NASA CONTRACTOR REPORT

NASA CR-2672

NASA CR-2672

DYNAMICS AND IDENTIFICATION OF FLEXIBLE AIRCRAFT

William R. Wells

Prepared by
UNIVERSITY OF CINCINNATI
Cincinnati, Ohio
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1976

1. Report No. NASA CR-2672	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Dynamics and Identification of Flexible Aircraft		5. Report Date April 1976
		6. Performing Organization Code
7. Author(s) William R. Wells		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Department of Aerospace Engineering University of Cincinnati Cincinnati, Ohio		11. Contract or Grant No. NGR 36-004-061
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		14. Sponsoring Agency Code
15. Supplementary Notes Langley technical monitor: M. J. Queijo Topical report		

16. Abstract

The equations of motion and a maximum likelihood parameter identification formulation are developed for a flexible aircraft. The various levels of approximation associated with the modal substitution representation of the elastic displacement field are discussed and illustrated when appropriate. The necessary extension of the parameter set of stability and control derivatives due to the aeroelastic effects is obtained.

17. Key Words (Suggested by Author(s)) Flexible aircraft, Parameter identification, Maximum Likelihood, Aircraft modeling	18. Distribution Statement UNCLASSIFIED-UNLIMITED Subject Category 02

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
UNCLASSIFIED	UNCLASSIFIED	66	$4.25

*For sale by the National Technical Information Service, Springfield, Virginia 22161

SYMBOLS

a_{x_i}, a_{z_i}	acceleration of particle i in the x and z directions, respectively (meters/sec^2)
A	sensitivity matrix
$[A_1], [A_2], [A_3], [A_4], [A_5]$	aerodynamic influence coefficients
B	matrix of stability and control derivatives
$\{B\}$	translation and rotation vector associated with point P_o
$\{B_p\}$	perturbation of $\{B\}$
$\bar{c}$	wing mean aerodynamic chord (meters)
$\bar{C}_{ij}$	dyadic of structural coefficients for m_i, m_j
C_{ij}	matrix of structural coefficients for m_i, m_j
$[C]$	matrix of structural coefficinets for all masses
$[\bar{C}]$	flexibility matrix
C_x	$f_x/\frac{1}{2}\rho U_1{}^2 S$

$$C_{x_u} = \frac{\partial C_x}{\partial (\frac{u}{U_1})}$$

$$C_{x_{\dot{u}}} = \frac{\partial C_x}{\partial (\frac{\dot{u}\bar{c}}{U_1{}^2})}$$

$$C_{x_q} = \frac{\partial C_x}{\partial (\frac{q\bar{c}}{2U_1})}$$

$C_{x_{\dot{q}}} = \dfrac{\partial C_x}{\partial \left(\dfrac{\dot{q}\bar{c}^2}{4U_1^2}\right)}$

$C_{x_{u_i}} = \dfrac{\partial C_x}{\partial \left(\dfrac{2u_i}{\bar{c}}\right)}$

$C_{x_{\dot{u}_i}} = \dfrac{\partial C_x}{\partial \left(\dfrac{\dot{u}_i}{U_1}\right)}$

$C_{x_{\ddot{u}_i}} = \dfrac{\partial C_x}{\partial \left(\dfrac{\ddot{u}_i \bar{c}}{2U_1^2}\right)}$

$C_{x_{\delta_i}} = \dfrac{\partial C_x}{\partial \delta_i}$

C_{xo} reference value of C_x

$C_{x_\alpha} = \dfrac{\partial C_x}{\partial \alpha}$

$C_{x_{\dot{\alpha}}} = \dfrac{\partial C_x}{\partial \left(\dfrac{\dot{\alpha}\bar{c}}{2U_1}\right)}$

C_z normal force coefficient f_z/qS

$$C_{z_u} = \frac{\partial C_z}{\partial (\frac{u}{U_1})}$$

$$C_{z_{\dot{u}}} = \frac{\partial C_z}{\partial (\frac{\dot{u}\bar{c}}{2U_1^2})}$$

$$C_{z_q} = \frac{\partial C_z}{\partial (\frac{q\bar{c}}{U_1})}$$

$$C_{z_{\dot{q}}} = \frac{\partial C_z}{\partial (\frac{\dot{q}\bar{c}^2}{4U_1^2})}$$

$$C_{z_{u_i}} = \frac{\partial C_z}{\partial (\frac{2u_i}{\bar{c}})}$$

$$C_{z_{\dot{u}_i}} = \frac{\partial C_z}{\partial (\frac{\dot{u}_i}{U_1})}$$

$$C_{z_{\ddot{u}_i}} = \frac{\partial C_z}{\partial (\frac{\ddot{u}_i\bar{c}}{2U_1^2})}$$

$$C_{z_{\delta_i}} = \frac{\partial C_z}{\partial \delta_i}$$

C_{zo} reference value of C_z

$C_{z_\alpha} = \frac{\partial C_z}{\partial \alpha}$

$C_{z_{\dot{\alpha}}} = \frac{\partial C_z}{\partial (\frac{\dot{\alpha}\bar{c}}{2U_1})}$

C_m pitching moment coeffcient $\frac{m_y}{\frac{1}{2}\rho U_1{}^2 S\bar{c}}$

$C_{m_u} = \frac{\partial C_m}{\partial (\frac{u}{U_1})}$

$C_{m_{\dot{u}}} = \frac{\partial C_m}{\partial (\frac{\dot{u}\bar{c}}{2U_1{}^2})}$

$C_{m_q} = \frac{\partial C_m}{\partial (\frac{q\bar{c}}{2U_1})}$

$C_{m_{\dot{q}}} = \frac{\partial C_m}{\partial (\frac{\dot{q}\bar{c}^2}{4U_1{}^2})}$

$C_{m_{u_i}} = \frac{\partial C_m}{\partial (\frac{2u_i}{\bar{c}})}$

$$C_{m_{\dot{u}_i}} = \frac{\partial C_m}{\partial \left(\frac{\dot{u}_i}{U_1}\right)}$$

$$C_{m_{\ddot{u}_i}} = \frac{\partial C_m}{\partial \left(\frac{\ddot{u}_i \bar{c}}{2U_1^2}\right)}$$

$$C_{m_{\delta_i}} = \frac{\partial C_m}{\partial \delta_i}$$

C_{mo} reference value of C_m

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m_{\dot{\alpha}}} = \frac{\partial C_m}{\partial \left(\frac{\dot{\alpha}\bar{c}}{2U_1}\right)}$$

C_{u_i} $f_i/\frac{1}{2}\,\rho U_1^{\,2} S$

$$C_{u_{i_u}} = \frac{\partial C_{u_i}}{\partial \left(\frac{u}{U_1}\right)}$$

$$C_{u_{i_{\dot{u}}}} = \frac{\partial C_{u_i}}{\partial \left(\frac{\dot{u}\bar{c}}{2U_1^2}\right)}$$

$$C_{u_{i_q}} = \frac{\partial C_{u_i}}{\partial\left(\frac{q\bar{c}}{2U_1}\right)}$$

$$C_{u_{i_{\dot{q}}}} = \frac{\partial C_{u_i}}{\partial\left(\frac{\dot{q}\bar{c}^2}{4U_1^2}\right)}$$

$$C_{u_{i_{u_j}}} = \frac{\partial C_{u_i}}{\partial\left(\frac{2u_j}{\bar{c}}\right)}$$

$$C_{u_{i_{\dot{u}_j}}} = \frac{\partial C_{u_i}}{\partial\left(\frac{\dot{u}_j}{U_1}\right)}$$

$$C_{u_{i_{\ddot{u}_j}}} = \frac{\partial C_{u_i}}{\partial\left(\frac{\ddot{u}_j\bar{c}}{2U_1^2}\right)}$$

$$C_{u_{i_{\delta_i}}} = \frac{\partial C_{u_i}}{\partial \delta_j}$$

$$C_{u_{i_\alpha}} = \frac{\partial C_{u_i}}{\partial \alpha}$$

$$C_{u_{i_{\dot{\alpha}}}} = \frac{\partial C_{u_i}}{\partial\left(\frac{\dot{\alpha}\bar{c}}{2U_1}\right)}$$

$\underline{d}$	elastic deformation vector
$\underline{d}_i$	deformation of particle m_i measured relative to a coordinate system with origin fixed at center of mass
$\underline{d}'_i$	deformation of particle m_i relative to a coordinate system fixed to point P_o
$\underline{d}o$	displacement of point P_o after deformation
$\{d'\} = [\underline{d}'_1, \underline{d}'_2, \ldots \underline{d}'_n]^T$	
$\{d\} = [\underline{d}_1, \underline{d}_2, \ldots \underline{d}_n]^T$	
$\{d_p\}$	perturbation in {d}
$\underline{F}$	body force
$\underline{F}_i$	aerodynamic force acting on m_i
$\underline{F}_{1i}$	nominal value of $\underline{F}_i$
$\underline{F}_{pi}$	perturbation in $\underline{F}_i$
$\{F_p\} = [\underline{F}_{p1}, \underline{F}_{p2}, \ldots \underline{F}_{pn}]^T$	
f_x, f_y, f_z	components of perturbations of $\underline{F}$
$f_{x_u} = \frac{1}{2}\rho S U_1 C_{x_u}$	
$f_{x_w} = \frac{1}{2}\rho S U_1 C_{x_\alpha}$	
$f_{x_q} = \frac{1}{4}\rho S U_1 \bar{c}\ C_{x_q}$	
$f_{x_{\dot{u}}} = \frac{1}{2}\rho S\ \bar{c}\ C_{x_{\dot{u}}}$	
$f_{x_{\dot{w}}} = \frac{1}{2}\rho S\ \bar{c}\ C_{x_{\dot{\alpha}}}$	

$$f_{x_{\dot{q}}} = \frac{1}{4}\rho S \bar{c}^2 C_{x_{\dot{q}}}$$

$$f_{x_{u_i}} = \frac{1}{2\bar{c}} \rho S U_1^2 C_{x_{u_i}}$$

$$f_{x_{\dot{u}_i}} = \frac{1}{2}\rho S U_1 C_{x_{\dot{u}_i}}$$

$$f_{x_{\ddot{u}_i}} = \frac{1}{2}\rho S \bar{c} C_{x_{\ddot{u}_i}}$$

$$f_{x_{\delta_i}} = \frac{1}{2}\rho S U_1^2 C_{x_{\delta_i}}$$

$$f_{z_u} = \frac{1}{2}\rho S U_1 C_{z_u}$$

$$f_{z_w} = \frac{1}{2}\rho S U_1 C_{z_\alpha}$$

$$f_{z_q} = \frac{1}{4} S U_1 \bar{c} C_{z_q}$$

$$f_{z_{\dot{u}}} = \frac{1}{2}\rho S \bar{c} C_{z_{\dot{u}}}$$

$$f_{z_{\dot{w}}} = \frac{1}{2}\rho S \bar{c} C_{z_{\dot{\alpha}}}$$

$$f_{z_{\dot{q}}} = \frac{1}{4}\rho S \bar{c}^2 C_{z_{\dot{q}}}$$

$$f_{z_{u_i}} = \frac{1}{2\bar{c}} \rho S U_1^2 C_{z_{u_i}}$$

$$f_{z_{\dot{u}_i}} = \frac{1}{2}\rho S U_1 C_{z_{\dot{u}_i}}$$

$$f_{z_{\ddot{u}_i}} = \frac{1}{2}\rho S \bar{c} C_{z_{\ddot{u}_i}}$$

$$f_{z_{\delta_i}} = \frac{1}{2}\rho S U_1^2 C_{z_{\delta_i}}$$

$$f_{i_u} = \frac{1}{2}\rho S U_1 C_{u_{i_u}}$$

$f_{i_w} = \frac{1}{2}\rho S U_1 \; C_{u_{i_\alpha}}$

$f_{i_q} = \frac{1}{4}\rho S \; \bar{c} \; C_{u_{i_q}}$

$f_{i_{\dot{u}}} = \frac{1}{2}\rho S \; \bar{c} \; C_{u_{i_{\dot{u}}}}$

$f_{i_{\dot{w}}} = \frac{1}{2}\rho S \; \bar{c} \; C_{u_{i_{\dot{\alpha}}}}$

$f_{i_{\dot{q}}} = \frac{1}{4}\rho S \; \bar{c} \; C_{u_{i_{\dot{q}}}}$

$f_{i_{\delta_j}} = \frac{1}{2}\rho S U_1^2 \; C_{u_{i_{\delta_j}}}$

$f_{i_{u_j}} = \frac{1}{2\bar{c}} \; \rho U_1^2 S \; C_{u_{i_{u_j}}}$

$f_{i_{\dot{u}_j}} = \frac{1}{2}\rho U_1 S \; C_{u_{i_{\dot{u}_j}}}$

$f_{i_{\ddot{u}_j}} = \frac{1}{2}\rho S \; \bar{c} \; C_{u_{i_{\ddot{u}_j}}}$

F	stability and control matrix
G	stability and control matrix
$\underline{g}$	acceleration of gravity
$\underline{g}_p$	perturbation in $\underline{g}$
H	coefficient matrix in measurement
$\underline{i}$, $\underline{j}$, $\underline{k}$	orthogonal unit base vectors
I	identity matrix
I_{xx}, I_{yy}, I_{zz}	moments of inertia about body axes
I_{xy}, I_{xz}, I_{yz}	products of inertia about body axes

