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Dynamics analysis and numerical simulations of a new 5D Lorenz-type chaos dynamical system

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Abstract

Ultimate bound sets of chaotic systems have important applications in chaos control and chaos synchronization. Ultimate bound sets can also be applied in estimating the dimensions of chaotic attractors. However, it is often a difficult work to obtain the bounds of high-order chaotic systems due to complex algebraic structure of high-order chaotic systems. In this paper, a new 5D autonomous quadratic chaotic system which is different from the Lorenz chaotic system is proposed and analyzed. Ultimate bound sets and globally exponential attractive sets of this system are studied by introducing the Lyapunov-like functions. To validate the ultimate bound estimation, numerical simulations are also investigated.

Keywords: Lorenz-type system, Lyapunov exponents, Lyapunov stability, chaotic attractor, ultimate bound estimation.
2010 MSC: 65P20, 65P30, 65P40, 34D40.

1. Introduction

Since Lorenz chaotic system was found in 1963 [16], chaotic systems have played an important role in a variety of engineering fields. Since then, some new chaotic systems which are not equivalent to Lorenz system were found, i.e., Rössler system [19], Chua system [2], Chen system [1], Lü system [17], Shimizu-Morioka system [8], etc. Not only basic properties and dynamical behaviors of these chaotic systems have been widely studied, but also chaotic control, chaotic synchronization, and other applications have been investigated by many researchers from various fields [3–22, 24, 26]. Various dynamical behaviors of these chaotic systems have been studied due to their various applications in the fields of population dynamics, atmospheric dynamics, electric circuits, image encryption, fluid dynamics, lasers, engineering, stock exchanges, chemical reactions, and so on [23, 25, 27–34]. In 2013, Wang et al. investigated a 5D

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chaotic system to describe the evolution of permanent magnet synchronous motor system in field-oriented rotor [25]:

$$\begin{cases} \frac{du_1}{dt} = a\,(u_3 - u_1)\,, \\ \frac{du_2}{dt} = a\,(u_4 - u_2)\,, \\ \frac{du_3}{dt} = bu_1 - u_3 - u_1u_5, \\ \frac{du_4}{dt} = bu_2 - u_4 - u_2u_5, \\ \frac{du_5}{dt} = u_1u_3 + u_2u_4 - u_5. \end{cases} \tag{1.1}$$

In this paper, we propose a generalized 5D chaotic system with five parameters as follows:

$$\begin{cases} \frac{du_1}{dt} = a\,(u_3 - u_1)\,, \\ \frac{du_2}{dt} = a\,(u_4 - u_2)\,, \\ \frac{du_3}{dt} = bu_1 - du_3 - u_1u_5, \\ \frac{du_4}{dt} = bu_2 - eu_4 - u_2u_5, \\ \frac{du_5}{dt} = u_1u_3 + u_2u_4 - cu_5, \end{cases} \tag{1.2}$$

where u_1, u_2, u_3, u_4, u_5 are state variables and a, b, c, d, e are positive parameters of system (1.2).

The Lyapunov exponents of the dynamical system (1.2) are calculated numerically for the parameter values $a = 35, b = 55, c = \frac{8}{3}, d = 1, e = 1$ with the initial state $(u_{10}, u_{20}, u_{30}, u_{40}, u_{50}) = (-1, -2, -3, -4, 1)$. System (1.2) has Lyapunov exponents as $\lambda_{LE_1} = 1.1041, \lambda_{LE_2} = 0, \lambda_{LE_3} = 0, \lambda_{LE_4} = -0.6473, \lambda_{LE_5} = -1.9854$ for the parameters listed above (see [5] and [26] for a detailed discussion of Lyapunov exponents of strange attractors in dynamical systems). When parameters $a = 35, b = 55, c = \frac{8}{3}, d = 1, e = 1$, system (1.2) has a chaotic attractor, as shown in Figure 1.

