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DYNAMICALLY EVOLVING GAUSSIAN SPATIAL FIELDS

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Abstract. We discuss general non-stationary spatio-temporal surfaces that involve dynamics governed by velocity fields. The approach formalizes and expands previously used models in analysis of satellite data of significant wave heights. We start with homogeneous spatial fields and by applying an extension of the standard moving average construction we arrive to stationary in time models. The obtained surface although changing in time can be considered dynamically inactive since its velocities when sampled across the field have distributions centered at zero. We introduce a dynamical evolution to such a field by composing it with a dynamical flow governed by a given velocity field. This leads to non-stationary models that are extensions of the previous discretized auto-regressions accounting for a local velocity of traveling surface. For such a surface we demonstrate that its dynamics is a combination of dynamics introduced by the flow and the dynamics resulting from the covariance structure of the underlying stochastic field. We extend this approach to fields that are only locally stationary and have their parameters varying over a larger spatio-temporal horizon.

1. Introduction

1.1. **Motivation and basic terminology.** Recent technological advances such as aerial laser and satelite scanning result in increasingly complex environmental data over large regions in space and over relatively long periods of time. Examples of such data, among others, come from marine climate, air quality, and vegetation surveys. Accounting for all aspects of such spatio-temporal data can be a challenging task thus a proper stochastic framework has to be carefully designed to capture important features of the considered data. For example in [4], Gaussian spatio-temporal fields have been successfully used to model non-stationary in time and space variation of the significant wave height data combined from

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vast satellite records as well as by a stationary buoy or systems of buoys. Here we elaborate on general mathematical models that, in a special case, lead to the models discussed in that work. In the analysis of such models, the focus is on dynamically evolving fields that locally represent the data. The considered fields account both for large scale variability and variability introduced by a dynamical flow that transports stochastic effects from one region to another.

In view of environmental applications, our main object of studies, a stochastic field $X(\mathbf{p}, t)$, has two arguments: space variable $\mathbf{p}$ representing position and time variable t. We limit ourselves to the case $\mathbf{p} = (x, y)$ although extension to higher dimensions is immediate. In connection with some invariance properties, we use the following terminology. We call a field *stationary* if it is invariant to shifts in time and space, i.e. for each fixed $\mathbf{p}_0$ and t_0: $X(\mathbf{p} + \mathbf{p}_0, t + t_0) \stackrel{d}{=} X(\mathbf{p}, t)$, where $\stackrel{d}{=}$ stands for the equality of underlying probability distributions of stochastic processes (in $\mathbf{p}$ and t variables). The invariance only to shift in space (time) will be referred to as *spatial* (*temporal*) *stationarity*. If the field is (in distribution) invariant on the space rotation, i.e. $X(\mathbf{R}_\phi \mathbf{p}, t) \stackrel{d}{=} X(\mathbf{p}, t)$, where $\mathbf{R}_\phi$ is the rotation by an angle ϕ, then we call X *isotropic*. Finally, a field that is isotropic and stationary in space is referred to as *homogeneous*.

1.2. **Description of the approach.** Gaussian stationary fields constitute a convenient class of models that found many applications. In this work they serve as building blocks for more general, non-stationary models. The need to reach beyond stationarity is usually due to two aspects observed in environmental records: dynamics and spatio-temporal variability due to different properties at different locations (or/and time instants). The nature of these deviations from stationarity is different and thus has to be treated differently. In our approach, the dynamics is introduced through a deterministic flow that transports independently and locally generated stochastic fields, while the long scale variability is represented by location and time dependent spectra of the underlying locally stationary fields. In the course of our presentation we start with a given *spatial-only* covariance and introduce temporal dependence following a classical time series approach in which independent spatial innovations have the assumed spatial covariance structure. For the so obtained fields under stationarity, we argue that properly defined velocities sampled randomly from the surface are centered at zero indicating that the fields are dynamically inactive. Then we introduce dynamics by means of a velocity field representing the motion of the surface. This velocity field transports the independent stochastic innovations created at each time point and weighted by a proper time dependent scaling that diminishes over time. These scaled innovations, when added up, result in a field that is stationary neither in space nor in time. This spatio-temporal variability is due only to different velocities at different locations and times and thus is dictated by the underlying flow. Analysis of the velocity distributions of these fields can be performed by a method of [5] and it is observed that velocities are centered at the value that is the sum of the flow velocity and an additional term that is due to the non-isotropic character of the underlying stochastic field.

As it was pointed out, from a practical point of view it is also important to account for the spatial variability that is due not only to the motion but also to geographic specificity of the location. The approach is extended to account for this type of non-stationarity in space by taking spectral representations corresponding to stationary fields but making spectra depending on parameters that vary from one location to another.

1.3. **Relation to previous work.** Our main inspiration is the model that was described in a discretized version in [4]. It starts with a selected *spatial* stationary Gaussian covariance function $r_S(\mathbf{p}) = \sigma^2 \exp(-|\mathbf{p}|^2/(2L^2))$. Then, *temporal* dependence is introduced by considering the recursive autoregression

$$X(\mathbf{p},t) = \rho X(\mathbf{p} - \mathbf{v}dt, t-dt) + \sqrt{1-\rho^2}\ \Phi_t(\mathbf{p}), \tag{1}$$

with independent (in t) innovations $\Phi_t(\mathbf{p})$ having the covariance r_S. The model has simple motivation: at each time step the past surface is moving forward to a new location with velocity $\mathbf{v}$ and is modified by an independent innovation with prescribed (fixed) spatial covariance structure. The resulting (stationary) covariance is of the form $r(\mathbf{p},t) = \rho^t r_S(\mathbf{p} - \mathbf{v}t)$.

In the quoted work the model has been also extended to account for a lack of stationarity in space. This has been accomplished at three different levels. Firstly, by taking the non-stationary innovation $\Phi_t(\mathbf{p})$ with a general covariance $r_S(\mathbf{p}', \mathbf{p})$. In particular, the dependence of L on location in the Gaussian covariances has been considered. Secondly, the auto-regression coefficient ρ has been made dependent on the location. Finally, it has been also assumed that the velocity $\mathbf{v}$ depends on both location $\mathbf{p}$ and time t. In this work, the case of equation (1) with $\mathbf{v} = 0$ will be referred to as the *underlying static field* and be denoted by X_0 in parallel to the dynamical field X with non-zero velocity. We note that the first two sources of non-stationarity mentioned above are due to the non-dynamical field while the latter is a result of dynamical flow represented here by velocities.

This work is extending the above model in several aspects. Firstly, we depart from the discretization and provide with a fully continuous set-up by means of properly defined moving averages in time of independent spatial fields

$$X(\mathbf{p}, t) = \int_{-\infty}^{\infty} f(t-s)\ \Phi(\mathbf{p};\ ds).$$

Secondly, and more importantly, we notice that the construction is independent of the form of spatial covariance r_S and can lead to fairly general time dependence as defined by correlations $\rho(t) \sim f * \tilde{f}(t)$, where $*$ stands for the operation of convolution and $\tilde{f}(u) = f(-u)$. Essentially, for each covariance in space $r_S(\mathbf{p})$ and a general class of correlations in time $\rho(t)$, we explicitly represent Gaussian fields with covariance structure given by the product $r_S(\mathbf{p}) \cdot \rho(t)$.

The model coincides with the one that in the discretized version was given by (1). Dynamics is expressed by an arbitrary time varying velocity field that generates a flow given by $\boldsymbol{\psi}_{t,h}(\mathbf{p})$ which is the location at time $t+h$ of a point that at time t is at $\mathbf{p}$. Such a flow is incorporated into a stochastic framework by means of the stochastic integral

$$Y(\mathbf{p}, t) = \int f(t, t-s; \mathbf{p})\ \Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p});\ ds),$$

where for fixed $\mathbf{p}$ and t the value $f(t, t-s; \mathbf{p})$ represent the weight with which the innovation $\Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), ds)$ that is introduced at time s contributes to the value $Y(\mathbf{p}, t)$. As a result we obtain a large class of Gaussian spatio-temporal fields that incorporate dynamical evolution of a random surface. A general scheme of fitting to the actual spatio-temporal data can be obtained by extension of the approach presented before in [4].

The concept of integration that is used above, is based on the general methods of defining integrals with respect to spectral measures of orthogonal projections in Hilbert spaces (see, for example, [10]) and can be considered standard in mathematical literature, so here we only sketch fundamentals in the Appendix. The generality of the approach allows a natural extension for second order models that goes beyond Gaussian distribution but this is not explored here. Steps in this direction have been undertaken for the fields driven by Laplace motion in [1] and will be continued in future research.

2. Spatio-temporal static fields

2.1. **Locally stationary spatial fields.** Before we turn to the building of spatio-temporal structures let us briefly discuss a way to obtain a fairly general class of spatial non-stationary fields. The starting point is the following spectral representation of a stationary process

$$X(\mathbf{p}) \stackrel{d}{=} \int_{\mathbb{R}^n} \exp(i\mathbf{p}\cdot\boldsymbol{\omega})\sqrt{S(\boldsymbol{\omega})}\; dB(\boldsymbol{\omega}), \tag{2}$$

where the symmetric $S(\boldsymbol{\omega}) \geq 0$ is a spectral density and dB is a random Gaussian measure whose variance coincides with the Lebesgue measure in $\mathbb{R}^n$.

A natural extension of (2) to non-stationary fields is by considering spectra that depend on location. More precisely, for a family of symmetric spectral densities $S_{\mathbf{p}}(\boldsymbol{\omega}) \geq 0$ parameterized by $\mathbf{p}$, we define

$$X(\mathbf{p}) \stackrel{d}{=} \int_{\mathbb{R}^n} \exp(i\mathbf{p}\cdot\boldsymbol{\omega})\sqrt{S_{\mathbf{p}}(\boldsymbol{\omega})}\; dB(\boldsymbol{\omega}).$$

The non-stationary covariance of X is given by

$$r_S(\mathbf{p},\mathbf{p}') = \mathbb{C}ov(X(\mathbf{p}),X(\mathbf{p}')) = \int_{\mathbb{R}^n} \exp(i(\mathbf{p}-\mathbf{p}')\cdot\boldsymbol{\omega})\sqrt{S_{\mathbf{p}}(\boldsymbol{\omega})S_{\mathbf{p}'}(\boldsymbol{\omega})}\; d\boldsymbol{\omega}.$$

If $S_{\mathbf{p}}(\cdot) \approx S_{\mathbf{p}_0}(\cdot)$ in some neighborhood of $\mathbf{p}_0$, then $X(\cdot)$ in this neighborhood can be approximately viewed as a realization of

$$X(\mathbf{p}) = \int_{\mathbb{R}^n} \exp(i\mathbf{p}\cdot\boldsymbol{\omega})\sqrt{S_{\mathbf{p}_0}(\boldsymbol{\omega})}\; dB(\boldsymbol{\omega}).$$

Thus such random fields provide a convenient way of modelling phenomena that are locally stationary in space.

