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Dynamical systems method for solving operator equations

A.G. Ramm *

Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA

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Abstract

Consider an operator equation $F(u) = 0$ in a real Hilbert space. The problem of solving this equation is ill-posed if the operator $F'(u)$ is not boundedly invertible, and well-posed otherwise. A general method, dynamical systems method for solving linear and non-linear ill-posed problems in a Hilbert space is presented. This method consists of the construction of a non-linear dynamical system, that is, a Cauchy problem, which has the following properties: (1) it has a global solution, (2) this solution tends to a limit as time tends to infinity, (3) the limit solves the original linear or non-linear problem. New convergence and discretization theorems are obtained. Examples of the applications of this approach are given. The method works for a wide range of well-posed problems as well.

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1. Introduction

This paper contains a recent development of the theory of dynamical systems method (DSM) earlier developed in papers [1–11], see also [12,13]. DSM is a general method for solving operator equations, especially non-linear, ill-posed, but also well-posed operator equations. The author hopes that DSM will demonstrate its practical efficiency and allow one to solve ill-posed problems. This paper is intended for a broad audience: the presentation is simplified considerably, and is non-technical in its present form. Most of the results are presented in a new way. Some of the results and/or proofs are new (Theorems 1–3, 5, Theorems 9–11, Remarks 1, 2, and the discussion of the stopping rules). We try to emphasize the basic ideas and methods of the proofs.

* Tel.: +1-334-9116-4286; fax: +1-334-9116-4270.
E-mail address: ramm@math.ksu.edu (A.G. Ramm).

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What is the dynamical systems method for solving operator equations?

Consider an equation

$$F(u) := B(u) - f = 0, \quad f \in H, \tag{1}$$

where B is a linear or non-linear operator in a real Hilbert space H. Some of our results can be generalized to more general spaces, though these generalizations are not discussed here. Throughout the paper we assume that:

$$\sup_{u \in B(u_0,R)} \|F^{(j)}(u)\| \leqslant M_j, \quad j = 1, 2, \tag{2}$$

where $B(u_0, R) := \{u : \|u - u_0\| \leqslant R\}$, $F^{(j)}(u)$ is the Fréchet derivative, and

$$F(y) = 0, \quad y \in B(u_0, R), \tag{3}$$

that is, we assume existence of a solution to (1), not necessarily unique globally.

Assumptions (2) and (3) are our standard assumptions below, unless otherwise stated. Only for well-posed problems in Section 2 we do not assume existence of a solution, but prove it, and sometimes we can assume in these problems $j = 1$ in (2), rather than $j = 2$. In all the ill-posed problems we assume existence of the solution to (1).

Let $\dot{u}$ denote derivative with respect to time. Consider the following dynamical system (the Cauchy problem):

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \tag{4}$$

where $\Phi(t, u)$ is locally Lipschitz with respect to $u \in H$ and continuous with respect to $t \geqslant 0$:

$$\sup_{u,v \in B(u_0,R), t \in [0,T]} \|\Phi(t, u) - \Phi(t, v)\| \leqslant c\|u - v\|, \quad c = c(R, u_0, T) > 0. \tag{5}$$

One can relax "locally Lipschitz" assumption about Φ (for example, use one-sided inequalities) but we do not discuss this point. Problem (4) has a unique local solution if (5) holds. The DSM for solving (1) consists of solving (4), where Φ is so chosen that the following three conditions hold:

$$\exists u(t) \forall t > 0; \quad \exists u(\infty) := \lim_{t \to \infty} u(t); \quad F(u(\infty)) = 0. \tag{6}$$

Some of the basic results of this paper are the Theorems which provide the choices of Φ for which (6) holds, and the technical tools (Theorems 4, 8) basic for our proofs.

Problem (1) with noisy data f_δ, $\|f_\delta - f\| \leqslant \delta$, given in place of f, generates the problem:

$$\dot{u}_\delta = \Phi_\delta(t, u_\delta), \quad u_\delta(0) = u_0, \tag{7}$$

the solution u_δ to (7), calculated at $t = t_\delta$, will have the property

$$\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{8}$$

The choice of t_δ with this property is called the stopping rule. One has usually $\lim_{\delta \to 0} t_\delta = \infty$.

In Section 2 we discuss well-posed problems (1), that is, the problems for which

$$\sup_{u \in B(u_0,R)} \|[F'(u)]^{-1}\| \leqslant m_1, \tag{9}$$

and in the other sections ill-posed problems (1), for which (9) fails, are discussed.

The motivations for this work are:

(1) to develop a general method for solving operator equations, especially non-linear and ill-posed, and
(2) to develop a general approach to constructing convergent iterative schemes for solving these equations.

If (6) holds, and if one constructs a convergent discretization scheme for solving Cauchy problem (4), then one will get a convergent iterative scheme for solving the original Eqs. (1).

2. Well-posed problems

Consider (1), let (2) hold, and assume

$$(F'(u)\Phi(t,u),F(u)) \leqslant -g_1(t)\|F(u)\|^a \quad \forall u \in B(u_0,R), \qquad \int_0^\infty g_1\,\mathrm{d}t = \infty, \tag{10}$$

where $g_1 > 0$ is an integrable function, $a > 0$ is a constant. Assume

$$\|\Phi(t,u)\| \leqslant g_2(t)\|F(u)\|, \quad \forall u \in B(u_0,R), \tag{11}$$

where $g_2 > 0$ is such that

$$G(t) := g_2(t)\exp\left(-\int_0^t g_1\,\mathrm{d}s\right) \in L^1(\mathbb{R}_+). \tag{12}$$

Remark. Sometimes assumption (11) can be used in the following modified form:

$$\|\Phi(t,u)\| \leqslant g_2(t)\|F(u)\|^b, \quad \forall u \in B,$$

where $b > 0$ is a constant. The statement and proof of Theorem 1 can be easily adjusted to this assumption.

Our first basic result is the following:

Theorem 1.
(i) *If* (10)–(12) *hold, and*

$$\|F(u_0)\|\int_0^\infty G(t)\,\mathrm{d}t \leqslant R, \quad a = 2, \tag{13}$$

then (4) *has a global solution*, (6) *holds*, (1) *has a solution* $y = u(\infty) \in B(u_0,R)$, *and*

$$\|u(t)-y\| \leqslant \|F(u_0)\|\int_t^\infty G(x)\,\mathrm{d}x, \quad \|F(u(t))\| \leqslant \|F(u_0)\|\exp\left(-\int_0^t g_1(x)\,\mathrm{d}x\right). \tag{14}$$

(ii) *If* (10)–(12) *hold*, $0 < a < 2$, *and*

$$\|F(u_0)\| \int_0^T g_2 \, \mathrm{d}s \leqslant R,$$

where $T > 0$ *is defined by the equation*

$$\int_0^T g_1(s) \, \mathrm{d}s = \|F(u_0)\|^{2-a}/(2-a),$$

then (4) *has a global solution*, (6) *holds*, (1) *has a solution* $y = u(\infty) \in B(u_0, R)$, *and* $u(t) = y$ *for* $t \geqslant T$.

(iii) *If* (10)–(12) *hold*, $a > 2$, *and*

$$\int_0^\infty g_2(s) h(s) \, \mathrm{d}s \leqslant R,$$

where

$$\left[\|F(u_0)\|^{2-a} + (a-2) \int_0^t g_1(s) \, \mathrm{d}s \right]^{\frac{1}{2-a}} := h(t), \quad \lim_{t\to\infty} h(t) = 0,$$

then (4) *has a global solution*, (6) *holds*, (1) *has a solution* $y = u(\infty) \in B(u_0, R)$, *and*

$$\|u(t) - u(\infty)\| \leqslant \int_t^\infty g_2(s) h(s) \, \mathrm{d}s \to 0$$

as $t \to \infty$.

Let us sketch the proof.

Proof. The assumptions about Φ imply local existence and uniqueness of the solution $u(t)$ to (4). To prove global existence of u, it is sufficient to prove a uniform with respect to t bound on $\|u(t)\|$. Indeed, if the maximal interval of the existence of $u(t)$ is finite, say $[0, T)$, and $\Phi(t, u)$ is locally Lipschitz with respect to u, then $\|u(t)\| \to \infty$ as $t \to T$.