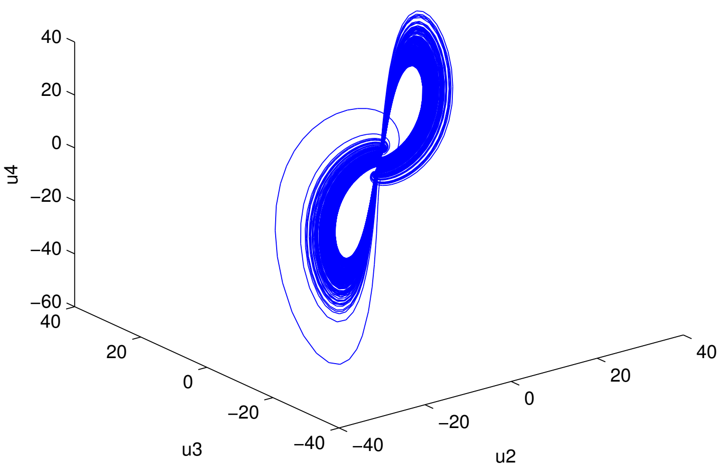

Figure 1: Projection of chaotic attractor of system (1.2) in (u_2, u_3, u_4) space.

In this paper, all the simulations are carried out by using fourth order Runge-Kutta Method with time-step $h = 0.01$.

Recently, ultimate bound sets of chaotic systems and hyperchaotic systems have been discussed in many research papers [7, 27, 28, 30, 31]. It is well-known that there is a bounded ellipsoid in R^3 for the famous Lorenz system, which all orbits of the Lorenz system will eventually enter for all positive parameters of the Lorenz system [7, 34]. The ultimate bound sets of chaotic systems can be used in chaos control and synchronization [32]. Furthermore, the ultimate bound sets of chaotic systems can be used for estimation of the fractal dimension of chaotic and hyperchaotic attractors [6, 18]. Usually, the approach to investigate the ultimate bound set of a given chaotic system is to construct the Lyapunov-like functions for the given system. However, the approach to construct Lyapunov-like functions in each system is only suitable for that particular system. It is very difficult to propose a universal approach to estimate the ultimate bound sets for an arbitrary chaotic system. Furthermore, it is often a difficult work to obtain the bounds of high-order chaotic systems due to complex algebraic structure of high-order chaotic systems. As far as the authors know, very little work has been done on ultimate bound sets of n-dimensional high-order chaotic systems with $n \geqslant 4$.

Motivated by the above discussion, we will investigate the bounds of the new 5D Lorenz-type chaotic system (1.2) in this paper. The innovation of the paper is that this paper not only proves this 5D system is global bounded for all parameters of this system according to Lyapunov stability theory of dynamical systems but also gives a family of mathematical expressions of global exponential attractive sets for this 5D system with respect to the parameters of this system.

The rest of this paper is organized as follows. The ultimate bound sets of chaotic system (1.2) are studied in Section 2. To validate the ultimate bound estimation, numerical simulations are also provided. In Section 3, the globally attractive sets for the chaotic attractors in (1.2) are studied based on Lyapunov stability theory. Finally, the conclusion is drawn in Section 4.

2. Ultimate bound estimation

Lemma 2.1. *Define the set*

$$\Gamma_0 = \left\{ (x_1, x_2, y_1, y_2, z) | \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{(z-c)^2}{c^2} + \frac{y_1^2}{d^2} + \frac{y_2^2}{e^2} = 1, \ abcde \neq 0 \right\},$$

and two functions

$$\begin{aligned} &G_1(x_1, x_2, y_1, y_2, z) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + z^2, \\ &H_1(x_1, x_2, y_1, y_2, z) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + (z-2c)^2, \quad (x_1, x_2, y_1, y_2, z) \in \Gamma_0. \end{aligned}$$

Then, we can get

$$\max_{(x_1,x_2,y_1,y_2,z)\in\Gamma_0} G_1 = \max_{(x_1,x_2,y_1,y_2,z)\in\Gamma_0} H_1 = \begin{cases} \frac{a^4}{a^2-c^2}, & a \geqslant b, \ \ a \geqslant d, \ \ a \geqslant e, \ \ a \geqslant \sqrt{2}c, \\ \frac{b^4}{b^2-c^2}, & b > a, \ \ b \geqslant d, \ \ b > e, \ \ b \geqslant \sqrt{2}c, \\ \frac{d^4}{d^2-c^2}, & d > a, \ \ d > b, \ \ d \geqslant e, \ \ d \geqslant \sqrt{2}c, \\ \frac{e^4}{e^2-c^2}, & e > a, \ \ e \geqslant b, \ \ e > d, \ \ e \geqslant \sqrt{2}c, \\ 4c^2, & a < \sqrt{2}c, \ \ b < \sqrt{2}c, \ \ d < \sqrt{2}c, \ \ e < \sqrt{2}c. \end{cases}$$

Proof. It can be easily proved by the Lagrange multiplier method. □

According to Lemma 2.1, we can get the following theorem for system (1.2).