Example 1 (Non-stationary locally isotropic covariance). In this example, n is an arbitrary natural number although in this work we are mainly interested in $n = 1, 2$. In [4], we considered isotropic spectra that locally had the so-called Gaussian form

$$S_{\mathbf{p}}(\boldsymbol{\omega}) = \frac{s^2(\mathbf{p})L^n(\mathbf{p})}{2\pi^{n/2}} \exp\left(-L^2(\mathbf{p})|\boldsymbol{\omega}|^2/2\right),$$

where $s^2(\mathbf{p})$ is the variance at a location $\mathbf{p}$. The covariance of processes with such spectra can be evaluated and is given by

$$\begin{aligned} r_S(\mathbf{p},\mathbf{p}') &= \frac{s(\mathbf{p})s(\mathbf{p}')}{2}\left(\frac{L(\mathbf{p})L(\mathbf{p}')}{\pi}\right)^{n/2}\int_{\mathbb{R}^n}\exp\left(i(\mathbf{p}-\mathbf{p}')\cdot\boldsymbol{\omega}-\frac{L^2(\mathbf{p})+L^2(\mathbf{p}')}{4}|\boldsymbol{\omega}|^2\right)\,d\boldsymbol{\omega} \\ &= \frac{s(\mathbf{p})s(\mathbf{p}')}{2}\left(\frac{L(\mathbf{p})L(\mathbf{p}')}{\pi}\right)^{n/2}\int_{\mathbb{R}^n}\exp(i(\mathbf{p}-\mathbf{p}')\cdot\boldsymbol{\omega})\exp\left(-\boldsymbol{\omega}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}/2\right)\,d\boldsymbol{\omega}, \end{aligned}$$

where $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}(\mathbf{p},\mathbf{p}')=2/\left(L^2(\mathbf{p})+L^2(\mathbf{p}')\right)\cdot\mathbf{I}$, with $\mathbf{I}$ being the identity matrix in $\mathbb{R}^n$. Using the formula for the characteristic function of n-dimensional Gaussian vector we obtain

$$r_S(\mathbf{p},\mathbf{p}')=s(\mathbf{p})s(\mathbf{p}')\left(\frac{2L(\mathbf{p})L(\mathbf{p}')}{L^2(\mathbf{p})+L^2(\mathbf{p}')}\right)^{-n/2}\exp\left(-(\mathbf{p}-\mathbf{p}')^T\boldsymbol{\Sigma}(\mathbf{p}-\mathbf{p}')/2\right). \tag{3}$$

Consequently, if $s(\mathbf{p})$ and $L(\mathbf{p})$ are approximately constant in some region, then the correlation is approximately invariant with respect to isometries of $\mathbb{R}^n$. In such a case, we refer to the field as *locally isotropic*. Obviously, by taking an arbitrary positive definite $\boldsymbol{\Sigma}$ we can obtain an anisotropic extension of the model. Here and in the rest of this paper we follow the convention that vectors are column matrices and for a matrix $\mathbf{A}$ its transpose is denoted by $\mathbf{A}^T$. □

2.2. **Building spatio-temporal dependence.** In the Appendix 5.1, a notion of integral has been introduced to give a proper meaning to the following general spatio-temporal field

$$X(\mathbf{p},t)=\int f(t,s;\mathbf{p})\;\Phi(\mathbf{p};ds), \tag{4}$$

for a deterministic kernel f and a Gaussian field valued measure $\Phi(\boldsymbol{\cdot};ds)$ that is uniquely characterized by time dependent spatial covariances $r_S(\mathbf{p},\mathbf{p}';s)$. Here, they will be referred to as *spatial covariances governing* X. As an example of $r_S(\mathbf{p},\mathbf{p}';s)$, one can consider the covariances of the previous subsection where dependence on time can be introduced quite arbitrarily by making spectra $S_{\mathbf{p}}$ also dependent on time t. The above model is the most general form of *static fields* discussed in this work. Since the fields we are interested in are centered Gaussian fields, to compare the different models is enough to compare their covariance functions,

$$r(\mathbf{p},\mathbf{p}';t,t')=\mathbb{C}ov(X(\mathbf{p},t),X(\mathbf{p}',t'))=\int f(t,s;\mathbf{p})f(t',s;\mathbf{p}')\cdot r_S(\mathbf{p},\mathbf{p}';s)\;ds. \tag{5}$$

One important simplification of the model is obtained by taking $r_S(\mathbf{p},\mathbf{p}';s) = r_S(\mathbf{p}-\mathbf{p}';s)$ – the case which will be referred to as the *spatially stationary innovation model*, while the case $r_S(\mathbf{p},\mathbf{p}';s) = r_S(\mathbf{p},\mathbf{p}')$ defines the *time independent innovation model*. Several other specifications of the above model will be important throughout the paper and they are listed below in order of increasing complexity. Before we commence with the presentation we need some notation. We write $r(\mathbf{p},\mathbf{p}';t,t')$ to denote the covariance between $X(\mathbf{p},t)$ and $X(\mathbf{p}',t')$. In the presence of spatial stationarity we write $r(\mathbf{p};t,t')$ while if the field is stationary in time $r(\mathbf{p},\mathbf{p}';t)$. Finally if the field X is stationary both in space and time we write $r(\mathbf{p};t)$.

Stationary moving average: This case is defined by taking $f(t,s;\mathbf{p}) = f(t-s)$ and thus the kernel f does not depend on $\mathbf{p}$, while the stationary spatial covariance $r_S(\mathbf{p}-\mathbf{p}')$ is independent of time. In this case, we obtain the complete stationary case with covariance given by

$$r(\mathbf{p};t) = r_S(\mathbf{p}) \cdot r_T(t),$$

where $r_T(t) = \int f(t-s)f(-s)\ ds$. If additionally we assume $r_S(\mathbf{p})$ to be isotropic we obtain the special subcase of a homogeneous moving average field.

Separable stationary in time moving average: A generalization of the previous case is when the spatial stationarity of $r_S(\mathbf{p},\mathbf{p}')$ is dropped so that the covariance is given by

$$r(\mathbf{p},\mathbf{p}';t) = r_S(\mathbf{p},\mathbf{p}') \cdot r_T(t). \tag{6}$$

Observe the temporal stationarity of the model.

Separable covariance model: This case corresponds to kernel f independent of the space variable $\mathbf{p}$, and spatial covariance independent of the time variable t. In this case the covariance can be still presented as a product of the spatial and temporal covariances which sometimes is referred to as *multiplicative separability* of the model,

$$r(\mathbf{p},\mathbf{p}';t,t') = r_S(\mathbf{p},\mathbf{p}') \cdot r_T(t,t'),$$

where $r_T(t,t') = \int_{-\infty}^{\infty} f(t,s) \cdot f(t',s)\ ds$. Notice that the covariance of both the stationary moving average and the separable stationary in time moving average are also of (multiplicative) separable models.

Heteroscedastic moving average: This corresponds to the case of time dependent spatial covariance structure with space independent kernel $f(t-t')$ for which

$$r(\mathbf{p},\mathbf{p}';t,t') = \int_{-\infty}^{\infty} f(t-s)\cdot f(t'-s)\cdot r_S(\mathbf{p},\mathbf{p}';s)\ ds. \tag{7}$$

We note that typically this model is non-stationary both in time and space. The terminology is borrowed from the general theory of time series as a spatial analog of the non-constant variance innovation case. We also consider a special subcase of space stationary innovation that is referred to as *heterodscedastic, space-stationary moving average* and which is defined by stationary covariances $r_S(\mathbf{p}-\mathbf{p}';s)$. Clearly in this case, the stationarity in space holds.

Temporal stationary moving average: Here we assume time independent (homoscedastic) spatial covariance structure with space dependent kernel $f(t,s;\mathbf{p}) = f(t-s;\mathbf{p})$ for which

$$r(\mathbf{p},\mathbf{p}';t) = r_S(\mathbf{p},\mathbf{p}')\cdot f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'}(t), \tag{8}$$

where $f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'}$ is the convolution of $f_{\mathbf{p}}(s) = f(s;\mathbf{p})$ with $\tilde{f}_{\mathbf{p}'}(s) = f(-s;\mathbf{p}')$. We note stationarity in the time direction at any fixed spatial position $\mathbf{p}$ while temporal models differ at various locations.

Remark 1 (Temporal moving averages). The temporal moving average field is introduced by (6) by taking $f(t,s) = f(t-s)$ and assuming that the spatial covariance function r_S is independent of time. The relation to moving averages appearing in time series analysis can be more explicitly seen through the approximation of the field by a sum. Let $s = k\Delta t, \quad k = -M,\dots,M$ for some large M and $t = n\Delta t$. Then,

$$X(\mathbf{p},t) \approx \sum_{k=-M}^{M} f((n-k)\Delta t)\cdot \epsilon_k(\mathbf{p})\cdot\sqrt{\Delta t}, \tag{9}$$

where $\epsilon_k(\mathbf{p})$ are independent (in time) Gaussian fields with $\mathbb{C}ov(\epsilon_k(\mathbf{p}),\epsilon_k(\mathbf{p}')) = r_S(\mathbf{p},\mathbf{p}')$ that are related to the fields Φ by

$$\epsilon_k(\mathbf{p}) = \frac{\Phi(\mathbf{p};(k\Delta t,(k+1)\Delta t])}{\sqrt{\Delta t}}.$$

Relation (9) can be rewritten as

$$X_n(\mathbf{p}) = \lim_{M\to\infty,\Delta t\to 0} \sum_{k=-M}^{M} \alpha_k \epsilon_{n-k}(\mathbf{p}), \tag{10}$$

with $\alpha_k = \sqrt{\Delta t} \cdot f(k\Delta t)$, which is the well known form of a discrete moving average time series.

Example 2 (Temporal Ornstein-Uhlenbeck field). A special case of the separable temporal moving average model (6) is obtained by taking $f(t) = e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t)$. In this case,

$$X(\mathbf{p}, t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\ \Phi(\mathbf{p}; ds) \tag{11}$$

and since additionally $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p}, \mathbf{p}')$, its covariance is given by

$$r(\mathbf{p}, \mathbf{p}'; t) = r_S(\mathbf{p}, \mathbf{p}') \cdot \frac{1}{2\lambda} e^{-\lambda|t|}. \tag{12}$$

This example corresponds to the case considered in [4], where the autoregression model of order one

$$X(\mathbf{p}, t) = \rho X(\mathbf{p}, t - \Delta t) + \sqrt{1-\rho^2}\ \Phi_t(\mathbf{p}),$$

has been discussed. It is clear from Remark 1 that the above is a discretization of the Ornstein-Uhlenbeck model with $\rho = e^{-\lambda\Delta t}$.

The space dependent Ornstein-Uhlenbeck model is obtained as a special case of temporal stationary moving average (8) by taking a space dependent $\lambda(\mathbf{p})$ in which case we obtain

$$r(\mathbf{p}, \mathbf{p}'; t) = \frac{r_S(\mathbf{p}, \mathbf{p}')}{\lambda(\mathbf{p}) + \lambda(\mathbf{p}')} \begin{cases} e^{-\lambda(\mathbf{p}')\cdot t}\ ; & \text{if } t > 0, \\ e^{-\lambda(\mathbf{p})\cdot t}\ ; & \text{if } t < 0. \end{cases} \tag{13}$$

We note stationarity in time as in any other space dependent moving average.