$[K_{11}] = [C]^{-1}$	
$[\bar{K}]$	stiffness matrix
$[\grave{}\bar{K}\lrcorner]$	generalized stiffness matrix
$[\grave{}\bar{K}_1\lrcorner]$	generalized stiffness matrix for retained modes
$[\grave{}\bar{K}_2\lrcorner]$	generalized stiffness matrix for deleted modes
m_i	mass of ith lumped mass
M	total mass of aircraft
$\underline{M}$	applied external moments about center of mass
m_x, m_y, m_z	components in perturbation of $\underline{M}$
$[\grave{}m\lrcorner]$	diagonal mass matrix
$[M_1]$	matrix of nominal linear and angular velocities
$[M_2]$	matrix of nominal gravity terms

$[M] = [\bar{\phi}]^T[\grave{}m\lrcorner][\bar{\phi}]$

$[M'] = [\bar{\phi}]^T\{[I] + ([I] - [A_3][\bar{C}])^{-1}[A_3][\bar{C}]\}[\grave{}m\lrcorner][\bar{\phi}]$

$[\grave{}\bar{m}\lrcorner]$	generalized mass matrix
$[\grave{}m_2\lrcorner]$	generalized mass matrix for deleted modes

$M_u = \frac{1}{2}\rho S \bar{c} U_1 (C_{m_u} + 2C_{mo})$

$M_w = \frac{1}{2}\rho S \bar{c} U_1 C_{m_\alpha}$

$M_q = \frac{1}{4}\rho S \bar{c}^2 U_1 C_{m_q}$

$M_{\dot{u}} = \frac{1}{2}\rho S \bar{c}^2 C_{m_{\dot{u}}}$

$M_{\dot{w}} = \frac{1}{2}\rho S \bar{c}^2 C_{m_{\dot{\alpha}}}$

$M_{\dot{q}} = \frac{1}{4}\rho S \bar{c}^3 C_{m_{\dot{q}}}$

$M_{\delta_i} = \frac{1}{2}\rho S \bar{c} U_1^2 C_{m_{\delta_i}}$

$M_{u_i} = \frac{1}{2}\rho S U_1^2 C_{m_{u_i}}$

$M_{\dot{u}_i} = \frac{1}{2}\rho S \bar{c} U_1 C_{m_{\dot{u}_i}}$

$M_{\ddot{u}_i} = \frac{1}{2}\rho S \bar{c}^2 C_{m_{\ddot{u}_i}}$

P	angular velocity about x body axis
P_1	nominal value of P
p	perturbation in P
$\underline{p}$	parameter vector
$\hat{\underline{p}}$	estimate of $\underline{p}$
Q	angular velocity about y body axis
Q_1	nominal value of Q
q	perturbation in Q
$\underline{Q}_i$	applied and inertia forces acting on particle m_i
$\{Q\} = [\underline{Q}_1, \underline{Q}_2, \ldots \underline{Q}_n]^T$	
$\underline{Q}_{1i}$	nominal value of $\underline{Q}_i$
Q_{pi}	perturbation in $\underline{Q}_i$
$\{Q_p\} = [\underline{Q}_{p1}, \underline{Q}_{p2}, \ldots \underline{Q}_{pn}]^T$	

$\underline{r}$, $\underline{r}_i$	position vectors of mass element prior to deformation
$\underline{r}$, $\underline{r}_i$	position vectors of mass element after deformation
$\underline{r}'$, $\underline{r}_i'$	position vectors of mass element after deformation relative to inertial system
$\underline{r}_o'$	location of center of mass relative to inertial system
$\underline{R}$	body force
R	angular velocity about z body axis
R_1	nominal value of R
r	perturbation in R
$\{r_{op}'\} = [0,0,0,\phi_p,\theta_p,\psi_p]^T$	
$\hat{R}$	covariance of the measurement error
S	surface area
t	time
U	component of center of mass velocity along x body axis
U_1	nominal value of U
u	perturbation of U
u_i	generalized displacement associated with ith mode
$\{u\} = [u_1, u_2, \ldots u_{3n-6}]^T$	
$\{u_1\}$	displacement vector of dynamically retained modes
$\{u_2\}$	displacement vector of dynamically deleted modes
$\underline{V}_c$	absolute velocity of center of mass
V	component of center of mass velocity along y body axis
V	volume

V_1	nominal value of V
v	perturbation in V
$\underline{V}_{c1}$	nominal value of $\underline{V}_c$
$\{V_{cp}\} = [u,v,w,p,q,r]^T$	
W	component of center mass velocity along z body axis
W_1	nominal value of W
w	perturbation in W
x,y,z	coordinates relative to body fixed axis system with origin at P_o
x',y',z'	coordinates relative to inertia axis system
X,Y,Z	coordinates relative to axes fixed to center of mass
$\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$	coordinates of undeformed mass particle relative to axis system with origin at center of mass
$\underline{y}$	vector of measurements
$\underline{z} = [u,w,\theta,\dot{\theta},u_1,u_2,...u_m,\dot{u}_1,\dot{u}_2,...\dot{u}_m]^T$	
α	angle of attack
δ_{ij}	Kronecker delta
δ_i	control surface deflection
$\underline{\delta} = [\delta_1,\delta_2,...\delta_c]^T$	
$\underline{\eta}$	measurement noise vector
θ_o	rotation vector associated with axis system fixed to point P_o
θ,ϕ,ψ	Euler angles

θ_1, ϕ_1, ψ_1	nominal values of Euler angles
θ_p, ϕ_p, ψ_p	perturbation in Euler angles
λ_i	eigenvalue associated with ith mode
$\underline{\nu} = \underline{Y} - \underline{z}$	
ρ	air density
$\tilde{\rho}$	aircraft density
$[\bar{\phi}]$	rigid body mode shape matrix
$[\bar{\phi}'] = [\bar{\phi}]^T([I] - [A_3][\bar{C}])^{-1}$	
$\{\phi_i\}$	free vibration matrix of ith mode
$[\phi] = [\{\phi_1\},\{\phi_2\},\ldots\{\phi_{3n-6}\}]$	
$[\phi_1]$	dynamically retained mode matrix
$[\phi_2]$	dynamically deleted mode matrix
ϕ_{xij}	element of $[\phi]$ corresponding to ith particle, ith mode, and x direction
ϕ_{zij}	element of $[\phi]$ correspondint to ith particle, jth mode, and z direction

DYNAMICS AND IDENTIFICATION OF FLEXIBLE AIRCRAFT

William R. Wells
University of Cincinnati

INTRODUCTION

The practical necessity of removing all excessive structural weight through either conventional design practices or an active aerodynamic control system, has resulted in vehicles that are more aeroelastic. However, most existing parameter estimation methods used in the study of the stability and control and handling properties of the aircraft stop short of explicity identifying important aeroelastic parameters that affect the aircraft dynamics. Previously, structural motion was considered as part of the measurement noise and filtered rather than modeled dynamically.

The purpose of this paper is twofold: (1) to present, in detail, a development of the equations of motion for an elastic aircraft indicating, when appropriate, the difference and similarities with the dynamics of the rigid airplane, (2) discuss the additional computational difficulties due to the inclusion of added stability derivatives arising from consideration of aeroelastic effects.

EQUATIONS OF MOTION

Rigid Body Motion

The first set of equations to be obtained are the "rigid body equations of motion" for the elastic airplane. These result from the application of Newton's laws of motion to the aircraft as a whole and describe the motion of the aircraft center of mass relative to an inertial system.

Consider the elemental mass and surface area shown in Figure 1. The externally applied forces acting on these are a body force, $\underline{R}$, and a surface force $\underline{F}$. The body force will be assumed to be gravitational and the surface force aerodynamic. The thrust and control surface forces can be added independently after the equations are formulated under the assumption that these quantities do not contribute to the elastic deformation of the structure. In this figure, $\tilde{\underline{r}}$ is the location of the mass element relative to the center of mass before the aircraft distorts and $\underline{d}$ represents the elastic displacement of the particle due to distortion.

Conservation of linear and angular momentum dictate that the following equations be satisfied:

$$\frac{d}{dt} \int_V \tilde{\rho} \frac{d\underline{r}'}{dt} dV = \int_V \underline{R} \, dV + \int_S \underline{F} \, dS \tag{1}$$

$$\frac{d}{dt} \int_V \tilde{\rho} \underline{r}' \times \frac{d\underline{r}'}{dt} dV = \int_V \underline{r}' \times \underline{R} \, dV + \int_S \underline{r}' \times \underline{F} \, dS \tag{2}$$

where V and S refer to volume and surface quantities respectively. The time rate of change of the vectors in the left-hand sides of equations (1) - (2) are relative to the inertial earth fixed system.

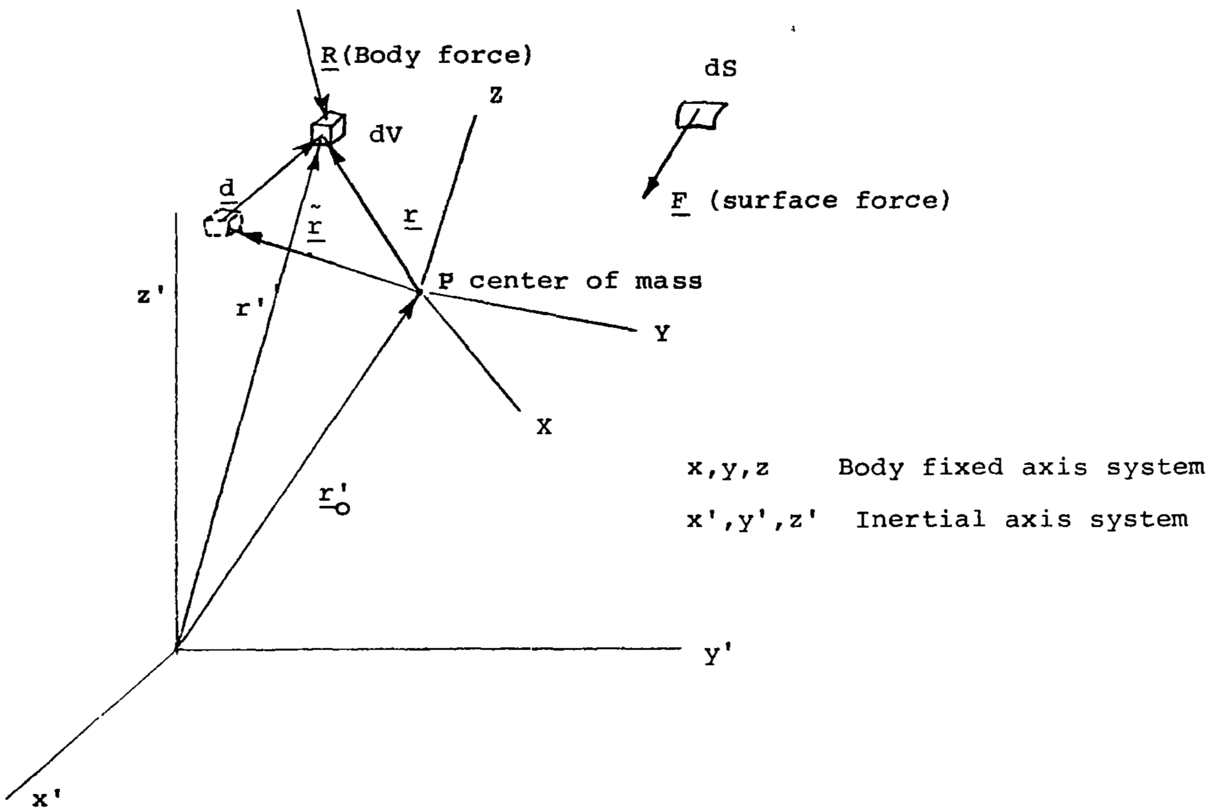

Figure 1. - Definition of axis systems used to describe the aircraft deformation.

