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Dynamical Systems Method and a Homeomorphism Theorem

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1 INTRODUCTION.

The aim of this paper is to demonstrate the power of the dynamical systems method (DSM) as a tool for proving theoretical results. The DSM was introduced and applied to solving nonlinear operator equations in [4]-[8], where the emphasis was on convergence and stability of the DSM-based algorithms for solving operator equations, especially nonlinear and ill-posed equations. The DSM for solving an operator equation $F(u) = 0$ consists of finding a nonlinear map $u \mapsto \Phi(t, u)$, depending on a parameter t in $[0, \infty)$, that has the following three properties:

(1) the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0 \quad (\dot{u} := \frac{du(t)}{dt})$$

has a unique global solution $u(t)$ for a given initial approximation u_0;

(2) the limit $u(\infty) = \lim_{t\to\infty} u(t)$ exists; and

(3) this limit solves the original equation $F(u) = 0$: $F(u(\infty)) = 0$.

The operator $F : H \to H$ is a nonlinear map of a Hilbert space H. It is assumed that the equation $F(u) = 0$ has a solution, possibly nonunique.

The problem is to find a Φ such that the properties (1), (2), and (3) hold. Various choices of Φ for which these properties hold are proposed in [4], where the DSM is justified for wide classes of operator equations, in particular, for some classes of nonlinear ill-posed equations (i.e., equations $F(u) = 0$ for which the linear operator $F'(u)$ is not boundedly invertible). By $F'(u)$ we denote the Fréchet derivative of the nonlinear map F at the element u. For the purposes of this note we assume that F is a map of a Hilbert space, but similar techniques can be applied to a study of operator equations in Banach spaces (see [7]).

In this note the DSM is used as a tool for proving certain theoretical results. Namely, we give a proof of a Hadamard-type theorem on global homeomorphisms and a sufficient condition for the surjectivity of a nonlinear map in a Hilbert space.

Although the global homeomorphism theorem that we prove is not new, its proof is shorter and simpler than the published ones ([1] or [3], for example). This proof yields also a new result, formulated in Remark in Section 2, at the end of the paper. J. Hadamard (see [2]) proved that a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property

$$\|[F'(u)]^{-1}\| \leq b \qquad (\forall u \in \mathbb{R}^n),$$

for a positive constant b is a global homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. This result has been generalized to Hilbert and Banach spaces under the weaker assumption that:

$$\|[F'(u)]^{-1}\| \leq a\|u\| + b, \tag{1.1}$$

where a and b are positive constants (see [1], [3], and the references therein).

We denote by $F^{(j)}$ the Frèchet derivative of F of order j. Our aim is to apply the DSM to obtain a proof of the following:

Theorem 1.1. *Assume that $F : H \to H$ is a twice-differentiable map of a real Hilbert space H such that*

$$\sup_{u \in B(u_0,R)} \|F^{(j)}(u)\| \leq M_j(R) \qquad (1 \leq j \leq 2) \tag{1.2}$$

and

$$\sup_{u \in B(u_0,R)} \|[F'(u)]^{-1}\| \leq m(R), \tag{1.3}$$

where $B(u_0, R) = \{u : ||u - u_0|| \leq R\}$, R is in $(0, \infty)$, and u_0 is a fixed element of H. If

$$\sup_{R>0} \frac{R}{m(R)} = \infty, \tag{1.4}$$

then F is surjective. If (1.1) holds, then F is a global homeomorphism of H onto H.

Remark. Condition (1.4) is essential. For example, if $H = \mathbb{R}^1$ and $F(u) = e^u$, then the equation $e^u = 0$ does not have a solution, conditions (1.2) and (1.3) hold, but (1.4) does not hold: $m(R) = e^R$.

2 PROOFS.

Consider the problem

$$\dot{u} = -[F'(u)]^{-1}[F(u) - f], \quad u(0) = u_0, \tag{2.1}$$

where f is an arbitrary given element of H. Problem (2.1) is an example of the DSM with the following choice of Φ:

$$\Phi := -[F'(u)]^{-1}[F(u) - f].$$

From our arguments it will follow that, assuming (1.3) and (1.4), we can justify the DSM with this choice of Φ, that is, we can prove that the three properties (1), (2), and (3), mentioned in the introduction hold.