Example 3 (Temporal Gaussian dependence). Another case of a temporal stationary moving average model (8) can be obtained by taking the Gaussian kernel $f_{\mathbf{p}}(t) = f(t; \mathbf{p}) = \pi^{-1/4} \cdot e^{-t^2/L^2(\mathbf{p})}$. By (8) we have

$$r(\mathbf{p}, \mathbf{p}'; t) = r_S(\mathbf{p}, \mathbf{p}') \cdot f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'}(t),$$

and since the convolution of Gaussian kernels is again a Gaussian kernel, the resulting covariance is stationary in t and given by

$$r(\mathbf{p},\mathbf{p}';t) = r_S(\mathbf{p},\mathbf{p}') \cdot \left(\frac{1}{L^2(\mathbf{p})} + \frac{1}{L^2(\mathbf{p}')} \right)^{-1/2} \cdot e^{-\frac{(t-t')^2}{L^2(\mathbf{p})+L^2(\mathbf{p}')}}. \tag{14}$$

Use of space dependent kernel $f(t,s;\mathbf{p})$ is natural for building non-stationary spatio-temporal correlation. However, by an analogy to the approach of Subsection 2.1, one can alternatively consider for this purpose space dependent temporal spectra. More specifically, one can consider the model

$$\int_{\mathbb{R}^{n+1}} e^{i(\mathbf{p},t)\cdot(\boldsymbol{\omega},\tau)} \sqrt{S_{\mathbf{p}}(\boldsymbol{\omega}) S^T_{\mathbf{p}}(\tau)}\, dB(\boldsymbol{\omega},\tau), \tag{15}$$

where $S^T_{\mathbf{p}}(\tau)$ is a location dependent temporal spectrum. It must be realized that this alternative construction is for the most part equivalent to the one based on space dependent symmetric kernels as stated in the following result.

Theorem 1. *Let a Gaussian random field $X_1(\mathbf{p},t)$ be defined through (15) and a Gaussian space dependent moving average $X_2(\mathbf{p},t)$ be defined through (8). Let us assume that $\int_{\mathbb{R}} e^{it\tau} S^T_{\mathbf{p}}(\tau)\, d\tau = f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}}(t)$, where $\tilde{f}_{\mathbf{p}}(t) = f_{\mathbf{p}}(-t)$, so that the covariances in time at a fixed point $\mathbf{p}$ are the same for X_1 and X_2. If we assume that the kernels $f_{\mathbf{p}}$ are symmetric and have non-negative Fourier transform, then the spatio-temporal covariances for both processes agree and consequently $X_1 \overset{d}{=} X_2$.*

Proof. The spatio-temporal covariance of (15) is given by

$$\begin{aligned}
\mathbb{C}ov(X_1(\mathbf{p},t), X_1(\mathbf{p}',t')) &= \int_{\mathbb{R}^n} e^{i(\mathbf{p}-\mathbf{p}')\boldsymbol{\omega}} \sqrt{S_{\mathbf{p}}(\boldsymbol{\omega}) \cdot S_{\mathbf{p}'}(\boldsymbol{\omega})}\, d\boldsymbol{\omega} \cdot \\
&\quad \cdot \int_{\mathbb{R}} e^{i(t-t')\tau} \sqrt{S^T_{\mathbf{p}}(\tau) \cdot S^T_{\mathbf{p}'}(\tau)}\, d\tau \\
&= r_S(\mathbf{p},\mathbf{p}') \cdot \int_{\mathbb{R}} e^{i(t-t')\tau} \sqrt{S^T_{\mathbf{p}}(\tau) \cdot S^T_{\mathbf{p}'}(\tau)}\, d\tau,
\end{aligned}$$

while in the space dependent moving average case the covariance is given by

$$\mathbb{C}ov(X_2(\mathbf{p},t), X_2(\mathbf{p}',t')) = r_S(\mathbf{p},\mathbf{p}') \cdot f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'}(t-t').$$

By taking the Fourier transform $\mathcal{F}h(\tau) = \int e^{-i\tau t} h(t)\, dt$, we obtain

$$\mathcal{F} f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}} = (\mathcal{F} f_{\mathbf{p}})^2 = 2\pi S^T_{\mathbf{p}},$$

so $\mathcal{F} f_{\mathbf{p}} = \sqrt{2\pi S^T_{\mathbf{p}}}$. On the other hand we note that

$$\mathcal{F}^{-1}(2\pi\sqrt{S^T_{\mathbf{p}} \cdot S^T_{\mathbf{p}'}})(t-t') = \int_{\mathbb{R}} e^{i(t-t')\tau} \sqrt{S^T_{\mathbf{p}}(\tau) \cdot S^T_{\mathbf{p}'}(\tau)}\; d\tau$$

Consequently, by the inverse Fourier theorem and

$$\begin{aligned} \mathcal{F} f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'} &= \mathcal{F} f_{\mathbf{p}} \cdot \mathcal{F} \tilde{f}_{\mathbf{p}'} \\ &= 2\pi \sqrt{S^T_{\mathbf{p}}} \sqrt{S^T_{\mathbf{p}'}} \end{aligned}$$

we obtain equality of the covariances. □

Remark 2. The above result assumes symmetric kernels. The case of Ornstein-Uhlenbeck process is not covered by it since this process is not represented by a symmetric kernel. The spectra are given by

$$S^T_{\mathbf{p}}(\tau) = \frac{1}{\lambda^2(\mathbf{p}) + \tau^2},$$

while the kernel approach leads to the correlation given in (13). Thus the equivalence of the models would mean that $e^{-\lambda(\mathbf{p}')t} / \left(\lambda(\mathbf{p}) + \lambda(\mathbf{p}')\right)$ is equal to

$$\int_{\mathbb{R}} e^{-it\tau} \frac{1}{(\lambda^2(\mathbf{p}) + \tau^2)^{1/2}} \cdot \frac{1}{(\lambda^2(\mathbf{p}') + \tau^2)^{1/2}}\; d\tau.$$

which obviously is not true. We conclude that the symmetry of kernels can not be dropped from the assumptions.

2.3. **Velocities of a random field.** Defining motion of a surface is a non-trivial task. A proper definition of velocity emerges as a fundamental issue in describing dynamics of surface. There is no unique approach to this problem and for a comprehensive treatment we refer to [5], [8], and [11]. Below we focus on a conceptually simple surface velocity that was first introduced in the pioneering work of Longuet-Higgins [8], and allows us to investigate the field dynamics or the lack thereof.

Let $X(\mathbf{p}, t)$ be as before, a Gaussian random field defined through the stochastic integral in (4), for a sufficiently smooth kernel f so that the process has well-defined partial derivative fields. We introduce velocity in an arbitrary but fixed direction. Since a simple rotation would allow us to obtain velocity in any direction in what follows we focus on the velocity along the direction of the x-axis. Indeed, for

$$\mathbf{R}_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

being a rotation matrix by angle ϕ, we can consider

$$X(\mathbf{R}_\phi \mathbf{p}, t) = \int f(t, s; \mathbf{R}_\phi \mathbf{p})\ \Phi(\mathbf{R}_\phi \mathbf{p};\ ds), \tag{16}$$

which is the field $X(\mathbf{p}, t)$ rotated by angle ϕ and $\Phi(\mathbf{R}_\phi \mathbf{p};\ ds)$ is the Gaussian field measure governed by covariance $r_S^\phi(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{R}_\phi \mathbf{p}, \mathbf{R}_\phi \mathbf{p}'; s)$. Thus to obtain the results in an arbitrary direction ϕ from the ones given along the x-axis one needs to substitute r_S^ϕ for r_S and $f^\phi(t, s; \mathbf{p}) = f(t, s; \mathbf{R}_\phi \mathbf{p})$ for $f(t, s; \mathbf{p})$. Later when the dynamical flow is governed by a velocity field $\mathbf{v}(\mathbf{p}, t)$, one needs to substitute instead the rotated velocity field $\mathbf{v}^\phi(\mathbf{p}, t) = \mathbf{v}(\mathbf{R}_\phi \mathbf{p}, t)$.

Hence, let us consider a zero-upcrossing in the x-direction at (x, y, t). Then, the zero upcrossing speed at the x-direction V is the x-coordinate of the slope of the tangent plane to the up-crossing contour attached at the point (x, y, t). Clearly we have

$$V = -\frac{X^t}{X^x}, \tag{17}$$

at the points (x, y, t) such that $X^x = X^x(x, y, t) > 0$ and $X = X(x, y, t) = 0$. Here $X^x = X^x(x, y, t)$ and $X^t = X^t(x, y, t)$ are the first order partial derivatives of $X = X(\mathbf{p}, t)$ with respect to x and t, respectively and $\mathbf{p} = (x, y)$.

Remark 3. Let us mention here that from scalar velocities along directions we can get to the vector velocity by integrating them along all directions. For this, at any point (x, y), we define a new velocity $\mathbf{V}(x, y) = \int_0^\pi (\cos(\phi), \sin(\phi)) \cdot V(x, y, \phi)\ d\phi$, where $V(x, y, \phi)$ for the special case $\phi = 0$ is the velocity defined in (17) and for other ϕ is its analog in the ϕ direction as described above.

To obtain the one-dimensional marginal distribution of the velocity V defined in (17), we just need to use some standard facts from the theory of Gaussian random vectors. For a jointly Gaussian vector (X^x, X^t), we can write

$$X^t = \mathbb{E}(X^t | X^x) + s_X \cdot Z,$$

where $\mathbb{E}(X^t|X^x) = \mathbb{C}ov(X^x, X^t)/\mathbb{V}ar(X^x) \cdot X^x$ and $Z = X^t - \mathbb{E}(X^t|X^x)/s_X$ is a standard Gaussian random variable independent of X^x while

$$s_X^2 = \frac{\mathbb{V}ar(X^x)\mathbb{V}ar(X^t) - \mathbb{C}ov(X^x, X^t)^2}{\mathbb{V}ar(X^x)}.$$

Hence, since $X^x = \sqrt{\mathbb{V}ar(X^x)}Z_1$, where Z_1 is a standard normal variable independent of Z, we get

$$V = -\frac{X^t}{X^x} = -\frac{\mathbb{C}ov(X^x, X^t)}{\mathbb{V}ar(X^x)} - \frac{s_X}{\sqrt{\mathbb{V}ar(X^x)}} \cdot C \tag{18}$$

where C is a random Cauchy variable defined as the ratio Z/Z_1 of two independent normal variables. We use the above velocity to describe local dynamics of a stochastic field. Although it is properly defined only if the partial derivatives of the process exist, the previous definition could be extended to more irregular fields by applying a proper filtering that would smooth the process so that the derivatives are well-defined. We skip obvious details of such an approach. We say that a Gaussian stochastic field $X(\mathbf{p}, t)$ *does not exhibit any organized movement* at the point (x, y) and at time t in the direction x, if the median of the distribution of V is equal to zero, i.e. if

$$\mathbb{C}ov(X^x(\mathbf{p}, t), X^t(\mathbf{p}, t)) = 0.$$

Theorem 2. *A heteroscedastic space-stationary moving average $X(\mathbf{p}, t)$ that is defined by (7) with $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p} - \mathbf{p}'; s)$ does not exhibit any organized movement.*

Proof. Indeed, since the field $X(\mathbf{p}, t)$ is governed by stationary r_S, this follows from Lemma 2 in the Appendix.

□

Notice that contrary to the above case, when the kernel is space dependent the resulting field may exhibit some non-trivial dynamics as their velocities are no-longer centered at zero.