Theorem 2.2. *For any* $m > 0, \ \lambda > 0, \ a > 0, \ b > 0, \ c > 0, \ d > 0, \ e > 0$, *the following set*

$$\Omega_{\lambda,m} = \left\{ (u_1, u_2, u_3, u_4, u_5) | \ mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda\left(u_5 - \frac{\lambda b + am}{\lambda}\right)^2 \leqslant R^2 \right\}, \tag{2.1}$$

is the ultimate bound set and positively invariant set of system (1.2), *where*

$$R^2 = \begin{cases} \frac{c^2(b\lambda+am)^2}{4\lambda a(c-a)}, & d \geqslant a, \ \ e \geqslant a, \ \ c \geqslant 2a, \\ \frac{c^2(b\lambda+am)^2}{4\lambda d(c-d)}, & a > d, \ \ e \geqslant d, \ \ c \geqslant 2d, \\ \frac{c^2(b\lambda+am)^2}{4\lambda e(c-e)}, & a > e, \ \ d > e, \ \ c \geqslant 2e, \\ \frac{(b\lambda+am)^2}{\lambda}, & c < 2d, \ \ c < 2a, \ \ c < 2e. \end{cases}$$

Proof. Define the Lyapunov-like function

$$V_{\lambda,m}(U)=V_{\lambda,m}(u_1,u_2,u_3,u_4,u_5)=mu_1^2+mu_2^2+\lambda u_3^2+\lambda u_4^2+\lambda\left(u_5-\frac{b\lambda+am}{\lambda}\right)^2 \tag{2.2}$$

for all $\lambda>0$, and for all $m>0$, $U=(u_1,u_2,u_3,u_4,u_5)$. Differentiating the above Lyapunov-like function $V_{\lambda,m}(U)$ in (2.2) with respect time t along the trajectory of system (1.2) yields

$$\begin{aligned}\left.\frac{dV_{\lambda,m}(U)}{dt}\right|_{(1.2)}&=2mu_1\frac{du_1}{dt}+2mu_2\frac{du_2}{dt}+2\lambda u_3\frac{du_3}{dt}+2\lambda u_4\frac{du_4}{dt}+2\lambda\left(u_5-\frac{b\lambda+am}{\lambda}\right)\frac{du_5}{dt}\\&=2amu_1(u_3-u_1)+2amu_2(u_4-u_2)+2\lambda u_3(bu_1-du_3-u_1u_5)\\&\quad+2\lambda u_4(bu_2-eu_4-u_2u_5)+2\lambda\left(u_5-\frac{b\lambda+am}{\lambda}\right)(u_1u_3+u_2u_4-cu_5)\\&=-2amu_1^2-2amu_2^2-2\lambda du_3^2-2\lambda eu_4^2-2\lambda cu_5^2+2c(b\lambda+am)u_5\\&=-2amu_1^2-2amu_2^2-2\lambda du_3^2-2\lambda eu_4^2-2\lambda c\left(u_5-\frac{\lambda b+am}{2\lambda}\right)^2+\frac{c(\lambda b+am)^2}{2\lambda}.\end{aligned}$$

Obviously, the set Γ_1 that defined by

$$\left\{(u_1,u_2,u_3,u_4,u_5)\,|amu_1^2+amu_2^2+\lambda du_3^2+\lambda eu_4^2+c\lambda\left(u_5-\frac{\lambda b+am}{2\lambda}\right)^2=\frac{c(\lambda b+am)^2}{4\lambda}\right\}, \tag{2.3}$$

is an ellipsoid in R^5 for all $\lambda>0$, and for all $m>0$, $a>0$, $b>0$, $c>0$, $d>0$, $e>0$. Outside Γ_1, $\frac{dV_{\lambda,m}(U)}{dt}<0$, while inside Γ_1, $\frac{dV_{\lambda,m}(U)}{dt}>0$. Thus, the maximum value of $V_{\lambda,m}(U)$ can only be reached on Γ_1. Since the $V_{\lambda,m}(U)$ is a continuous function and Γ_1 is a bounded closed set, then the function (2.2) can reach its maximum value $\max V_{\lambda,m}(U)=R^2, (U\in\Gamma_1)$ on the set defined in (2.3).

