Dynamics and Stability of a Two Degree of Freedom Oscillator With an Elastic Stop

Madeleine Pascal

Dynamics and Stability of a Two Degree of Freedom Oscillator With an Elastic Stop

Madeleine Pascal
Laboratoire Systèmes Complexes,
Université d'Evry Val d'Essonne et CNRS FRE
2494, 40 rue du Pelvoux,
91020 Evry cedex, France
e-mail: mpascal@iup.univ-evry.fr

A two degree of freedom oscillator with a colliding component is considered. The aim of the study is to investigate the dynamic behavior of the system when the stiffness obstacle changes to a finite value to an infinite one. Several cases are considered. First, in the case of rigid impact and without external excitation, a family of periodic solutions are found in analytical form. In the case of soft impact, with a finite time duration of the shock, and no external excitation, the existence of periodic solutions, with an arbitrary value of the period, is proved. Periodic motions are also obtained when the system is submitted to harmonic excitation, in both cases of rigid or soft impact. The stability of these periodic motions is investigated for these four cases.

Keywords: Nonlinear Vibrations, Two Degree of Freedom Oscillator, Rigid and Soft Impact, Periodic Motion, Stability, Forced and Unforced System

1 Introduction

Vibrating systems with clearance between the moving parts are frequently encountered in technical applications. These systems with impacts are strongly nonlinear; they are usually modeled as a spring-mass system with amplitude constraint. Such systems have been the subject of several investigations, mainly in the simplest case of a one degree of freedom system [1–4] and more seldom for multidegree of freedom systems [5–8]. On the other hand, the system behavior during the contact between the moving parts can be described as rigid impact, usually associated with a restitution coefficient, or modeled as soft impact, with a finite time duration of the shock. Several other parameters like damping, external excitation, influence the behavior of the system.

The work is the continuation of a previous paper [9], in which a two degree of freedom oscillator is considered. The nonlinearity in this case comes from the presence of two fixed stops limiting the motion of one mass. Assuming no damping and no external excitation, the behavior of the system is investigated when the obstacles stiffness changes from a finite value to an infinite one. In both cases, a family of symmetrical periodic solutions, with two impacts per period, is obtained in analytical form.

In the present paper, a two degree of freedom system in the presence of one fixed obstacle is considered. Assuming that no damping occurs, we investigate four cases: Unforced system with rigid impact, unforced system with soft impact, forced system (with harmonic excitation) with rigid impact and, at last, forced system with soft impact. In all cases, periodic solutions are found and stability results of these particular motions are obtained.

2 Problem Formulation

The system under consideration (Fig. 1) is a generalization of the double oscillator investigated in the paper [9]. It consists of two masses m_1 and m_2 connected by linear springs of stiffness k_1 and k_2. The displacement z_1 of the mass m_1 is limited by the presence of a fixed stop. When z_1 is greater than the clearance, a contact of the first mass with the stop occurs; this contact gives rise to a restoring force associated to a spring stiffness k_3. The mathematical model of the system is given by:

$$M\ddot{z} + Kz = F + P\cos(\omega t + \varphi), \quad z = (z_1, z_2)^t, \quad F = \begin{pmatrix} f(z_1) \\ 0 \end{pmatrix}, \tag{1}$$

$$f(z_1) = \begin{cases} -k_3(z_1 - 1) & z_1 > 1 \\ 0 & z_1 < 1 \end{cases}, \quad M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$
$$K = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix}$$

$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ and φ are the amplitude and the phase angle of the harmonic excitation.

3 Unforced System

Let us consider the system without external excitation ($P=0$).

3.1 Rigid Impact. When the stiffness obstacle k_3 tends to infinity, a rigid impact of the first mass against the stop occurs. Starting from initial positions $z_0 = \begin{pmatrix} 1 \\ y \end{pmatrix}$ and initial velocities $\dot{z}_0 = \begin{pmatrix} u \\ w \end{pmatrix}$ ($u > 0$), corresponding to a contact of the first mass against the stop, assuming a perfect elastic impact, the new positions z_c and the new velocities $\dot{z}_c$ after the shock are obtained by:

$$\begin{pmatrix} z_c \\ \dot{z}_c \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} z_0 \\ \dot{z}_0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

After the impact, the system performs a free motion defined by:

$$\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = C(t)\begin{pmatrix} z_c \\ \dot{z}_c \end{pmatrix}, \quad C(t) = \begin{pmatrix} \Gamma_1(t) & \Gamma_2(t) \\ \Gamma_3(t) & \Gamma_1(t) \end{pmatrix}, \quad \Gamma_i = \Lambda B_i(t)\Lambda^{-1}$$
$$(i = 1,2,3) \tag{3}$$

$$\Lambda = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad B_1(t) = \begin{pmatrix} C_1(t) & 0 \\ 0 & C_2(t) \end{pmatrix},$$

$$B_2(t) = \begin{pmatrix} \dfrac{S_1(t)}{\omega_1} & 0 \\ 0 & \dfrac{S_2(t)}{\omega_2} \end{pmatrix}, \tag{4}$$

$B_3(t) = \dot{B}_1(t)$, $C_i(t) = \cos \omega_i t$, $S_i(t) = \sin \omega_i t$ $(i = 1,\ 2)$.

In these formulas, (ω_1, ω_2) are the roots of the characteristic equation: $\Delta(\omega^2) \equiv \det(K - M\omega^2) = 0$ while $\Psi_i = \binom{1}{\lambda_i}$ are defined by $(K - M\omega_i^2)\Psi_i = 0$ $(i = 1, 2)$.

The following properties for the Γ_i matrices hold:

$$\Gamma_1^2(t) - \Gamma_2(t)\Gamma_3(t) = I, \quad \Gamma_i(t)\Gamma_j(t) = \Gamma_j(t)\Gamma_i(t) \text{ for } i,j = 1,2,3 \tag{5}$$

Moreover, the coefficients $C_{ij}(t)$ of the 4 by 4 matrix $C(t)$ satisfy the property:

$$C_{ij}(t) = C_{i-2,j-2}(t), \quad (i,j = 3,4) \tag{6}$$

Let us investigate if for a set of initial conditions $Z_0 = \binom{z_0}{\dot{z}_0}$, $z_0 = \binom{1}{y}$, $\dot{z}_0 = \binom{u>0}{w}$, it is possible to obtain a periodic solution of period T, with one impact per period.

The free motion performed by the system after the rigid impact finishes at time $t = T$ when $z_1(T) = 1$ and $\dot{z}_1(T) > 0$. Let us denote by $Z_f = \binom{z_f}{\dot{z}_f}$ the positions and the velocities reached by the system at $t = T$.

The condition to obtain such a periodic motion is given by:

$$Z_f \equiv C(T)Z_c = Z_0, \quad Z_c = H_0 Z_0, \quad H_0 = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \tag{7}$$

Let us introduce the position Z_s reached by the system from the initial position Z_0 after a backward motion of duration T: $Z_s = C(-T)Z_0$. The condition (7) is equivalent to: $Z_s = Z_c$.