Theorem 3. *For $X(\mathbf{p}, t)$ defined by (4) with space-stationary innovations defined through covariances $r_S(\mathbf{p}, t)$, the center of the velocity in (17) equals*

$$-\int \frac{f^x(t, s; \mathbf{p})}{A} \cdot f^t(t, s; \mathbf{p}) \cdot r_S(\mathbf{0}; s)\ ds, \tag{19}$$

where $A = \int |f^x(t, s; \mathbf{p})|^2 r_S(\mathbf{0}; s) + |f(t, s; \mathbf{p})|^2 r_S^{xx}(\mathbf{0}; s)\ ds$.

Proof. This is a direct consequence of (56) in the Appendix, Subsection 5.2, since the first order partial derivative with respect to x of the covariance $r_S(\mathbf{p}; s)$ equals zero when evaluated at $\mathbf{p} = \mathbf{0}$ for the reasons explained in Theorem 1 of the Appendix.

□

Another way of obtaining dynamics is through space varying scaling. That type of "dynamics" may be not desirable and the following example shows that eliminating the space scaling variability (in terms of variance) should probably precede analysis of the 'real' dynamics.

Example 4 (Deterministic rescaling). Let $X(\mathbf{p}, t)$ be a heteroscedastic space-stationary moving average $X(\mathbf{p}, t)$ that is defined by (7) with $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p}-\mathbf{p}'; s)$. By Theorem 2, it does not exhibit any organized motion since $r_X^{xt}(\mathbf{p}, t) = 0$. Consider a positive deterministic field $A(\mathbf{p}, t)$ and define $Y(\mathbf{p}, t) = A(\mathbf{p}, t)X(\mathbf{p}, t)$. Then the covariance of $Y(\mathbf{p}, t)$ is given by

$$r_Y(\mathbf{p}, \mathbf{p}'; t, t') = A(\mathbf{p}, t)A(\mathbf{p}', t')r_X(\mathbf{p} - \mathbf{p}'; t - t').$$

Consequently by Lemma 2, we have

$$\mathbb{C}ov(Y^x(\mathbf{p}, t), Y^t(\mathbf{p}, t)) = A^x(\mathbf{p}, t)A^t(\mathbf{p}, t) \cdot r_X(\mathbf{0}; 0) + A^x(\mathbf{p}, t)A(\mathbf{p}, t) \cdot r_X^t(\mathbf{0}; 0),$$
$$\mathbb{V}ar(Y^x(\mathbf{p}, t)) = (A^x)^2(\mathbf{p}, t) \cdot r_X(\mathbf{0}; 0) + A^2(\mathbf{p}, t) \cdot r_X^{xx}(\mathbf{0}; 0),$$

The center of velocity is thus given by

$$-\frac{A^x A^t r_X + A^x A r_X^t}{(A^x)^2 r_X + A^2 r_X^{xx}}.$$

In general, the field Y has non-trivial dynamics (unless A does not depend on space variable) identified by the velocity center as given above. Since the underlying field X has no dynamics we conclude that the organized movement of Y is only due to the deterministic rescaling A.

To avoid the dynamics illustrated in the above example, in practice, the variable space rescaling can be eliminated by dividing the data by local in space standard deviation, i.e. by replacing $X(\mathbf{p}, t)$ by $X(\mathbf{p}, t)/\sqrt{\mathbb{V}ar(X(\mathbf{p}, t))}$ (for example by local estimation of the variance). Therefore, typically, we consider a version of the presented models for which $r_S(\mathbf{p}, \mathbf{p}; t) = \sigma^2(t)$ or even $r_S(\mathbf{p}, \mathbf{p}; t) = 1$, i.e. the variance of innovations is space independent. In such a case, as long as the kernel $f(t, s)$ is independent of the space variable $\mathbf{p}$, the variance of $X(\mathbf{p}, t)$ is only time dependent. For this more general model the thesis of Theorem 2 remains valid as stated in the following result with the proof being a direct consequence of Lemma 3 of the Appendix.

Theorem 4. *A heteroscedastic moving average $X(\mathbf{p}, t)$ with innovations having space independent variance does not exhibit any organized movement.*

Despite a possibility of introducing dynamics through the space variable kernels as shown in Theorem 3, it is difficult to give a natural interpretation to the obtained center of velocities. In the next section we turn to a more direct method of imposing dynamics on stochastic fields that is based on using deterministic flow generated by velocity fields.

3. Dynamics in the models

3.1. **Constant Velocity Dynamics.** The constant velocity field $\mathbf{v}(\mathbf{p}, t) = \mathbf{v} = (v_1, v_2)$ applied to stochastic fields with space independent kernels f results in

$$Y(\mathbf{p}, t) = \int_{-\infty}^{\infty} f(t, s)\ \Phi(\mathbf{p} + \mathbf{v}(s - t); ds)$$

with covariance

$$r(\mathbf{p}, \mathbf{p}'; t, t') = \int_{-\infty}^{\infty} f(t, s) \cdot f(t', s) \cdot r_S(\mathbf{p} + \mathbf{v} \cdot (s - t), \mathbf{p}' + \mathbf{v} \cdot (s - t'); s)\ ds.$$

A notable special case is given by a spatial stationary innovation covariance r_S as

$$r(\mathbf{p} - \mathbf{p}'; t, t') = \int_{\infty}^{\infty} f(t, s) \cdot f(t', s) \cdot r_S(\mathbf{p} - \mathbf{p}' + \mathbf{v}(t - t'); s)\ ds$$

in which we observe that the dynamic field is equivalent to the static field subordinated to the deterministic dynamics, i.e. $Y(\mathbf{p}, t) = X(\mathbf{p} + \mathbf{v} \cdot t, t)$.

Theorem 5. *The center of velocities in the x-direction of the field Y that is driven by constant velocity $\mathbf{v} = (v_1, v_2)$ is given by*

$$v_1 + v_2 \frac{\int |f(t, s)|^2 r_S^{xy}(\mathbf{0}; s)\ ds}{\int |f(t, s)|^2 r_S^{xx}(\mathbf{0}; s)\ ds}.$$

If additionally the innovations are homogeneous (isotropic and stationary), then the above velocity equals the constant flow velocity component v_1.

Proof. By Lemma 1 of the Appendix we have

$$\begin{aligned}\mathbb{C}ov(Y^x(\mathbf{p},t),Y^t(\mathbf{p},t)) =& \int_{-\infty}^{\infty} f(t,s)\cdot f^t(t,s)\cdot r_S^x(\mathbf{0};s)\ ds+\\ &+v_1\int_{-\infty}^{\infty}|f(t,s)|^2\cdot r_S^{xx}(\mathbf{0};s)\ ds+v_2\int_{-\infty}^{\infty}|f(t,s)|^2\cdot r_S^{xy}(\mathbf{0};s)\ ds\\ =&v_1\int_{-\infty}^{\infty}|f(t,s)|^2\cdot r_S^{xx}(\mathbf{0};s)\ ds+v_2\int_{-\infty}^{\infty}|f(t,s)|^2\cdot r_S^{xy}(\mathbf{0};s)\ ds.\end{aligned}$$

For the homogeneous field it is enough to use Lemma 2 to conclude that $r_S^{xy}(\mathbf{0};s)=0$ and the result follows.

$\square$

Example 5. Consider the Ornstein-Uhlenbeck type time dependence. Then

$$r(\mathbf{p},\mathbf{p}';t,t') = e^{-\lambda(t+t')}\int_{-\infty}^{t\wedge t'} e^{2\lambda s} r_S(\mathbf{p}-\mathbf{p}'-\mathbf{v}(t-t');s)\ ds.$$

If additionally r_S does not depend on time, then

$$r(\mathbf{p},\mathbf{p}';t,t') = \frac{1}{2\lambda} r_S(\mathbf{p}-\mathbf{p}'-\mathbf{v}(t-t'))\cdot e^{-\lambda|t-t'|}.$$

3.2. **Spatio-temporal dynamical models.** In the static scheme described in Section 2, a stochastic field $X(\mathbf{p})$ has been built from independent innovation fields $\Phi(\mathbf{p};ds)$ that occurred at time s and were summed up while weighted by $f(s)$. Dynamics can be introduced to this model by assuming that the contribution to a field $Y(\mathbf{p},t)$ from the innovation field $\Phi(\cdot;ds)$ that occurred at time s is not evaluated at the point $\mathbf{p}$ but at the point $\boldsymbol{\psi}_{t,s-t}(\mathbf{p})$ that corresponds to the location at time s of what at time t is at $\mathbf{p}$. This has been presented in the previous section for the constant velocity dynamics where $\boldsymbol{\psi}_{t,h}(\mathbf{p})=\mathbf{p}+\mathbf{v}\cdot h$. It can be generalized as follows.

Let us consider a flow $\boldsymbol{\psi}_{t,h}(\mathbf{p})$ obtained from a velocity field $\mathbf{v}(\mathbf{p},t)$ satisfying the transport equation

$$\boldsymbol{\psi}_{t,h}(\mathbf{p}) = \mathbf{p} + \int_t^{t+h} \mathbf{v}(\boldsymbol{\psi}_{t,u-t}(\mathbf{p}),u)\ du = \mathbf{p}+\int_0^h \mathbf{v}(\boldsymbol{\psi}_{t,s}(\mathbf{p}),t+s)\ ds, \tag{20}$$

i.e. a point with the initial location $\mathbf{p}$ at t relocates after h time units to $\boldsymbol{\psi}_{t,h}(\mathbf{p})$. In what follows it will be convenient to use $\boldsymbol{\psi}(\mathbf{p},t,h)$ for $\boldsymbol{\psi}_{t,h}(\mathbf{p})$ and $\boldsymbol{\psi}^t(\mathbf{p},t,h)=\boldsymbol{\psi}^t_{t,h}(\mathbf{p})$ for the partial derivative of $\boldsymbol{\psi}$ with respect to t. We note the following basic properties

$$\begin{aligned}
&\boldsymbol{\psi}(\mathbf{p}, t, 0) = \mathbf{p}, \\
&\boldsymbol{\psi}(\boldsymbol{\psi}(\mathbf{p}, t, h), t+h, \tilde{h}) = \boldsymbol{\psi}(\mathbf{p}, t, h+\tilde{h}), \\
&\frac{\partial \boldsymbol{\psi}}{\partial h}(\mathbf{p}, t, h) = \mathbf{v}(\boldsymbol{\psi}_{t,h}(\mathbf{p}), t+h), \\
&\frac{\partial \boldsymbol{\psi}}{\partial h}(\mathbf{p}, t, 0) = \mathbf{v}(\mathbf{p}, t).
\end{aligned} \tag{21}$$

Construction of the stochastic field

$$Y(\mathbf{p}, t) = \int_a^b f(t, s)\ \Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}); ds) \tag{22}$$

with dynamics driven by $\boldsymbol{\phi}$ is obtained at the fixed location and fixed time t from the following elements

- $\Phi(\mathbf{p}, (s, s+ds])$ – the field generated at time s with assumed independence between different s,
- $f(t, s)$ – a weight function defining how much of contribution should come from the spatial field $\Phi(\mathbf{p}, (s, s+ds])$,
- $\boldsymbol{\psi}_{t,s-t}(\mathbf{p})$ – the location at time s of a flow element that at time t resides at $\mathbf{p}$,
- $\Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), (s, s+ds])$ – the value of the field at time s that contributes to $Y(\mathbf{p}, t)$ after accounting on the flow movement,
- $f(t, s) \cdot \Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), (s, s+ds])$ – contribution to $Y(\mathbf{p}, t)$ accounted for the weight function $f(t, s)$.