Obviously, $\{U|\,V_{\lambda,m}(U)\leqslant\max V_{\lambda,m}(U), U\in\Gamma_1\}$ contains solutions of system (1.2). By solving the following conditional extremum problem, one can get the maximum value of the function (2.2) as follows:

$$\begin{cases}\max\ V_{\lambda,m}(U)=\max\left\{mu_1^2+mu_2^2+\lambda u_3^2+\lambda u_4^2+\lambda\left(u_5-\frac{b\lambda+am}{\lambda}\right)^2\right\},\\ \text{s.t.}\ \ amu_1^2+amu_2^2+\lambda du_3^2+\lambda eu_4^2+c\lambda\left(u_5-\frac{b\lambda+am}{2\lambda}\right)^2=\frac{c(b\lambda+am)^2}{4\lambda}.\end{cases}$$

That is to say,

$$\begin{cases}\max\ V_{\lambda,m}(U)=\max\left\{mu_1^2+mu_2^2+\lambda u_3^2+\lambda u_4^2+\lambda\left(u_5-\frac{\lambda b+am}{\lambda}\right)^2\right\},\\ \text{s.t.}\ \frac{mu_1^2}{\frac{c(\lambda b+am)^2}{4a\lambda}}+\frac{mu_2^2}{\frac{c(\lambda b+am)^2}{4a\lambda}}+\frac{\lambda u_3^2}{\frac{c(\lambda b+am)^2}{4d\lambda}}+\frac{\lambda u_4^2}{\frac{c(\lambda b+am)^2}{4e\lambda}}+\frac{\lambda\left(u_5-\frac{\lambda b+am}{2\lambda}\right)^2}{\frac{(\lambda b+am)^2}{4\lambda}}=1.\end{cases} \tag{2.4}$$

Let us take $\sqrt{m}u_1=\tilde{u}_1, \sqrt{m}u_2=\tilde{u}_2, \sqrt{\lambda}u_3=\tilde{u}_3, \sqrt{\lambda}u_4=\tilde{u}_4, \sqrt{\lambda}u_5=\tilde{u}_5$ as the new variables, then conditional extremum problem (2.4) is transformed into the following form:

$$\begin{cases}\max V_{\lambda,m}(U)=\max\left\{\tilde{u}_1^2+\tilde{u}_2^2+\tilde{u}_3^2+\tilde{u}_4^2+\left(\tilde{u}_5-\frac{\lambda b+am}{\sqrt{\lambda}}\right)^2\right\},\\ \text{s.t.}\ \frac{\tilde{u}_1^2}{\frac{c(b\lambda+am)^2}{4a\lambda}}+\frac{\tilde{u}_2^2}{\frac{c(b\lambda+am)^2}{4a\lambda}}+\frac{\tilde{u}_3^2}{\frac{c(b\lambda+am)^2}{4d\lambda}}+\frac{\tilde{u}_4^2}{\frac{c(b\lambda+am)^2}{4e\lambda}}+\frac{\left(\tilde{u}_5-\frac{b\lambda+am}{2\sqrt{\lambda}}\right)^2}{\frac{(b\lambda+am)^2}{4\lambda}}=1.\end{cases} \tag{2.5}$$

According to Lemma 2.1, we can easily get the above conditional extremum problem (2.5) as

$$\max_{U\in\Gamma_1}V_{\lambda,m}(U)=\begin{cases}\frac{c^2(b\lambda+am)^2}{4\lambda a(c-a)},\ d\geqslant a,\ e\geqslant a,\ c\geqslant 2a,\\ \frac{c^2(b\lambda+am)^2}{4\lambda d(c-d)},\ a>d,\ e\geqslant d,\ c\geqslant 2d,\\ \frac{c^2(b\lambda+am)^2}{4\lambda e(c-e)},\ a>e,\ d>e,\ c\geqslant 2e,\\ \frac{(b\lambda+am)^2}{\lambda},\ c<2d,\ c<2a,\ c<2e.\end{cases}$$

Finally, it is easy to show that (2.1) is the ultimate bound set and positively invariant set of system (1.2). □

Remark 2.3.