Assume $a = 2$. Let $g(t) := \|F(u(t))\|$. Since H is real, one uses (4) and (10) to get $g\dot{g} = (F'(u)\dot{u}, F) \leqslant -g_1(t) g^2$, so $\dot{g} \leqslant -g_1(t) g$, after integrating one gets the second inequality (14), because $g(0) = \|F(u_0)\|$. Using (11), (4) and the second inequality (14), one gets:

$$\|u(t) - u(s)\| \leqslant g(0) \int_s^t G(x) \, \mathrm{d}x, \quad G(x) := g_2(x) \exp\left(-\int_0^x g_1(z) \, \mathrm{d}z \right). \tag{15}$$

Because $G \in L^1(R_+)$, it follows from (15) that the limit $y := \lim_{t\to\infty} u(t) = u(\infty)$ exists, and $y \in B$ by (13). From the second inequality (14) and the continuity of F one gets $F(y) = 0$, so y solves (1). Taking $t \to \infty$ and setting $s = t$ in (15) yields the first inequality (14). The inclusion $u(t) \in B$ for all $t \geqslant 0$ follows from (13) and (15). The first part of Theorem 1 is proved. The proof of the other parts is similar. □

There are many applications of this theorem. We mention just a few, and assume that $g_1 = c_1 = \text{const} > 0$ and $g_2 = c_2 = \text{const} > 0$ (see [18]).

Example 1 (*Continuous Newton-type method* (*Gavurin*, 1958)). $\Phi = -[F'(u)]^{-1}F(u)$. Assume that (9) holds, then $c_1 = 1, c_2 = m_1$, (13) takes the form $m_1(R)\|F(u_0)\| \leqslant R$. This inequality implies that (4) has a global solution, (6) and (14) hold, and (1) has a solution in $B(u_0, R)$.

Example 2 (*Continuous simple iterations method*). Let $\Phi = -F$, and assume $F'(u) \geqslant c_1(R) > 0$ for all $u \in B(u_0, R)$. Then $c_2 = 1$, $c_1 = c_1(R)$, (13) is: $[c_1(R)]^{-1}\|F(u_0)\| \leqslant R$, and the conclusions of Example 1 hold.

Example 3 (*Continuous gradient method*). Let $\Phi = -[F']^*F$, (2) and (9) hold, $c_1 = m_1^{-2}$, $c_2 = M_1(R)$, (13) is $M_1 m_1^2\|F(u_0)\| \leqslant R$. This inequality implies the conclusions of Example 1.

Example 4 (*Continuous Gauss–Newton method*). Let $\Phi = -([F']^*F')^{-1}[F']^*F$, (2) and (9) hold, $c_1 = 1$, $c_2 = m_1^2 M_1$, (13) is $M_1 m_1^2\|F(u_0)\| \leqslant R$. This inequality implies the conclusions of Example 1.

Example 5 (*Continuous modified Newton method*). Let $\Phi = -[F'(u_0)]^{-1}F(u)$. Assume $\|[F'(u_0)]^{-1}\| \leqslant m_0$, and let (2) hold. Then $c_2 = m_0$. Choose $R = (2M_2 m_0)^{-1}$, and $c_1 = 0.5$. Then (13) is $2m_0\|F(u_0)\| \leqslant (2M_2 m_0)^{-1}$, that is, $4m_0^2 M_2\|F(u_0)\| \leqslant 1$. Thus, if $4m_0^2 M_2\|F(u_0)\| \leqslant 1$, then the conclusions of Example 1 hold.

Example 6 (*Descent methods*). Let $\Phi = -(f/(f', h))h$, where $f = f(u(t))$ is a differentiable functional $f : H \to [0, \infty)$, and h is an element of H. From (4) one gets $\dot{f} = (f', \dot{u}) = -f$. Thus $f = f_0 e^{-t}$, where $f_0 := f(u_0)$. Assume $\|\Phi\| \leqslant c_2|f|^b$, $b > 0$. Then $\|\dot{u}\| \leqslant c_2|f_0|^b e^{-bt}$. Therefore $u(\infty)$ does exist, $f(u(\infty)) = 0$, and $\|u(\infty) - u(t)\| \leqslant c e^{-bt}$, $c = \text{const} > 0$.

If $h = f'$, and $f = \|F(u)\|^2$, then $f'(u) = 2[F']^*(u)F(u)$, $\Phi = -(f/\|f'\|^2)f'$, and (4) is a descent method. For this Φ one has $c_1 = 1/2$, and $c_2 = m_1/2$, where m_1 is defined in (9). Condition (13) is: $m_1\|F(u_0)\| \leqslant R$. If this inequality holds, then the conclusions of Example 1 hold.

In Example 6 we have obtained some results from [14]. Our approach is more general than the one in [14], since the choices of f and h do not allow one, for example, to obtain Φ used in Example 5.

3. Linear ill-posed problems

We assume that (9) fails. Consider

$$Au = f. \tag{16}$$

Let us denote by $(\mathscr{A})$ the following assumption:

$(\mathscr{A})$: *A is a linear, bounded operator in H, defined on all of H, the range R(A) is not closed, so* (16) *is an ill-posed problem, there is a y such that* $Ay = f$, $y \perp N$, *where N is the null-space of A.*

Let $B := A^*A$, $q := By = A^*f$, A^* is the adjoint of A. Every solution to (16) solves

$$Bu = q, \tag{17}$$

and, if $f = Ay$, then every solution to (17) solves (16). Choose a continuous, monotonically decaying to zero function $\epsilon(t) > 0$, on $\mathbb{R}_+$.

Sometimes it is convenient to assume that

$$\lim_{t\to\infty}(\dot{\epsilon}\epsilon^{-2}) = 0. \tag{18}$$

For example, the functions $\epsilon = c_1(c_0 + t)^{-b}$, $0 < b < 1$, where c_0 and c_1 are positive constants, satisfy (18). There are many such functions. One can prove [3,7] the following:

Claim. *If* $\epsilon(t) > 0$ *is a continuous monotonically decaying function on* $\mathbb{R}_+$, $\lim_{t\to\infty}\epsilon(t) = 0$, *and* (18) *holds, then*

$$\int_0^\infty \epsilon \, \mathrm{d}s = \infty. \tag{19}$$

In this section we do not use assumption (18): in the proof of Theorem 2 one uses only the monotonicity of a continuous function $\epsilon > 0$ and (19). One can drop assumption (19), but then convergence is proved in Theorem 2 to some element of N, not necessarily to the normal solution y, that is, to the solution orthogonal to N, or, which is the same, to the minimal norm solution to (16). However, (18) is used (in a slightly weaker form) in Section 4.

Consider problems (4) and (7) with

$$\Phi := -[Bu + \epsilon(t)u - q], \quad \Phi_\delta = -[Bu_\delta + \epsilon(t)u_\delta - q_\delta], \tag{20}$$

where $\|q - q_\delta\| \leqslant \|A^*\|\delta := C\delta$. Without loss of generality one may assume this $C = 1$, which we do in what follows. Our main result in Section 3, is Theorem 2, stated below. It yields the following:

Conclusion. *Given noisy data* f_δ, *every linear ill-posed problem* (16) *under assumptions* ($\mathscr{A}$) *can be stably solved by the DSM.*

The result presented in Theorem 2 is essentially obtained in [7], but our proof is different and much shorter.

Theorem 2. *Problem* (4) *with* Φ *from* (20) *has a unique global solution* $u(t)$, (6) *holds, and* $u(\infty) = y$. *Problem* (7) *with* Φ_δ *from* (20), *has a unique global solution* $u_\delta(t)$, *and there exists* t_δ, *such that*

$$\lim_{\delta\to 0}\|u_\delta(t_\delta) - y\| = 0. \tag{21}$$

This t_δ *can be chosen, for example, as a root of the equation*

$$\epsilon(t) = \delta^b, \quad b \in (0, 1), \tag{22}$$

or the equation (27) below.