If the vector $\underline{r}'$ is replaced by $\underline{r}_0' + \underline{r}$ in equation (1), the result is

$$\frac{d}{dt}\int_V \tilde{\rho}\,\frac{d\underline{r}_0'}{dt}\,dV + \frac{d}{dt}\int_V \tilde{\rho}\,\frac{d\underline{r}}{dt}\,dV = \int_V \underline{R}\,dV + \int_S \underline{F}\,dS$$

The center of mass is located by the vector

$$\underline{r}_0' = \frac{1}{M}\int_V \tilde{\rho}\,\underline{r}'\,dV$$

From the definition of the center of mass location, it follows that

$$\int_V \tilde{\rho}\,\underline{r}\,dV = 0$$

and

$$\frac{d}{dt}\int_V \tilde{\rho}\,\underline{r}\,dV = \int_V \tilde{\rho}\,\frac{d\underline{r}}{dt}\,dV$$

These results allow equation (1), the linear momentum equation, to be expressed as

$$M\,\frac{d\underline{V}_c}{dt} = \int_S F\,d\,S + M\,\underline{g} \qquad (3)$$

where

$$\underline{V}_c \equiv \frac{d\underline{r}_0'}{dt} \quad ; \quad \underline{R} = \tilde{\rho}\,\underline{g}$$

Similarly, equation (2) can be written in the form

$$\underline{r}_0' \times [M \frac{d\underline{V}_c}{dt} - M\underline{g} - \int_S \underline{F}\, dS] + \frac{d}{dt}\int_V \tilde{\rho}\, \underline{r} \times \frac{d\underline{r}}{dt}\, dV = \int_S \underline{r} \times \underline{F}\, dS$$

The bracketed term in this equation vanishes due to equation (3); consequently, the conservation of angular momentum equation reduces to

$$\frac{d}{dt}\int_V \tilde{\rho}\, \underline{r} \times \frac{d\underline{r}}{dt}\, dV = \int_S \underline{r} \times \underline{F}\, dS \tag{4}$$

At this point in the development it is of interest to note how coupling between the elastic deformation and rigid body motions occurs. In equations (3)-(4) both vectors $\underline{F}$ and $\underline{r}$ depend on the shape of the aircraft and, hence, upon the elastic deformation represented by $\underline{d}$. An objective of the analysis is to minimize, as much as possible, this coupling without sacrificing accuracy.

Next, equations (3)-(4) will be written in terms of velocities and accelerations relative to the body axis system which translates and rotates relative to the inertial system with angular velocity $\underline{\omega}$. The reason for doing this is to illustrate the similarity of this overall motion or axis system motion to the motion of the rigid aircraft and to indicate how elastic effects are treated.

If $\frac{d}{dt}$ and $(\dot{\ })$ represent the rate operators as measured in inertial and body axis systems, respectively, then

$$\frac{d\underline{V}_c}{dt} = \dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c$$

$$\frac{d\underline{r}}{dt} = \dot{\underline{r}} + \underline{\omega} \times \underline{r}$$

Further, let

$$\underline{\omega} = P\underline{i} + Q\underline{j} + R\underline{k}$$

$$\underline{V}_c = U\underline{i} + V\underline{j} + W\underline{k}$$

$$\underline{g} = -g\sin\theta\underline{i} + g\cos\theta\sin\phi\underline{j} + g\cos\theta\cos\phi\underline{k}$$

$$\int_S \underline{F}\, dS = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

$$\int_S \underline{r} \times \underline{F}\, dS = M_x\underline{i} + M_y\underline{j} + M_z\underline{k}$$

where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are unit base vectors in the body fixed axis system.

In scalar form, the linear momentum equations (Equation (3)) are written as

$$M\dot{U} + M(QW-RV) = -Mg\sin\theta + F_x$$

$$M\dot{V} + M(RU-PW) = Mg\cos\theta\sin\phi + F_y$$

$$M\dot{W} + M(PV-QU) = Mg\cos\theta\cos\phi + F_z \qquad (5)$$

It can be seen that in these equations, the rigid body motion and elastic deformations are dynamically uncoupled; that is, the effects of elastic deformation enter the equations only in the right-hand side through the applied force terms.

The angular momentum equation, from equation 4, can be rewritten as

$$\frac{d}{dt} \int_V \tilde{\rho} [\underline{r} \times \dot{\underline{d}} + \underline{r} \times (\underline{\omega} \times \underline{r})] \, dV = \underline{M} \tag{6}$$

where the identities $\dot{\underline{r}} = \dot{\underline{d}}$ and $\underline{M} = \int_S \underline{r} \times \underline{F} \, dS$ have been used. Note that the rigid body motion and elastic deformation are dynamically coupled in this equation due to the presence of $\underline{r}$ and $\dot{\underline{d}}$ in the left hand side of the equation.

The dynamic coupling expressed in equation (6) can be eliminated by the adoption of two approximations: the rate of elastic displacement is slow relative to the velocity $\underline{\omega} \times \underline{r}$ and the total angular momentum about the center of mass of the deformed aircraft can be computed with $\underline{r} = \tilde{\underline{r}}$ as if the aircraft is undeformed.

The first approximation assumes that $\int_V \tilde{\rho} \, \underline{r} \times \dot{\underline{d}} \, dV = 0$ which is also an approximation consistent with the normal mode method to be discussed later. This allows the angular momentum equation to be written as

$$\frac{d}{dt} \int_V \tilde{\rho} \, \underline{r} \times (\underline{\omega} \times \underline{r}) \, dV = \underline{M} \tag{7}$$

The second approximation, $\underline{r} = \tilde{\underline{r}}$ allows the angular momentum equations to be written in terms of moments of inertia of the jig shape and allows equation (7) to be written as

$$\frac{d}{dt} \int_V \tilde{\rho} \, \tilde{\underline{r}} \times (\underline{\omega} \times \tilde{\underline{r}}) \, dV = \underline{M} \tag{8}$$

For a constant mass, symmetric aircraft with the x y z axis as principal axes, Equation (8) can be expressed in scalar form as

$$I_{xx}\dot{P} - I_{xz}\dot{R} - I_{xz} P Q + (I_{zz} - I_{yy}) R Q = M_x$$

$$I_{yy}\dot{Q} + (I_{xx} - I_{zz}) P R + I_{xz} (P^2 - R^2) = M_y \qquad (9)$$

$$I_{zz}\dot{R} - I_{xz}\dot{P} + (I_{yy} - I_{xx}) P Q + I_{xz} Q R = M_z$$

In either equation (8) or (9) it is noted that the effects of elasticity enter through the applied moments.

Equations (5) and (9) can be expressed in terms of perturbed state variables u, r, w, p, q, etc. from a reference flight condition U_1, V_1, W_1, P_1, Q_1, etc. For sake of brevity, the perturbed form of equations (10) and (11) are presented for the special reference conditions of zero roll ($P_1 = 0$) and level flight ($W_1 = 0$) as

$$M\dot{u} + M(Q_1 w - R_1 v - V_1 r) + Mg\theta_p \cos\theta_1 = f_x$$

$$M\dot{v} + M(R_1 u + V_1 r) + Mg(\theta_p \sin\theta_1 \sin\phi_1 - \phi_p \cos\theta_1 \cos\phi_1) = f_y \qquad (10)$$

$$M\dot{w} + M(V_1 p - Q_1 u - U_1 q) + Mg(\theta_p \sin\theta_1 \cos\phi_1 + \phi_p \cos\theta_1 \sin\phi_1) = f_z$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} - I_{xz} Q_1 p + (I_{zz} - I_{yy})(Q_1 r + R_1 q) = m_x$$

$$I_{yy}\dot{q} - 2I_{xz} R_1 r + (I_{xx} - I_{zz}) R_1 p = m_y \qquad (11)$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} + I_{xz}(Q_1 r + R_1 q) + (I_{yy} - I_{xx}) Q_1 p = m_z$$

In equations (10) - (11), the quantities with subscript "1" refer to reference values and the quantities $\{f_x, f_y, f_z\}$, $\{m_x, m_y, m_z\}$

refer to perturbed values of the externally applied forces and moments respectively.

Equations (10) - (11) represent the uncoupled equations of perturbed motion in body axes of an elastic airplane consistent with the assumptions necessary to achieve decoupling. They are to be considered further in another section.

Elastic Body Motion

In addition to the six equations for the rigid body motion for the body axis system, there exist 3n equations of motion which reflect the internal equilibrium of the n lumped masses making up the total aircraft mass. Here, it is convenient to introduce influence coefficients into the analysis which are obtained from a solution ot the Navier equations of elasticity. This requires that we consider the deformations $\underline{d}_i'$ measured relative to a coordinate system (x, y, z) with origin fixed to a point P_o coincident with the location of the aircraft center of mass before deformation (see figure 2). After deformation, this point translates a distance given by the vector $\underline{d}_o$ relative to the center of mass which is assumed to have its position unaltered by elastic deformation. The coordinate system fixed to point P_o also rotates through an angle which can be represented by a rotation vector $\underline{\theta}_o$ relative to the initial orientation of the coordinate system before deformation or loading. Then, the deformation $\underline{d}_i$ measured relative to a coordinate system X Y Z with origin fixed at the center of mass of the deformed aircraft P, consists of the deformation $\underline{d}_i'$ measured in the x, y, z system in addition to the translation $\underline{d}_o$ and rotation $\underline{\theta}_o$ x $\underline{\tilde{r}}_i$.

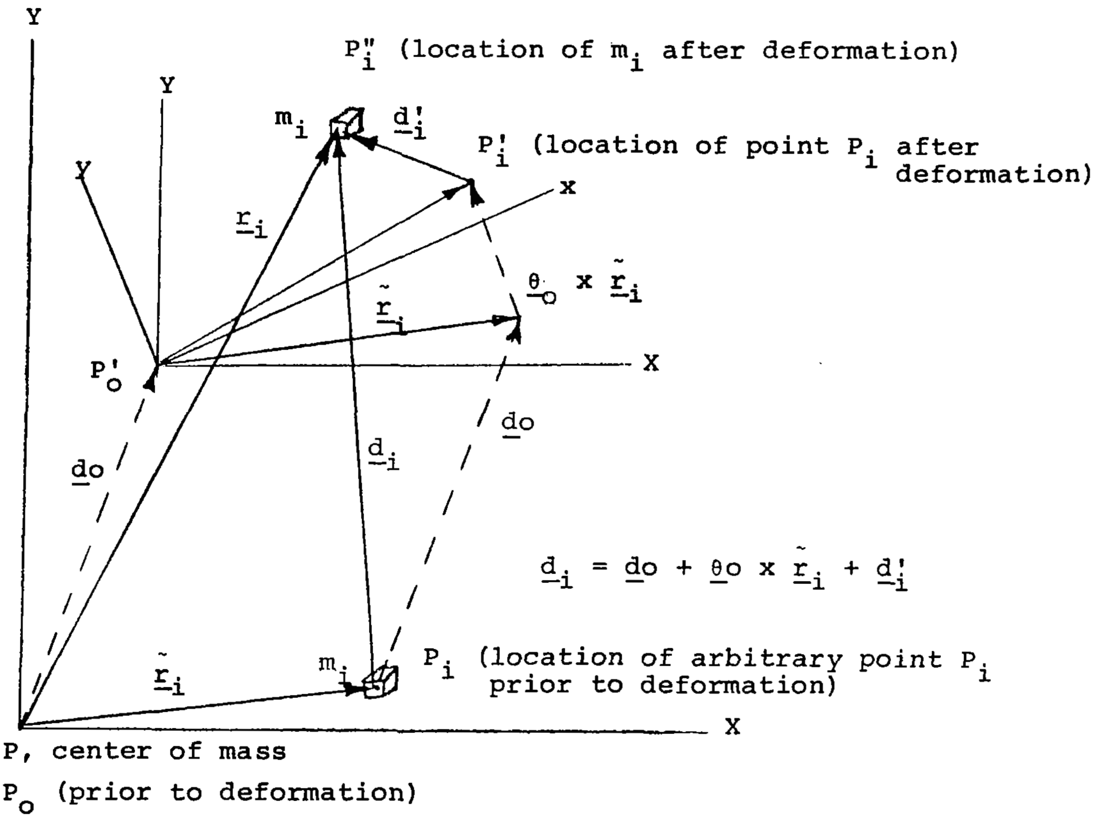

Figure 2. - Vector Diagram illustrating relationship of $\underline{d}_i$ to $\underline{d}'_i$

That is,

$$\underline{d}_i = \underline{d}_i' + \underline{d}_o + \underline{\theta}_o \times \underline{\tilde{r}}_i \quad , \ (i=1,2,\ldots n) \qquad (12)$$

where $\underline{\tilde{r}}_i$ is the location of the ith particle relative to the center of mass before deformation.