From (1.2) and (1.3) it follows that the right-hand side of the first equation in (2.1) satisfies a Lipschitz condition as a function of u. Therefore, by a standard result, problem (2.1) has a unique local solution. Using (1.4), we prove that this solution is global, that is, that $u(t)$ exists on $[0,\infty)$. This is done by establishing a uniform bound

$$\sup_{0\le t<T}\|u(t)\|<c. \tag{2.2}$$

Here and later we denote by c various positive constants. The supremum in (2.2) extends over $[0,T)$ where $[0,T)$ is the maximal interval of existence of the local solution to (2.1). If such an estimate is established, then it follows that $T=\infty$, which means that the solution to (2.1) exists globally. Reason: if the maximal interval of existence of the solution u is finite, say $[0,T)$ with $T<\infty$, then $\lim_{t\to T^-}||u(t)||=\infty$, and the bound (2.2) does not hold. Indeed, assuming that $\lim_{t\to T^-}||u(t)||\le c$, where $[0,T)$ is the maximal interval of existence of the solution u, one has a bound on the Lipschitz constant of the right-hand side of equation (2.1) in the ball $||u||\le c$. Consequently the local solution u exists on an interval of a fixed length, say $\ell>0$. Therefore equation (2.1) with the initial data $u(T-0.5\ell)$ at the point $T-0.5\ell$ exists on the interval $[T-0.5\ell,T+0.5\ell)$, so that u exists on the interval $[0,T+0.5\ell)$. This contradicts the maximality of the interval of existence $[0,T)$ and proves that $\lim_{t\to T^-}||u(t)||=\infty$ if $T<\infty$.

Furthermore, we prove that the limit $u(\infty)=\lim_{t\to\infty}u(t)$ exists, and that $F(u(\infty))=f$, so that the properties (1), (2), and (3) hold. Since f in H is arbitrary, F is surjective.

We now give the arguments in detail. Write

$$g(t)=\|F(u(t))-f\|.$$

Equation (2.1) implies that $g\dot g=-g^2$. Thus,

$$g(t)\le g(0)e^{-t},\quad \|\dot u\|\le m(R)g(0)e^{-t}, \tag{2.3}$$

where $m(R)$ is the constant from (1.3). If the solution $u(t)$ does not leave the ball $B(u_0,R)$ for any positive t, then by an earlier remark $u(t)$ exists on $(0,\infty)$. Integrating the second inequality in (2.3) yields

$$\|u(t)-u(0)\|\le\int_0^t||\dot u(s)||ds\le m(R)g(0).$$

If there is an $R>0$ such that

$$m(R)g(0)\le R, \tag{2.4}$$

then $u(t)$ lies in $B(u_0,R)$ for every positive t. Therefore, equation (2.4) implies that $u(t)$ is the global solution to (2.1). Condition (1.4) guarantees that *for any fixed u_0 there is* *an R in $(0,\infty)$ such that* (2.4) *holds.* For this R one has $u(t)$ in $B(u_0,R)$ for all positive t, from which we infer that $u(\infty)=\lim_{t\to\infty}u(t)$ exists and that the following estimate holds:

$$\|u(t)-u(\infty)\|\leq m(R)g(0)e^{-t}. \tag{2.5}$$

Using the second inequality (2.3), invoking estimate (2.5), and letting $t\to\infty$ in (2.1), one concludes that $F(u(\infty))=f$ because $\lim_{t\to\infty}||F'(u(t))\dot{u}(t)||=0$, as we have proved, the limit $\lim_{t\to\infty}u(t)=u(\infty)$ exists, and $F(u)$ is continuous. This establishes the surjectivity of F.

Remark. Our argument shows that for any u_0 for which (2.4) holds one can assert that the element $u(\infty):=u(\infty,u_0)$ solves the equation $F(u(\infty))=f$. By the continuity of F, one concludes that if $m(R)g(u_0)<R$ for some u_0, then $m(R)g(\tilde{u}_0)<R$ for any $\tilde{u}_0$ sufficiently close to u_0. Therefore, small perturbations of u_0 lead to elements $u(\infty,\tilde{u}_0)$ that also solve the equation $F(u(\infty,\tilde{u}_0))=f$. If $u(\infty,\tilde{u}_0)$ differs by a sufficiently small amount from $u(\infty,u_0)$, then, in fact, the two are equal because F is a local homeomorphism.

If (1.1) holds, then (2.3) is replaced with the inequality

$$\|\dot{u}\|\leq(a\|u(t)\|+b)g(0)e^{-t}. \tag{2.6}$$

Let $h(t)=\|u(t)\|$. Then $\dot{h}\leq||\dot{u}||$, because $h\dot{h}=\Re(\dot{u},u)\leq||\dot{u}||||u||=||\dot{u}||h$. Therefore, (2.6) yields (with $g_0=g(0)$ and $p=b/a$)