Proof. Linear Eqs. (4) with bounded operators have unique global solutions. If $\Phi = -[Bu + \epsilon(t)u - q]$, then the solution u to (4) is

$$u(t) = h^{-1}(t)U(t)u_0 + h^{-1}(t)\int_0^{\|B\|}\exp(-t\lambda)\int_0^t \mathrm{e}^{s\lambda}h(s)\,\mathrm{d}s\lambda\,\mathrm{d}E_\lambda y, \tag{23}$$

where $h(t) := \exp(\int_0^t \epsilon(s)\,ds) \to \infty$ as $t \to \infty$, E_λ is the resolution of the identity corresponding to the selfadjoint operator B, and $U(t) := e^{-tB}$ is a non-expansive operator, because $B \geqslant 0$. Actually, (23) can be used also when B is unbounded, $\|B\| = \infty$.

Using L'Hôspital's rule one checks that

$$\lim_{t\to\infty} \frac{\lambda \int_0^t e^{s\lambda} h(s)\,ds}{e^{t\lambda} h(t)} = \lim_{t\to\infty} \frac{\lambda e^{t\lambda} h(t)}{\lambda e^{t\lambda} h(t) + e^{t\lambda} h(t)\epsilon(t)} = 1 \quad \forall \lambda > 0, \tag{24}$$

provided only that $\epsilon(t) > 0$ and $\lim_{t\to\infty} \epsilon(t) = 0$. From (23), (24), and the Lebesgue dominated convergence theorem, one gets $u(\infty) = y - Py$, where P is the orthogonal projection operator on the null-space of B. Under our assumptions ($\mathscr{A}$), $Py = 0$, so $u(\infty) = y$. If $v(t) := \|u(t) - y\|$, then $\lim_{t\to\infty} v(t) = 0$. In general, the rate of convergence of v to zero can be arbitrarily slow for a suitably chosen f. Under an additional a priori assumption on f (for example, the source type assumptions), this rate can be estimated.

Let us describe a method for deriving a stopping rule. One has:

$$\|u_\delta(t) - y\| \leqslant \|u_\delta(t) - u(t)\| + v(t).$$

Since $\lim_{t\to\infty} v(t) = 0$, any choice of t_δ such that

$$\lim_{t_\delta\to\infty} \|u_\delta(t_\delta) - u(t_\delta)\| = 0,$$

gives a stopping rule: for such t_δ one has $\lim_{\delta\to 0} \|u_\delta(t) - y\| = 0$.

To prove that (22) gives such a rule, it is sufficient to check that

$$\|u_\delta(t) - u(t)\| \leqslant \frac{\delta}{\epsilon(t)}. \tag{25}$$

Let us prove (25). Denote $w := u_\delta - u$. Then

$$\dot{w} = -[Bw + \epsilon w - p], \quad w(0) = 0, \quad \|p\| \leqslant \mathrm{d}. \tag{26}$$

Integrating (26), and using the property $B \geqslant 0$, one gets (25).

Alternatively, multiply (26) by w, let $\|w\| := g$, use $B \geqslant 0$, and get $\dot{g} \leqslant -\epsilon(t) g + \delta$, $g(0) = 0$. Thus,

$$g(t) \leqslant \delta \exp\left(-\int_0^t \epsilon\,ds\right) \int_0^t \exp\left(\int_0^s \epsilon\,d\tau\right) ds \leqslant \frac{\delta}{\epsilon(t)}.$$

A more precise estimate, also used at the end of the proof of Theorem 3 below, yields:

$$\|u_\delta(t) - u(t)\| \leqslant \frac{\delta}{2\sqrt{\epsilon(t)}},$$

and the corresponding stopping time t_δ can be taken as the root of the equation:

$$2\sqrt{\epsilon(t)} = \delta^b, \quad b \in (0, 1). \tag{27}$$

Theorem 2 is proved. □.

If the rate of decay of v is known, then a more efficient stopping rule can be derived: t_δ is the minimizer of the problem:

$$v(t) + \delta[\epsilon(t)]^{-1} = \min. \tag{28}$$

For example, if $v(t) \leqslant c\epsilon^a(t)$, then t_δ is the root of the equation

$$\epsilon(t) = \left(\frac{\delta}{ca}\right)^{\frac{1}{1+a}},$$

which one gets from (28) with $v = c\epsilon^a$.

One can also use a stopping rule based on an a posteriori choice of the stopping time, for example, the choice by a discrepancy principle.

A method, much more efficient numerically than Theorem 2, *is given below in Theorem* 5.

For linear Eq. (17) with exact data this method uses (4) with

$$\Phi = -(B + \epsilon(t))^{-1}[Bu + \epsilon(t)u - q] = -u + (B + \epsilon(t))^{-1}q, \tag{29}$$

and for noisy data it uses (7) with $\Phi_\delta = -u_\delta + (B + \epsilon(t))^{-1}q_\delta$. The linear operator $B \geqslant 0$ is monotone, so Theorem 5 is applicable. For exact data (4) with Φ, defined in (29), yields:

$$\dot{u} = -u + (B + \epsilon(t))^{-1}q, \quad u(0) = u_0, \tag{30}$$

and (6) holds if $\epsilon(t) > 0$ is monotone, continuous, decreasing to 0 as $t \to \infty$.

Let us formulate the result:

Theorem 3. *Assume* ($\mathscr{A}$), *and let* $B := A^*A$, $q := A^*f$. *Assume* $\epsilon(t) > 0$ *to be a continuous, monotonically decaying to zero function on* $[0, \infty)$. *Then, for any* $u_0 \in H$, *problem* (30) *has a unique global solution,* $\exists u(\infty) = y$, $Ay = f$, *and* y *is the minimal-norm solution to* (16). *If* f_δ *is given in place of* f, $\|f - f_\delta\| \leqslant \delta$, *then* (21) *holds, with* $u_\delta(t)$ *solving* (30) *with* q *replaced by* $q_\delta := A^*f_\delta$, *and* t_δ *should be chosen, for example, as the root of* (27) (*or by a discrepancy principle*).

Proof. One has $q = Bz$, where $Az = f$, and the solution to (30) is

$$u(t) = u_0\mathrm{e}^{-t} + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s(B + \epsilon(s))^{-1}Bz\,\mathrm{d}s := u_0\mathrm{e}^{-t} + \int_0^{\|B\|} j(\lambda, t)\,\mathrm{d}E_\lambda z, \tag{31}$$

where

$$j(\lambda, t) := \int_0^t \frac{\lambda\mathrm{e}^s}{[\lambda + \epsilon(s)]\mathrm{e}^t}\,\mathrm{d}s, \tag{32}$$

and E_λ is the resolution of the identity of the selfadjoint operator B. One has

$$0 \leqslant j(\lambda, t) \leqslant 1, \quad \lim_{t\to\infty} j(\lambda, t) = 1 \quad \lambda > 0, \quad j(0, t) = 0. \tag{33}$$

From (31)–(33) it follows that $\exists u(\infty)$, $u(\infty) = z - P_N z = y$, where y is the minimal-norm solution to (16), $N := N(B) = N(A)$ is the null-space of B and of A, and P_N is the orthoprojector onto N in H. This proves the first part of Theorem 3.

To prove the second part, denote $w := u_\delta - u$, $g := f_\delta - f$, where we dropped the dependence on δ in w and g for brevity. Then $\dot{w} = -w + (B + \epsilon(t))^{-1}A^*g$, $w(0) = 0$. Thus

$$w = e^{-t} \int_0^t e^s (B + \epsilon(s))^{-1} A^* g \, ds,$$

so

$$\|w\| \leqslant \delta e^{-t} \int_0^t \frac{e^s}{2\sqrt{\epsilon(s)}} ds \leqslant \frac{\delta}{2\sqrt{\epsilon(t)}},$$

where the known estimate (see e.g. [4,7]) was used: $\|(B+\epsilon)^{-1}A^*\| \leqslant 1/(2\sqrt{\epsilon})$. Theorem 3 is proved. □

4. Non-linear ill-posed problems with monotone operators

There is a large literature on Eqs. (1) and (4) with monotone operators. In the result we present the problem is non-linear and ill-posed, the new technical tool, Theorem 4, is used, and the stopping rules are discussed.