The displacement field $\underline{d}_i'$ is linearly related to the applied force field $\underline{Q}_i$ (consisting of inertial, gravitational, and aerodynamic forces) for elastic deformations which satisfy a linear stress-strain relationship (reference 1). That is,

$$\underline{d}_i' = \sum_{j=1}^{n} \overline{C}_{ij} \cdot \underline{Q}_j \quad , \ (i=1,2,\ldots n) \qquad (13)$$

where n is the number of lumped masses and $\overline{C}_{ij}$ is a dyadic of structural coefficients associated with the mass pair m_i, m_j. The components of $\overline{C}_{ij}$ are expressed according to

$$\begin{aligned} \overline{C}_{ij} = {} & \underline{i}[C_{xxij}\,\underline{i} + C_{xyij}\,\underline{j} + C_{xzij}\,\underline{k}] \\ & + \underline{j}[C_{yxij}\,\underline{i} + C_{yyij}\,\underline{j} + C_{yzij}\,\underline{k}] \qquad (14) \\ & + \underline{k}[C_{zxij}\,\underline{i} + C_{zyij}\,\underline{j} + C_{zzij}\,\underline{k}] \end{aligned}$$

The physical meaning of a typical coefficient C_{xyij} is that it represents the component of displacement of the ith lumped mass in the x direction due to a unit force in the y direction at the jth lumped mass.

The result expressed by equation (13) can be rewritten as

$$\{d'\} = [C]\{Q\}, \text{ where}$$

$$\{d'\} = [d'_{x1}, d'_{y1}, d'_{z1}, d'_{x2}, d'_{y2}, d'_{z2}, \ldots d'_{zn}]^T$$

$$\{Q\} = [Q_{x1}, Q_{y1}, Q_{z1}, Q_{x2}, Q_{y2}, Q_{z2}, \ldots Q_{zn}]^T$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} \cdots & C_{1n} \\ C_{21} & C_{22} \cdots & C_{2n} \\ & & \\ C_{n1} & C_{n2} \cdots & C_{nn} \end{bmatrix} \tag{15}$$

where

$$C_{ij} = \begin{bmatrix} C_{xxij} & C_{xyij} & C_{xzij} \\ C_{yxij} & C_{yyij} & C_{yzij} \\ C_{zxij} & C_{zyij} & C_{zzij} \end{bmatrix}$$

Substitution for $\underline{d}'_i$ from equation (13) into equation (12) results in the elastic displacement field measured relative to the center of mass of the aircraft

$$\underline{d}_i = \underline{d}_o + \underline{\theta}_o \times \tilde{\underline{r}}_i + \sum_{j=1}^{n} \overline{C}_{ij} \cdot \underline{Q}_j \quad , \ (i=1,2,\ldots n) \tag{16}$$

where

$$\underline{Q}_i = -m_i\left(\frac{d^2\underline{r}'_i}{dt^2} - \underline{g}\right) + \underline{F}_i \quad , \ (i=1,2,\ldots n) \tag{17}$$

In equations (16) - (17), $\underline{r}_i'$ is the position vector relative to an inertial system (figure 1) of the ith mass particle after deformation has occurred and $\underline{F}_i$ is the aerodynamic force acting on this particle.

In order to develop a system of differential equations from equation (16) we must express the acceleration $\frac{d^2\underline{r}_i'}{dt^2}$ in terms of rigid body quantities (center of mass motion) as well as elastic displacements. Since $\underline{r}_i' = \underline{r}_0' + \tilde{\underline{r}}_i + \underline{d}_i$, where $\underline{r}_0'$ is the position vector to the center of mass relative to the inertial system, we have

$$\frac{d^2\underline{r}_i'}{dt^2} = \frac{d^2\underline{r}_0'}{dt^2} + \frac{d^2}{dt^2}(\underline{r}_i + \underline{d}_i)$$

$$= \frac{d\underline{V}_c}{dt} + \frac{d}{dt}\left[\frac{d}{dt}(\tilde{\underline{r}}_i + \underline{d}_i)\right]$$

$$= \dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c + \frac{d}{dt}[\dot{\underline{d}}_i + \underline{\omega} \times (\tilde{\underline{r}}_i + \underline{d}_i)]$$

which further reduces to

$$\frac{d^2\underline{r}_i'}{dt^2} = \dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c + \ddot{\underline{d}}_i + 2\,\underline{\omega} \times \dot{\underline{d}}_i$$

$$+ \dot{\underline{\omega}} \times (\tilde{\underline{r}}_i + \underline{d}_i) + \underline{\omega} \times [\underline{\omega} \times (\tilde{\underline{r}}_i + \underline{d}_i)] \qquad (18)$$

In equation (18), the accelerations $\underline{\omega} \times \dot{\underline{d}}_i$ and $\underline{\omega} \times [\underline{\omega} \times (\tilde{\underline{r}}_i + \underline{d}_i)]$ are assumed small compared to the other terms which make up the absolute acceleration of particle m_i. Another simplification is realized by computing the acceleration $\dot{\underline{\omega}} \times (\tilde{\underline{r}}_i + \underline{d}_i)$ as $\dot{\underline{\omega}} \times \tilde{\underline{r}}_i$ which utilizes the small elastic deformation assumption. These simplifications allow equation (18) to be written as

$$\frac{d^2\underline{r}_i}{dt^2} \simeq \dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c + \dot{\underline{\omega}} \times \tilde{\underline{r}}_i + \ddot{\underline{d}}_i \tag{19}$$

which results in a value of $\underline{Q}_i$ approximated as

$$\underline{Q}_i \simeq \underline{F}_i + m_i\underline{g} - m_i[\dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c + \dot{\underline{\omega}} \times \tilde{\underline{r}}_i + \ddot{\underline{d}}_i] \tag{20}$$

The governing equation for the elastic displacement field is then

$$\underline{d}_i = \underline{d}_o + \underline{\theta}_o \times \tilde{\underline{r}}_i$$

$$+ \sum_{j=1}^{n} \overline{C}_{ij} \cdot [\underline{F}_j + m_j\underline{g} - m_j(\dot{\underline{V}}_c + \underline{\omega} \times \underline{V}_c + \dot{\underline{\omega}} \times \tilde{\underline{r}}_j + \ddot{\underline{d}}_j)] \tag{21}$$

To be consistent with the previous rigid body results (equations (10) - (11)) we now will write the equations in their perturbed form; that is, we let

$$\underline{Q}_i = \underline{Q}_{1i} + \underline{Q}_{pi}$$

$$\underline{F}_i = \underline{F}_{1i} + \underline{F}_{pi}$$

$$\underline{d}_i = \underline{d}_{1i} + \underline{d}_{pi}$$

$$\underline{V}_c = \underline{V}_{c1} + \underline{V}_{cp}$$

$$\underline{\omega} = \underline{\omega}_1 + \underline{\omega}_p$$

$$\underline{g} = \underline{g}_1 + \underline{g}_p$$

$$\theta = \theta_1 + \theta_p$$

$$\phi = \phi_1 + \phi_p$$

$$\psi = \psi_1 + \psi_p \tag{22}$$

where the subscripts "1" and "p" denote steady-state or nominal values and perturbed values, respectively. Isolation of nominal and perturbed values in equation (20) results in the perturbed force

$$\underline{Q}_{pi} = \underline{F}_{pi} + m_i \underline{g}_p - m_i \dot{\underline{V}}_{cp} - m_i \underline{\omega}_p \times \underline{V}_{c1}$$

$$- m_i \underline{\omega}_1 \times \underline{V}_{cp} - \ddot{\underline{d}}_{pi} m_i - m_i \dot{\underline{\omega}}_p \times \tilde{\underline{r}}_i \quad , \ (i=1,2,\ldots n) \tag{23}$$

This result is further expanded into component form through the use of the relations

$$\underline{V}_{c1} = U_1\underline{i} + V_1\underline{j} + W_1\underline{k}$$

$$\underline{\omega}_1 = P_1\underline{i} + Q_1\underline{j} + R_1\underline{k}$$

$$\tilde{\underline{r}}_i = \tilde{x}_i\underline{i} + \tilde{y}_i\underline{j} + \tilde{z}_i\underline{k} \quad , \ (i=1,2,\ldots n) \tag{24}$$

$$\underline{V}_{cp} = u\underline{i} + v\underline{j} + w\underline{k}$$

$$\underline{\omega}_p = p\underline{i} + q\underline{j} + r\underline{k}$$

$$g_p = g[\theta_p \cos\theta_1 \underline{i} + (\phi_p \cos\theta_1 \cos\phi_1 - \theta_p \sin\theta_1 \sin\phi_1) i$$

$$-(\theta_p \sin\theta_1 \cos\phi_1 + \phi_p \sin\phi_1 \cos\theta_1)\underline{k}] \qquad (25)$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit base vectors in the principal body axis system.

Substitution from equation (24) into equation (23) results in

$$\begin{bmatrix} Q_{pix} \\ Q_{piy} \\ Q_{piz} \end{bmatrix} = \begin{bmatrix} F_{pix} \\ F_{piy} \\ F_{piz} \end{bmatrix} - m_i \begin{bmatrix} \ddot{d}_{pix} \\ \ddot{d}_{piy} \\ \ddot{d}_{piz} \end{bmatrix} - m_i \begin{bmatrix} 1 & 0 & 0 & 0 & \tilde{z}_i & -\tilde{y}_i \\ 0 & 1 & 0 & -\tilde{z}_i & 0 & \tilde{x}_i \\ 0 & 0 & 1 & \tilde{y}_i & -\tilde{x}_i & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}$$

$$- m_i \begin{bmatrix} 0 & -R_1 & Q_1 & 0 & W_1 & -V_1 \\ R_1 & 0 & -P_1 & -W_1 & 0 & U_1 \\ -Q_1 & P_1 & 0 & V_1 & -U_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

$$- m_i \begin{bmatrix} 0 & 0 & 0 & 0 & -g\cos\theta_1 & 0 \\ 0 & 0 & 0 & -g\cos\theta_1\cos\phi_1 & g\sin\theta_1\sin\phi_1 & 0 \\ 0 & 0 & 0 & g\sin\phi_1\cos\theta_1 & g\sin\theta_1\cos\phi_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \phi_p \\ \theta_p \\ \psi_p \end{bmatrix}$$

$$(i=1,2,\ldots n) \qquad (26)$$

This result can be expressed compactly as

$$\{Q_p\} = \{F_p\} - [\lceil m \rfloor]\{\ddot{d}_p\} - [\lceil m \rfloor][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{27}$$

where

$$\{Q_p\} = [Q_{p1x}, Q_{p1y}, Q_{piz}, Q_{p2x}, \ldots, Q_{pny}, Q_{pnz}]^T \tag{28}$$

$$\{F_p\} = [F_{p1x}, F_{p1y}, F_{p1z}, F_{p2x}, \ldots, F_{pny}, F_{pnz}]^T \tag{29}$$

$$\{d_p\} = [d_{p1x}, d_{p1y}, d_{p1z}, d_{p2x}, \ldots, d_{pnz}]^T \tag{30}$$

$$\{V_{cp}\} = [u, v, w, p, q, r]^T \tag{31}$$

$$\{r'_{op}\} = [0, 0, 0, \phi_p, \theta_p, \psi_p]^T \tag{32}$$

$$[\lceil m \rfloor] = \begin{bmatrix} m_1 & & & & & & & & & \\ & m_1 & & & & & & 0 & & \\ & & m_1 & & & & & & & \\ & & & m_2 & & & & & & \\ & & & & m_2 & & & & & \\ & & & & & m_2 & & & & \\ & & & & & & \ddots & & & \\ & & 0 & & & & & m_n & & \\ & & & & & & & & m_n & \\ & & & & & & & & & m_n \end{bmatrix} \tag{33}$$

$$[\bar{\phi}] = \begin{bmatrix} 1 & 0 & 0 & 0 & \tilde{z}_1 & -\tilde{y}_1 \\ 0 & 1 & 0 & -\tilde{z}_1 & 0 & \tilde{x}_1 \\ 0 & 0 & 1 & \tilde{y}_1 & -\tilde{x}_1 & 0 \\ \hline 1 & 0 & 0 & 0 & \tilde{z}_2 & -\tilde{y}_2 \\ 0 & 1 & 0 & -\tilde{z}_2 & 0 & \tilde{x}_2 \\ 0 & 0 & 1 & \tilde{y}_2 & -\tilde{x}_2 & 0 \\ \hline \\ 1 & 0 & 0 & 0 & \tilde{z}_n & -\tilde{y}_n \\ 0 & 1 & 0 & -\tilde{z}_n & 0 & \tilde{x}_n \\ 0 & 0 & 1 & \tilde{y}_n & -\tilde{x}_n & 0 \end{bmatrix} \tag{34}$$

$$[M_1] = \begin{bmatrix} 0 & -R_1 & Q_1 & 0 & W_1 & -V_1 \\ R_1 & 0 & -P_1 & -W_1 & 0 & U_1 \\ -Q_1 & P_1 & 0 & V_1 & -U_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$[M_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & -g\cos\theta_1 & 0 \\ 0 & 0 & 0 & -g\cos\theta_1\cos\phi_1 & g\sin\theta_1\sin\phi_1 & 0 \\ 0 & 0 & 0 & g\sin\phi_1\cos\theta_1 & g\sin\theta_1\cos\phi_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (36)$$

The matrices $[\ulcorner m \lrcorner]$ and $[\bar{\phi}]$ are referred to as the diagonal mass matrix and rigid body mode shape matrix, respectively.

