}(U)}\mu(V)=\inf_{\mu\in\mathscr{E}(U)}\mu(V)\ \text{ for any }\ V\in C_p(\Omega),\ p=1,2,\cdots\ .$$

This implies $\mu(V)=\mu_U(V)$ for any $V\in C_p(\Omega)$ $(p=1,2,\cdots)$, and any $\mu\in\mathscr{E}(U)$ and hence $\mu=\mu_U$. Therefore we get $\mathscr{E}(U)=\{\mu_U\}$.

§ 3. The cluster property of the measure μ_U.

In this section we will show the mixing property of μ_U. Moreover if U is of finite range, we will show μ_U is weak Bernoulli. Since each point x in Ω_0 has a base of neighborhood $\{V_n(x)\}$ such that the boundary of $V_n(x)$ is of measure zero with respect to ν_0, we can find a sequence $\{\xi_m\}$ of finite measurable partitions of Ω_0 such that diameters of atoms of ξ_m converge to zero as $m\to\infty$ and moreover each atom A of ξ_m satisfies

$$\nu(A)=\inf_{f\in C(\Omega_0)}\{\nu_0(f);\ 0\leqq\chi_A\leqq f\}\ . \tag{3.1}$$

THEOREM 3.1. *Suppose that U is of finite range. Then μ_U is weak Bernoulli.*

PROOF. It is sufficient to show the weak Bernoulli property of μ_U with respect to the partition $\eta=\{\{\omega\in\Omega:\ \omega_1\in A\};\ A\in\xi\}$ for any ξ whose atoms satisfy (3.1). For any $\varepsilon>0$ and $A\in\bigvee_{i=0}^{p}\sigma^{-i}\eta$, we can take $f_A\in C_{p+1}(\Omega)$ such that

$$\chi_A\leqq f_A \quad \text{and} \quad \mu_U(f_A-\chi_A)\leqq\varepsilon/k^p .$$

It is sufficient to show

$$\lim_{n\to\infty}\sup_{p\geq 1}\sum_{A,B\in\bigvee_{j=0}^{p}\sigma^{-j}\eta}|\mu_U(f_A(\chi_B\circ\sigma^{n+p}))-\mu_U(f_A)\mu_U(\chi_B)|=0 . \tag{3.2}$$

We get by (1.25)

$$\begin{aligned}
&\sum_{A,B}|\mu_U(f_A(\chi_B\circ\sigma^{n+p}))-\mu_U(f_A)\mu_U(\chi_B)|\\
&\quad=\sum_{A,B}|\rho_U(h_Uf_A(\chi_B\circ\sigma^{n+p}))-\mu_U(f_A)\rho_U(h_U\chi_B)|\\
&\quad=\sum_{A,B}|\rho_U(L^{n+p}(h_Uf_A)\chi_B)-\mu_U(f_A)\rho_U(h_U\chi_B)|\\
&\quad=\sum_{A,B}|\rho_U\{(L^{n+p}(h_Uf_A)-\mu_U(f_A)h_U)\chi_B\}|\\
&\quad\leqq\sum_{A,B}\rho_U\{|L^{n+p}(h_Uf_A-\mu_U(f_A)h_U)|\chi_B\}\\
&\quad\leqq\sum_{A}\rho_U\{|L^{n+p}(h_Uf_A-\mu_U(f_A)h_U)|\} .
\end{aligned}$$

From the assumption we get $U\in C_q(\Omega)$ for some $q>0$. Since L^p maps $C_p(\Omega)$ into $C_q(\Omega)$, we get

$$L^p(h_Uf_A-\mu_U(f_A)h_U)\in C_q(\Omega) .$$

Let $n=m(q+r)+q'$, $0\leqq q'\leqq q+r$. Then we get by Lemma 1.5.

$$\rho_U\{|L^{n+p}(h_Uf_A-\mu_U(f_A)h_U)\}\leqq(1-C^{-1})^m\rho_U\{|L^{p+q'}(h_Uf_A-\mu_U(f_A)h_U)|\} .$$

On the other hand,

$$\begin{aligned}
&\sum_{A}\rho_U\{|L^{p+q'}(h_Uf_A-\mu_U(f_A)h_U)|\}\\
&\qquad\leqq\rho_U\{\sum_{A}|(L^{p+q'}(h_Uf_A)+\mu_U(f_A)h_U)|\}\\
&\qquad\leqq 2(1+\varepsilon) .
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\sum_{A,B}|\mu_U(f_A(f_B\circ\sigma^{n+p}))-\mu_U(f_A)\mu_U(f_B)|\\
&\qquad\leqq 4(1-C^{-1})^m .
\end{aligned}$$

This completes the proof.

THEOREM 3.2. *μ_U is mixing for any U satisfying the condition* D).

PROOF. By Lemma 1.7, we have

$$\mu_U((f\circ\sigma^n)g)=\mu_U(f(L^n g))\rightarrow\mu_U(f\circ\mu_U(g)h_U)=\mu_U(f)\mu_U(g)$$

for $f, g\in C(\Omega)$.

References

[1] G. GALLAVOTTI and S. MIRACLE-SOLE, Absence of phase transitions in hard core one dimensional systems with long range interactions, J. Mathematical Phys., **11**, No 1 (1970), 147-154.

[2] H. MORI, Invariant measures of shift transformations, Master Thesis of Tokyo University of Education, 1974 (in Japanese).

[3] M. MORI AND M. KOWADA, Relative free energy and its application to speed change model, Tsukuba J. Math., **5**, No. 1 (1981), 1-14.

[4] PINSKER, Information and Information Stability of Random Variables and Processes, Holden-day, U.S.A., 1964.

[5] D. RUELLE, Statistical mechanics of a one dimensional lattice gas, Comm. Math. Phys., **9** (1968), 267-278.

[6] Y. TAKAHASHI, Shift transformations and free energy, private communication.

Present Address:
INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
SAKURA-MURA, NIIHARI-GUN
IBARAKI 305
AND
DEPARTMENT OF FINE MEASUREMENT
NAGOYA INSTITUTE OF TECHNOLOGY
GOKISO-CHO, SHOWA-KU,
NAGOYA 466