Consider (4) with monotone F under standard assumptions (2) and (3), and

$$\Phi = -A^{-1}_{\epsilon(t)}(u)[F(u(t)) + \epsilon(t)(u(t) - \tilde{u}_0)], \tag{34}$$

where $A = A(u) := F'(u)$, A^* is its adjoint, $\epsilon(t)$ is the same as in Theorem 3, and in Theorem 5 $\epsilon(t)$ is further specified, $\tilde{u}_0 \in B(u_0, R)$ is an element we can choose to improve the numerical performance of the method. If noisy data are given, then, as in Section 3, we take

$$F(u) := B(u) - f, \quad \Phi_\delta = -A^{-1}_{\epsilon(t)}(u_\delta)[B(u_\delta(t)) - f_\delta + \epsilon(t)(u_\delta(t) - \tilde{u}_0)],$$

where $\|f_\delta - f\| \leqslant \delta$, B is a monotone non-linear operator, $B(y) = f$, and u_δ solves (7).

To prove that (4) with the above Φ has a global solution and (6) holds, we use the following:

Theorem 4. *Let $\gamma(t)$, $\sigma(t)$, $\beta(t) \in C[t_0, \infty)$ for some real number t_0. If there exists a positive function $\mu(t) \in C^1[t_0, \infty)$ such that*

$$0 \leqslant \sigma(t) \leqslant \frac{\mu(t)}{2}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \quad \beta(t) \leqslant \frac{1}{2\mu(t)}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \quad g_0\mu(t_0) < 1, \tag{35}$$

where g_0 is the initial condition in (36), *then a non-negative solution g to the following differential inequality*:

$$\dot{g}(t) \leqslant -\gamma(t)g(t) + \sigma(t)g^2(t) + \beta(t), \quad g(t_0) = g_0, \tag{36}$$

satisfies the estimate:

$$0 \leqslant g(t) \leqslant \frac{1 - \nu(t)}{\mu(t)} < \frac{1}{\mu(t)}, \tag{37}$$

for all $t \in [t_0, \infty)$, *where*

$$0 < v(t) = \left(\frac{1}{1 - \mu(t_0) g(t_0)} + \frac{1}{2} \int_{t_0}^{t} \left(\gamma(s) - \frac{\dot{\mu}(s)}{\mu(s)} \right) \mathrm{d}s \right)^{-1}.$$

There are several novel features in this result. First, differential equation, which one gets from (36) by replacing the inequality sign by the equality sign, is a Riccati equation, whose solution may blow up in a finite time, in general. Conditions (35) guarantee the global existence of the solution to this Riccati equation with the initial condition (36). Secondly, this Riccati differential equation cannot be integrated analytically by separation of variables. Thirdly, the coefficient $\sigma(t)$ may grow to infinity as $t \to \infty$, so that the quadratic term does not necessarily have a small coefficient, or the coefficient smaller than $\gamma(t)$. Without loss of generality one may assume $\beta(t) \geqslant 0$ in Theorem 4. The proof of Theorem 4 is given in [2].

The main result of this Section is new. It claims a global convergence in the sense that no assumptions on the choice of the initial approximation u_0 are made. Usually one assumes that u_0 is sufficiently close to the solution of (1) in order to prove convergence. We take $\tilde{u}_0 = 0$ in Theorem 5, because in this theorem $\tilde{u}_0$ does not play any role. The proof is valid for any choice of $\tilde{u}_0$, but then the definition of r in Theorem 5 is changed.

Theorem 5. *If* (2) *and* (3) *hold*, $\tilde{u}_0 = 0$, $R = 3r$, *where* $r := \|y\| + \|u_0\|$, *and* $y \in N := \{z : F(z) = 0\}$ *is the* (*unique*) *minimal norm solution to* (1), *then, for any choice of* u_0, *problem* (4) *with* Φ *defined in* (34), $\tilde{u}_0 = 0$, *and* $\epsilon(t) = c_1(c_0 + t)^{-b}$ *with some positive constants* c_1, c_0, *and* $b \in (0, 1)$, *specified in the proof of Theorem* 5, *has a global solution*, *this solution stays in the ball* $B(u_0, R)$ *and* (6) *holds. If* $u_\delta(t)$ *solves* (4) *with* Φ_δ *in place of* Φ, *then there is a* t_δ *such that* $\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0$.

Proof. Let us sketch the steps of the proof. Let V solve the equation

$$F(V) + \epsilon(t)V = 0. \tag{38}$$

Under our assumptions on F, it is well known that: (i) (38) has a unique solution for every $t > 0$, and (ii) $\sup_{t \geqslant 0} \|V\| \leqslant \|y\|$, (cf. [2]). If F is Fréchet differentiable, then V is differentiable, and $\|\dot{V}(t)\| \leqslant \|y\| |\dot{\epsilon}(t)| / \epsilon(t)$. It is also known that if (3) holds, then $\lim_{t\to\infty} \|V(t) - y\| = 0$. We will show that the global solution u to (4), with the Φ from (34), does exist, and $\lim_{t\to\infty} \|u(t) - V(t)\| = 0$. This is done by deriving a differential inequality for $w := u - V$, and by applying Theorem 4 to $g = \|w\|$. Since $\|u(t) - y\| \leqslant \|u(t) - V(t)\| + \|V(t) - y\|$, it then follows that (6) holds. We also check that $u(t) \in B(u_0, R)$, where $R := 3(\|y\| + \|u_0\|)$, for any choice of u_0 and a suitable choice of ϵ.

Let us derive the differential inequality for w. One has

$$\dot{w} = -\dot{V} - A^{-1}_{\epsilon(t)}(u)[F(u(t)) - F(V(t)) + \epsilon(t) w], \tag{39}$$

and $F(u) - F(V) = Aw + K$, where $\|K\| \leqslant M_2 g^2/2$, $g := \|w\|$ and M_2 is the constant from (2). Multiply (39) by w, use the monotonicity of F, that is, the property $A \geqslant 0$, and the estimate $\|\dot{V}\| \leqslant \|y\| |\dot{\epsilon}|/\epsilon$, and get:

$$\dot{g} \leqslant -g + \frac{0.5Mg^2}{\epsilon} + \|y\| \frac{|\dot{\epsilon}|}{\epsilon}, \tag{40}$$

where $M := M_2$. Inequality (40) is of the type (36): $\gamma = 1$, $\sigma = 0.5M/\epsilon$, $\beta(t) = \|y\||\dot{\epsilon}|/\epsilon$. Choose

$$\mu(t) = \frac{2M}{\epsilon(t)}.$$

Clearly $\mu \to \infty$ as $t \to \infty$. Let us check three conditions (35). One has $\dot{\mu}(t)/\mu(t) = |\dot{\epsilon}|/\epsilon$. Take $\epsilon = c_1(c_0 + t)^{-b}$, where $c_j > 0$ are constants, $0 < b < 1$, and choose these constants so that $|\dot{\epsilon}|/\epsilon < 1/2$, for example, $b/c_0 = 1/4$. Then the first condition (35) is satisfied. The second condition (35) holds if

$$8M\|y\||\dot{\epsilon}|\epsilon^{-2} \leqslant 1. \tag{41}$$

One has $\epsilon(0) = c_1 c_0^{-b}$. Choose

$$\epsilon(0) = 4Mr.$$

Then

$$|\dot{\epsilon}|\epsilon^{-2} = bc_1^{-1}(c_0 + t)^{b-1} \leqslant bc_0^{-1}c_1^{-1}c_0^b = \frac{1}{4\epsilon(0)} = \frac{1}{16Mr},$$

so (41) holds. Thus, the second condition (35) holds. The last condition (35) holds because

$$\frac{2M\|u_0 - V_0\|}{\epsilon(0)} \leqslant \frac{2Mr}{4Mr} = \frac{1}{2} < 1.$$

By Theorem 4 one concludes that $g = \|w(t)\| < \epsilon(t)/2M \to 0$ when $t \to \infty$, and

$$\|u(t) - u_0\| \leqslant g + \|V - u_0\| \leqslant g(0) + r \leqslant 3r. \tag{42}$$

This estimate implies the global existence of the solution to (4), because if $u(t)$ would have a finite maximal interval of existence, $[0, T)$, then $u(t)$ could not stay bounded when $t \to T$, which contradicts the boundedness of $\|u(t)\|$, and from (42) it follows that $\|u(t)\| \leqslant 4r$. We have proved the first part of Theorem 5, namely properties (6).