Consequently, the contribution to $Y(\mathbf{p}, t)$ at time s is coming from

$$\Phi_t(\mathbf{p}; ds) := \int_s^{s+ds} \Phi(\boldsymbol{\psi}_{t,s-t}(\mathbf{p});\ ds) \tag{23}$$

multiplied by $f(t, s)$ and the integral $Y(\mathbf{p}, t) = \int_{-\infty}^{\infty} f(s, t)\ \Phi_t(\mathbf{p};\ ds)$ in its essence does not differ from the one defined in (4). One has simply to consider

$$r_S^t(\mathbf{p}, \mathbf{p}'; s) = r_S(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), \boldsymbol{\psi}_{t,s-t}(\mathbf{p}'))$$

instead of $r_S(\mathbf{p}, \mathbf{p}'; s)$. Thus if we have two fields $X(\mathbf{p}, t)$ and $Y(\mathbf{p}', t')$ with corresponding functions f and g, we obtain the cross-correlation formula

$$\mathbb{C}ov(X(\mathbf{p}, t), Y(\mathbf{p}', t')) = \int_{-\infty}^{\infty} f(s, t) \cdot g(s, t') \cdot r_S(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), \boldsymbol{\psi}_{t',s-t'}(\mathbf{p}'); s)\ ds. \tag{24}$$

The technical but standard details of the above construction which for the most part hinge on the definition (23) are omitted.

The median the velocity of the so defined $Y(\mathbf{p}, t)$ is given in the next result.

Theorem 6. *Let the field measure $\Phi(\mathbf{p}; ds)$ be driven by stationary in space innovations, so $r_S^x(\mathbf{0}; s) = 0$. Then the center of the velocity in the x-direction is given by*

$$V = \frac{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{t,s-t}(\mathbf{p})}{\partial x}^T \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \left(\boldsymbol{\psi}_{t,s-t}^t(\mathbf{p}) - \mathbf{v}(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), s)\right)\, ds}{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{t,t-s}(\mathbf{p})}{\partial x}^T \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \frac{\partial \boldsymbol{\psi}_{t,t-s}(\mathbf{p})}{\partial x}\, ds}, \tag{25}$$

where $r_S^{xx}, r_S^{yy}, r_S^{xy}, r_S^{yx}$ are all evaluated at $\boldsymbol{\psi}_{t,s-t}(\mathbf{p})$. If additionally it is assumed that the innovations are isotropic, then

$$V = \frac{\displaystyle\int |f(t,s)|^2 \cdot \left(x_{t,s-t}^x(\mathbf{p}) \cdot \alpha_{t,s-t}(\mathbf{p}) \cdot r_S^{xx} + y_{t,s-t}^x(\mathbf{p}) \cdot \beta_{t,s-t}(\mathbf{p}) \cdot r_S^{yy}\right)\, ds}{\displaystyle\int |f(t,s)|^2 \cdot \left((x_{t,s-t}^x)^2(\mathbf{p}) \cdot r_S^{xx} + (y_{t,s-t}^x)^2(\mathbf{p}) \cdot r_S^{yy}\right)\, ds}, \tag{26}$$

where

$$\alpha_{t,h}(\mathbf{p}) = x_{t,h}^t(\mathbf{p}) - v_1(\boldsymbol{\psi}_{t,h}(\mathbf{p}), t+h),$$

$$\beta_{t,h}(\mathbf{p}) = y_{t,h}^t(\mathbf{p}) - v_2(\boldsymbol{\psi}_{t,h}(\mathbf{p}), t+h).$$

Here $\boldsymbol{\psi}_{t,h}(\mathbf{p}) = (x_{t,h}(\mathbf{p}), y_{t,h}(\mathbf{p}))$, $\boldsymbol{\psi}_{t,h}^t(\mathbf{p}) = (x_{t,h}^t(\mathbf{p}), y_{t,h}^t(\mathbf{p}))$, and $\mathbf{v}(\mathbf{p}, t) = (v_1(\mathbf{p}, t), v_2(\mathbf{p}, t))$.

Proof. The proof is a direct consequence of (18) after applying Lemma 2 and the formulas for covariances given in (56) both in the Appendix.

□

Example 6 (Temporal Dynamic Ornstein-Uhlenbeck). A dynamic modification of the Ornstein-Uhlenbeck model discussed in Example 2 is obtained by taking

$$Y(\mathbf{p}, t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\ \Phi_t(\mathbf{p}; ds) \tag{27}$$

with covariance

$$\mathbb{C}ov(Y(\mathbf{p}, t), Y(\mathbf{p}', t')) = e^{-\lambda(t+t')} \int_{-\infty}^{t \wedge t'} e^{-2\lambda s} r_S(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}), \boldsymbol{\psi}_{t',s-t'}(\mathbf{p}'); s)\, ds. \tag{28}$$

We note that the covariance is no longer separable even if $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p}, \mathbf{p}')$.

Example 7 (Dynamic Autoregression Field of Order One). A discretized way of introducing dynamics represented by a flow $\boldsymbol{\psi}$ for some suitably chosen time lag dt is through the following recurrence

$$Y(\mathbf{p},t) = \rho Y(\boldsymbol{\psi}_{t,-dt}(\mathbf{p}), t-dt) + \sqrt{1-\rho^2}\epsilon_t(\mathbf{p}), \tag{29}$$

where $\rho = \rho(dt) = e^{-\lambda dt}$ for some $\lambda > 0$ and $\epsilon_t(\mathbf{p})$ represent independent in time t fields with the spatial covariance $\mathbb{C}ov(\epsilon_t(\mathbf{p}), \epsilon_t(\mathbf{p}')) = r_S(\mathbf{p},\mathbf{p}';t)$. This example has been discussed in detail in [4].

In the next result, it is shown that the last two examples describe in fact the same model.

Theorem 7. *Consider a spatio-temporal centered Gaussian field, $Y(\mathbf{p},t)$ defined by the recursive formula (29). Then, it has a covariance function that converges with time increment dt decreasing to zero to the covariance function (28) scaled by 2λ.*

Proof. Let $t = kdt$ for some $k \in \mathbb{Z}$. Using the flow properties given in (21), the recursive formula in (29) can be rewritten in a non-recursive way assuming that the series below is convergent

$$\begin{aligned} Y(\mathbf{p},kdt) &= \sum_{j=0}^{\infty} \rho^j \sqrt{1-\rho^2}\epsilon_{(k-j)dt}(\boldsymbol{\psi}_{kdt,-jdt}(\mathbf{p})) \\ &= \sqrt{1-\rho^2} \sum_{l=-\infty}^{k} \rho^{k-l}\epsilon_{ldt}(\boldsymbol{\psi}_{kdt,(l-k)dt}(\mathbf{p})). \end{aligned} \tag{30}$$

The covariance function if $t \le t' = k'dt$ is given by

$$\mathbb{C}ov(Y(\mathbf{p},t), Y(\mathbf{p}',t')) = (1-\rho^2)\sum_{l=-\infty}^{k} \rho^{k+k'-2l} \cdot r_S(\boldsymbol{\psi}_{kdt,(l-k)dt}(\mathbf{p}), \boldsymbol{\psi}_{k'dt,(l-k')dt}(\mathbf{p}'); ldt). \tag{31}$$

For small values of dt, $1-\rho^2 \approx 2\lambda \cdot dt$, and therefore letting $dt \to 0$ we have

$$\lim_{dt\to 0} \mathbb{C}ov(Y(\mathbf{p},t), Y(\mathbf{p}',t')) = 2\lambda \int_{-\infty}^{t} e^{-\lambda(t+t'-2u)} r_S(\boldsymbol{\psi}_{t,u-t}(\mathbf{p}), \boldsymbol{\psi}_{t',u-t'}(\mathbf{p}'); u)\, du. \tag{32}$$

□

The most complicated dynamics appears when the temporal dependence of the field $Y(\mathbf{p},t)$ varies in space and additionally there is dynamics introduced by the deterministic flow $\boldsymbol{\psi}$. A model like this can be written as

$$Y(\mathbf{p},t) = \int_{-\infty}^{\infty} f(t,s;\mathbf{p})\ \Phi_t(\mathbf{p};ds),$$

with the covariance

$$\mathbb{C}ov(Y(\mathbf{p},t),Y(\mathbf{p}',t')) = \int_{-\infty}^{\infty} f(t,s;\mathbf{p})\cdot f(t',s,\mathbf{p}')\cdot r_S(\boldsymbol{\psi}_{t,s-t}(\mathbf{p})\boldsymbol{\psi}_{t',s-t'}(\mathbf{p}');s)\ ds.$$

Example 8 (Space varying Ornstein-Uhlenbeck field driven by a dynamical flow). The model is an extension of (27) that can be obtained by letting in (11) the parameter λ depend on $\mathbf{p}$, i.e. by considering

$$Y(\mathbf{p},t) = \int_{-\infty}^{t} e^{-\lambda(\mathbf{p})(t-s)}\ \Phi_t(\mathbf{p},t;ds). \tag{33}$$

Then

$$\mathbb{C}ov(Y(\mathbf{p},t),Y(\mathbf{p}',t')) = e^{\lambda(\mathbf{p}')t'+\lambda(\mathbf{p})t}\int_{-\infty}^{t\wedge t'} e^{-(\lambda(\mathbf{p})+\lambda(\mathbf{p}'))s} r_S(\boldsymbol{\psi}_{t,s-t}(\mathbf{p}),\boldsymbol{\psi}_{t',s-t'}(\mathbf{p}'))\ ds. \tag{34}$$

In particular, when the flow is generated by a constant velocity $\mathbf{v}$ (the case that is important in local approximations of more general fields) and $r_S(\mathbf{p},\mathbf{p}';s) = r_S(\mathbf{p}-\mathbf{p}')$, we obtain for $t<t'$

$$\mathbb{C}ov(Y(\mathbf{p},t),Y(\mathbf{p}',t')) = r_S\left(\mathbf{p}'-\mathbf{p}-\mathbf{v}(t'-t)\right)\cdot\frac{1}{\lambda(\mathbf{p})+\lambda(\mathbf{p}')}\cdot e^{-\lambda(\mathbf{p}')(t'-t)}. \tag{35}$$

4. Fatigue damage of a vessel - an application

4.1. Introduction. Material fatigue is one of the most important safety issues for structures subjected to cyclic loads and the cause of failure in a majority of cases. Fatigue is a two phase process that starts with the initiation of microscopic cracks in the material and continues with these cracks growing to a critical size at which a fracture occurs. Often in large structures, cracks initiate at the construction phase in which case their growth is computed using fracture mechanics. There is a number of factors, depending both on the component and the material the component is made of, that influence the fatigue life. Such factors are geometry, size of the structure, surface smoothness, surface coating, residual tensions, material grain size and inner defects. Furthermore, the nature of the load process is also of importance. The complex dependence between these factors and the fatigue life makes predictions uncertain.

Experiments from fatigue life tests, even during controlled laboratory experiments, exhibit considerable scatter especially for high cycle life.