It results for the determination of the four scalar parameters $(y,\ u,\ w,\ T)$ the four scalar equations:

$$(-H_0 + C(-T))Z_0 = 0, \quad Z_0 = (1, y, u, w)^t$$

or equivalently:

$$\begin{aligned} (\Gamma_1 - I)z_0 - \Gamma_2 \dot{z}_0 &= 0 \\ -\Gamma_3 z_0 + (\Gamma_1 - E)\dot{z}_0 &= 0 \end{aligned} \quad \Gamma_i = \Gamma_i(T) \quad (i = 1,2,3) \tag{8}$$

Taking into account the properties (5) of the Γ_i matrices, the system (8) leads to:

$$\begin{aligned} z_0 &= (\Gamma_1 - I)^{-1}\Gamma_2 \dot{z}_0 \\ (E + I)\dot{z}_0 &= 0 \end{aligned} \tag{9}$$

The last equation of (9) reduces to $w = 0$. From the first one, y and u are obtained in terms of the period:

$$y = \frac{(\omega_2 t_2 - \omega_1 t_1)\lambda_1\lambda_2}{\lambda_2\omega_2 t_2 - \lambda_1\omega_1 t_1}, \quad u = \frac{(\lambda_1 - \lambda_2)\omega_1\omega_2 t_1 t_2}{\lambda_2\omega_2 t_2 - \lambda_1\omega_1 t_1},$$

$$t_i = \tan\left(\frac{\omega_i T}{2}\right) \quad (i = 1,2) \tag{10}$$

In case of rigid impact, a family of periodic solutions is obtained for which the initial conditions are defined in terms of the period and the initial velocity w of the nonimpacting mass is zero. For these particular motions, the conditions giving the positions and the velocities after the shock can be formulated by:

$$z_c = z_0, \quad \dot{z}_c = -\dot{z}_0 \tag{11}$$

These results are similar to the results obtained in Ref. [9]. The system considered in this previous paper is a symmetrical system with respect to the position z_1 of the first mass and it can be expected that the obtained results are due to this property. But it is not the real explanation because the system investigated now is not symmetrical.

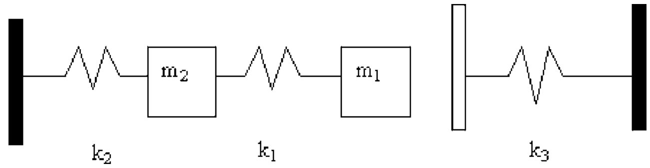

Fig. 1 Double oscillator

3.2 Soft Impact. Let us assume that the stiffness obstacle is bounded. The mathematical model of the system is given by: For $z_1 < 1$

$$M\ddot{z} + Kz = 0 \tag{12}$$

For $z_1 > 1$

$$M\ddot{z} + K_1 z = K_3 \quad K_1 = \begin{pmatrix} k_1 + k_3 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \quad K_3 = \begin{pmatrix} k_3 \\ 0 \end{pmatrix} \tag{13}$$

Let us assume that the initial conditions are given by: $Z_0 = (1, y, u, w)^t$ $u > 0$.

A periodic solution is defined in two steps:

- For $0 \leqslant t \leqslant \tau$, $z_1 > 1$, the system is defined by the motion equations (13). The time duration τ of this constraint motion is defined by the condition:

$$z_1(\tau) = 1 \tag{14}$$

  Let us denote by $Z_c = Z(\tau) = (1, y_c, u_c, w_c)^t$ the value of the parameters at the end of shock, with the condition $u_C < 0$.
- For $\tau \leqslant t \leqslant \tau + T$, a free motion obtained from Eqs. (12) and initial conditions Z_c occurs. This motion finishes when $z_1(\tau + T) = 1$. Let us denote by $Z_f = Z(\tau + T) = (1, y_f, u_f, w_f)^t$ the value of the parameters at the end of free motion $(u_f > 0)$. The condition to obtain a periodic orbit of period $\tau + T$ is given by

$$Z_f = Z_0 \tag{15}$$

  The piecewise linear systems (12) and (13) give the two parts of the motion in analytical form.
- For $0 \leqslant t \leqslant \tau$, the constraint motion is deduced from a modal analysis of system (13):

$$Z(t) = H(t)(Z_0 - d) + d, \quad d = (d_1, d_2, 0, 0)^t$$

$$d_1 = \frac{k_3(k_1 + k_2)}{k_1 k_2 + k_3(k_1 + k_3)}, \quad d_2 = \frac{k_3 k_1}{k_1 k_2 + k_3(k_1 + k_2)} \tag{16}$$

$$H(t) = \begin{pmatrix} H_1(t) & H_2(t) \\ H_3(t) & H_1(t) \end{pmatrix}, \quad H_i(t) = \Sigma G_i(t) \Sigma^{-1}$$

$$(i = 1,2,3), \quad \Sigma = \begin{pmatrix} 1 & \mu_2 \\ \mu_1 & 1 \end{pmatrix} \tag{17}$$

$$G_1(t) = \begin{pmatrix} c_1(t) & 0 \\ 0 & c_2(t) \end{pmatrix}, \quad G_2 = \begin{pmatrix} \dfrac{s_1(t)}{\sigma_1} & 0 \\ 0 & \dfrac{s_2(t)}{\sigma_2} \end{pmatrix}, \quad G_3 = \dot{G}_1, \tag{18}$$

$$c_i(t) = \cos(\sigma_i t), \quad s_i(t) = \sin(\sigma_i t), \quad (i = 1,2)$$

In these formulas, σ_1, σ_2, $\Phi_1 = \binom{1}{\mu_1}$, $\Phi_2 = \binom{\mu_2}{1}$ define the characteristic frequencies and the eigenvectors of the constraint system (13). For the H_i matrices, the properties (5) obtained for the Γ_i matrices hold, together with the property $H_{ij}(t) = H_{i-2,j-2}(t)$, $(i,\ j=3,4)$ for the coefficients $H_{ij}(t)$ of the matrix H(t).

– For $\tau \leqslant t \leqslant \tau+T$, the free motion is obtained from $Z(t)=C(t-\tau)Z_c$ where the matrix C is defined by formulas (3) and (4) and $Z_c=H(\tau)(Z_0-d)+d$.