To derive a stopping rule we argue as in Section 3. One has:

$$\|u_\delta(t) - y\| \leqslant \|u_\delta(t) - V(t)\| + \|V(t) - y\|.$$

We have already proved that $\lim_{t\to\infty} v(t) := \lim_{t\to\infty} \|V(t) - y\| = 0$. The rate of decay of v can be arbitrarily slow, in general. Additional assumptions, for example, the source-type ones, can be used to estimate the rate of decay of $v(t)$. One derives differential inequality (36) for $g_\delta := \|u_\delta(t) - V(t)\|$, and estimates g_δ using (37). The analog of (40) for g_δ contains additional term δ/ϵ on the right-hand side. If $\delta/\epsilon^2 \leqslant 1/16M$, then conditions (35) hold, and $g_\delta < \epsilon(t)/2M$. Let t_δ be the root of the equation $\epsilon^2(t) = 16M\delta$. Then $\lim_{\delta\to 0} t_\delta = \infty$, and (8) holds because $\|u_\delta(t_\delta) - y\| \leqslant v(t_\delta) + g_\delta$, $\lim_{t_\delta\to\infty} g_\delta(t_\delta) = 0$ and $\lim_{t_\delta\to\infty} v(t_\delta) = 0$, but the convergence in (8) can be slow. See [3,4] for the rate of convergence under source assumptions. If the rate of decay of $v(t)$ is known, then one chooses t_δ as the minimizer of the problem, similar to (28),

$$v(t) + g_\delta(t) = \min,$$

where the minimum is taken over $t > 0$ for a fixed small $\delta > 0$. This yields a quasioptimal stopping rule. Theorem 5 is proved. □

In [3] a local convergence result, similar to the first part of Theorem 4, was obtained, that is, $\|u_0 - y\|$ was assumed sufficiently small, and no discussion of noisy data was given.

Let us give another result:

Theorem 6. *Assume that* $\Phi = -F(u) - \epsilon(t)u$, *F is monotone*, $\epsilon(t)$ *as in Theorem* 3, *and* (18), (2) *and* (3) *hold. Then* (6) *holds.*

Proof. As in the proof of Theorem 5, it is sufficient to prove that $\lim_{t\to\infty} g(t) = 0$, where g, w, and V are the same as in Theorem 5, and u solves (4) with the Φ defined in Theorem 6. Similarly to the derivation of (39), one gets:

$$\dot{w} = -\dot{V} - [F(u) - F(V) + \epsilon(t)w]. \tag{43}$$

Multiply (43) by w, use the monotonicity of F, the estimate $\|\dot{V}\| \leqslant (|\dot{\epsilon}(t)|/\epsilon(t))\|y\|$, which was used also in the proof of Theorem 5, and get:

$$\dot{g} \leqslant -\epsilon(t)g + \frac{|\dot{\epsilon}(t)|}{\epsilon(t)}\|y\|.$$

This implies

$$g(t) \leqslant \mathrm{e}^{-\int_0^t \epsilon(s)\mathrm{d}s}\left[g(0) + \int_0^t \mathrm{e}^{\int_0^s \epsilon(x)\,\mathrm{d}x}\frac{|\dot{\epsilon}(s)|}{\epsilon(s)}\|y\|\,\mathrm{d}s\right]. \tag{44}$$

From our assumptions relation (19) follows, and (44) together with (18) and (19) imply $\lim_{t\to\infty} g(t) = 0$. Theorem 6 is proved. □

Remark 1. One can drop assumption (2) in Theorem 6 and assume only that F is a monotone hemicontinuous operator defined on all of H.

Claim 1. *If* $\epsilon(t) = \epsilon = \text{const} > 0$, *then* $\lim_{\epsilon\to 0}\|u(t_\epsilon) - y\| = 0$, *where* $u(t)$ *solves* (4) *with* $\Phi := -F(u) - \epsilon u$, *and* t_ϵ *is any number such that* $\lim_{\epsilon\to 0}\epsilon t_\epsilon = \infty$.

Proof. One has $\|u(t) - y\| \leqslant \|u(t) - V_\epsilon\| + \|V_\epsilon - y\|$, where V_ϵ solves (38) with $\epsilon(t) = \epsilon = \text{const} > 0$. Under our assumptions on F, Eq. (38) has a unique solution, and $\lim_{\epsilon\to 0}\|V_\epsilon - y\| = 0$. So, to prove the claim, it is sufficient to prove that $\lim_{\epsilon\to 0}\|u(t_\epsilon) - V_\epsilon\| = 0$, provided that $\lim_{\epsilon\to 0}\epsilon t_\epsilon = \infty$. Let $g := \|u(t) - V_\epsilon\|$, and $w := u(t) - V_\epsilon$. Because $\dot{V}_\epsilon = 0$, one has the equation: $\dot{w} = -[F(u) - F(V_\epsilon) + \epsilon w]$. Multiplying this equation by w, and using the monotonicity of F, one gets $\dot{g} \leqslant -\epsilon g$, so $g(t) \leqslant g(0)\mathrm{e}^{-\epsilon t}$. Therefore $\lim_{\epsilon\to 0} g(t_\epsilon) = 0$, provided that $\lim_{\epsilon\to 0}\epsilon t_\epsilon = \infty$. The claim is proved. □

Remark 2. One can prove claims (i) and (ii), formulated below formula (38), using DSM version presented in Theorem 11 below.

Claim 2. *Assume that F is monotone*, (2) *holds, and* $F(y) = 0$. *Then claims* (*i*) *and* (*ii*), *formulated below formula* (38), *hold.*

Proof. First, note that (ii) follows from (i) easily, because the assumptions $F(y) = 0$, F is monotone, and $\epsilon > 0$, imply, after multiplying $F(V) - F(y) + \epsilon V = 0$ by $V - y$, the inequality $(V, V - y) \leqslant 0$, from which claim (ii) follows. Claim (i) follows from Theorem 11, proved below. □

Claim 3. *Assume that the operator F is monotone, hemicontinuous, defined on all of H, $F(y) = 0$, y is the minimal norm element of $N_F := \{z : F(z) = 0\}$, $\Phi = -F(u) - \epsilon(t)u$, $\epsilon(t) > 0$, is monotone, decaying to zero, and* (18) *holds. Then* (6) *holds for the solution to* (4).

Proof. Existence of the unique global solution to (4) under our assumptions is known (see e.g. [15]). Let $w := u - V_b$, $g := \|w\|$, where V_b solves $F(V_b) + bV_b = 0$, $b = \text{const} > 0$. It is known [15] that $\|V_b\| \leqslant \|y\|$, and $\lim_{b\to 0} \|V_b - y\| = 0$. One has $\|u(t) - y\| \leqslant \|u(t) - V_b\| + \|V_b - y\|$. Thus, to prove $\lim_{t\to\infty} \|u(t) - y\| = 0$ it is sufficient to prove that $\lim_{t\to\infty} g(t) = 0$. One has $\dot{w} = -[F(u) + \epsilon(t)u - F(V_b) - bV_b]$. Multiply this equation by w, use the monotonicity of F and get: $\dot{g} \leqslant -\epsilon(t)w + |\epsilon(t) - b|\|y\|$. Denote $h(t) := \exp(\int_0^t \epsilon(s)\,\mathrm{d}s)$. Then,

$$g(\xi) \leqslant g(0)h^{-1}(\xi) + h^{-1}(\xi) \int_0^{\xi} h(s)|\epsilon(s) - b|\,\mathrm{d}s\|y\|.$$

Clearly, $\lim_{\xi\to\infty} g(0)h^{-1}(\xi) = 0$, because $\lim_{t\to\infty} h(t) = \infty$. In fact, $\epsilon^{-1} \leqslant ct + c_0$, where $c_0 := \epsilon^{-1}(0) > 0$, and one can choose $0 < c < 1$ because of (18), so $h \geqslant (ct + c_0)^{1/c}$, and $\lim_{t\to\infty} \epsilon(t)h(t) = \infty$. Choose $b = \epsilon(\xi)$ and apply L'Hôspital's rule to the last term in the above inequality for $g(\xi)$. L'Hôspital's rule is applicable, and one gets:

$$\lim_{\xi\to\infty} g(\xi) = \lim_{\xi\to\infty} \frac{|\dot{\epsilon}(\xi)|}{\epsilon^2(\xi)} \frac{\epsilon(\xi)\int_0^{\xi} h\,\mathrm{d}s}{h(\xi)} = 0.$$

Claim 3 is proved. □

The result in Claim 3 contains the result from [16], where additional assumptions are made on $\epsilon(t)$, global existence of the solution to (4) is assumed, and the proof contains a gap, because it is not shown that the L'Hôspital's rule can be applied twice.