(i) Let us take $\lambda > 0$, and $m > 0$ arbitrarily in Theorem 2.2, then we can get a series of ultimate bounds sets and positively invariant sets of system (1.2).

(ii) Let us take $m = 1$ in Theorem 2.2, then we can get that

$$\Omega_{\lambda,1}=\left\{(u_1,u_2,u_3,u_4,u_5)\,|\,u_1^2+u_2^2+\lambda u_3^2+\lambda u_4^2+\lambda\left(u_5-\frac{\lambda b+a}{\lambda}\right)^2\leqslant r^2,\ \forall\lambda>0\right\},$$

is the ultimate bound set and positively invariant set of system (1.2), where

$$r^2=\begin{cases}\frac{c^2(b\lambda+a)^2}{4\lambda a(c-a)}, & d\geqslant a,\ e\geqslant a,\ c\geqslant 2a,\\ \frac{c^2(b\lambda+a)^2}{4\lambda d(c-d)}, & a>d,\ e\geqslant d,\ c\geqslant 2d,\\ \frac{c^2(b\lambda+a)^2}{4\lambda e(c-e)}, & a>e,\ d>e,\ c\geqslant 2e,\\ \frac{(b\lambda+a)^2}{\lambda}, & c<2d,\ c<2a,\ c<2e.\end{cases}$$

(iii) Let us take $\lambda = 1$ in Theorem 2.2, then we can get that

$$\Omega_{1,m}=\left\{(u_1,u_2,u_3,u_4,u_5)\,|\,mu_1^2+mu_2^2+u_3^2+u_4^2+(u_5-b-am)^2\leqslant l^2,\ \forall m>0\right\},$$

is the ultimate bound set and positively invariant set of system (1.2), where

$$l^2=\begin{cases}\frac{c^2(b+am)^2}{4a(c-a)}, & d\geqslant a,\ e\geqslant a,\ c\geqslant 2a,\\ \frac{c^2(b+am)^2}{4d(c-d)}, & a>d,\ e\geqslant d,\ c\geqslant 2d,\\ \frac{c^2(b+am)^2}{4e(c-e)}, & a>e,\ d>e,\ c\geqslant 2e,\\ (b+am)^2, & c<2d,\ c<2a,\ c<2e.\end{cases}$$

(iv) Let us take $\lambda = 1, m = 1$ in Theorem 2.2, then we can get that

$$\Delta=\left\{(u_1,u_2,u_3,u_4,u_5)\,|\,u_1^2+u_2^2+u_3^2+u_4^2+(u_5-b-a)^2\leqslant h^2\right\},$$

is the ultimate bound set and positively invariant set of system (1.2), where

$$h^2=\begin{cases}\frac{c^2(b+a)^2}{4a(c-a)}, & d\geqslant a,\ e\geqslant a,\ c\geqslant 2a,\\ \frac{c^2(b+a)^2}{4d(c-d)}, & a>d,\ e\geqslant d,\ c\geqslant 2d,\\ \frac{c^2(b+a)^2}{4e(c-e)}, & a>e,\ d>e,\ c\geqslant 2e,\\ (b+a)^2, & c<2d,\ c<2a,\ c<2e.\end{cases}$$

When $a = 35,\ b = 55,\ c = \frac{8}{3},\ d = 1,\ e = 1$, we can obtain that

$$\Delta=\left\{(u_1,u_2,u_3,u_4,u_5)\,|\,u_1^2+u_2^2+u_3^2+u_4^2+(u_5-90)^2\leqslant 92.95^2\right\},$$

is the ultimate bound set and positively invariant set of system (1.2). In Figure 2, we give the localization of the chaotic attractor of system (1.2) in the (u_2, u_3, u_4) space formed by Δ.

(v) Let us take $c = 1, d = 1, e = 1$ in Theorem 2.2, then we can obtain the ultimate bound set and positively invariant set of system (1.1).

Though Theorem 2.2 give the ultimate bound set and positively invariant set of the 5D Lorenz-type system (1.2), it does not gives the global exponential attractive set of system (1.2). The global exponential attractive set of system (1.2) is described by the following Theorem 3.1.