Let us introduce the positions and the velocities $Z_s = (z_{1s}, z_{2s}, \dot{z}_{1s}, \dot{z}_{2s})^t$ of the system after a backward motion of duration T from the initial position Z_0. The condition (15) of periodicity is reformulated as

$$Z_s \equiv C(-T)Z_0 = Z_c \tag{19}$$

$$Z_c = \begin{pmatrix} z_c \\ \dot{z}_c \end{pmatrix} = \begin{pmatrix} H_1(z_0-d_0)+H_2\dot{z}_0+d_0 \\ H_3(z_0-d_0)+H_1\dot{z}_0 \end{pmatrix}, \quad d_0=(d_1,d_2)^t$$

$$Z_s = \begin{pmatrix} z_s \\ \dot{z}_s \end{pmatrix} = \begin{pmatrix} \Gamma_1 z_0 - \Gamma_2\dot{z}_0 \\ -\Gamma_3 z_0 + \Gamma_1\dot{z}_0 \end{pmatrix}, \quad H_i = H_i(\tau), \quad \Gamma_i = \Gamma_i(T)$$
$$(i=1,2,3) \tag{20}$$

The condition (19) is equivalent to

$$X_1 = X_2, \quad Y_1 = Y_2 \tag{21}$$

$$X_1 \equiv z_c - z_0 = (H_1-I)(z_0-d_0) + H_2\dot{z}_0, \quad X_2 \equiv z_s - z_0 = (\Gamma_1 - I)z_0 - \Gamma_2\dot{z}_0$$

$$Y_1 \equiv \dot{z}_c + \dot{z}_0 = H_3(z_0-d_0) + (H_1+I)\dot{z}_0, \quad Y_2 \equiv \dot{z}_s + \dot{z}_0 = -\Gamma_3 z_0 + (\Gamma_1+I)\dot{z}_0 \tag{22}$$

From the properties (5) of Γ_i and H_i, we deduce:

$$Y_i = P_i X_i \quad (i=1,2)$$

$$P_1 = H_2^{-1}(H_1+I), \quad P_2 = -\Gamma_2^{-1}(\Gamma_1+I) \tag{23}$$

The condition (21) leads to

$$X_1 = X_2, \quad P_1X_1 = P_2X_2 \tag{24}$$

Two possible cases of periodic solutions can be deduced from (24), namely:

$$X_1 = X_2 = 0 \quad \text{or } X_1 = X_2, \quad \det(P_1 - P_2) = 0 \tag{25}$$

3.3 Existence of Periodic Motions (Soft Impact). Let us discuss the first conditions (25). In this case, from (22), we deduce:

$$z_c = z_0, \quad \dot{z}_c = -\dot{z}_0 \tag{26}$$

The condition (14) is fulfilled and the initial conditions are obtained from the equations:

$$(H_1 - I)(z_0 - d_0) + H_2\dot{z}_0 = 0$$

$$(\Gamma_1 - I)z_0 - \Gamma_2\dot{z}_0 = 0 \tag{27}$$

This system provides four scalar equations for the determination of the five parameters $(y,\ u,\ w,\ \tau,\ T)$. It results that, as in the case of rigid impact, T and hence the period can be chosen arbitrarily. Moreover, the conditions (11) and (26) obtained at the end of the shock are the same for both rigid and soft impacts. From (27), we deduce:

$$\dot{z}_0 = -H_2^{-1}(H_1-I)(z_0-d_0)$$

$$[\Gamma_1 - I + \Gamma_2 H_2^{-1}(H_1-I)]z_0 = \Gamma_2 H_2^{-1}(H_1-I)d_0 \tag{28}$$

The last equation (28), after the elimination of y, provides a relation $F(\tau,T)=0$ between the time duration τ of the shock and the time duration T of the free motion.

-The other case $X_1=X_2$, $\det(P_1-P_2)=0$ leads to no solution (see Appendix A).

In both cases (soft or rigid impact), a family of periodic motions is obtained, with an arbitrary value of the period. Moreover, the conditions (26) obtained at the end of the shock for soft impact are consistent with Newton rules of rigid impact (11) with a restitution coefficient equal to one, i.e., with assumption of ideal elastic impact. This rather remarkable result has already been obtained for the symmetrical system of Ref. [9].

4 Forced System

Let us assume that the two masses are subjected to harmonic external excitations of period $2\pi/\omega$, constant amplitudes P_1, P_2 and constant phase angle φ. From the results obtained in the previous paragraph, where a family of periodic orbits is found with an arbitrary value of the period, it can be expected that for the forced system, periodic solutions of period $2\pi/\omega$ exist.

4.1 Rigid Impact. Let us investigate the case of rigid impact. Starting from the initial conditions $Z_0=(1,y,u,w)^t$ $(u>0)$, the conditions $Z_c=(1,y_c,u_c,w_c)^t$ after the shock are obtained from (2) and the free motion performed by the system is given by:

$$z = \Gamma_1(t)(z_0 - R\cos\varphi) + \Gamma_2(t)(\dot{z}_c + R\omega\sin\varphi) + R\cos(\omega t+\varphi)$$

$$\dot{z} = \Gamma_3(t)(z_0 - R\cos\varphi) + \Gamma_1(t)(\dot{z}_c + R\omega\sin\varphi) - R\omega\sin(\omega t+\varphi) \tag{29}$$

where $R=(R_1,R_2)^t$ is the amplitude of the response defined by:

$$R_1 = A_1 + A_2, \quad R_2 = \lambda_1 A_1 + \lambda_2 A_2$$

$$A_i = \frac{P_1 + \lambda_i P_2}{(\omega_i^2 - \omega^2)(m_1 + \lambda_i^2 m_2)}, \quad (i=1,2) \tag{30}$$

The free motion finishes at time $t=T$ when $z_1(T)=1$, $\dot{z}_1(T)>0$. Let us denote by $Z_f=(z_f,\dot{z}_f)^t$ the positions and the velocities reached by the system at this time. The condition to obtain a periodic motion of period T is:

$$Z_f = Z_0$$

or :

$$z_0 = \Gamma_1(T)(z_0 - R\cos\varphi) + \Gamma_2(T)(\dot{z}_c + R\omega\sin\varphi) + R\cos(\omega T+\varphi)$$

$$\dot{z}_0 = \Gamma_3(T)(z_0 - R\cos\varphi) + \Gamma_1(T)(\dot{z}_c + R\omega\sin\varphi) - R\omega\sin(\omega T+\varphi)$$

$$\dot{z}_c = E\dot{z}_0 \tag{31}$$

Let us assume that $T=2\pi/\omega$, $\varphi=0$, $\dot{z}_c=-\dot{z}_0$. A periodic motion of period $2\pi/\omega$ is obtained if the initial conditions $Z_0=(1,y,u,0)^t$ are defined by the system:

$$(\Gamma_1 - I)(z_0 - R) - \Gamma_2\dot{z}_0 = 0 \tag{32}$$

$$\Gamma_3(z_0 - R) - (\Gamma_1 + I)\dot{z}_0 = 0$$

$$\Gamma_i = \Gamma_i(2\pi/\omega), \quad (i=1,2,3)$$

Taking into account the properties (5) of the Γ_i matrices, this system reduces to

$$\dot{z}_0 = \Gamma_2^{-1}(\Gamma_1 - I)(z_0 - R) \tag{33}$$

and the corresponding values of y and u are obtained:

$$y = R_2 + \frac{(\omega_2 t_2 - \omega_1 t_1)\lambda_1\lambda_2}{\lambda_2\omega_2 t_2 - \lambda_1\omega_1 t_1}(1 - R_1), \tag{34}$$

$$u = (1-R_1)\frac{(\lambda_1 - \lambda_2)\omega_1\omega_2 t_1 t_2}{\lambda_2\omega_2 t_2 - \lambda_1\omega_1 t_1}, \quad t_i = \tan\left(\frac{\omega_i \pi}{\omega}\right) \quad (i=1,2)$$

Remark: In more general cases, the impact is described by a restitution coefficient r $(0<r<1)$. The initial conditions and the phase angle related to a periodic solution of period $2\pi/\omega$ can also be obtained in analytical form [10]. A similar solution has been studied in paper [6].