Further, if we combine the vectors $\underset{\sim}{d}_o$ and $\underset{\sim}{\theta}_o$ into a vector $\{B\} = [d_{xo}, d_{yo}, d_{zo}, \theta_{xo}, \theta_{yo}, \theta_{zo}]^T$, we can express the elastic displacement field from equation (16) as

$$\{d\} = [\bar{\phi}]\ B\ + [C]\{Q\} \quad (37a)$$

where $\{d\} = [d_{x1}, d_{y1}, d_{z1}, d_{x2}, d_{y2}, \ldots d_{zn}]^T$

and

$$\{Q\} = [Q_{x1}, Q_{y1}, Q_{z1}, Q_{x2}, \ldots Q_{zn}]^T$$

In terms of perturbed quantities, we have

$$\{d_p\} = [\bar{\phi}]\{B_p\} + [C]\{Q_p\} \quad (37b)$$

where $\{B_p\}$ represents the perturbed value of $\{B\}$ from the reference condition.

The vector $\{B_p\}$ can be eliminated from the analysis by assuming the mean body axis system coincides with the principal axes system.

According to Milne (references 2, 3), the mean axis system is defined by the equations

$$\sum_{i=1}^{n} m_i \, \tilde{\underline{r}}_i \times \underline{d}_i = 0 \tag{38}$$

and

$$\sum_{i=1}^{n} m_i \, \underline{d}_i = 0 \tag{39}$$

or equivalently

$$[\bar{\phi}]^T [\lceil m \rfloor] \{d\} = 0 \tag{40}$$

Then, the perturbation in elastic displacement also satisfies this relationship or

$$[\bar{\phi}]^T [\lceil m \rfloor] \{d_p\} = 0 \tag{41}$$

Equations (38) - (39) state that the deformation motion has zero linear and angular momentum relative to the body axis system. Ashley (reference 4) interprets the assumption of equations (38) - (39) as a statement that the overall motion of the body is described by a set of principal axes for the deforming body.

If both sides of equation (37b) are premultiplied by the matrix $[\bar{\phi}]^T [\lceil m \rfloor]$, the result is

$$[\bar{\phi}]^T[\grave{}m\lrcorner]\{d_p\} = [\bar{\phi}]^T[\grave{}m\lrcorner][\bar{\phi}]\{B_p\} + [\bar{\phi}]^T[\grave{}m\lrcorner][C]\{Q_p\} = 0$$

from which $\{B_p\}$ can be solved for as

$$\{B_p\} = -[M]^{-1}[\bar{\phi}]^T[\grave{}m\lrcorner][C]\{Q_p\} \tag{42}$$

where

$$[M] = [\bar{\phi}]^T[\grave{}m\lrcorner][\bar{\phi}] \tag{43}$$

Substitution of $\{B_p\}$ from equation (42) into equation (37b) gives

$$\{d_p\} = [\bar{C}]\{Q_p\} \tag{44}$$

where

$$[\bar{C}] = (I - [\bar{\phi}][M]^{-1}[\bar{\phi}]^T[\grave{}m\lrcorner])[C] \tag{45}$$

If $\{Q_p\}$ from equation (27) is substituted into equation (44), we find that the perturbation elastic displacement field satisfies the differential equation

$$\{d_p\} = [\bar{C}]\{F_p\} - [\bar{C}][\grave{}m\lrcorner]\{\ddot{d}_p\}$$
$$- [\bar{C}][\grave{}m\lrcorner][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{46}$$

It is possible to express the equation for $\{d_p\}$ in terms of the stiffness matrix $[K_{11}] = [C]^{-1}$ by substitution of $\{Q_p\}$ from equation

(27) into the expression for $\{d_p\}$ as expressed by equation (37b). This results in

$$\{d_p\} = [\bar{\phi}]\{B_p\} + [C]\{F_p\} - [C][\text{`}m\text{.}]\{\ddot{d}_p\}$$

$$- [C][\text{`}m\text{.}][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \qquad (47)$$

Again, we wish to eliminate $\{B_p\}$ from equation (47). This can be accomplished by the following operations. First, premultiply both sides of equation (47) by $[K_{11}] = [C]^{-1}$ to get

$$[K_{11}]\{d_p\} = [K_{11}][\bar{\phi}]\{B_p\} + \{F_p\} - [\text{`}m\text{.}]\{\ddot{d}_p\}$$

$$- [\text{`}m\text{.}][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \qquad (48)$$

Note that the rigid body perturbation equations expressed by equations (10) and (11) can be written as

$$[M](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) = [\bar{\phi}]^T\{F_p\} \qquad (49)$$

where

$$[\bar{\phi}]^T\{F_p\} = [f_x, f_y, f_z, m_x, m_y, m_z]^T$$

Also note that $[\bar{\phi}]^T[\text{`}m\text{.}]\{\ddot{d}_p\} = 0$ since $[\bar{\phi}]^T[\text{`}m\text{.}]\{d_p\} = 0$.

These two results reduce equation (48) to the form

$$[\bar{\phi}]^T[K_{11}]\{d_p\} = [\bar{\phi}]^T[K_{11}][\bar{\phi}]\{B_p\}$$

from which $\{B_p\}$ is solved for as

$$\{B_p\} = ([\bar{\phi}]^T[K_{11}][\bar{\phi}])^{-1}[\bar{\phi}]^T[K_{11}]\{d_p\} \tag{50}$$

Substitution of this value for $\{B_p\}$, which is equivalent to the value expressed by equation (42), into equation (48) results in the equation

$$[\diagdown m \diagdown]\{\ddot{d}_p\} + [\bar{K}]\{d_p\} = \{F_p\} - [\diagdown m \diagdown][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{51}$$

where

$$[\bar{K}] = [K_{11}] - [K_{11}][\bar{\phi}]([\bar{\phi}]^T[K_{11}][\bar{\phi}])^{-1} [\bar{\phi}]^T[K_{11}] \tag{52}$$

Equation (51) represents 3n scalar equations of motion for the n lumped masses. Only 3n - 6 of them are independent; however, since only 3n - 6 components of $\{d_p\}$ are linearly independent due to the constraint equation (40) defining mean axes. These 3n - 6 independent equations for internal motion together with the rigid body or body axes motion from equation (49) make up the necessary 3n equations of motion to describe the dynamics of the elastic airplane.

The results represented in equations (49) and (51) are very difficult to treat mathematically without further simplification. Before preceding to the various approximate forms for these equations, it is of interest to consider them in more detail. To demonstrate the roles of the stability derivatives associated with rigid versus elastic airplanes, we now develop the right hand side of equation (49). The perturbed aerodynamic force associated with the n lumped masses is generally assumed to be a linear function of the velocity and acceleration of the aircraft center of mass and of the elastic displacement field and its velocity and accelerations fields, i.e., from reference 5,

$$\{F_p\} = [A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3]\{d_p\} + [A_4]\{\dot{d}_p\} + [A_5]\{\ddot{d}_p\} \qquad (53)$$

The matrices $[A_1]$, ... $[A_5]$ are referred to as aerodynamic influence coefficients which contribute to the stability derivatives. Substitution of $\{F_p\}$ from equation (53) into equation (49) result in the following aerodynamic force and moments due to rigid and elastic body motions

$$[\bar{\phi}]^T[A_1]\{V_{cp}\},\ [\bar{\phi}]^T[A_2]\{\dot{V}_{cp}\},\ [\bar{\phi}]^T[A_3]\{d_p\},$$

$$[\bar{\phi}]^T[A_4]\{\dot{d}_p\},\ [\bar{\phi}]^T[A_5]\{\ddot{d}_p\}$$

The nature of the elements of these quantities is demonstrated in the following example

$$[\bar{\phi}]^T[A_1]\{V_{cp}\} = \left[\begin{array}{ccc|ccc|c|ccc} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \hline 0 & -\tilde{z}_1 & \tilde{y}_1 & 0 & -\tilde{z}_2 & \tilde{y}_2 & \cdots & 0 & -\tilde{z}_n & \tilde{y}_n \\ \tilde{z}_1 & 0 & -\tilde{x}_1 & \tilde{z}_2 & 0 & -\tilde{x}_2 & \cdots & \tilde{z}_n & 0 & -\tilde{x}_n \\ -\tilde{y}_1 & \tilde{x}_1 & 0 & -\tilde{y}_2 & \tilde{x}_2 & 0 & \cdots & -\tilde{y}_n & \tilde{x}_n & 0 \end{array}\right] \cdot$$

$$\left[\begin{array}{l} a_{11xu}\ a_{11xv}\ a_{11xw}\ a_{11xp}\ a_{11xq}\ a_{11xr} \\ a_{11yu}\ a_{11yv}\ \cdots\cdots\cdots \\ a_{11zu}\cdots\cdots\cdots \\ \hline a_{12xu}\ a_{12xv}\cdots\cdots\cdots \\ a_{12yu}\ \cdots\cdots\cdots \\ a_{12zu}\cdots\cdots\cdots \\ \hline \cdots\cdots\cdots\cdots\cdots \\ \hline a_{1nxu}\cdots\cdots\cdots a_{1nxr} \\ a_{1nyu}\cdots\cdots\cdots \\ a_{1nzu}\cdots\cdots\cdots a_{1nzr} \end{array}\right] \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \qquad (54)$$

where the first subscript denotes an element of $[A_1]$, the second subscript denotes the element of mass, the third subscript denotes the component of force and the fourth subscript denotes the component of velocity it multiplies. Equation (54), on multiplication of $[\bar{\phi}]^T[A_1]$, becomes

$$[\bar{\phi}]^T[A_1]\{V_{cp}\} = \begin{bmatrix} \sum_{i=1}^{n} a_{1ixu} & \sum_{i=1}^{n} a_{1ixv} & \cdots & \sum_{i=1}^{n} a_{1ixr} \\ \sum_{i=1}^{n} a_{1iyu} & \cdots & & \\ \sum_{i=1}^{n} a_{1izu} & \cdots & & \\ \sum_{i=1}^{n} (\tilde{y}_i a_{1izu} - \tilde{z}_i a_{1iyu}) & \cdots & & \\ \sum_{i=1}^{n} (\tilde{z}_i a_{1ixu} - \tilde{x}_i a_{1izu}) & \cdots & & \\ \sum_{i=1}^{n} (\tilde{x}_i a_{1iyu} - \tilde{y}_i a_{1ixu}) & \cdots & & \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

or,

$$[\bar{\phi}]^T[A_1]\{V_{cp}\} = \begin{bmatrix} f_{x_u} & f_{x_v} & f_{x_w} & f_{x_p} & f_{x_q} & f_{x_r} \\ f_{y_u} & \cdots & & & & \\ f_{z_u} & \cdots & & & & \\ M_{x_u} & \cdots & & & & \\ M_{y_u} & \cdots & & & & \\ M_{z_u} & \cdots & & & & \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \qquad (55)$$

where, for instance,

$$f_{x_u} = 1/2\rho SU_1(C_{x_u} + 2\ C_{xo}) \tag{56}$$

$$M_{x_u} = 1/2\rho SU_1\bar{c}(C_{l_u} + 2\ C_{lo}) \tag{57}$$

The quantities f_{x_u} and M_{x_u} can be interpreted as the force in the x direction per unit change in velocity u, and moment about the x axis per unit change in u, respectively. Also, as a second example, consider

$$[\bar{\phi}]^T[A_3]\{d_p\} =$$

$$[\bar{\phi}]^T \begin{bmatrix} a_{311xx} & a_{311xy} & a_{311xz} & a_{312xx}\cdots a_{31nxz} \\ a_{311yx} & a_{311yy} & \cdots\cdots & \cdots\cdots a_{31nyz} \\ a_{311zx} & \cdots\cdots & \cdots\cdots & \cdots\cdots a_{31nzz} \\ \hline a_{321xx} & \cdots\cdots & \cdots\cdots & \cdots\cdots a_{32nxz} \\ a_{321yz} & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{321zx} & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \hline\hline a_{3n1xx} & \cdots\cdots & \cdots\cdots & \cdots\cdots a_{3nnxz} \\ a_{3n1yx} & \cdots\cdots & \cdots\cdots & \cdots\cdots a_{3nnyz} \\ a_{3n1zx} & \cdots\cdots & \cdots\cdots & \cdots\cdots a_{3nnzz} \end{bmatrix} \{d_p\}$$

(58)

The subscripts on the elements of $[A_3]$ have the following meaning: the first subscript indicates an element of $[A_3]$, the second and third indicates a mass element pair, the fourth indicates the component of force and the fifth denotes the component of displacement. For instance, a_{321zy} represents the force acting on panel 2 in the z direction due to a unit displacement of panel 1 in the y direction. We will defer discussion of the stability derivatives contained in the term $[\bar{\phi}]^T[A_3]\{d_p\}$ until we have introduced the modal substitution concept.

