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DYNAMICAL ZETA FUNCTIONS FOR MAPS OF THE INTERVAL

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ABSTRACT. A dynamical zeta function ζ and a transfer operator $\mathscr{L}$ are associated with a piecewise monotone map f of the interval $[0, 1]$ and a weight function g. The analytic properties of ζ and the spectral properties of $\mathscr{L}$ are related by a theorem of Baladi and Keller under an assumption of "generating partition". It is shown here how to remove this assumption and, in particular, extend the theorem of Baladi and Keller to the case when f has negative Schwarzian derivative.

Let $0 = a_0 < a_1 < \cdots < a_N = 1$. We write $X = [0, 1] \subset \mathbb{R}$ and assume that f is continuous $X \to X$ and strictly monotone on the intervals $J_i = [a_{i-1}, a_i]$. Furthermore, let $g: X \to \mathbb{C}$ have bounded variation. A *transfer operator* $\mathscr{L}$ acting on functions $\Phi: X \to \mathbb{C}$ of bounded variation is defined by

$$(\mathscr{L}\Phi)(x) = \sum_{y\,:\,fy=x} g(y)\Phi(y),$$

and we let

$$\theta = \lim_{m\to\infty} \sup_{x\in X} \left|\prod_{k=0}^{m-1} g(f^k x)\right|^{1/m}.$$

It is known (see [1]) that the essential spectral radius of $\mathscr{L}$ is $\leq \theta$. Assuming that $(J_1, \ldots, J_N)$ is generating (i.e., if $f^k x$ and $f^k y$ are in the same $J_{i(k)}$ for all $k \geq 0$, then $x = y$), Baladi and Keller [1] have proved the following remarkable result (referred to as B-K theorem in what follows):

The function

$$d(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in \mathrm{Fix}\, f^m} \prod_{k=0}^{m-1} g(f^k x)$$

is holomorphic for $|z|\theta < 1$, *and its zeros there are the inverses* λ^{-1} *of the eigenvalues* λ *of* $\mathscr{L}$ *such that* $|\lambda| > \theta$, *with the same multiplicity.*

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Note that $\zeta(z) = 1/d(z)$ is the *dynamical zeta function* associated with the weight g in the manner suggested by statistical mechanics (see [8]); Milnor and Thurston [7] have discussed the case $g = 1$. If $(J_1, \ldots, J_N)$ is not generating, the set $\operatorname{Per} f$ of f-periodic points may be uncountable, so $\zeta(z)$ is not even defined. The purpose of the present note is to indicate how to remove this obstruction to the B-K theorem.

There is a map α defined on $\operatorname{Per} f\backslash$finite set such that $\alpha x = (\xi(k))_{k\geq 0}$ is the unique sequence of symbols $\xi(k) \in \{1, \ldots, N\}$ for which $f^k x \in J_{\xi(k)}$. We say that an f-invariant subset S of $\operatorname{Per} f$ is a *representative set of periodic points* if the restriction of α to $S\backslash$finite set is a bijection to $(\alpha(\operatorname{Per}\backslash\text{finite set}))\backslash$finite set and preserves the period. Such a set S always exists, and the B-K theorem remains true provided $\zeta(z)$ is replaced by

$$\zeta_S(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in S\cap \operatorname{Fix} f^m} \prod_{k=0}^{m-1} g(f^k x).$$

There are several cases of interest where $\operatorname{Per} f$ itself is a representative set of periodic points: when $(J_1, \ldots, J_N)$ is generating (case considered in [1]), or when f has negative Schwarzian derivative: $Sf = f'''/f' - \frac{3}{2}(f''/f')^2 < 0$, or when f is affine on each J_i, and the slopes σ_i are such that $\prod \sigma_i^{m_i} \neq 1$ when $m_i \geq 0$, $\sum m_i > 0$. In those cases the B-K theorem remains true in its original form.

To prove these results, it is convenient to work in the more general setup where X is a compact subset of $\mathbb{R}$ and to perform geometric changes on $(X, f, g, (J_1, \ldots, J_N))$, observing the effects of those changes on the spectrum of $\mathscr{L}$, θ, and on the zeta function. By doubling the points $a_1, \ldots, a_{N-1}$ (see Hofbauer and Keller [5]), collapsing certain intervals (see Baladi and Ruelle [2]), and embedding (X, f) into a full shift on N symbols (see Ruelle [9]), one reduces to a particularly simple situation where one can use a form (due to Baladi and Keller) of an argument due originally to Haydn [3]. This avoids the use of the Markov extension (an infinite Markov partition) of Hofbauer [4] (see also [6]). The effect of the geometric changes mentioned above on θ and ζ is relatively easy to determine; the effect on the spectrum of $\mathscr{L}$ takes more effort to analyze.

Using analytic completion (see Ruelle [9]), one can extend the analyticity properties of $d(z)$ to situations when g is not a function with bounded variation (but a quotient of such functions) and one recovers results of Keller and Nowicki [6].

Let r be the spectral radius of $\mathscr{L}$, and define the *pressure*

$$P(\log|g|) = \sup_{\rho\in\mathbf{I}} \left(h(\rho) + \int \rho(dx) \log|g(x)| \right)$$

where $h(\cdot)$ is the entropy of an element of the set $\mathbf{I}$ of f-invariant probability measures. With the above methods one can show that

$$\theta \leq r \leq \max(\theta, \exp P(\log|g|)).$$

If $g \geq 0$ and $r > \theta$, then* $r = \exp P(\log g)$ and r is an eigenvalue of $\mathscr{L}$;

* This result is obtained by Baladi and Keller [1] with a different definition of the pressure.

the set

$$\Delta = \left\{ \rho \in \mathbf{I} : h(\rho) + \int \rho(dx) \log g(x) = P(\log g) \right\}$$

of equilibrium measures is nonempty, Δ is a Choquet simplex, and its vertices are ergodic measures.

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