5. Non-linear ill-posed problems with non-monotone operators

Assume that $F(u) := B(u) - f$, B is a non-monotone operator, $A := F'(u)$, $\tilde{A} := F'(y)$, $T := A^*A$, $\widetilde{T} := \tilde{A}^*\tilde{A}$, $T_\epsilon := T + \epsilon I$, where I is the identity operator, ϵ is as in Theorem 3 and $|\dot{\epsilon}(t)|/\epsilon(t) < 1$,

$$\Phi := -T_\epsilon^{-1}(u)[A^*(B(u) - f) + \epsilon(u - \tilde{u}_0)], \quad \epsilon = \epsilon(t) > 0, \tag{45}$$

and Φ_δ is defined similarly, with f_δ replacing f and u_δ replacing u.

The main result of this section is:

Theorem 7. *If* (2) *and* (3) *hold, $u, u_0 \in B(y, R)$, $y - \tilde{u}_0 = \widetilde{T}z$, $\|z\| \ll 1$, and $R \ll 1$, then problem* (4) *has a unique global solution and* (6) *holds. If $u_\delta(t)$ solves* (7), *then there exists a t_δ such that* $\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0$.

The derivation of the stopping rule, that is, the choice of t_δ, is based on the ideas presented in Section 4 (cf. [7,4]). $R \ll 1$ means that R is sufficiently small.

Sketch of proof. Proof of Theorem 7 consists of the following steps.

First we prove that $g := \|w\| := \|u(t) - y\|$ satisfies a differential inequality (36), and, applying (37), conclude that $g(t) < \mu^{-1}(t) \to 0$ as $t \to \infty$. A new point in this derivation (compared with the one for monotone operators) is the usage of the source assumption $y - u_0 = \widetilde{T}z$.

Secondly, we derive the stopping rule using the ideas from Section 4. The source assumption allows one to get a rate of convergence (see [1,4]). Details of the proof are technical and are not included. One can see [4] for some proofs.

Let us sketch the derivation of the differential inequality for g. Write $B(u) - f = B(u) - sB(y) = Aw + K$, where $\|K\| \leqslant M_2 g^2/2$, and $\epsilon(u - \tilde{u}_0) = \epsilon w + \epsilon(y - \tilde{u}_0) = \epsilon w + \epsilon \widetilde{T} z$. Then (45) can be written as

$$\Phi = -w - T_\epsilon^{-1} A^* K - \epsilon T_\epsilon^{-1} \widetilde{T} z, \quad \epsilon := \epsilon(t). \tag{46}$$

Multiplying (4), with Φ defined in (46), by w, one gets:

$$g\dot{g} \leqslant -g^2 + \frac{M_2}{2} \|T_{\epsilon(t)}^{-1} A^*\| g^3 + \epsilon(t) \|T_{\epsilon(t)}^{-1} \widetilde{T}\| \|z\| g.$$

Since $g \geqslant 0$, one obtains:

$$\dot{g} \leqslant -g + \frac{M_2}{4\sqrt{\epsilon(t)}} g^2 + \epsilon(t) \|T_\epsilon^{-1} \widetilde{T}\| \|z\|, \tag{47}$$

where the estimate $\|T_\epsilon^{-1} A^*\| \leqslant 1/(2\sqrt{\epsilon})$ was used. Clearly,

$$\|T_\epsilon^{-1} \widetilde{T}\| \leqslant \|(T_\epsilon^{-1} - \widetilde{T}_\epsilon^{-1}) \widetilde{T}\| + \|\widetilde{T}_\epsilon^{-1} \widetilde{T}\|, \quad \|\widetilde{T}_\epsilon^{-1} \widetilde{T}\| \leqslant 1, \quad \epsilon \|T_\epsilon^{-1}\| \leqslant 1,$$

and

$$T_\epsilon^{-1} - \widetilde{T}_\epsilon^{-1} = T_\epsilon^{-1} (A^* A - \tilde{A}^* \tilde{A}) \widetilde{T}_\epsilon^{-1}.$$

One has:

$$\|A^* A - \tilde{A}^* \tilde{A}\| \leqslant 2 M_2 M_1 g, \quad \|z\| \ll 1.$$

Let $2M_1 M_2 \|z\| \leqslant 1/2$. This is possible since $\|z\| \ll 1$. Using the above estimates, one transforms (47) into the following inequality:

$$\dot{g} \leqslant -\frac{1}{2} g + \frac{M_2}{4\sqrt{\epsilon(t)}} g^2 + \|z\| \epsilon. \tag{48}$$

Now, apply Theorem 4 to (48), choosing

$$\mu = \frac{2M_2}{\sqrt{\epsilon}}, \quad \frac{|\dot{\epsilon}|}{\epsilon} < \frac{1}{2}, \quad 16 M_2 \|z\| \sqrt{\epsilon(0)} < 1, \quad \text{and} \quad \frac{2M_2 \|u_0 - y\|}{\sqrt{\epsilon(0)}} < 1.$$

Then conditions (35) are satisfied, and Theorem 4 yields the estimate:

$$g(t) < \frac{\sqrt{\epsilon(t)}}{2M_2}.$$

This is the main part of the proof of Theorem 7. □

6. Non-linear ill-posed problems: avoiding inverting of operators in the Newton-type continuous schemes

In the Newton-type methods for solving well-posed non-linear problems, for example, in the continuous Newton method (4) with $\Phi = -[F'(u)]^{-1}F(u)$, the difficult and expensive part of the solution is inverting the operator $F'(u)$. In this section we give a method to avoid inverting of this operator. This is especially important in the ill-posed problems, where one has to invert some regularized versions of F', and to face more difficulties than in the well-posed problems.

Consider problem (1) and assume (2), (3) and (9). Thus, we discuss our method in the simplest well-posed case.

Replace (4) by the following Cauchy problem (dynamical system):

$$\dot{u} = -QF, \quad u(0) = u_0, \tag{49}$$

$$\dot{Q} = -TQ + A^*, \quad Q(0) = Q_0, \tag{50}$$

where $A := F'(u)$, $T := A^*A$, and $Q = Q(t)$ is a bounded operator in H.

First let us state our new technical tool: an operator version of the Gronwall inequality (cf. [8]).

Theorem 8. *Let*

$$\dot{Q} = -T(t)Q(t) + G(t), \quad Q(0) = Q_0,$$

where $T(t)$, $G(t)$, *and* $Q(t)$ *are linear bounded operators on a real Hilbert space* H. *If there exists* $\epsilon(t) > 0$ *such that*

$$(T(t)h, h) \geqslant \epsilon(t)\|h\|^2 \quad \forall h \in H,$$

then

$$\|Q(t)\| \leqslant e^{-\int_0^t \epsilon(s)\,ds}\left[\|Q(0)\| + \int_0^t \|G(s)\| e^{\int_0^s \epsilon(x)\,dx}\,ds\right]. \tag{51}$$

Let us turn now to a proof of Theorem 9, formulated at the end of this section. This theorem is the main result of Section 6.

Applying (51) to (50), and using (2) and (9), which implies

$$(T(t)h, h) \geqslant c\|h\|^2 \quad \forall h \in H, \quad c = \text{const} > 0,$$

one gets:

$$\|Q(t)\| \leqslant e^{-ct}\left[\|Q(0)\| + \int_0^t M_1 e^{st}\,ds\right] \leqslant [\|Q_0\| + M_1 c^{-1}] := c_1,$$

as long as $u(t) \in B(u_0, R)$.