Quasi-Static Approximation

The quasi-static approximation considers only those aeroelastic forces dependent upon $\{d_p\}$ but not upon the time rate of change of this displacement field. For example, the inertial forces $[\grave{}m\lrcorner]\{\ddot{d}_p\}$ and $[\bar{\phi}]^T[A_5]\{\ddot{d}_p\}$ and the aerodynamic damping force $[\bar{\phi}]^T[A_4]\{\dot{d}_p\}$ are considered to be equal to zero. This assumption allows for the elastic displacement field to be eliminated from the analysis by solving equation (46) for $\{d_p\}$ as

$$\{d_p\} = [\bar{C}]\{F_p\}-[\bar{C}][\grave{}m\lrcorner][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \qquad (59)$$

Substitution of $\{d_p\}$ from equation (59) into equation (53) results in $\{F_p\}$ given as

$$\{F_p\} = (I - [A_3][\bar{C}])^{-1} \{[A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\}$$

$$- [A_3][\bar{C}][\lceil m \rfloor][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\})\} \qquad (60)$$

Substitution for the aerodynamic force given by equation (60) into the right hand side of the rigid body equations of motion, equation (49), results in the quasi-static form for the rigid body motion as

$$[M'](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) = [\bar{\phi}']([A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\}) \qquad (61)$$

where

$$[M'] = [\bar{\phi}]^T\{[I] + ([I] - [A_3][\bar{C}])^{-1} [A_3][\bar{C}]\}[\lceil m \rfloor][\bar{\phi}] \qquad (62)$$

$$[\bar{\phi}'] = [\bar{\phi}]^T([I] - [A_3][\bar{C}])^{-1} \qquad (63)$$

Comparison of equation (61) with equation (49) shows that in the quasi-static approximation, the effect of elasticity is to modify the mass matrix $[\lceil m \rfloor]$ and the rigid body mode shape matrix, $\bar{\phi}$ by amounts depending upon the elastic properties of the aircraft.

Modal Substitution

The method of modal substitution is a means of reducing the large numbers of equations of motion associated with the "exact" formulation of equations (49) and (51) while at the same time, uncoupling the degrees of freedom of the elastic motion. To this end, we introduce the transformation

$$\{d_p\} = \{\phi(x,y,z)\}\ u(t) \qquad (64)$$

where $\{\phi(x,y,z)\}$ is the free vibration mode matrix of constants of dimension 3n x 1, and u(t) is the generalized elastic scalar displacement associated with this mode shape. The values of $\{\phi\}$ are determined from the eigenvalue problem associated with the invacuum motion of the aircraft; that is, the deformation of the aircraft vibrating in the absence of external forces. The values of u, however, are evaluated from the full equation of elastic motion expressed by equation (51).

The elastic displacement field for invacuum motion, neglecting structural damping, is governed by the equation

$$[\ulcorner m \lrcorner]\{\ddot{d}_p\} + [\bar{K}]\{d_p\} = 0 \tag{65}$$

In terms of $\{\phi\}$ and u, equation (65) for each mode is written as

$$\frac{\ddot{u}}{u}[\ulcorner m \lrcorner]\{\phi\} = - [\bar{K}]\{\phi\}$$

which requires that $\frac{\ddot{u}}{u}$ be set to a constant say $-\lambda^2$, then

$$[\bar{K}]\{\phi\} = \lambda^2 [\ulcorner m \lrcorner]\{\phi\} \tag{66}$$

Associated with equation (66) are 3n-6 distinct eigenvalues λ_i^2 and corresponding eigenvectors $\{\phi_i\}$. The elastic displacement field is assumed to be a linear combination of these modes as

$$\{d_p\} = \sum_{i=1}^{3n-6} \{\phi_i\} u_i \tag{67}$$

where u_i is the amplitude of the mode shape $\{\phi_1\}$. This is further written as

$$\{d_p\} = [\phi]\{u\} \tag{68}$$

where

$$[\phi] = [\{\phi_1\},\ \{\phi_2\},\ldots\{\phi_{3n-6}\}]$$

$$\{u\} = [u_1,\ u_2,\ldots u_{3n-6}]^T$$

Hence $[\phi]$ has dimension (3n) x (3n-6) and $\{u\}$ has dimension (3n-6)x1.

Since the mode shapes correspond to free vibration, it is found that the eigenvectors are orthogonal as indicated by the equations

$$\begin{aligned} \{\phi_j\}^T[K]\{\phi_i\} &= \bar{K}_i\delta_{ij} \\ \{\phi_j\}^T[\ulcorner m \lrcorner]\{\phi_i\} &= \bar{m}_i\delta_{ij} \end{aligned} \tag{69}$$

or

$$\begin{aligned} [\phi]^T[K][\phi] &= [\ulcorner \bar{K} \lrcorner] \\ [\phi]^T[\ulcorner m \lrcorner][\phi] &= [\ulcorner \bar{m} \lrcorner] \end{aligned} \tag{70}$$

The matrices $[\ulcorner \bar{m} \lrcorner]$ and $[\ulcorner \bar{K} \lrcorner]$ are the generalized mass and stiffness matrices, respectively.

Substitution for $\{d_p\}$ from equation (68) into equation (51) results in the expression

$$[\backslash m_\backslash]\ [\phi]\{\ddot{u}\} + [\bar{K}][\phi]\{u\} = \{F_p\}$$

$$- [\backslash m_\backslash][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{71}$$

If both sides of this expression are multiplied by $[\phi]^T$, it follows that

$$[\phi]^T[\backslash m_\backslash][\phi]\{\ddot{u}\} + [\phi]^T[\bar{K}][\phi]\{u\} = [\phi]^T\{F_p\}$$

$$-[\phi]^T[\backslash m_\backslash][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{72}$$

The last term on the right-hand side of equation (72) vanishes due to the definition of the mean axis. This reduces the equation of motion for the generalized displacements to the form

$$[\backslash \bar{m}_\backslash]\{\ddot{u}\} + [\backslash \bar{K}_\backslash]\{u\} = [\phi]^T\{F_p\} \tag{73}$$

where

$$\{F_p\} = [A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3][\phi]\{u\}$$

$$+ [A_4][\phi]\{\dot{u}\} + [A_5][\phi]\{\ddot{u}\} \tag{74}$$

The term $[A_3][\phi]\{u\}$, occurring in equation (74) is now considered to show the form of some of the additional stability derivatives due to elastic effects. The force $[\overline{\phi}]^T[A_3][\phi]\{u\}$ discussed in equation (58) in terms of the physical displacements $\{d_p\}$ is now written in terms of generalized displacements as

$$[\overline{\phi}]^T[A_3][\phi]\{u\} = \begin{bmatrix} f_{x_{u_1}} & f_{x_{u_2}} & \cdots & f_{x_{u_{3n-6}}} \\ f_{y_{u_1}} & f_{y_{u_2}} & \cdots & f_{y_{u_{3n-6}}} \\ f_{z_{u_1}} & \cdots & \cdots & \cdots \\ M_{x_{u_1}} & M_{x_{u_2}} & \cdots & M_{x_{u_{3n-6}}} \\ M_{y_{u_1}} & \cdots & \cdots & \cdots \\ M_{z_{u_1}} & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ \cdot\cdot \\ \cdot\cdot \\ u_{3n-6} \end{bmatrix}$$

where, for example,

$$f_{x_{u_i}} = 1/2\rho U_1^{\;2} S/\overline{c}\; C_{x_{u_i}} \quad , \qquad (i=1,2,\ldots 3n-6)$$

$$M_{y_{u_i}} = 1/2\rho U_1^{\;2} S\; C_{m_{u_i}}$$

To summarize the results of this section, we outline the sequence of computations to arrive at the elastic displacements $\{d_p\}$:

1) Compute $[\phi]$, the mode shape matrix, from the 3n-6 eigenfunctions of the relation

$$[\bar{K}]\{\phi\} = \lambda^2 [\diagdown m \diagup]\{\phi\}$$

If $\{\phi_i\}$ is the ith eigenfunction, then

$$[\phi] = [\{\phi_1\}\{\phi_2\}\ldots\{\phi_{3n-6}\}]$$

2) Compute the generalized mass and stiffness matrices $[\diagdown \bar{m} \diagup]$ and $[\diagdown \bar{K} \diagup]$ from the orthogonality equations

$$[\phi]^T [\diagdown m \diagup][\phi] = [\diagdown \bar{m} \diagup]$$

$$[\phi]^T [K][\phi] = [\diagdown \bar{K} \diagup]$$

3) Compute the generalized elastic displacement vector $\{u\}$ from the equation

$$[\diagdown \bar{m} \diagup]\{\ddot{u}\} + [\diagdown \bar{K} \diagup]\{u\} = [\phi]^T \{F_p\}$$

4) Compute the physical elastic displacement vector $\{d_p\}$ from the equation

$$\{d_p\} = [\phi]\{u\}$$

Residual Flexibility

In this formulation, the free vibration mode shapes matrix and amplitudes are separated into dynamically retained modes $[\phi_1]$ and dynamically deleted modes $[\phi_2]$. This means we take the modal substitution formulation and omit terms or forces such as $[\grave{}\bar{m}_2\lrcorner]\{\ddot{u}_2\}$, $[A_4]\cdot[\phi_2]\{\dot{u}_2\}$, and $[A_5][\phi_2]\{\ddot{u}_2\}$, where $[\grave{}\bar{m}_2\lrcorner]$ the generalized mass matrix associated with the deleted modes. Here, we have treated the deleted modes as we did the total displacement vector in the quasi-static formulation.

We write $\{d_p\}$ as

$$\{d_p\} = [\phi_1]\{u_1\} + [\phi_2]\{u_2\} \tag{75}$$

where

$$[\phi] = [[\phi_1], [\phi_2]] \; ; \{u\} = \begin{bmatrix} \{u_1\} \\ \{u_2\} \end{bmatrix}$$

Further, let

$$[\grave{}\bar{K}\lrcorner] = \begin{bmatrix} [\grave{}\bar{K}_1\lrcorner] & [0] \\ [0] & [\grave{}\bar{K}_2\lrcorner] \end{bmatrix} , \quad [\ulcorner\bar{m}\lrcorner] = \begin{bmatrix} [\ulcorner\bar{m}_1\lrcorner] & [0] \\ [0] & [\ulcorner\bar{m}_2\lrcorner] \end{bmatrix}$$

Then, we have (substituting the above relations into equation (73))

$$\{u\} = \begin{bmatrix} \{u_1\} \\ \{u_2\} \end{bmatrix} = \begin{bmatrix} [\grave{}\bar{K}_1\lrcorner]^{-1} & [0] \\ [0] & [\grave{}\bar{K}_2\lrcorner]^{-1} \end{bmatrix} \begin{bmatrix} [\phi_1]^T \\ [\phi_2]^T \end{bmatrix} \tag{76}$$

$$\{\{F_p\} - [\grave{}m\lrcorner][\phi_1]\{\ddot{u}_1\} - [\grave{}m\lrcorner][\phi_2]\{\ddot{u}_2\}\}$$

Solve equation (76) for $\{u_2\}$ to get

$$\{u_2\} = -[\diagdown \bar{K}_2 \diagdown]^{-1} [\phi_2]^T [\diagdown m \diagdown] [\phi_1] \{\ddot{u}_1\}$$

$$-[\diagdown \bar{K}_2 \diagdown]^{-1} [\phi_2]^T [\diagdown m \diagdown] [\phi_2] \{\ddot{u}_2\} + [\diagdown \bar{K}_2 \diagdown]^{-1} [\phi_2]^T \{F_p\} \qquad (77)$$

This equation is simplified by noting the following

$$[\phi_2]^T [\diagdown m \diagdown] [\phi_1] = 0$$

$$- [\diagdown \bar{K}_2 \diagdown] [\phi_2]^T [\diagdown m \diagdown] [\phi_2] \{\ddot{u}_2\} \cong 0$$

The latter result follows from the basic assumption of residual flexibility. This results in the approximation

$$\{u_2\} \cong [\diagdown \bar{K}_2 \diagdown]^{-1} [\phi_2]^T \{F_p\} \qquad (78)$$

where

$$\{F_p\} \cong [A_1] \{V_{cp}\} + [A_2] \{\dot{V}_{cp}\} + [A_3] [\phi_1] \{u_1\}$$

$$+ [A_3] [\phi_2] \{u_2\} + [A_4] [\phi_1] \{\dot{u}_1\} + [A_5] [\phi_1] \{\ddot{u}_1\} \qquad (79)$$

Substitution of $\{u_2\}$ from equation (78) into equation (79) results in

$$\{F_p\} = [A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3][\phi_1]\{u_1\}$$

$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\} \tag{80}$$

$$+ [A_3][\phi_2][\urcorner\bar{K}_2\lrcorner]^{-1}[\phi_2]^T\{F_p\}$$

The last term in equation (80) which explicitly relates to the deleted mode shape can be rewritten in terms of the identity

$$[\phi_2][\urcorner\bar{K}_2\lrcorner]^{-1}[\phi_2]^T\{F_p\} = [\bar{C}_R][\{F_p\} - [\urcorner m\lrcorner][\phi_1]]\{\ddot{u}_1\}$$

$$- [\bar{C}][\urcorner m\lrcorner][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \tag{81}$$

where

$$[\bar{C}_R] = [\bar{C}] - [\phi_1][\urcorner\bar{K}_1\lrcorner]^{-1}[\phi_1]^T \tag{82}$$

This identity is obtained by equating the expression for $\{d_p\}$ from equation (46) to the expression obtained by using equation (76).