In this example we are interested in computing the fatigue at some critical point of a vessel. For simplicity, we consider dependence only on the load variability and the rest of the uncertainties are represented by a “quality” factor k. Empirically it is known that the damage rate of the material, $d(t)$, is proportional to the average wave energy raised to the power 1.5. Then, the damage rate $d(t)$ is defined

$$d(t) = k \cdot f_z(t) \cdot H_s(t)^3$$

where $f_z(t)$ is the intensity of the encountered waves at time t, $H_s(t)$ is the significant wave height at time t, which equals four times the standard deviation of the sea surface elevation and k is a generic constant. In [9], the authors have used the more realistic damage rate

$$d(t) = k_1(\beta) \cdot H_s(t)^{2.5} + k_2(\beta) \cdot H_s(t)^2, \tag{36}$$

where β is the heading angle, while the constants $k_1(\beta), k_2(\beta)$ depend strongly on the location of the structural detail on a ship, the carried load and some additional factors. Summarizing, in order to study the fatigue damage accumulation process one needs to be able to compute the distribution of integrals of polynomials in $H_s(t)$, such as the total damage

$$D = \int_{t_0}^{t_1} d(t)\, dt, \tag{37}$$

where $d(t)$ is given by (36).

Already at the design stage it is important to have some estimates of the variability of the total damage, D, for different possible routes during the operation life of a vessel. More precisely, let t_0 be the starting date for a voyage $\mathbf{p}(t) = (x(t), y(t))$, $t \in [t_0, t_1]$, where $x(t), y(t)$ are the coordinates of the vessel at time t. (Alternatively, a route could be specified using the starting location $\mathbf{p}(t_0)$ and the ship velocity $\mathbf{v}^s(t) = (v_x^s(t), v_y^s(t))$ for the duration of the trip $[t_0, t_1]$.) In the following subsection we provide the means for simulating the total damage D for different routes.

4.2. **Encountered significant wave heights during a voyage.** As reported in [6], the significant wave height H_s at position $\mathbf{p}$ and time t is accurately modelled by means of a log-normal distribution. Let $X(\mathbf{p}, t) = \ln(H_s(\mathbf{p}, t))$ denote the field of logarithmic values of the significant wave height that evolves with time. Let also t_0 denote the starting date of a voyage, $\mathbf{p}(t) = (x(t), y(t)), t \in [t_0, t_1]$ the planned route, and $\mathbf{v}(t) = (v_x(t), v_y(t))$ the velocity the ship moves with. Additionally let $z(t) = X(\mathbf{p}(t), t)$ be the encountered logarithms of the significant wave height field. (The encountered significant wave heights are $H_s(t) = \exp(z(t))$.) Then the process $z(t)$ is non stationary Gaussian with moments which we will compute next.

Locally stationary field: Suppose that for a fixed geographical region and season $X(\mathbf{p}, t)$ is modelled as a stationary Gaussian field that drifts (moves) with constant velocity $\mathbf{V} = (V_x, V_y)$, has mean m, variance σ^2 and separable correlation structure, which can attain the form

$$\mathbb{C}ov(X(\mathbf{p}_1, t_1), X(\mathbf{p}_2, t_2)) = \sigma^2 \rho_S(x_2 - x_1 - V_x(t_2 - t_1), y_2 - y_1 - V_y(t_2 - t_1)) \cdot \rho_T(t_2 - t_1),$$

where as before ρ_S denotes the spatial correlation and ρ_T the temporal correlation. In our particular case, the correlation ρ_S could be estimated using a map of significant wave heights derived by means of reanalysis data (ERA-40) or using satellite measurements, while the temporal correlation ρ_T can be computed using buoy measurements. If additionally the vessel is sailing with constant velocity (v_x, v_y), then the process $z(t)$ is also stationary Gaussian with the same mean as the field and covariance function given by

$$\mathbb{C}ov(z(t_1), z(t_2)) = \sigma^2 \rho_S(v_1 (t_2 - t_1), v_2 (t_2 - t_1))\rho_T(t_2 - t_1) = r_z(t_2 - t_1), \tag{38}$$

where $v_1 = v_x - V_x$ and $v_2 = v_y - V_y$. In [3] the authors have used in formula (38),

$$\rho_S(x, y) = \exp(-(x^2 + y^2)/2L^2), \qquad \rho_T(t) = \exp(-\lambda|t|), \tag{39}$$

where t was measured in hours and the parameters L and λ were slowly varying over the oceans and the different seasons.

Since the z is stationary it has a power spectral density $S(\omega)$ which depends on the parameters σ^2, L, λ as well as the relative ship velocity $\mathbf{v} = (v_1, v_2)$. The parameters σ^2, L were estimated by means of satellite observations while λ was estimated using buoy measuremets, see [3] where the variability of the parameters in space and time over the globe is presented.

However, since the statistical properties of the sea change with the geographical region, the parameters m, σ^2, L, λ and the velocity $\mathbf{v}$ vary with space and hence the encountered process

$z(t)$ cannot be assumed stationary during the whole voyage. Since though the properties of the encountered process z change slowly we can model it by means of locally stationary processes to the definition of which we turn next.

Let $S_t(\omega)$ be the spectrum of a stationary process z with the covariance function defined by formulas (38-39) where the parameters $\sigma^2(t), L(t), \lambda(t)$ and $\mathbf{v}(t)$ are functions of the vessel's position $\mathbf{p}(t)$. If S_t is known for all $t \in [t_0, t_1]$ then a "locally stationary" process z can be defined by means of spectral representation and moving average construction, as follows

$$z(t) = \int \exp(-it\,\omega)\sqrt{S_t(\omega)}\,dB(\omega), \tag{40}$$

where $B(\omega)$ is a Brownian motion. This technical construction results in a non-stationary Gaussian model for z, with $e^{z(t)} = m(t)$ and covariance

$$\mathbb{C}ov(z(t_1), z(t_2)) = \int \exp(-i(t_2 - t_1)\,\omega)\sqrt{S_{t_1}(\omega)S_{t_2}(\omega)}\,d\omega = r_z(t_1, t_2). \tag{41}$$

As a result of the fundamental property of the Gaussian models, the process $H_s(t) = \exp(z(t))$ is uniquely defined by the encountered local spectra $S_t(\omega)$ and the mean values $m(t)$.

Having estimates of the first two moments of the encountered process z, one can compute, employing the methods presented in [7], the mean $\mathbb{E}[D_j]$ and the variance $\mathbb{V}ar(D_j)$ of the damage during the j^{th} trip assuming that the heading angle $\beta(t)$ and the vessel's speed are known. Alternatively, the distribution of the damage D_j can be computed using Monte Carlo methods. For this, one generates sequences of the encountered significant wave height processes during the j^{th} trip and computes the damage D_j. More precisely, let t_i, $i = 0, \dots, n$, with $t_{i+1} - t_i = \Delta t$ equal 30 minutes, be the times a vessel is at the locations $(x_i, y_i) = (x(t_i), y(t_i))$ and $H_s(i) = \exp(z_i)$ be the corresponding significant wave height values, where $z_i = z(t_i)$ are correlated normal variables. Then it is a simple task to generate a sequence of z_i with means $\mathbf{m} = [m_i]$, $m_i = m(t_i)$ and covariance matrix $\Sigma = [r_{ij}] = [r_z(t_i, t_j)]$. If instead of the integral (41) one has an explicit formula for the covariance r_z, that would improve considerably the speed of the calculations.

However in order to make computation fast one would like to have explicit formula for covariance r_z instead of the that has to be evaluated numerically. In the following subsection, we modify the autocorrelation function ρ_T in such a way that the covariance r_z is given by an explicit algebraic expression that depends only on easily interpretable parameters.

Approximation of $r_z(t_1,t_2)$. In [4], the authors have used formula (38) with $\rho_T(t) = \exp(-\lambda|t|)$ to model the temporal correlation structure of the significant wave field at a fixed position. A typical value for the parameter λ, estimated using buoy measurements, is 0.0125. That means that the correlation length τ_T, that is the time it takes for the temporal correlation at a position to drop to 0.6, is about 40 hours. The computations are simplified considerably in one instead approximates $\rho_T(t) = \exp(-0.0125\,|t|)$, with the Gaussian covariance with the same correlation length $\rho(t) = \exp(-0.5(t/\tau_T)^2)$ $\tau_T = 2/\lambda$. Then, formula (38) and some simple algebra gives

$$r_z(t) = \sigma^2 \exp(-0.5t^2/C^2), \quad C = \frac{\tau_T\,\tau_S}{\sqrt{\tau_T^2+\tau_S^2}}, \quad \tau_S = \sqrt{v_1^2+v_2^2}/L. \tag{42}$$

Note that the space correlation length, τ_S, has an interpretation analogous to that of τ_T, as the time it takes for a vessel to travel between the points $\mathbf{p}_1$ and $\mathbf{p}_2$ for which the spatial correlation of the logarithmic values of the significant wave heights drops to 0.6. The parameters τ_S and τ_T characterize the spatial and temporal sizes of storms, respectively. The covariance (42) is particularly convenient since now the power spectrum S_t, used in (40), attains the explicit formula

$$S_t(\omega) = \sigma^2 \frac{C}{\sqrt{2\pi}} \exp(-\omega^2\, C^2/2).$$

The spectrum depends on t since the parameters σ^2 and C depend on the location $\mathbf{p}(t)$. Assuming that $\sigma(t)$ and $C(t)$ are known the integral in (41) can be computed resulting to

$$r_z(t,s) = 2\sigma(t)\sigma(s)\frac{C(t)C(s)}{C(t)^2+C(s)^2}\, e^{-(t-s)^2/(C(s)^2+C(t)^2)}. \tag{43}$$

Example 9. This example is based on results presented in [9] where the authors studied the fatigue damage of a container ship sailing between Europe and Canada during the first half of 2008. For a detailed description see [12]. In Figure 9 (left) the routes along which the stresses were measured are presented. Using these stresses the damage during the different voyages were estimated. The empirical distribution of the observed damages is presented in Figure 9 (right). The dotted line corresponds to the distribution derived using the covariance model (43) with parameter values taken from [3] and the ship's specific constants $K_i(\beta)$. As heading angles we used those observed during the voyages. A passage was taken at random and the sequence of the logarithmic values of significant wave heights were simulated using the derived model. Then the damage D was computed by means of (37-36). The procedure

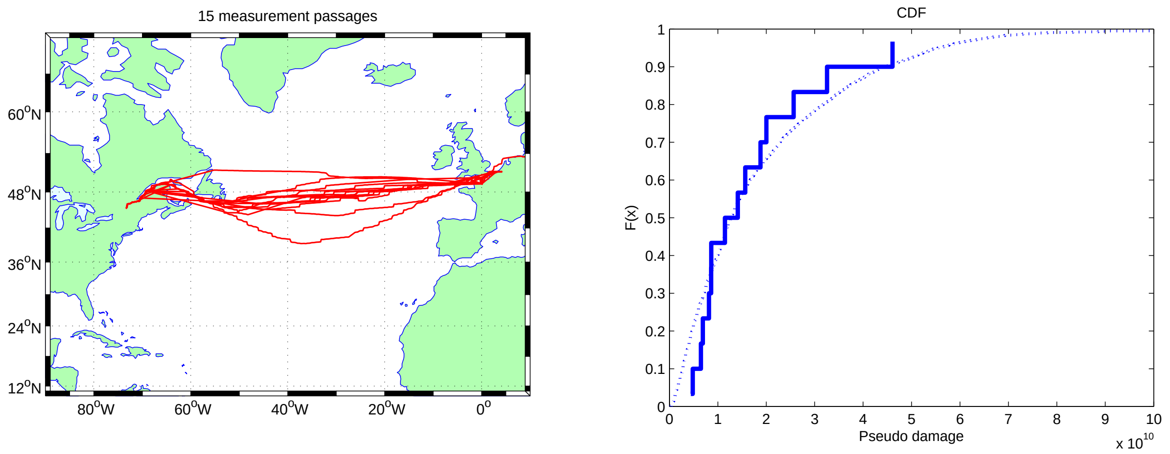

FIGURE 1. The 15 routes for which the damage was measured *(Left)*. Comparison between empirical cumulative distributions of the observed (computed from the measured stresses) damages for the 15 voyages and the cumulative distribution of D (dotted line). Here D has distribution derived by means of parametric bootstrap derived using and (43)*(Right)*.

was repeated 10000 times and the empirical distribution of the data resulted to the dotted line in Figure 9 (right). We can see that the agreement between the observed damage distribution and the distribution derived from the model is excellent, at least for this particular data set.