4.2 Soft Impact. When the stiffness obstacle is bounded, the motion equations of the system are given by:

$$M\ddot{z}+K_1z=K_3+P\cos(\omega t+\varphi)\quad z_1>1$$

$$M\ddot{z}+Kz=P\cos(\omega t+\varphi)\quad z_1<1 \tag{35}$$

From the initial condition $Z_0=(1,y,u,w)^t$ $(u>0)$, the solution is defined in two parts:

– For $0<t<\tau$, $z_1>1$, the solution is given by:

$$z=H_1(t)(z_0-d_0-Q\cos\varphi)+H_2(t)(\dot{z}_0+Q\omega\sin\varphi)+d_0+Q\cos(\omega t+\varphi)$$

$$\dot{z}=H_3(t)(z_0-d_0-Q\cos\varphi)+H_1(t)(\dot{z}_0+Q\omega\sin\varphi)-Q\omega\sin(\omega t+\varphi) \tag{36}$$

$Q=(Q_1,Q_2)^t$ is the response amplitude defined by:

$$Q_1=\frac{P_1(1+\mu_2^2)+P_2(\mu_1+\mu_2)}{(\sigma_1^2-\omega^2)(m_1+m_2\mu_1^2)},\qquad Q_2=\frac{P_1(\mu_1+\mu_2)+P_2(1+\mu_1^2)}{(\sigma_2^2-\omega^2)(m_2+m_1\mu_2^2)} \tag{37}$$

The time duration τ of this motion is obtained from the condition $z_1(\tau)=1$. Let us denote $Z_c=(1,y_c,u_c,w_c)^t$ the value of the parameters at $t=\tau$ $(u_c<0)$.

– For $\tau<t<\tau+T$, the motion of the system is defined by:

$$z=\Gamma_1(t-\tau)(z_c-R\cos\psi)+\Gamma_2(t-\tau)(\dot{z}_c+R\omega\sin\psi)+R\cos(\omega t+\varphi)$$

$$\dot{z}=\Gamma_3(t-\tau)(z_c-R\cos\psi)+\Gamma_1(t-\tau)(\dot{z}_c+R\omega\sin\psi)-R\omega\sin(\omega t+\varphi) \tag{38}$$

where $R=(R_1,R_2)^t$ is defined by (30) and $\psi=\omega\tau+\varphi$. This motion finishes at time $t=\tau+T$ when $z_1(\tau+T)=1$, $\dot{z}_1(\tau+T)>0$. If $Z_f=(z_f,\dot{z}_f)^t$ denote the positions and the velocities reached by the system at this time, the condition to obtain a periodic motion of period $\tau+T$ is $Z_f=Z_0$. Let us assume that $\tau+T=2\pi/\omega$, $\varphi=-\omega\tau/2$. At the end of the first part of the motion $(t=\tau)$, the positions and the velocities are given by:

$$z_c=H_1(z_0-d_0-Q\cos\tilde{\varphi}_0)+H_2(\dot{z}_0-Q\omega\sin\tilde{\varphi}_0)+d_0+Q\cos\tilde{\varphi}_0$$

$$\dot{z}_c=H_3(z_0-d_0-Q\cos\tilde{\varphi}_0)+H_1(\dot{z}_0-Q\omega\sin\tilde{\varphi}_0)-Q\omega\sin\tilde{\varphi}_0$$

$$\tilde{\varphi}_0=\omega\tau/2,\quad H_i=H_i(\tau)\quad(i=1,2,3) \tag{39}$$

At the end of the second part of the motion $(t=2\pi/\omega)$, we obtain:

$$z_f=\Gamma_1(z_c-R\cos\tilde{\varphi}_0)+\Gamma_2(\dot{z}_c+R\omega\sin\tilde{\varphi}_0)+R\cos\tilde{\varphi}_0$$

$$\dot{z}_f=\Gamma_3(z_c-R\cos\tilde{\varphi}_0)+\Gamma_1(\dot{z}_c+R\omega\sin\tilde{\varphi}_0)+R\omega\sin\tilde{\varphi}_0$$

$$\Gamma_i=\Gamma_i(2\pi/\omega-\tau)\quad(i=1,2,3) \tag{40}$$

The conditions of periodicity are reformulated as:

$$\tilde{X}_1=\tilde{X}_2,\quad\tilde{Y}_1=\tilde{Y}_2 \tag{41}$$

$$\tilde{X}_1\equiv z_c-z_0=(H_1-I)(z_0-d_0-Q\cos\tilde{\varphi}_0)+H_2(\dot{z}_0-Q\omega\sin\tilde{\varphi}_0)$$

$$\tilde{Y}_1\equiv\dot{z}_c+\dot{z}_0=H_3(z_0-d_0-Q\cos\tilde{\varphi}_0)+(H_1+I)(\dot{z}_0-Q\omega\sin\tilde{\varphi}_0) \tag{42}$$

$$\tilde{X}_2\equiv z_s-z_0=(\Gamma_1-I)(z_0-R\cos\tilde{\varphi}_0)-\Gamma_2(\dot{z}_0-R\omega\sin\tilde{\varphi}_0)$$

$$\tilde{Y}_2\equiv\dot{z}_s+\dot{z}_0=-\Gamma_3(z_0-R\cos\tilde{\varphi}_0)+(\Gamma_1+I)(\dot{z}_0-R\omega\sin\tilde{\varphi}_0)$$

$$Z_s=(z_s,\dot{z}_s)^t=C(-T)Z_0 \tag{43}$$

As in the case of an unforced system taking into account the properties of the H_i and Γ_i matrices, the solution of system (41) is given by:

$$\tilde{X}_1=\tilde{X}_2=0,\quad\tilde{Y}_1=\tilde{Y}_2=0 \tag{44}$$

We deduce for the forced system the existence of a periodic motion of period $2\pi/\omega$ for which the conditions at the end of the shock are $z_c=z_0$, $\dot{z}_c=-\dot{z}_0$. The time duration τ of the shock and the initial conditions (y,u,w) are obtained from the first part of system (44):

$$\dot{z}_0=Q\omega\sin\tilde{\varphi}_0-H_2^{-1}(H_1-I)(z_0-d_0-Q\cos\tilde{\varphi}_0)$$

$$[\Gamma_2^{-1}(\Gamma_1-I)+H_2^{-1}(H_1-I)]z_0=(Q-R)\omega\sin\tilde{\varphi}_0+\Gamma_2^{-1}(\Gamma_1-I)R\cos\tilde{\varphi}_0+H_2^{-1}(H_1-I)(d_0+Q\cos\tilde{\varphi}_0) \tag{45}$$

These formulas give the relations (28) when $Q=R=0$ (unforced system).