Substitution of equation (81) into equation (80) results in the following expression for $\{F_p\}$.

$$\{F_p\} = ([I] - [A_3][\bar{C}_R])^{-1}[[A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3][\phi_1]\{u_1\}$$

$$- [A_3]([\bar{C}][\,^\frown m_\smile][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) \qquad (83)$$

$$+ [\bar{C}_R][\,^\frown m_\smile][\phi_1]\{\ddot{u}_1\}) + [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}]$$

This form for the perturbed aerodynamic force results in the following equations of motion for the residual flexibility approximation to the elastic airplane:

Rigid Body Motion:

$$[M](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) =$$

$$[\bar{\phi}]^T([I] - [A_3][\bar{C}_R])^{-1}\{[A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\}$$

$$+ [A_3][\phi_1]\{u_1\} - [A_3][\bar{C}_R][\,^\frown m_\smile][\phi_1]\{u_1\} \qquad (84)$$

$$- [A_3][\bar{C}][\,^\frown m_\smile][\bar{\phi}](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\})$$

$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}\}$$

Elastic Displacement:

$$\lceil \bar{m}_1 \rfloor \{\ddot{u}_1\} + \lceil \bar{K}_1 \rfloor \{u_1\} =$$

$$[\phi_1]^T ([I] - [A_3][\bar{C}_R])^{-1} \{[A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\}$$

$$+ [A_3][\phi_1]\{u_1\} - [A_3][\bar{C}_R]\lceil m \rfloor [\phi_1]\{\ddot{u}_1\} \tag{85}$$

$$- [A_3][\bar{C}]\lceil m \rfloor [\bar{\phi}] (\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\})$$

$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}\}$$

Modal Truncation

The modal truncation form of the equations of motion is obtained by representing the modal matrix in terms of only r mode shapes where $r < 3n-6$. That is, we let $\{d_p\} = [\phi_1]\{u_1\}$ where

$$[\phi_1] = [\{\phi_1\}, \{\phi_2\} \ldots \{\phi\}_r]$$

$$\{u_1\} = [u_1, u_2, \ldots u_r]^T$$

This results in the following equations of motion:

Rigid Body Motion:

$$[M](\{\dot{V}_{cp}\} + [M_1]\{V_{cp}\} + [M_2]\{r'_{op}\}) =$$

$$[\bar{\phi}]^T([A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3][\phi_1]\{u_1\}$$

$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\})$$

Elastic Displacement:

$$[\lceil \bar{m}_1 \rfloor]\{\ddot{u}_1\} + [\lceil \bar{K}_1 \rfloor]\{u_1\} =$$

$$[\phi]^T([A_1]\{V_{cp}\} + [A_2]\{\dot{V}_{cp}\} + [A_3][\phi_1]\{u_1\}$$

$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\})$$

PARAMETER IDENTIFICATION

The parameter extraction process will be discussed for the simpler case of an elastic aircraft perturbed from symmetric, steady, level, non-accelerating flight. In addition, the equations of motion used to represent the aircraft dynamics will be the modal truncation formulation where invacuum modes are retained. For this case, the rigid body motion or mean axis motion from equation (84) is given as

$$M\dot{u} + Mg\theta_p = f_{x_u} u + f_{x_w} w + f_{x_q} \dot{\theta}_p + f_{x_{\dot{u}}} \dot{u}$$

$$+ f_{x_{\dot{w}}} \dot{w} + f_{x_{\dot{q}}} \ddot{\theta}_p + \sum_{i=1}^{c} f_{x_{\delta_i}} \delta_i$$

$$+ \sum_{i=1}^{m} (f_{x_{u_i}} u_i + f_{x_{\dot{u}_i}} \dot{u}_i + f_{x_{\ddot{u}_i}} \ddot{u}_i) \qquad (86)$$

$$-MU_1 \dot{\theta}_p + M\dot{w} = f_{z_u} u + f_{z_w} w + f_{z_q} \dot{\theta}_p + f_{z_{\dot{u}}} \dot{u}$$

$$+ f_{z_{\dot{w}}} \dot{w} + f_{z_{\dot{q}}} \ddot{\theta}_p + \sum_{i=1}^{c} f_{z_{\delta_i}} \delta_i$$

$$+ \sum_{i=1}^{m} (f_{z_{u_i}} u_i + f_{z_{\dot{u}_i}} \dot{u}_i + f_{z_{\ddot{u}_i}} \ddot{u}_i)$$

$$I_{xx} \ddot{\theta}_p = M_u u + M_w w + M_q \dot{\theta}_p + M_{\dot{u}} \dot{u} + M_{\dot{w}} \dot{w} + M_{\dot{q}} \ddot{\theta}_p$$

$$+ \sum_{i=1}^{c} M_{\delta_i} \delta_i + \sum_{i=1}^{m} (M_{u_i} u_i + M_{\dot{u}_i} \dot{u}_i + M_{\ddot{u}_i} \ddot{u}_i)$$

and the elastic motion satisfies

$$m_i \ddot{u}_i + K_i u_i = f_{i_u} u + f_{i_w} w + f_{i_q} \dot{\theta}_p + f_{i_{\dot{u}}} \dot{u}$$

$$+ f_{i_{\dot{w}}} \dot{w} + f_{i_{\dot{q}}} \ddot{\theta}_p + \sum_{j=1}^{c} f_{i_{\delta_j}} \delta_j \qquad (87)$$

$$+ \sum_{j=1}^{m} (f_{i_{u_j}} u_j + f_{i_{\dot{u}_j}} \dot{u}_j + f_{i_{\ddot{u}_j}} \ddot{u}_j) \quad , \quad (i=1,2,\ldots m)$$

In equations (86) - (87) the forces and moments due to control surfaces are included.

The equations (86) - (87) are in dimensional form. For purposes of estimation, they are put in nondimensional form as follows:

$$M\dot{u} + Mg\theta_p = 1/2\rho SU_1^2[(C_{x_u} + 2\,C_{xo})\,u/U_1 + C_{x_\alpha}\frac{w}{U_1}$$

$$+ C_{x_q}\frac{\bar{c}\dot{\theta}_p}{2U_1} + C_{x_{\dot{u}}}\frac{\bar{c}\dot{u}}{2U_1^2} + C_{x_{\dot{\alpha}}}\frac{\dot{w}\bar{c}}{2U_1^2} + C_{x_{\dot{q}}}\frac{\ddot{\theta}_p\bar{c}^2}{4U_1^2}$$

$$+\sum_{i=1}^{c} C_{x_{\delta_i}}\delta_i + \sum_{i=1}^{m}(C_{x_{u_i}}u_i/\bar{c} + C_{x_{u_i}}\frac{2\dot{u}_i}{U_1} + C_{x_{\ddot{u}_i}}\frac{\bar{c}\ddot{u}_i}{2U_1^2})]$$

$$-MU_1\dot{\theta}_p + M\dot{w} = 1/2\rho SU_1^2[(C_{z_u} + 2C_{zo})\frac{u}{U_1} + C_{z_\alpha}\frac{w}{U_1}$$

$$+ C_{z_q}\frac{\bar{c}\dot{\theta}_p}{U_1} + C_{z_{\dot{u}}}\frac{\bar{c}\dot{u}}{2U_1^2} + C_{z_{\dot{\alpha}}}\frac{\bar{c}\dot{w}}{2U_1^2} + C_{z_{\dot{q}}}\frac{\bar{c}^2\ddot{\theta}_p}{4U_1^2} \qquad (88)$$

$$+\sum_{i=1}^{c} C_{z_{\delta_i}}\delta_i + \sum_{i=1}^{m}(C_{z_{u_i}}\frac{2u_i}{\bar{c}} + C_{z_{\dot{u}_i}}\frac{\dot{u}_i}{U_1} + C_{z_{\ddot{u}_i}}\frac{\ddot{u}_i\bar{c}}{2U_1^2})]$$

$$I_{xx}\ddot{\theta}_p = 1/2\rho S\bar{c}U_1^{\,2}[(C_{m_u} + 2\ C_{mo})\ {}^{u}/_{U_1} + C_{m_\alpha}\frac{w}{U_1}$$

$$+ C_{m_q}\frac{\dot{\theta}_p\bar{c}}{2U_1} + C_{m_{\dot{u}}}\frac{\dot{u}\bar{c}}{2U_1^{\,2}} + C_{m_{\dot{\alpha}}}\frac{\dot{w}\bar{c}}{2U_1^{\,2}} + C_{m_{\dot{q}}}\frac{\ddot{\theta}_p\bar{c}^2}{4U_1^{\,2}}$$

$$+\sum_{i=1}^{c} C_{m_{\delta_i}}\ \delta_i + \sum_{i=1}^{m}(C_{m_{u_i}}\frac{2u_i}{\bar{c}} + C_{m_{\dot{u}_i}}\frac{\dot{u}_i}{U_1} + C_{m_{\ddot{u}_i}}\frac{\ddot{u}_i\bar{c}}{2U_1^{\,2}})]$$

$$m_i\ddot{u}_i + K_iu_i = 1/2\rho U_1^{\,2}S[C_{u_{i_u}}\ {}^{u}/_{U_1} + C_{u_{i_\alpha}}\ {}^{w}/_{U_1}$$

$$+ C_{u_{i_q}}\frac{\dot{\theta}_p\bar{c}}{2U_1^{\,2}} + C_{u_{i_{\dot{u}}}}\frac{\dot{u}\bar{c}}{2U_1^{\,2}} + C_{u_{i_{\dot{\alpha}}}}\frac{\dot{w}\bar{c}}{2U_1^{\,2}} + C_{u_{i_{\dot{q}}}}\frac{\ddot{\theta}_p\bar{c}}{4U_1^{\,2}} \qquad (89)$$

$$+\sum_{j=1}^{c} C_{u_{i_{\delta j}}} + \sum_{j=1}^{m}(C_{u_{i_{u_j}}}\frac{2u_j}{\bar{c}} + C_{u_{i_{\dot{u}_j}}}\frac{\dot{u}_j}{U_1} + C_{u_{i_{\ddot{u}_j}}}\frac{\ddot{u}_j\bar{c}}{2U_1^{\,2}})]$$

The stability and control parameters that are to be estimated from flight test data are the components of a vector

$$\underline{p} = [C_{xo}, C_{x_u}, C_{x_\alpha}, C_{x_q}, C_{x_{\dot{u}}}, C_{x_{\dot{\alpha}}}, C_{x_{\dot{q}}},$$

$$C_{x_{\delta_k}}, C_{x_{u_i}}, C_{x_{\dot{u}_i}}, C_{x_{\ddot{u}_i}}, C_{zo}, C_{z_u},$$

$$C_{z_\alpha}, C_{z_q}, C_{z_{\dot{u}}}, C_{z_{\dot{\alpha}}}, C_{z_{\dot{q}}}, C_{z_{\delta_k}},$$

$$C_{z_{u_i}}, C_{z_{\dot{u}_i}}, C_{z_{\ddot{u}_i}}, C_{m_u}, C_{m_\alpha}, C_{m_q},$$

$$C_{m_{\dot{u}}}, C_{m_{\dot{\alpha}}}, C_{m_{\dot{q}}}, C_{m_{\delta_k}}, C_{m_{u_i}}, C_{m_{\dot{u}_i}},$$

$$C_{m_{\ddot{u}_i}}, C_{u_{i_u}}, C_{u_{i_\alpha}}, C_{u_{i_q}}, C_{u_{i_{\dot{u}}}}, C_{u_{i_{\dot{\alpha}}}},$$

$$C_{u_{i_{\dot{q}}}}, C_{u_{i_{\delta_k}}}, C_{u_{i_{u_j}}}, C_{u_{i_{\dot{u}_j}}}, C_{u_{i_{\ddot{u}_j}}}]^T$$

where (k=1,2,...c) , (i,j=1,2,...m).