Let $u(t) - y := w$, $\|w\| := g$, $\tilde{A} := F'(y)$. Since $F(y) = 0$, one has $F(u) = \tilde{A}w + K$, where $\|K\| \leqslant 0.5M_2 g^2 := c_0 g^2$, and M_2 is the constant from (2). Rewrite (49) as

$$\dot{w} = -Q[\tilde{A}w + K]. \tag{52}$$

Let $\Lambda := I - Q\tilde{A}$. Multiply (52) by w and get

$$g\dot{g} \leqslant -g^2 + (\Lambda w, w) + c_0 g^3, \quad c_0 = \text{const} > 0. \tag{53}$$

We prove below that

$$\sup_{t \geqslant 0} \|\Lambda\| \leqslant \lambda < 1. \tag{54}$$

From (53) and (54) one gets the following differential inequality:

$$\dot{g} \leqslant -\gamma g + c_0 g^2, \quad 0 < \gamma < 1, \quad \gamma := 1 - \lambda, \tag{55}$$

which implies:

$$g(t) \leqslant r\mathrm{e}^{-\gamma t}, \quad r := g(0)[1 - g(0)c_0]^{-1}, \tag{56}$$

provided that

$$g(0)c_0 < 1. \tag{57}$$

Inequality (57) holds if u_0 is sufficiently close to y.

From (56) and (55) it follows that $u(\infty) = y$. Thus, (6) holds.

The trajectory $u(t) \in B(u_0, R)$, $\forall t > 0$, provided that

$$\int_0^\infty \|\dot{u}\| \mathrm{d}t = \int_0^\infty \|\dot{w}\| \, \mathrm{d}t \leqslant r + \frac{c_0 r^2}{2\gamma} \leqslant R.$$

This inequality holds if u_0 is sufficiently close to y, that is, r is sufficiently small.

To complete the argument, let us prove (54). One has:

$$\dot{\Lambda} = -\dot{Q}\tilde{A} = -T\Lambda + A^*(A - \tilde{A}).$$

One has $\|A - \tilde{A}\| \leqslant M_2 g$. Using (56) and Theorem 8, one gets

$$\|\Lambda\| \leqslant \mathrm{e}^{-ct}\left[\|\Lambda_0\| + rM_1M_2 \int_0^t \mathrm{e}^{(c-\gamma)s} \, \mathrm{d}s\right].$$

Thus,

$$\|\Lambda\| \leqslant \|\Lambda_0\| + Cr := \lambda, \quad C := M_1 M_2 \sup_{t>0} \frac{\mathrm{e}^{-\gamma t} - \mathrm{e}^{-ct}}{c - \gamma}.$$

If u_0 is sufficiently close to y and Q_0 is sufficiently close to $\tilde{A}^{-1}$, then $\lambda > 0$ can be made arbitrary small. We have proved:

Theorem 9. *If* (2), (3) *and* (9) *hold*, Q_0 *and* u_0 *are sufficiently close to* $\tilde{A}^{-1}$ *and* y, *respectively*, *then problem* (49) *and* (50) *has a unique global solution*, (6) *holds*, *and* $u(t)$ *converges to* y, *which solves* (1), *exponentially fast*.

In [8] a generalization of Theorem 9 is given for ill-posed problems.

7. Iterative schemes

In this section we present a method for constructing convergent iterative schemes for a wide class of well-posed Eq. (1). Some methods for constructing convergent iterative schemes for a wide class of ill-posed problems are given in [2]. There is an enormous literature on iterative methods.

Consider a discretization scheme for solving (4) with $\Phi = \Phi(u)$, so that we assume no explicit time dependence in Φ:

$$u_{n+1} = u_n + h\Phi(u_n), \quad u_0 = u_0, \quad h = \text{const} > 0. \tag{58}$$

One of our results from [2], concerning the well-posed Eqs. (1) is Theorem 10, formulated below. Its proof is shorter and simpler than in [2].

Theorem 10. *Assume* (2), (3), (9)–(13) *with* $a = 2$, $g_1 = c_1 = \text{const} > 0$, $g_2 = c_2 = \text{const} > 0$, $\|\Phi'(u)\| \leqslant L_1$, *for* $u \in B(y, R)$. *Then, if* $h > 0$ *is sufficiently small, and* u_0 *is sufficiently close to* y, *then* (58) *produces a sequence* u_n *for which*

$$\|u_n - y\| \leqslant R\mathrm{e}^{-chn}, \quad \|F(u_n)\| \leqslant \|F_0\|\mathrm{e}^{-chn}, \tag{59}$$

where $R := c_2\|F_0\|/c_1$, $F_0 = F(u_0)$, $c = \text{const} > 0$, *and* $c < c_1$.

Proof. The proof is by induction. For $n = 0$ estimates (59) are clear. Assuming these estimates for $j \leqslant n$, let us prove them for $j = n + 1$. Let $F_n := F(u_n)$, and let $w_{n+1}(t)$ solve problem (4) on the interval (t_n, t_{n+1}), $t_n := nh$, with $w_{n+1}(t_n) = u_n$. By (14) (with $G = c_2\mathrm{e}^{-c_1 t}$) and (59) one gets:

$$\|w_{n+1}(t) - y\| \leqslant \frac{c_2}{c_1}\|F_n\|\mathrm{e}^{-c_1 t} \leqslant R\mathrm{e}^{-cnh - c_1 t}, \quad t_n \leqslant t \leqslant t_{n+1}. \tag{60}$$

One has:

$$\|u_{n+1} - y\| \leqslant \|u_{n+1} - w_{n+1}(t_{n+1})\| + \|w_{n+1}(t_{n+1}) - y\|, \tag{61}$$

and

$$\begin{aligned}\|u_{n+1} - w_{n+1}(t_{n+1})\| &\leqslant \int_{t_n}^{t_{n+1}} \|\Phi(u_n) - \Phi(w_{n+1}(s))\|\,\mathrm{d}s \\ &\leqslant L_1 c_2 h \int_{t_n}^{t_{n+1}} \|F(w_{n+1}(t))\|\,\mathrm{d}t \\ &\leqslant L_1 c_1 h^2 R\mathrm{e}^{-cnh},\end{aligned} \tag{62}$$

where we have used the formula $R := c_2\|F_0\|/c_1$, and the estimate:

$$\|F(w_{n+1}(t))\| \leqslant \|F_n\|\mathrm{e}^{-c_1(t-t_n)} \leqslant \|F_0\|\mathrm{e}^{-cnh - c_1(t-t_n)}. \tag{63}$$

From (60)–(63) it follows that:

$$\|u_{n+1} - y\| \leqslant R\mathrm{e}^{-cnh}(\mathrm{e}^{-c_1 h} + c_1 L_1 h^2) \leqslant R\mathrm{e}^{-c(n+1)h},$$

provided that

$$e^{-c_1 h} + c_1 L_1 h^2 \leqslant e^{-ch}. \tag{64}$$

Inequality (64) holds if h is sufficiently small and $c < c_1$. So, the first inequality (59), with $n+1$ in place of n, is proved if h is sufficiently small and $c < c_1$.

Now

$$\|F(u_{n+1})\| \leqslant \|F(u_{n+1}) - F(w_{n+1}(t))\| + \|F(w_{n+1}(t))\|, \quad t_n \leqslant t \leqslant t_{n+1}. \tag{65}$$

Using (2) and (62), one gets:

$$\|F(u_{n+1}) - F(w_{n+1}(t_{n+1}))\| \leqslant M_1 \|u_{n+1} - w_{n+1}(t_{n+1})\| \leqslant M_1 c_2 L_1 h^2 \|F_0\| e^{-cnh}. \tag{66}$$

From (65) and (66) it follows that:

$$\|F(u_{n+1})\| \leqslant \|F_0\| e^{-cnh}(e^{-c_1 h} + M_1 c_2 L_1 h^2) \leqslant \|F_0\| e^{-c(n+1)h},$$

provided that

$$e^{-c_1 h} + M_1 c_2 L_1 h^2 \leqslant e^{-ch}. \tag{67}$$

Inequality (67) holds if h is sufficiently small and $c < c_1$. So, the second inequality (59) with $n+1$ in place of n is proved if h is sufficiently small and $c < c_1$. Theorem 10 is proved. □

In the well-posed case, if $F(y) = 0$, the discrete Newton's method

$$u_{n+1} = u_n - [F'(u_n)]^{-1} F(u_n), \quad u_0 = u(0),$$

converges superexponentially if u_0 is sufficiently close to y. Indeed, if $v_n := u_n - y$, then $v_{n+1} = v_n - [F'(u_n)]^{-1}[F'(u_n)v_n + K]$ where $\|K\| \leqslant M_2 \|v_n\|^2/2$. Thus, $g_n := \|v_n\|$ satisfies the inequality: $g_{n+1} \leqslant q g_n^2$, where $q := m_1 M_2/2$. Therefore $g_n \leqslant q^{2^n - 1} g_0^{2^n}$, and if $0 < q g_0 < 1$, then the method converges superexponentially.