5 Stability of Periodic Motions (Rigid Impact)

5.1 Unforced System. Let us consider a periodic motion of period T related to initial conditions $z_{00}=\binom{1}{y_0}$, $\dot{z}_{00}=\binom{u_0}{0}$, where z_{00}, $\dot{z}_{00}$ are defined by:

$$(\Gamma_1-I)z_{00}-\Gamma_2\dot{z}_{00}=0\quad\Gamma_i=\Gamma_i(T)$$

$$-\Gamma_3z_{00}+(\Gamma_1-E)\dot{z}_{00}=0 \tag{46}$$

Let us consider the perturbed motion defined by a set of new initial conditions

$$\begin{matrix}z_0=z_{00}+dz_0\\ \dot{z}_0=\dot{z}_{00}+d\dot{z}_0\end{matrix}\quad\text{where } dz_0=\begin{pmatrix}0\\ y\end{pmatrix},\quad d\dot{z}_0=\begin{pmatrix}u\\ w\end{pmatrix} \tag{47}$$

This motion is defined for t>0, by:

$$z=\Gamma_1(t)z_0+\Gamma_2(t)E\dot{z}_0$$

$$\dot{z}=\Gamma_3(t)z_0+\Gamma_1(t)E\dot{z}_0$$

This motion ends at $t=T+dT$, when $z_1(T+dT)=1$ and $\dot{z}_1(T+dT)>0$. Let us denote by z_f, $\dot{z}_f$ the positions and the velocities of the system at this time.

$$z_f=\Gamma_1(T+dT)z_0+\Gamma_2(T+dT)E\dot{z}_0$$

$$\dot{z}_f=\Gamma_3(T+dT)z_0+\Gamma_1(T+dT)E\dot{z}_0$$

Assuming small perturbations dz_0, $d\dot{z}_0$ of the initial conditions,

$$dz_f=z_f-z_{00}=\Gamma_1dz_0+\Gamma_2Ed\dot{z}_0+L_1dT$$

$$d\dot{z}_f=\dot{z}_f-\dot{z}_{00}=\Gamma_3dz_0+\Gamma_1Ed\dot{z}_0+L_2dT \tag{48}$$

$$L_1=\dot{\Gamma}_1z_{00}+\dot{\Gamma}_2E\dot{z}_{00}$$

$$L_2 = \dot{\Gamma}_3 z_{00} + \dot{\Gamma}_1 E\dot{z}_{00}, \quad dz_f = \begin{pmatrix} 0 \\ y_f \end{pmatrix}, \quad d\dot{z}_f = \begin{pmatrix} u_f \\ w_f \end{pmatrix}$$

$$\dot{\Gamma}_i = \dot{\Gamma}_i(T) \tag{49}$$

The relations $\dot{\Gamma}_1 = \Gamma_3$, $\dot{\Gamma}_2 = \Gamma_1$, $\dot{\Gamma}_3 = \Gamma_2^{-1}\Gamma_1\Gamma_3$, $E\dot{z}_{00} = -\dot{z}_{00}$ and (46) leads to the relations:

$$L_1 = \dot{z}_{00}, \quad L_2 = \Gamma_2^{-1}(\Gamma_1 + I)\dot{z}_{00}$$

From (48), after the elimination of dT, we deduce:

$$y_f = C_{22}y - C_{23}u + C_{24}w$$

$$C_{12}y_f - C_{13}u_f - C_{14}w_f = C_{12}y - C_{13}u + C_{14}w$$

$$(C_{22}+1)y_f - C_{23}u_f - C_{24}w_f = (C_{22}+1)y - C_{23}u + C_{24}w \tag{50}$$

or $AX_f = BX$, $X_f = {}^t(y_f, u_f, w_f)$, $X = {}^t(y, u, w)$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ C_{22} & -C_{23} & -C_{24} \\ C_{12} & -C_{13} & -C_{14} \end{pmatrix}, \quad B = \begin{pmatrix} C_{22} & -C_{23} & C_{24} \\ 1 & 0 & 0 \\ C_{12} & -C_{13} & C_{14} \end{pmatrix}$$

The stability of the periodic impact solution is determined by the eigenvalues of the matrix $A^{-1}B$. If all the eigenvalues are inside the unit circle, the periodic solution is stable. If one of them is outside the unit circle, the solution is unstable. Critical cases occur if some eigenvalues lie on the unit circle, the other ones being strictly inside this circle.

Let us introduce the characteristic polynomial $P(\lambda)$ of the matrix $A^{-1}B$:

$$P(\lambda) \equiv \det(A^{-1}B - \lambda I_3) = -(\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)$$

(I_3 unitarian matrix of order 3)

$$P(\lambda) = 0 \text{ is equivalent to } D(\lambda) = 0, \quad D(\lambda) \equiv \det(B - \lambda A)$$

From the properties: $D(1) \equiv \det(B-A) = 0$, $\det(A) = \det(B)$, we deduce that one eigenvalue of $A^{-1}B$ is 1 and $b_0 \equiv -\det(A^{-1}B) = -1$. The characteristic polynomial of $A^{-1}B$ takes the following form:

$$P(\lambda) = (1-\lambda)[\lambda^2 + (1+b_2)\lambda + 1] \tag{51}$$

It results that when $\delta = (b_2-1)(b_2+3)$ is positive, the two other eigenvalues of $A^{-1}B$ are real leading to the instability of the periodic solution. For $\delta < 0$, $A^{-1}B$ has a complex conjugate pair of eigenvalues on the unit circle.

5.2 Forced System. A periodic solution of period $2\pi/\omega$ is obtained for initial conditions $z_{00} = \binom{1}{y_0}$, $\dot{z}_{00} = \binom{u_0}{0}$, and phase angle $\varphi_0 = 0$, (y_0, u_0) defined by (34).