The number of parameters listed above is $3m^2 + mc + 3c + 15m + 20$. With the exception of the control derivatives, unindexed quantities are the derivatives associated with rigid airplanes while the indexed quantities are the additional derivatives introduced through aeroelastic effects.

Equations (88)-(89) are now rewritten in matrix notation by defining a state vector

$$\underline{z} = [u, w, \theta_p, \dot{\theta}_p, u_1, u_2, \ldots u_m, \dot{u}_1, \dot{u}_2, \ldots \dot{u}_m]^T \tag{90}$$

and a control vector

$$\underline{\delta} = [\delta_1, \delta_2, \ldots \delta_c]^T \tag{91}$$

The equations of motion then become

$$F\dot{\underline{z}} = B\underline{z} + G\underline{\delta} \tag{92}$$

where F, B, and G are matrices of stability and control derivatives which are given in Appendix A. In the case of rigid airplanes, all state components are available for direct measurement. Additionally, linear accelerometers are used to augment the linear speed and angular velocities to provide for more efficient parameter extraction. In the case of elastic aircraft, additional accelerometer placement is desirable since the generalized elastic displacements are not available for direct measurement. The analytical form of the acceleration recorded by the ith sensor is

$$\begin{aligned} a_{x_i} &= \dot{u} + \tilde{z}_i \ddot{\theta}_p + g\,\theta_p + \sum_{j=1}^{m} \phi_{x_{ij}} \ddot{u}_j \\ a_{z_i} &= \dot{w} - \tilde{x}_i \ddot{\theta}_p - U_1\dot{\theta} + \sum_{j=1}^{m} \phi_{z_{ij}} \ddot{u}_j \end{aligned} \tag{93}$$

The form of the sensor measurements, in general, is taken as a linear function of the state, that is,

$$\underline{y} = H\underline{z} + \underline{n} \tag{94}$$

where $\underline{n}$ is assumed to be a white noise process of zero mean and covariance R, and H is either the identity matrix or some variation on it depending on whether or not the state vector is augmented to include the linear accelerations.

The maximum likelihood estimate of the parameter vector $\underline{p}$ is given as (ref. (6))

$$\hat{\underline{p}} = \underline{p}^{\circ} + \left[\sum_{i=1}^{N} A^T(t_i) H^T \hat{R}^{-1} H\, A(t_i)\right]^{-1} \left[\sum_{i=1}^{N} H\, A(t_i)\, \hat{R}^{-1}\, \underline{\nu}(t_i)\right] \qquad (95)$$

$$\hat{R} = \frac{1}{N} \sum_{i=1}^{N} \underline{\nu}(t_i)\, \underline{\nu}^T(t_i) \qquad (96)$$

In these equations, N is the number of measurements, A is the sensitivity matrix $\partial \underline{z} / \partial \underline{p}$ and $\underline{\nu}(t)$ the difference $\underline{y}(t) - \underline{z}(p^{\circ}, t)$. The asymptotic value of the covariance of the error in the estimate is

$$E[(\underline{p}-\hat{\underline{p}})(\underline{p}-\hat{\underline{p}})^T] = \left[\sum_{i=1}^{N} A^T(t_i) H^T \hat{R}^{-1} H\, A(t_i)\right]^{-1} \qquad (97)$$

which is the Cramer-Rao lower bound on the covariance of the estimate (reference (7)).

Several practical problems seem likely in the implementation of a computer program to estimate the parameters of elastic aircraft. Even for the simple case of symmetric, level flight the number of parameters is extensive and is likely to contribute to convergence problems due to lack of understanding of the physical significance of the aeroelastic derivatives. In addition, considerable computer effort is required to provide supporting information in the way of initial estimates on the parameters as well as the flexibility matrix and normal mode shapes.

CONCLUDING REMARKS

This effort has presented in a tutorial fashion the manner in which the complex phenomena of aeroelastic motion alters the equations of motion and stability parameters extraction processes of conventional rigid body aircraft. The various degrees of approximation in the representation of the motion as provided by the quasi-static, modal substitution and residual flexibility have been discussed in some detail. The corresponding physical parameters which enter into the analysis by each of these approximations are indicated and illustrated for the case of symmetric, level, unaccelerated flight. It is anticipated that potential problem areas exist in the implementation of a computer program to extract the large number of stability and control derivatives.

APPENDIX

ELEMENTS OF STABILITY AND CONTROL MATRICES

The elements of the matrices F, B, and G are listed below:

F: Dimension (4+2m) x (4+2m)

$F_{11} = M - \frac{1}{2}\rho S \bar{c} C_{x_{\dot{u}}}$

$F_{12} = - \frac{1}{2}\rho S \bar{c} C_{x_{\dot{\alpha}}}$

$F_{13} = 0$

$F_{14} = - \frac{1}{4}\rho \bar{c}^2 S C_{x_{\dot{q}}}$

$F_{1,4+i} = 0, \quad (i=1,2,...m)$

$F_{1, 4+m+i} = \frac{1}{2}\rho S \bar{c} C_{x_{\ddot{u}_i}}, \quad (i=1,2,...m)$

$F_{21} = - \frac{1}{2}\rho S \bar{c} C_{z_{\dot{u}}}$

$F_{22} = M - \frac{1}{2}\rho S \bar{c} C_{z_{\dot{\alpha}}}$

$F_{23} = 0$

$F_{24} = - \frac{1}{4}\rho S \bar{c}^2 C_{z_{\dot{q}}}$

$F_{2, 4+i} = 0 , \quad (i=1,2,...m)$

$F_{2,4+m+i} = - \frac{1}{2}\rho S \bar{c} C_{z_{\ddot{u}_i}} , \quad (i=1,2,...m)$

$F_{31} = \frac{1}{2}\rho S \bar{c}^2 C_{m_{\dot{u}}}$

$F_{32} = - \frac{1}{2}\rho S \bar{c}^2 C_{m_{\dot{\alpha}}}$

$$F_{33} = 0$$

$$F_{34} = I_{xx} - \frac{1}{4}\rho\bar{c}^3\ S\ C_{m_{\dot{q}}}$$

$$F_{3,4+i} = 0 \quad , \quad (i=1,2,\ldots m)$$

$$F_{3,4+m+i} = -\frac{1}{2}\rho S\ \bar{c}^2\ C_{m_{\ddot{u}_i}} \quad , \quad (i=1,2,\ldots m)$$

$$F_{i,1} = -\frac{1}{2}\rho S\ \bar{c}\ C_{u_{i_{\dot{u}}}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$F_{i,2} = -\frac{1}{2}\rho S\ \bar{c}\ C_{u_{i_{\dot{\alpha}}}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$F_{i,3} = 0$$

$$F_{i,4} = -\frac{1}{4}\ \rho S\ \bar{c}^2\ C_{u_{i_{\dot{q}}}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$F_{i,4+j} = 0 \quad , \quad (i=4,5,\ldots 3+m;\ j=1,2,\ldots m)$$

$$F_{i,4+m+j} = m\ \delta_{ij} - \frac{1}{2}\rho\bar{c}\ S\ C_{u_{i_{\ddot{u}_j}}} \quad (i=4,5,\ldots 3+m;\ j=1,2,\ldots m)$$

$$F_{4+m,j} = \delta_{3j}, \quad (j=1,2,\ldots 4+2m)$$

$$F_{4+m+i,j} = \delta_{4+m+i,j} \quad , \quad (i=1,2,\ldots m,\ j=1,2,\ldots 4+2m)$$

B: Dimension (4+2m) x (4+2m)

$$B_{11} = \frac{1}{2}\rho SU_1\ (2\ C_{xo} + C_{x_u})$$

$$B_{12} = \frac{1}{2}\rho SU_1\ C_{x_\alpha}$$

$$B_{13} = -\ Mg$$

$$B_{14} = \frac{1}{4}\rho SU_1\ \bar{c}\ C_{x_q}$$

$$B_{1,i+4} = \frac{\rho U_1^2 S}{2\bar{c}}\ C_{x_{u_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{1,4+m+i} = \frac{1}{2}\rho U_1 S \; C_{x_{\dot{u}_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{21} = \frac{1}{2}\rho S U_1 \; (2C_{zo} + C_{z_u})$$

$$B_{22} = \frac{1}{2}\rho S U_1 \; C_{z_\alpha}$$

$$B_{23} = 0$$

$$B_{24} = (M + \frac{1}{4}\rho S \; \bar{c} \; C_{z_q})U_1$$

$$B_{2,4+i} = \frac{\rho U_1^2 S}{2\bar{c}} \; C_{z_{u_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{2,4+m+i} = \frac{1}{2}\rho U_1 S \; C_{z_{\dot{u}_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{31} = \frac{1}{2}\rho U_1 \; S \; \bar{c} \; C_{m_u}$$

$$B_{32} = \frac{1}{2}\rho U_1 S \; \bar{c} \; C_{m_\alpha}$$

$$B_{33} = 0$$

$$B_{34} = \frac{1}{4}\rho S U_1 \; \bar{c}^2 \; C_{m_q}$$

$$B_{3,4+i} = \frac{1}{2}\rho U_1^2 S \; C_{m_{u_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{3,4+m+i} = \frac{1}{2}\rho S \; \bar{c}^2 \; C_{m_{\dot{u}_i}} \quad , \quad (i=1,2,\ldots m)$$

$$B_{i,1} = \frac{1}{2}\rho U_1 S \; C_{u_{i_u}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$B_{i,2} = \frac{1}{2}\rho U_1 S \; C_{u_{i_\alpha}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$B_{i,3} = 0 \quad , \quad (i=4,5,\ldots 3+m)$$

$$B_{i,4} = \frac{1}{4}\rho U_1 S \; \bar{c} \; C_{u_{i_q}} \quad , \quad (i=4,5,\ldots 3+m)$$

$$B_{i',j+4} = \frac{\rho U_1^2 S}{2\bar{c}} C_{u_{i_{u_j}}} - K_i \delta_{ij}, \quad (i=4,5,\ldots 3+m;\ j=1,2,\ldots m)$$

$$B_{i',j+4+m} = \frac{1}{2}\rho U_1 S\ C_{u_{i_{u_j}}}, \quad (i=4,5,\ldots 3+m;\ j=1,2,\ldots m)$$

$$B_{4+m,j} = \delta_{4j}, \quad (j=1,2,\ldots 4+2m)$$

$$B_{i',j} = \delta_{ij}, \quad (i=5+m,\ldots 4+2m;\ j=1,2,\ldots 4+2m)$$

G: Dimension (4+2m) x c

$$G_{1j} = \frac{1}{2}\rho U_1^2 S\ C_{x_{\delta_j}}, \quad (j=1,2,\ldots c)$$

$$G_{2j} = \frac{1}{2}\rho U_1^2 S\ C_{z_{\delta_j}}, \quad (j=1,2,\ldots c)$$

$$G_{3j} = \frac{1}{2}\rho U_1^2 S\ \bar{c}\ C_{m_{\delta_j}}, \quad (j=1,2,\ldots c)$$

$$G_{i+3,j} = \frac{1}{2}\rho U_1^2\ S\ C_{u_{i_{\delta_j}}}, \quad (i=1,2,\ldots m;\ j = 1,2,\ldots c)$$

$$G_{i+4,j} = 0, \quad (i=m,m+1,\ldots 2m;\ j=1,2,\ldots c)$$

REFERENCES

1. Bisplinghoff, R.L., and Ashley, H., "Principles of Aeroelasticity", John Wiley and Sons, Inc., 1962.

2. Milne, R.D., "Dynamics of the Deformable Airplane, Parts I and II", Her Majesty's Stationery Office, London, 1964.

3. Milne, R.D., "Some Remarks on the Dynamics of Deformable Bodies." AIAA Journal, Vol. 6, March 1968.

4. Ashley, H., "Observations on the Dynamic Behavior of Large Flexible Bodies in Orbit", AIAA Journal, Vol. 5, No. 3, March 1967.

5. Dusto, Arthur," Approximate Forms of the Equations of Motion for Analyzing Dynamic Stability of Large Flexible Airplanes," Transactions of the 3rd Technical Workshop on Dynamic Stability Problems, NASA-Ames Research Center, November 4-7, 1968.

6. Wells, W.R., "An Algorith for the Estimation of Aircraft Parameters in a Nonlinear Aerodynamic Model," Proceedings of the Third IFAC Conference on Identification and System Parameter Estimation, The Hague/Delft, The Netherlands, June 12-15, 1973.

7. Nahi, N.E., "Estimation Theory and Application," John Wiley, 1969.

NASA-Langley, 1976 CR-2672