5. Appendix

5.1. **Integration with respect to field valued random measure.** The following is a short account of how to combine independent spatial fields that are obtained at time points according to various spatial covariances into a single spatial field. This is an example of a standard construction known in general theory of integration and functional analysis, see for example [10]. For each $t \in \mathbb{R}$, let $r_S(\mathbf{p}, \mathbf{p}'; t)$ be a spatial covariance in $\mathbf{p}$ and $\mathbf{p}'$ (non-negative definite function). We interpret it as a spatial covariance of independent innovations created at time t. We assume that for each $a < b$, the following integral is well defined as a function of $\mathbf{p}$ and $\mathbf{p}'$:

$$r_{(a,b]}(\mathbf{p}, \mathbf{p}') = \int_a^b r_S(\mathbf{p}, \mathbf{p}'; s)\ ds,$$

and thus corresponds to a certain spatial covariance function.

It follows from the additivity of the covariance function with respect to independent fields and its correspondence to the additivity of the integral that there exists a family of Gaussian spatial fields $\Phi(\mathbf{p}; (a, b])$ centered at zero such that

(i) For each $a < b, c < d \in \mathbb{R}$ we have

$$r_{(a,b]\cap(c,d]}(\mathbf{p}, \mathbf{p}') = \mathbb{C}ov(\Phi(\mathbf{p}; (a, b]), \Phi(\mathbf{p}'; (c, d])).$$

(ii) For $(a, b] = \bigcup_{i=1}^{\infty}(a_i, b_i]$, where $(a_i, b_i]$ are disjoint intervals, we have with probability one

$$\Phi(\mathbf{p}; (a, b]) = \sum_{i=1}^{\infty} \Phi(\mathbf{p}; (a_i, b_i]).$$

Thus Φ is a σ-additive measure having as values Gaussian random fields and (ii) is evocative of Lebesgue integration. Consequently, for a step function

$$f(t) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{(a_i,b_i]}(t), \tag{44}$$

where $(a_i, b_i]$ are disjoint we define $X(\mathbf{p}) := \int f(s)\Phi(\mathbf{p}; ds)$ as

$$X(\mathbf{p}) = \sum_{i=1}^{n} \alpha_i \Phi(\mathbf{p}; (a_i, b_i]). \tag{45}$$

The function $f(t)$ can be viewed as the weights with which the independent fields $\Phi(\mathbf{p}; ds)$ are weighted and add up to build the field $X(\mathbf{p})$. It follows immediately from (i) - (ii), that the integral is a Gaussian centered field with covariance $\int_{-\infty}^{\infty} f^2(s) r_S(\mathbf{p}, \mathbf{p}'; s)\ ds = \sum_{i=1}^{n} \alpha_i^2 r_{(a_i,b_i]}(\mathbf{p}, \mathbf{p}')$.

The remainder of the construction of $X(\mathbf{p}) = \int f(s)\Phi(\mathbf{p}; ds)$ extends it for any complex-valued function f that satisfies for each $\mathbf{p}$, $\mathbf{p}'$

$$\int_{-\infty}^{\infty} |f|^2(s) \cdot r_S(\mathbf{p}, \mathbf{p}'; s)\ ds < \infty,$$

which can be done using standard measure theoretic arguments that are skipped here. In particular, it follows that for the fields X and Y with corresponding f and g satisfying the above condition, we have

$$r_{X,Y}(\mathbf{p}, \mathbf{p}') = \mathbb{C}ov(X(\mathbf{p}), Y(\mathbf{p}')) = \int_{-\infty}^{\infty} f(s) \cdot \overline{g(s)} \cdot r_S(\mathbf{p}, \mathbf{p}'; s)\ ds. \tag{46}$$

5.2. **Partial derivative fields.** In this section we derive the partial mean square derivatives of the field $X(\mathbf{p},t)$ defined as the stochastic integral (4). For simplicity in presentation, we consider only the case $\mathbf{p} \in \mathbb{R}^2$. Generalization to higher dimensions is straightforward.

Let $\mathbf{p}$ and $\mathbf{p}'$ be two points in $\mathbb{R}^2$ with first coordinate x and x' respectively. The covariance functions that are considered here can in general depend on six variables: x, y, x', y', t, t' and partial derivatives of these covariances with respect to these variables will be indicated by upper-scripts. So for example, $r^{xx'}(\mathbf{p},\mathbf{p}';s)$ stands for the second order partial derivative $\frac{\partial^2}{\partial x \partial x'} r(\mathbf{p},\mathbf{p}';s)$. For a field $X(\mathbf{p},t)$ that depends only on three variables with generic names x, y, t, the derivatives are marked in a similar manner. For example $X^x(\mathbf{p}',t') = \frac{\partial X}{\partial x}(x',y',t')$.

The second order derivative of the covariance function with respect to the first spatial coordinates x and x' is given by

$$r^{xx'}_{(a,b]}(\mathbf{p},\mathbf{p}') = \frac{\partial^2}{\partial x \partial x'} r_{(a,b]}(\mathbf{p},\mathbf{p}') = \int_a^b r^{xx'}_S(\mathbf{p},\mathbf{p}';s)ds, \tag{47}$$

which is again a covariance function. It follows from the additivity of the covariance function with respect to independent fields and its correspondence to the additivity of the integral that there exists a family of Gaussian spatial fields $\Phi^x(\mathbf{p};(a,b])$ centered at zero with properties analogous to those of the field $\Phi(\mathbf{p};(a,b])$ with the governing covariance $r^{xx'}_S(\mathbf{p},\mathbf{p}';s)$.

By some standard arguments and under suitable regularity conditions it can be shown that

$$X^x(\mathbf{p}) = \int f(s)\Phi^x(\mathbf{p};ds). \tag{48}$$

To extend the above to calculus of partial derivatives for a process $X(\mathbf{p},t)$, basic facts about mean square derivatives can be employed and a detailed treatment of this can be found in [2]. Here we just present some basic principles and resulting formulas. For a field $X(\mathbf{t}), \mathbf{t} = (t_1,\dots,t_n) \in \mathbb{R}^n$ the mean square (partial) derivatives are defined as

$$X^{t_i} = \frac{\partial X}{\partial t_i}(\mathbf{t}) = \lim_{h\to 0} \frac{X(\mathbf{t}+h\cdot e_i) - X(\mathbf{t})}{h}, \tag{49}$$

where e_i is the vector with the i^{th} element 1 and all others zero while convergence is in the mean square sense. Derivatives of higher order are defined in an analogous way. It is straightforward that the covariance function of such partial derivatives must be given by

$$\mathbb{C}ov\left(\frac{\partial^k X(\mathbf{s})}{\partial s_{i_1}\dots\partial s_{i_k}}, \frac{\partial^k X(\mathbf{t})}{\partial t_{i_1}\dots\partial t_{i_k}}\right) = \frac{\partial^{2k} r(\mathbf{s},\mathbf{t})}{\partial s_{i_1}\partial t_{i_1}\dots\partial s_{i_k}\partial t_{i_k}}, \tag{50}$$

where $r(\mathbf{s},\mathbf{t}) = \mathbb{C}ov(X(\mathbf{s}), X(\mathbf{t}))$.

Lemma 1. *For a stationary field $X(\mathbf{t})$, the following relations between the field and its mean square derivatives (when they exist) are true.*
(i) The field and its first order partial derivatives, i.e., $X(\mathbf{t})$ and $X^{t_j}(\mathbf{t})$, when evaluated at the same point are uncorrelated.
(ii) The first and second order partial derivatives of the field evaluated at $\mathbf{t}$, i.e., $X^{t_i}(\mathbf{t})$ and $X^{t_j t_k}(\mathbf{t})$, are uncorrelated for any t_i, t_j and t_k.
(iii) If additionally the field is isotropic, i.e. the covariance function depends only on the Euclidean length $|\mathbf{t}|$ of the vector $\mathbf{t}$ so we can write $r(\mathbf{t}) = r(|\mathbf{t}|)$, then the first order derivatives of the field X^{t_i} and X^{t_j}, $i \neq j$, are uncorrelated.

Proof. For a stationary field $X(\mathbf{t})$ with (mean square) derivatives of orders $\alpha+\beta$ and $\gamma+\delta$ for $\alpha,\beta,\gamma,\delta \in \{0,1,2,\dots\}$, formula (50) takes the equivalent form

$$\mathbb{C}ov\left(\frac{\partial^{\alpha+\beta}X(\mathbf{t})}{\partial^\alpha t_i \partial^\beta t_j}, \frac{\partial^{\gamma+\delta}X(\mathbf{t})}{\partial^\gamma t_k \partial^\delta t_l}\right) = (-1)^{\alpha+\beta}\frac{\partial^{\alpha+\beta+\gamma+\delta}r(\mathbf{t})}{\partial^\alpha t_i \partial^\beta t_j \partial^\gamma t_k \partial^\delta t_l}|_{\mathbf{t}=\mathbf{0}}. \tag{51}$$

Remember that stationarity implies $r(\mathbf{t}) = r(-\mathbf{t})$, (with some abuse in notation) which in turn means that all odd ordered derivatives of the covariance r are identically zero. Hence in view of that, follows from (51) for $\beta=\gamma=\delta=0$ and $\alpha=1$ that $X(\mathbf{t})$ and $X^{t_j}(\mathbf{t})$ are uncorrelated for every j and all $\mathbf{t}$, since the first order derivative of the covariance at zero equals zero, $\frac{\partial r(\mathbf{t})}{\partial t_i}|_{\mathbf{t}=\mathbf{0}} = 0$, which in the Gaussian case is equivalent to independent. Also for $\beta=\gamma=\delta=1$ and $\alpha=0$, we obtain $X^{t_i}(\mathbf{t})$ and $X^{t_j t_k}(\mathbf{t})$ are uncorrelated for all i,j,k and every $\mathbf{t}$. This proves statements (i) and (ii).