Let us consider a perturbed motion related to initial conditions (47) and phase angle $\varphi = \varphi_0 + d\varphi$. The corresponding free motion performed by the system for $t > 0$, is obtained from (29), with $\dot{z}_c = E\dot{z}_0$. This motion ends at $t = 2\pi/\omega + dT$, when $z_1(2\pi/\omega + dT) = 1$ and $\dot{z}_1(2\pi/\omega + dT) > 0$. Let us denote by z_f, $\dot{z}_f$ the positions and the velocities of the system at this time. Assuming small perturbations of the initial conditions and of the phase angle:

$$dz_f = z_f - z_{00} = \Gamma_1 dz_0 + \Gamma_2(Ed\dot{z}_0 + R\omega d\varphi) + n_1 dT$$

$$d\dot{z}_f = \dot{z}_f - \dot{z}_{00} = \Gamma_3 dz_0 + \Gamma_1(Ed\dot{z}_0 + R\omega d\varphi) - R\omega d\varphi' + n_2 dT$$

$$d\varphi' = \omega dT + d\varphi$$

$$n_1 = \dot{z}_{00} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \quad n_2 = \Gamma_2^{-1}(\Gamma_1 + I)n_1, \quad \Gamma_j = \Gamma_j(2\pi/\omega) \quad (j = 1,2,3) \tag{52}$$

From (52), after the elimination of dT, we deduce the matrix N_0 giving the linear correspondence between the initial perturbations $Y = (y, u, w, d\varphi)^t$ and the final ones $Y_f = (y_f, u_f, w_f, d\varphi')^t$:

$$Y_f = N_0 Y, \quad N_0 = \tilde{A}^{-1}\tilde{B}$$

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ C_{12} & -C_{13} & -C_{14} & -\omega l_1 \\ C_{22} & -C_{23} & -C_{24} & -\omega l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} C_{22} & -C_{23} & C_{24} & \omega l_2 \\ C_{12} & -C_{13} & C_{14} & \omega l_1 \\ 1 & 0 & 0 & 0 \\ -\chi C_{12} & \chi C_{13} & -\chi C_{14} & 1-\omega\chi l_1 \end{pmatrix}$$

$$l_1 = C_{13}R_1 + C_{14}R_2, \quad l_2 = C_{23}R_1 + C_{24}R_2, \quad \chi = \frac{\omega}{u_0} \tag{53}$$

Let us introduce the characteristic polynomial of the matrix N_0:

$$\tilde{P}(\lambda) \equiv \det(N_0 - \lambda I_4) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{54}$$

From the property $\det(\tilde{A}) = \det(\tilde{B}) = \det(\Gamma_2)$, we deduce

$$a_0 = \tilde{P}(0) = \det(\tilde{A}^{-1}\tilde{B}) = 1$$

In this case it is impossible that all the eigenvalues of the matrix N_0 lie strictly inside the unit circle. The periodic solution is unstable except if all these eigenvalues lie on the unit circle. Moreover, for any λ (see Appendix B) $\det(\tilde{A} - \lambda\tilde{B}) \equiv \det(\tilde{B} - \lambda\tilde{A})$.

It results that $\tilde{P}(\lambda) \equiv \lambda^4 \tilde{P}(1/\lambda)$ and hence $a_1 = a_3$. The conditions under which all the eigenvalues of N_0 lie on the unit circle are (Appendix B):

$$|a_3| \leqslant 4, \quad a_2 - 2a_3 + 2 \geqslant 0, \quad a_2 + 2a_3 + 2 \geqslant 0,$$
$$a_3^2 - 4a_2 + 8 \geqslant 0 \tag{55}$$

In the more general case of a restitution coefficient r $(0 < r < 1)$, the stability conditions of periodic solutions have been obtained in Ref. [10].

6 Stability of Periodic Motions (Soft Impact)

6.1 Unforced System. When the stiffness of the obstacle is bounded and when there is no external excitation, the mathematical model of the system is given by (12) for the free motion and (13) for the constraint motion.

Let us consider a periodic motion of period $\tau_0 + T_0 = 2\pi/\omega$, where ω is an arbitrary positive value in this case. This periodic solution is related to the initial conditions $z_{00} = \binom{1}{y_0}$, $\dot{z}_{00} = \binom{u_0}{w_0}$, where $(y_0, u_0, w_0, \tau_0, T_0)$ are defined in terms of ω by (28) and the condition $\tau_0 + T_0 = 2\pi/\omega$.

Let us consider the perturbed motion defined by a set of new initial conditions (47).

This motion is defined in two steps:

- For $0 \leqslant t \leqslant \tau = \tau_0 + d\tau$, the system performs a constraint motion ending when $z_1(\tau) = 1$ and $\dot{z}_1(\tau) < 0$. Let us denote by

$$z_c = z_{00} + dz_c, \quad \dot{z}_c = -\dot{z}_{00} + d\dot{z}_c,$$
$$dz_c = \begin{pmatrix} 0 \\ y_c \end{pmatrix}, \quad d\dot{z}_c = \begin{pmatrix} u_c \\ w_c \end{pmatrix} \tag{56}$$

the positions and the velocities reached by the system at this time.

– For $\tau \leqslant t \leqslant 2\pi/\omega + d\theta$, the system performs a free motion finishing for $z_1(2\pi/\omega+d\theta)=1$, $\dot{z}_1(2\pi/\omega+d\theta)>0$. Let us denote by $z_f=z_{00}+dz_f$, $\dot{z}_f=\dot{z}_{00}+d\dot{z}_f$ the positions and the velocities reached by the system at this time.

Assuming small perturbations dz_0, $d\dot{z}_0$ of the initial conditions,

$$\begin{aligned} dz_c &= H_1 dz_0 + H_2 d\dot{z}_0 + p_1 d\tau \\ d\dot{z}_c &= H_3 dz_0 + H_1 d\dot{z}_0 + p_2 d\tau \end{aligned} \quad (H_i = H_i(\tau_0), i=1,2,3) \tag{57}$$

$$\begin{aligned} p_1 &= \dot{H}_1(z_{00}-d_0) + \dot{H}_2 \dot{z}_{00} = -\dot{z}_{00} \\ p_2 &= \dot{H}_3(z_{00}-d_0) + \dot{H}_1 \dot{z}_{00} = -H_2^{-1}(H_1+I)\dot{z}_{00} \end{aligned} \quad (\dot{H}_i = \dot{H}_i(\tau_0), i=1,2,3) \tag{58}$$

In a same way

$$dz_f = \Gamma_1 dz_c + \Gamma_2 d\dot{z}_c + p_3 d\theta, \quad p_3 = \dot{\Gamma}_1 z_{00} - \dot{\Gamma}_2 \dot{z}_{00} = \dot{z}_{00}$$

$$d\dot{z}_f = \Gamma_3 dz_c + \Gamma_1 d\dot{z}_c + p_4 d\theta, \quad p_4 = \dot{\Gamma}_3 z_{00} - \dot{\Gamma}_1 \dot{z}_{00} = \Gamma_2^{-1}(\Gamma_1+I)\dot{z}_{00}$$

$$(\Gamma_i = \Gamma_i(T_0), \dot{\Gamma}_i = \dot{\Gamma}_i(T_0), i=1,2,3) \tag{59}$$