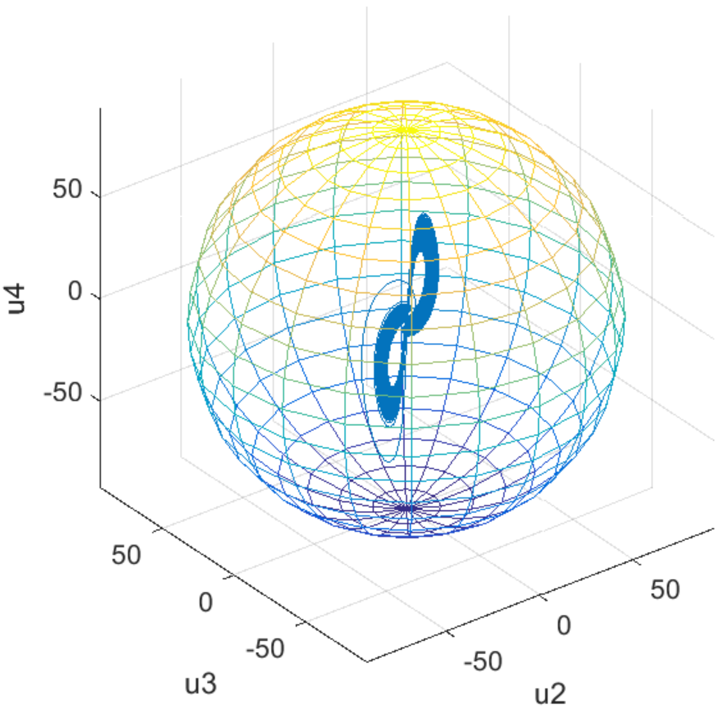

Figure 2: Localization of the chaotic attractor of system (1.2) in the (u_2, u_3, u_4) space formed by Δ.

3. Global domain of attraction

Theorem 3.1. *Suppose that* $m > 0, \lambda > 0,\ a > 0,\ b > 0,\ c > 0,\ d > 0,\ e > 0$, *and let*

$$U(t) = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)), \quad \theta = \min(a, c, d, e) > 0,$$

$$V_{\lambda,m}(U) = mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda\left(u_5 - \frac{b\lambda + am}{\lambda}\right)^2, \quad L_{\lambda,m} = \frac{c(b\lambda + am)^2}{\lambda\theta}.$$

When $V_{\lambda,m}(U(t)) > L_{\lambda,m}$, $V_{\lambda,m}(U(t_0)) > L_{\lambda,m}$, *then the estimation*

$$V_{\lambda,m}(U(t)) - L_{\lambda,m} \leqslant [V_{\lambda,m}(U(t_0)) - L_{\lambda,m}]\, e^{-\theta(t-t_0)},$$

holds for system (1.2), *and thus* $\Omega_{\lambda,m} = \{U |\, V_{\lambda,m}(U) \leqslant L_{\lambda,m}\}$ *is a globally exponential attractive set of system* (1.2), *i.e.*, $\overline{\lim\limits_{t \to +\infty}}\, V_{\lambda,m}(U(t)) \leqslant L_{\lambda,m}$.

Proof. Define the generalized Lyapunov-like function

$$V_{\lambda,m}(U) = V_{\lambda,m}(u_1, u_2, u_3, u_4, u_5) = mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda\left(u_5 - \frac{b\lambda + am}{\lambda}\right)^2 \tag{3.1}$$

for all $\lambda > 0$, and for all $m > 0$, $U = (u_1, u_2, u_3, u_4, u_5)$. Differentiating the Lyapunov-like function $V_{\lambda,m}(U)$ in (3.1) with respect time t along the trajectory of system (1.2) yields

$$\left.\frac{dV_{\lambda,m}(U)}{dt}\right|_{(1.2)} = 2mu_1\frac{du_1}{dt} + 2mu_2\frac{du_2}{dt} + 2\lambda u_3\frac{du_3}{dt} + 2\lambda u_4\frac{du_4}{dt} + 2\lambda\left(u_5 - \frac{b\lambda + am}{\lambda}\right)\frac{du_5}{dt}$$