If one uses the iterative method $u_{n+1} = u_n - h[F'(u_n)]^{-1}F(u_n)$, with $h \neq 1$, then, in the well-posed case, assuming that this method converges, it converges exponentially, that is, slower than in the case $h = 1$.

The continuous analog of the above method

$$\dot{u} = -a[F'(u)]^{-1} F(u), \quad u(0) = u_0,$$

where $a = \text{const} > 0$, converges at the rate $O(e^{-at})$. Indeed, if $g(t) := \|F(u(t))\|$, then $g\dot{g} = -ag^2$, so $g(t) = g_0 e^{-at}$, $\|\dot{u}\| \leqslant a m_1 g_0 e^{-at}$. Thus

$$\|u(t) - u(\infty)\| \leqslant m_1 g_0 e^{-at}, \quad \text{and} \quad F(u(\infty)) = 0.$$

In the continuous case one does not have superexponential convergence no matter what $a > 0$ is.

8. A spectral assumption (cf. [10])

In this section we introduce the spectral assumption which allows one to treat some non-linear non-monotone operators.

Assumption ($\mathscr{S}$). The set $\{r, \varphi : \pi - \varphi_0 < \varphi < \pi + \varphi_0, \varphi_0 > 0, 0 < r < r_0\}$, where φ_0 and r_0 are arbitrarily small, fixed numbers, consists of the regular points of the operator $A := F'(u)$ for all $u \in B(u_0, R)$.

Assumption ($\mathscr{S}$) implies the estimate:

$$\|(F'(u) + \epsilon)^{-1}\| \leqslant \frac{1}{\epsilon \sin \varphi_0}, \quad \epsilon < r_0(1 - \sin \varphi_0), \quad \epsilon = \text{const} > 0, \tag{68}$$

because $\|(A - z)^{-1}\| \leqslant 1/\text{dist}(z, s(A))$, where $s(A)$ is the spectrum of a linear operator A, and $\text{dist}(z, s(A))$ is the distance from a point z of a complex plane to the spectrum. In our case, $z = -\epsilon$, and $\text{dist}(z, s(A)) = \epsilon \sin \varphi_0$, if $\epsilon < r_0(1 - \sin \varphi_0)$.

Theorem 11. *If* (2) *and* (68) *hold, and* $0 < \epsilon < r_0(1 - \sin \varphi_0)$, *then problem* (38), *with* $\epsilon(t) = \epsilon = \text{const} > 0$, *is solvable, problem* (4), *with* Φ *defined in* (34) *and* $\tilde{u}_0 = 0$, *has a unique global solution,* $\exists u(\infty)$, *and* $F(u(\infty)) + \epsilon u(\infty) = 0$. *Every solution to the equation* $F(V) + \epsilon V = 0$ *is isolated.*

Proof. Let $g = g(t) := \|F(u(t)) + \epsilon u(t)\|$, where $u = u(t)$ solves locally (4), where Φ is defined in (34) and $\tilde{u}_0 = 0$. Then:

$$g\dot{g} = -((F'(u) + \epsilon)(F'(u) + \epsilon)^{-1}(F(u) + \epsilon u), F(u) + \epsilon u) = -g^2,$$

so

$$g = g_0 e^{-t}, \quad g_0 := g(0); \quad \|\dot{u}\| \leqslant \frac{g_0}{\epsilon \sin \varphi_0} e^{-t}.$$

Thus,

$$\|u(t) - u(\infty)\| \leqslant \frac{g_0}{\epsilon \sin \varphi_0} e^{-t}, \quad \|u(t) - u_0\| \leqslant \frac{g_0}{\epsilon \sin \varphi_0}, \quad F(u(\infty)) + \epsilon u(\infty) = 0.$$

Therefore equation

$$F(V) + \epsilon V = 0, \quad \epsilon = \text{const} > 0, \tag{69}$$

has a solution in $B(u_0, R)$, where $R = g_0/\epsilon \sin \phi_0$.

Every solution to Eqs. (69) is isolated. Indeed, if $F(W) + \epsilon W = 0$, and $\psi := V - W$, then $F(V) - F(W) + \epsilon \psi = 0$, so $[F'(V) + \epsilon]\psi + K = 0$, where $\|K\| \leqslant M_2 \|\psi\|^2/2$. Thus, using (68), one gets $\|\psi\| \geqslant 2\epsilon \sin \varphi_0 / M_2$. Consequently, if $\|\psi\|$ is sufficiently small, then $\psi = 0$. Theorem 11 is proved. □

The author used assumption ($\mathscr{S}$) in the theory of deconvolution [17].

References

[1] Airapetyan R, Ramm AG, Smirnova AB. Continuous analog of Gauss–Newton method. Math Models Meth Appl Sci 1999;9(3):463–74.

[2] Airapetyan R, Ramm AG. Dynamical systems and discrete methods for solving nonlinear ill-posed problems. In: Anastassiou G, editor. Applied mathematical reviews, vol. 1. World Scientific Publishers; 2000. p. 491–536.
[3] Airapetyan R, Ramm AG, Smirnova AB. Continuous methods for solving nonlinear ill-posed problems. In: Ramm AG, Shivakumar PN, Strauss AV, editors. Operator theory and applications. Providence, RI: American Mathematical Society, Fields Institute Communications; 2000. p. 111–38.
[4] Kaltenbacher B, Neubauer A, Ramm AG. Convergence rates of the continuous regularized Gauss–Newton method. J Inv Ill-Posed Prob 2002;10(3):261–80.
[5] Ramm AG, Smirnova AB. A numerical method for solving nonlinear ill-posed problems. Nonlinear Func Anal Opt 1999;20(3):317–32.
[6] Ramm AG, Smirnova AB. On stable numerical differentiation. Math Comput 2001;70:1131–53.
[7] Ramm AG. Linear ill-posed problems and dynamical systems. J Math Anal Appl 2001;258(1):448–56.
[8] Ramm AG, Smirnova AB. Continuous regularized Gauss–Newton-type algorithm for nonlinear ill-posed equations with simultaneous updates of inverse derivative. Int J Pure Appl Math 2002;2(1):23–34.
[9] Ramm AG. A numerical method for some nonlinear problems. Math Models Meth Appl Sci 1999;9(2):325–35.
[10] Ramm AG. Acceleration of convergence of a continuous analog of the Newton method. Appl Anal 2002;81(4):1001–4.
[11] Ramm AG. Regularization of ill-posed problems with unbounded operators. J Math Anal Appl 2001;271:547–50.
[12] Ramm AG. On equations of the first kind. Diff Equat 1968;4:2056–60.
[13] Ramm AG. Stable solutions of some ill-posed problems. Math Meth Appl Sci 1981;3:336–63.
[14] Alber Ya. Continuous Newton-type processes. Diff Uravn 1971;7(1):1931–45 (in Russian).
[15] Deimling K. Nonlinear functional analysis. Berlin: Springer; 1985. p. 99.
[16] Alber Ya, Ryasantseva I. On regularized evolution equations. Func Diff Equat 2000;7:177–87.
[17] Ramm AG, Galstian A. On deconvolution methods. Internal Jour of Engin Sci 2003;41(N1):31–43.
[18] Ramm AG, Smirnova AB, Favini A. Continuous modified Newton's-type method for nonlinear operator equations. Ann di Mat Pure Appl 2003;182(N1):37–52.