From (57) and (59), after the elimination of $d\tau$ and $d\theta$, we deduce the correspondence between the initial perturbations $(y\ u\ w)^t$ and the final ones $(y_f u_f w_f)^t$:

$$\begin{pmatrix} y_f \\ u_f \\ w_f \end{pmatrix} = A_3 \begin{pmatrix} y \\ u \\ w \end{pmatrix}, \quad A_3 = \bar{A}_2 \bar{A}_1, \quad \bar{A}_i = M_i^{-1} N_i, \quad (i=1,2) \tag{60}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ H_{12} & -H_{13} & -H_{14} \\ H_{22} & -H_{23} & -H_{24} \end{pmatrix},$$

$$N_1 = \begin{pmatrix} H_{22} - \tilde{\chi} H_{12} & H_{23} - \tilde{\chi} H_{13} & H_{24} - \tilde{\chi} H_{14} \\ H_{12} & H_{13} & H_{14} \\ 1 + \tilde{\chi} H_{12} & \tilde{\chi} H_{13} & \tilde{\chi} H_{14} \end{pmatrix}, \quad \tilde{\chi} = \frac{w_0}{u_0} \tag{61}$$

M_2 and N_2 are deduced, respectively, from the expression of M_1 and N_1 by substituting the terms H_{ij} by the terms C_{ij}. Let us introduce the characteristic polynomial $\bar{P}(\lambda)$ of the matrix A_3. From the property:

$$\det(M_i) = -\det(N_i), \quad (i=1,2) \tag{62}$$

we deduce that $\bar{P}(0)=\det(A_3)=1$. It is not possible in this case that all the eigenvalues of A_3 lie strictly inside the unit circle.

6.2 Forced System. Let us consider a periodic solution of system (35), of period $\tau_0+T_0=2\pi/\omega$, with $\varphi=-\tilde{\varphi}_0=-\omega\tau_0/2$, related to the initial conditions $z_{00}=\binom{1}{y_0}$, $\dot{z}_{00}=\binom{u_0}{w_0}$ where (τ_0, y_0, u_0, w_0) are deduced from the system (45).

The stability of this periodic solution is investigated by considering the motion related to the new initial conditions (47) and new phase angle $\varphi=-\omega\tau_0/2+d\varphi$.

This motion is defined in two steps:

– For $0 \leqslant t \leqslant \tau = \tau_0 + d\tau$: z, $\dot{z}$ are defined by (36). This motion ends when $z_1(\tau)=1$ and $\dot{z}_1(\tau)<0$. The positions and the velocities reached by the system at this final time are defined by (56).
– For $\tau \leqslant t \leqslant 2\pi/\omega + d\theta$, the motion is defined by (38). This motion ends when $z_1(2\pi/\omega+d\theta)=1$, $\dot{z}_1(2\pi/\omega+d\theta)>0$. Let us denote by $z_f=z_{00}+dz_f$, $\dot{z}_f=\dot{z}_{00}+d\dot{z}_f$, the positions and the velocities reached by the system at this time.

Assuming small perturbations dz_0, $d\dot{z}_0$ of the initial conditions,

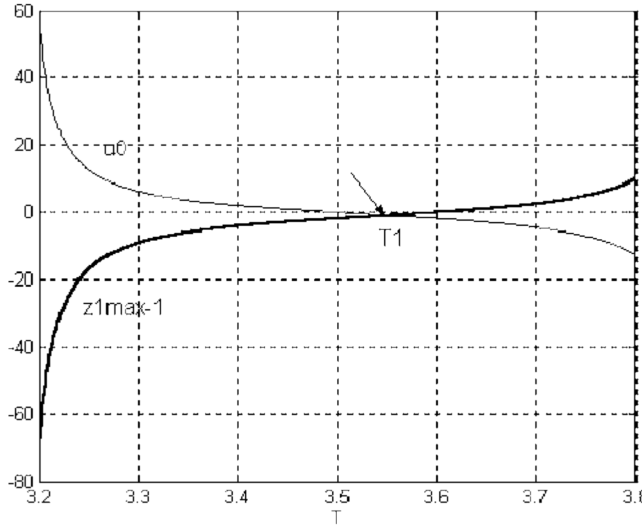

Fig. 2 Actual periodic impact solution (rigid impact, unforced system)

$$\begin{aligned} dz_c &= H_1 dz_0 + H_2 d\dot{z}_0 + \tilde{p}_1 d\tau + q_1 d\varphi \\ d\dot{z}_c &= H_3 dz_0 + H_1 d\dot{z}_0 + \tilde{p}_2 d\tau + q_2 d\varphi \end{aligned} \quad (H_i = H_i(\tau_0), i=1,2,3) \tag{63}$$

$$\tilde{p}_1 = -\dot{z}_{00}, \quad \tilde{p}_2 = -H_2^{-1}(H_1+I)(\dot{z}_{00} - Q\omega \sin \tilde{\varphi}_0) - Q\omega^2 \cos \tilde{\varphi}_0$$

$$\begin{aligned} q_1 &= -(H_1+I)Q \sin \tilde{\varphi}_0 + H_2 Q\omega \cos \tilde{\varphi}_0 \\ q_2 &= H_2^{-1}(H_1 - I) q_1 \end{aligned} \quad (q_1 = (q_{11}, q_{12})^t)$$

In a same way

$$\begin{aligned} dz_f &= \Gamma_1 dz_c + \Gamma_2 d\dot{z}_c + \tilde{p}_3 dT + q_3 d\varphi' \\ d\dot{z}_f &= \Gamma_3 dz_c + \Gamma_1 d\dot{z}_c + \tilde{p}_4 dT + q_4 d\varphi' \end{aligned} \quad (\Gamma_i = \Gamma_i(T_0), i=1,2,3) \tag{64}$$

$$dT = d\theta - d\tau, \quad d\varphi' = \omega d\theta + d\varphi$$

$$\begin{aligned} q_3 &= (\Gamma_1 + I) R \sin \tilde{\varphi}_0 + \Gamma_2 R\omega \cos \tilde{\varphi}_0 \\ q_4 &= \Gamma_2^{-1}(\Gamma_1 - I) q_3 \end{aligned} \quad (q_3 = (q_{31}, q_{32})^t)$$

$$\tilde{p}_3 = \dot{z}_{00} - \omega q_3$$

$$\tilde{p}_4 = \Gamma_2^{-1}(\Gamma_1 + I)(\dot{z}_{00} - R\omega \sin \tilde{\varphi}_0) - \omega \Gamma_2^{-1}(\Gamma_1 - I) q_3 - R\omega^2 \cos \tilde{\varphi}_0$$

$$(\tilde{p}_3 = (\tilde{p}_{31}, \tilde{p}_{32})^t)$$

From (63) and (64), after the elimination of $d\tau$ and dT, we deduce the matrix A_3 (see Appendix C) giving the correspondence between the initial perturbations and the final ones:

$$\begin{pmatrix} y_f \\ u_f \\ w_f \\ d\varphi' \end{pmatrix} = \tilde{A}_3 \begin{pmatrix} y \\ u \\ w \\ d\varphi \end{pmatrix} \quad \tilde{A}_3 = \begin{pmatrix} \tilde{A}_2 & \tilde{N}_2 \\ \tilde{M}_3 & \tilde{N}_3 \end{pmatrix} \tag{65}$$

The stability of the periodic motion is determined by the eigenvalues of the matrix $\tilde{A}_3$.