$$\dot{h}\leq(h+p)ag_0e^{-t}.$$

As a consequence

$$\sup_{t\geq0}h(t)\leq c_1 \qquad (c_1:=(\|u(0)\|+p)e^{ag_0}-p))$$

and

$$\|\dot{u}\|\leq c_2e^{-t}\quad (c_2:=(ac_1+b)g(0)). \tag{2.7}$$

Accordingly, $u(t)$ belongs to $B(u_0,c_2)$. It is well known and easy to prove that condition (1.3) implies that F is a *local homeomorphism* (i.e., F maps a neighborhood of any point from a sufficiently small neighborhood of an arbitrary point u homeomorphically onto a neighborhood of the point $F(u)$). From (2.1) and the estimate $\|\dot{u}\|\leq c_2e^{-t}$ we conclude as earlier that F is surjective. Therefore, in order to prove that F is a global homeomorphism of H onto H it is sufficient to prove that $F(u)=F(v)$ implies $u=v$.

The idea of our proof is to consider the path $w(s)=(1-s)u_0+sv$ from u_0 to v, to construct the solution $u(t,s)$ to problem (2.1) with the initial data $w(s)$ in place of u_0, and then to show that $u(\infty,s)=u$ for each s. If this is done, then we can conclude that $v=u(\infty,1)=u$. In the last step we use assumption (1.3), which implies that F is a local homeomorphism. Namely, if $F(u(\infty,s))=F(u(\infty,s+\sigma))=f$ and $||u(\infty,s)-u(\infty,s+\sigma)||$ is sufficiently small, then $u(\infty,s)=u(\infty,s+\sigma)$.

We now supply the details of the argument that we have sketched. If $s = 0$, then we have $u(\infty, 0) = u$ and $F(u) = f$. If σ is sufficiently small, then

$$\sup_{t \geq 0} \|u(t, s+\sigma) - u(t, s)\| \leq c\|u(0, s+\sigma) - u(0, s)\|, \tag{2.8}$$

where c does not depend on s, σ, or t, and σ does not depend on s. We return to inequality (2.8) after giving the rest of the proof.

If (2.8) holds, then $\|u(\infty, s+\sigma) - u(\infty, s)\|$ can be made arbitrarily small, provided that $\delta = \|u(0, s+\sigma) - u(0, s)\|$ is chosen sufficiently small. Since

$$F(u(\infty, s+\sigma)) = F(u(\infty, s)) = f$$

and since F *is a local homeomorphism*, it follows that

$$u(\infty, s+\sigma) = u(\infty, s).$$

Because $u(\infty, 0) = u$ and σ does not depend on s, we can get to the point $s + \sigma = 1$ in finitely many steps and conclude that

$$u = u(\infty, s) = u(\infty, 1) = v \quad (0 \leq s \leq 1).$$

Thus, to complete the proof we have only to check (2.8).

Denote

$$x(t) = u(t, s+\sigma) - u(t, s),$$

and let $\eta(t) = \|x(t)\|$. Then using (2.7) and (1.2), and writing $z = u(t, s+\sigma)$ and $y = u(t, s)$, we compute

$$\begin{aligned} \eta\dot{\eta} &= -([F'(z)]^{-1}(F(z) - f) - [F'(y)]^{-1}(F(y) - f), x(t)) \\ &= -(([F'(z)]^{-1} - [F'(y)]^{-1})(F(z) - f), x) - ([F'(y)]^{-1}(F(z) - F(y)), x) \\ &\leq ce^{-t}\eta^2 - \eta^2 + c\eta^3, \end{aligned} \tag{2.9}$$

where we have appealed to inequalities (1.2), (1.3), and (2.4), as well as the following relations:

$$||[F'(z)]^{-1} - [F'(y)]^{-1}|| \leq c||z-y||, \quad F(z) - F(y) = F'(y)(z-y) + K, \quad \|K\| \leq \frac{M_2}{2}\|z-y\|^2.$$

Since $\eta \geq 0$, we infer from (2.9) the inequality

$$\dot{\eta} \leq -\eta + c\eta^2 + ce^{-t}\eta, \quad \eta(0) = \delta. \tag{2.10}$$

Let $\eta = qe^{-t}$. Then

$$\dot{q} \leq ce^{-t}(q^2 + q), \quad q(0) = \delta. \tag{2.11}$$

We integrate (2.11) to obtain $q(t) \leq c_3\delta$, provided that $\delta > 0$ is sufficiently small. For such δ

$$\eta(t) \leq c_3 e^{-t}\delta,$$

which implies (2.8). Theorem 1.1 is thereby proved. □

Remark. If $\|[F'(u)]^{-1}\| \leq \psi(||u||)$, where ψ is a positive continuous function on $[0, \infty)$ such that $\int_0^\infty ds/\psi(s) = \infty$, then the conclusion of Theorem 1.1 still holds, and its proof is essentially the same.

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