$$
\begin{aligned}
&= 2amu_1\left(u_3-u_1\right)+2amu_2\left(u_4-u_2\right)+2\lambda u_3\left(bu_1-du_3-u_1u_5\right)\\
&\quad +2\lambda u_4\left(bu_2-eu_4-u_2u_5\right)+2\lambda\left(u_5-\frac{\lambda b+am}{\lambda}\right)\left(u_1u_3+u_2u_4-cu_5\right)\\
&= -2amu_1^2-2amu_2^2-2\lambda du_3^2-2\lambda eu_4^2-2\lambda cu_5^2+2c\left(b\lambda+am\right)u_5\\
&= -amu_1^2-amu_2^2-\lambda du_3^2-\lambda eu_4^2-\lambda cu_5^2+2c\left(b\lambda+am\right)u_5\\
&\quad -amu_1^2-amu_2^2-\lambda du_3^2-\lambda eu_4^2-\lambda cu_5^2\\
&= -amu_1^2-amu_2^2-\lambda du_3^2-\lambda eu_4^2-\lambda c\left(u_5-\frac{b\lambda+am}{\lambda}\right)^2+\frac{c(b\lambda+am)^2}{\lambda}\\
&\quad -amu_1^2-amu_2^2-\lambda du_3^2-\lambda eu_4^2-\lambda cu_5^2\\
&\leqslant -amu_1^2-amu_2^2-\lambda du_3^2-\lambda eu_4^2-\lambda c\left(u_5-\frac{b\lambda+am}{\lambda}\right)^2+\frac{c(b\lambda+am)^2}{\lambda}\\
&\leqslant -\theta V_{\lambda,m}\left(U\right)+\frac{c(b\lambda+am)^2}{\lambda}\\
&= -\theta\left(V_{\lambda,m}\left(U\right)-\frac{c(b\lambda+am)^2}{\lambda\theta}\right)<0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
V_{\lambda,m}\left(U\left(t\right)\right) &\leqslant V_{\lambda,m}\left(U_0\right)e^{-\theta(t-t_0)}+\int_{t_0}^{t}e^{-\theta(t-\tau)}\frac{c(b\lambda+am)^2}{\lambda\theta}d\tau\\
&= V_{\lambda,m}\left(U_0\right)e^{-\theta(t-t_0)}+L_{\lambda,m}\left(1-e^{-\theta(t-t_0)}\right).
\end{aligned}
$$

That is to say,

$$
V_{\lambda,m}\left(U\left(t\right)\right)-L_{\lambda,m}\leqslant\left[V_{\lambda,m}\left(U_0\right)-L_{\lambda,m}\right]e^{-\theta(t-t_0)}. \tag{3.2}
$$

Taking upper limit on both sides of the above inequality (3.2) as $t\to+\infty$ results in

$$
\overline{\lim_{t\to+\infty}}V_{\lambda,m}\left(U\left(t\right)\right)\leqslant L_{\lambda,m},
$$

which clearly shows that $\Omega_{\lambda,m}=\{U|\, V_{\lambda,m}\left(U\right)\leqslant L_{\lambda,m}\}$ is a globally exponential attractive set of system (1.2). □

Remark 3.2.

(i) In particular, let us take $m=1,\ \lambda=1$ in Theorem 3.1, we can get that

$$
\Pi=\left\{\left(u_1,u_2,u_3,u_4,u_5\right)|\, u_1^2+u_2^2+u_3^2+u_4^2+\left(u_5-b-a\right)^2\leqslant M\right\},
$$

is a globally exponential attractive set of system (1.2), where

$$
M=\frac{c(b+a)^2}{\theta},\quad \theta=\min\left(a,c,d,e\right)>0.
$$

(ii) Let us take $c=1,\ d=1,\ e=1$ in Theorem 3.1, then we can obtain the globally exponential attractive set of system (1.1).

4. Conclusion

A novel 5D autonomous chaotic system has been introduced and analyzed in this paper. Based on Lyapunov stability theory and the optimization method, the bounds of this 5D autonomous chaotic system are obtained. Numerical simulations have been used to verify the correctness of analytical results.

The approach presented in this paper can be applied to study other chaotic systems. These theoretical results obtained in this paper are useful in chaos control, chaos synchronization, estimating the fractal dimensions of chaotic attractors and other applications. Further analyses like chaos control, bifurcation analysis, and synchronization are interesting issues for future work.

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