To see (iii) notice that it is enough to consider only the two dimensional case $\mathbf{t} = (t_1, t_2)$. The spectral measure S of a stationary and isotropic field is also isotropic, i.e. if $\mathbf{R}_\phi$ is the rotation by an angle ϕ, then $S = S \circ \mathbf{R}_\phi$. The covariance between X^{t_1} and X^{t_2} is given by

$$r^{t_1 t_2}(\mathbf{t}) = \int_{\mathbb{R}^2} \omega_1 \omega_2 e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, dS(\boldsymbol{\omega}).$$

Thus for $\widetilde{\boldsymbol{\omega}} = \boldsymbol{R}_{\pi/2}\boldsymbol{\omega}$, we have

$$
\begin{aligned}
r^{t_1 t_2}(\mathbf{0}) &= \int_{\mathbb{R}^2} \omega_1\omega_2 \; dS(\boldsymbol{\omega}) \\
&= \int_{\omega_1\omega_2>0} \omega_1\omega_2 \; dS(\boldsymbol{\omega}) + \int_{\omega_1\omega_2<0} \omega_1\omega_2 \; dS(\boldsymbol{\omega}) \\
&= -\int_{\tilde{\omega}_1\tilde{\omega}_2<0} \tilde{\omega}_1\tilde{\omega}_2 \; dS \circ \mathbf{R}_{-\pi/2}(\widetilde{\boldsymbol{\omega}}) + \int_{\omega_1\omega_2<0} \omega_1\omega_2 \; dS(\boldsymbol{\omega}) \\
&= -\int_{\tilde{\omega}_1\tilde{\omega}_2<0} \tilde{\omega}_1\tilde{\omega}_2 \; dS(\widetilde{\boldsymbol{\omega}}) + \int_{\omega_1\omega_2<0} \omega_1\omega_2 \; dS(\boldsymbol{\omega}) \\
&= 0.
\end{aligned}
$$

□

Note that the statements (i), (ii) and (iii) do not imply that the field and its first order derivatives, or the first and second order derivatives or the first order derivatives with each other are uncorrelated as fields. For example, in general $\mathbb{C}ov(X(\mathbf{s}), X^{t_i}(\mathbf{t})) \neq 0$, if $\mathbf{s} \neq \mathbf{t}$.

Corollary 1. *If the considered field is Gaussian, the uncorrelated variables from the above results become independent.*

Let $X(\mathbf{p}, t) = \int f(t, s) \; \Phi(\mathbf{p}; ds)$ so the following relations hold

$$
\begin{aligned}
X^x(\mathbf{p}, t) &= \int f(t, s) \; \Phi^x(\mathbf{p}; ds) \\
X^y(\mathbf{p}, t) &= \int f(t, s) \; \Phi^y(\mathbf{p}; ds) \\
X^t(\mathbf{p}, t) &= \int f^t(t, s) \; \Phi(\mathbf{p}; ds).
\end{aligned} \tag{52}
$$

Using basic partial derivative calculus such as formula (50) we can obtain any covariance between the field and the different partial derivatives, like for example,

$$
\begin{aligned}
\mathbb{C}ov(X(\mathbf{p}, t), X^x(\mathbf{p}', t')) &= \int f(t, s) f(t', s) \cdot r_S^{x'}(\mathbf{p}, \mathbf{p}'; s) \; ds \\
\mathbb{C}ov(X(\mathbf{p}, t), X^t(\mathbf{p}', t')) &= \int f(t, s) f^t(t', s) \cdot r_S(\mathbf{p}, \mathbf{p}'; s) \; ds \\
\mathbb{C}ov(X^x(\mathbf{p}, t), X^t(\mathbf{p}', t')) &= \int f(t, s) f^t(t', s) \cdot r_S^x(\mathbf{p}, \mathbf{p}'; s) \; ds \\
\mathbb{C}ov(X^x(\mathbf{p}, t), X^y(\mathbf{p}', t')) &= \int f(t, s) f(t', s) \cdot r_S^{xy'}(\mathbf{p}, \mathbf{p}'; s) \; ds
\end{aligned} \tag{53}
$$

Lemma 2. *If the field $X(\mathbf{p},t)$ is governed by r_S that is stationary in space, then*

$$
\begin{aligned}
&\mathbb{C}ov(X(\mathbf{p},t), X^x(\mathbf{p},t')) = 0\\
&\mathbb{C}ov(X^x(\mathbf{p},t), X^t(\mathbf{p},t')) = 0 \qquad (54)\\
&\mathbb{C}ov(X^y(\mathbf{p},t), X^t(\mathbf{p},t')) = 0.
\end{aligned}
$$

If additionally $X(\mathbf{p},t)$ is governed by r_S that is homogeneous (stationary in space and isotropic), then

$$
\mathbb{C}ov(X^x(\mathbf{p},t), X^y(\mathbf{p},t')) = 0. \qquad (55)
$$

Proof. The result follows from the equations in (53) and Lemma 1 applied to governing correlations r_S.

□

In the general case of a space dependent kernel

$$
X(\mathbf{p},t) = \int f(t,s;\mathbf{p})\ \Phi(\mathbf{p};ds),
$$

the covariances become slightly more complicated. Here we list some cases where all fields are considered at the same $(\mathbf{p},t)$:

$$
\begin{aligned}
r^{xx} &= \int |f^x(t,s;\mathbf{p})|^2 \cdot r_S(\mathbf{p},\mathbf{p};s)\ ds + \int |f(t,s;\mathbf{p})|^2 \cdot r_S^{xx}(\mathbf{p},\mathbf{p};s)\ ds+\\
&\quad + 2\int f^x(t,s;\mathbf{p}) f(t,s;\mathbf{p}) r_S^x(\mathbf{p},\mathbf{p};s)\ ds\\
r^{xt} &= \int f^x(t,s;\mathbf{p}) f^t(t,s;\mathbf{p}) \cdot r_S(\mathbf{p},\mathbf{p};s)\ ds + \int f(t,s;\mathbf{p}) f^t(t,s;\mathbf{p}) \cdot r_S^x(\mathbf{p},\mathbf{p};s)\ ds\\
r^{yt} &= \int f^y(t,s;\mathbf{p}) f^t(t,s;\mathbf{p}) \cdot r_S(\mathbf{p},\mathbf{p};s)\ ds + \int f(t,s;\mathbf{p}) f^t(t,s;\mathbf{p}) \cdot r_S^y(\mathbf{p},\mathbf{p};s)\ ds.
\end{aligned}
$$

Also, in view of Lemma 1, if the field is assumed to be stationary in space the last two equations in formula (56) simplify since the second terms on the right hand side of the equations equal zero.

Lemma 3. *If the field $X(\mathbf{p},t)$ is governed by r_S having space constant variance $\sigma^2(t)$, then (54) of Lemma 2 still holds.*

Proof. Holding the dependence on t implicit, we have

$$
\mathbb{C}ov(X(\mathbf{p}), X^x(\mathbf{p}')) = \frac{\partial}{\partial x} r(\mathbf{p},\mathbf{p}')|_{\mathbf{p}=\mathbf{p}'} = \frac{\partial}{\partial x'} r(\mathbf{p},\mathbf{p}')|_{\mathbf{p}=\mathbf{p}'} = 0,
$$

since constant variance and the symmetry of covariance imply that

$$0 \equiv \frac{\partial r(\mathbf{p},\mathbf{p})}{\partial x} = \frac{\partial}{\partial x} r(\mathbf{p},\mathbf{p}')|_{\mathbf{p}=\mathbf{p}'} + \frac{\partial}{\partial x'} r(\mathbf{p},\mathbf{p}')|_{\mathbf{p}=\mathbf{p}'} = 2\frac{\partial}{\partial x} r(\mathbf{p},\mathbf{p}')|_{\mathbf{p}=\mathbf{p}'}.$$

Thus, the field and its first order derivatives in space have to be uncorrelated.

Similarly, using (56), we get $\mathbb{C}ov(X^x(\mathbf{p},t), X^t(\mathbf{p},t)) = \mathbb{C}ov(X^y(\mathbf{p},t), X^t(\mathbf{p},t)) = 0$ since $f^x \equiv 0$, the kernel f is independent of location, and by the same argument as above $r_S^x(\mathbf{p},\mathbf{p};s) = r_S^y(\mathbf{p},\mathbf{p};s) = 0$.

□

5.3. **Dynamic flow driven derivative fields.** The diffeomorphism $\boldsymbol{\psi}_t(\mathbf{p}) = \boldsymbol{\psi}(\mathbf{p};t)$ given by (20) takes values in $\mathbb{R}^2$ so let us write $\psi_t(\mathbf{p}) = (x_t(\mathbf{p}), y_t(\mathbf{p})) = (x(\mathbf{p};t), y(\mathbf{p};t))$. We will also use the following convention $\boldsymbol{\psi}_{t+s}(\mathbf{p}) = \boldsymbol{\psi}_s(\mathbf{p};t)$ to explicitly point out the time argument t. We note the relations

$$\mathbf{v}(\mathbf{p},t) = \frac{\partial \boldsymbol{\psi}_{-t}(\mathbf{p};t)}{\partial t}$$
$$\mathbf{v}(\boldsymbol{\psi}_{s+t}(\mathbf{p}),t) = \frac{\partial \boldsymbol{\psi}_{s}(\mathbf{p};t)}{\partial t}$$

The field $Y(\mathbf{p},t)$ as defined by (22), has the following partial derivatives:

$$Y^x(\mathbf{p},t) = \int f(t,s) \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}^T \begin{bmatrix} \Phi^x(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \\ \Phi^y(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \end{bmatrix},$$

$$Y^y(\mathbf{p},t) = \int f(t,s) \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial y}^T \begin{bmatrix} \Phi^x(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \\ \Phi^y(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \end{bmatrix},$$

$$Y^t(\mathbf{p},t) = \int f^t(t,s)\ \Phi(\psi_{s-t}(\mathbf{p}); ds) - \int f(t,s) \cdot \mathbf{v}(\boldsymbol{\psi}_{s-t}(\mathbf{p}); -t)^T \begin{bmatrix} \Phi^x(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \\ \Phi^y(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds) \end{bmatrix}.$$

Here the field valued stochastic measures Φ^x, Φ^y are driven by covariances $r^{xx'}(\mathbf{p},\mathbf{p}',t)$ and $r^{yy'}(\mathbf{p},\mathbf{p}';t)$, respectively. Moreover, we follow the convention that vectors are column matrices and $\mathbf{A}^T$ stands for the transpose of matrix $\mathbf{A}$ so $(\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})/\partial x)^T$ is the row vector of derivatives in x.

It is quite straightforward to compute the covariances between the different partial derivatives as shown next

(56)

$$\mathbb{C}ov(Y^x(\mathbf{p},t), Y^t(\mathbf{p},t)) =$$

$$= \int f(t,s)\cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}^T \left(f^t(t,s) \begin{bmatrix} r_S^x \\ r_S^y \end{bmatrix} - \overline{f(t,s)} \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \mathbf{v}(\boldsymbol{\psi}_{s-t}(\mathbf{p}), -t) \right) ds,$$

$$\mathbb{V}ar(Y^x(\mathbf{p},t)) =$$

$$= \int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}^T \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}\, ds$$

where $r_S^x, r_S^y, r_S^{xx}, r_S^{yy}, r_S^{xy}, r_S^{yx}$ are all evaluated at $(\boldsymbol{\psi}_{s-t}(\mathbf{p}), \boldsymbol{\psi}_{s-t}(\mathbf{p}); s)$.

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