7 Numerical Results

Some numerical investigations are performed for the following values of the parameters: $k_1=k_3=1$, $k_2=5$, $m_1=1$, $m_2=2$, $P_1=2/3$, $P_2=0$. The corresponding eigenvalues of the free system are: $\omega_1=1.7958$, $\omega_2=0.8805$, while the eigenvalues of the constraint system are: $\sigma_1=1.8347$, $\sigma_2=1.2783$. An example of peri-

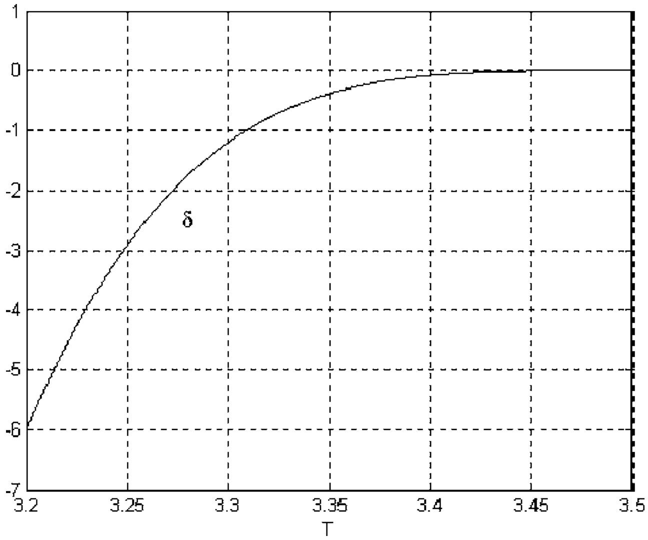

Fig. 3 Stability of the periodic solution (rigid impact, unforced system)

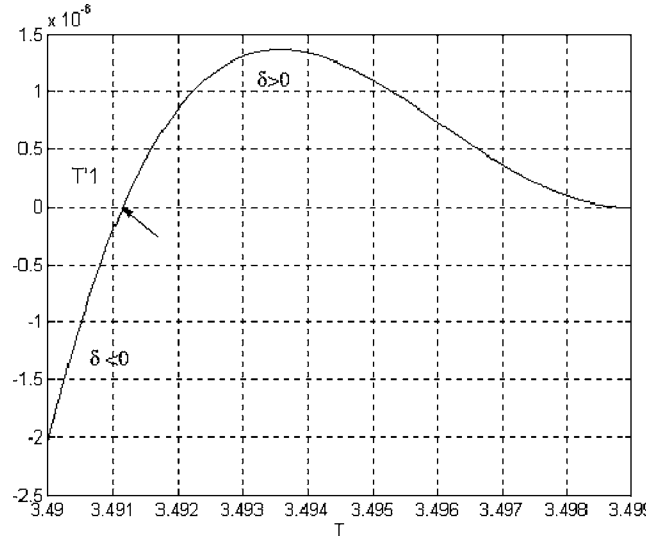

Fig. 4 Bifurcation value of the period (rigid impact, unforced system)

odic solutions obtained for forced and unforced systems is shown in Ref. [10] in the rigid impact and soft impact cases. Stability conditions are investigated in the case of rigid impact, for both unforced and forced systems. In both cases, the initial conditions (10) (unforced system) and (34) (forced system) are related to an actual periodic solution of period T only if $u_0>0$, $z_1(t)<1$, $0<t<T$. The initial velocity of the impacting mass is reformulated as:

$$u_{0\mathrm{NF}}=\frac{(\lambda_1-\lambda_2)x_1x_2}{\widetilde{D}}, \quad \widetilde{D}=\lambda_2x_2y_1-\lambda_1x_1y_2$$

$$\text{(unforced system)} \tag{66}$$

$$x_i=\omega_i\sin(\omega_iT/2), \quad y_i=\cos(\omega_iT/2)$$

$$u_{0\mathrm{F}}=(1-R_1)u_{0\mathrm{NF}}, \quad T=2\pi/\omega \text{ (forced system)} \tag{67}$$

In both systems, limiting cases are obtained for $x_i=0$, $(i=1, 2)$ or $T_i=2k\pi/\omega_i$, $(k=1,2,\ldots)$. On the other hand, $u_{0\mathrm{NF}}$ and $u_{0\mathrm{F}}$ are not defined if $\widetilde{D}=0$. The corresponding motion of the impacting mass is given by:

$$z_{1\mathrm{NF}}(t)=\frac{\lambda_2x_2\cos\omega_1(t-T/2)-\lambda_1x_1\cos\omega_2(t-T/2)}{\widetilde{D}} \text{ (unforced system)}$$

$$z_{1\mathrm{F}}(t)=R_1\cos\omega t+(1-R_1)z_{1\mathrm{NF}}(t), \quad T=2\pi/\omega \text{ (forced system)} \tag{68}$$

The motion of the impacting mass is symmetrical with respect to $T/2$ (π/ω for the forced system) and the maximum value of $z_1(t)$ is obtained at this point. For the unforced system, the behavior of u_0,$z_{1\,\max}-1$ and δ (stability condition) are investigated in terms of the period T. Figure 2 is related to the interval [3.2,3.8] which contains the value $T_1=2\pi/\omega_1\simeq 3.4989$ for which $u_0=0$. Figure 3 shows the behavior of δ for $3.2<T<T_1$. Figure 4 is a zoom of the last figure in the interval $[3.49,T_1]$ which includes a bifurcation value $T_1'\simeq 3.4911$ of the period for which a change of the stability occurs. (For this value, two eigenvalues of the matrix $A^{-1}B$ are equal to -1.) For the forced system, the behavior of u_0,$z_{1\,\max}-1$ (Fig. 5) and the stability conditions (Fig. 6) depends on the value of ω or equivalently on the value of $T=2\pi/\omega$. Figure 5 is related to the interval $5<T<6$ ($1.0472<\omega<1.2566$) while in Fig. 6, the functions

$$n_1'=16-a_3^2, \quad n_2'=a_2-2a_3+2,$$
$$n_3'=a_2+2a_3+2, \quad n_4'=a_3^2-4a_2+8$$

occurring in the stability conditions (55) are plotted in this interval.

Three bifurcation values ($T_2'\simeq 5.0809$, $T_3'\simeq 5.2002$, and $T_4'\simeq 5.8109$) appear in this interval: The two first values are related to the case of two eigenvalues of the matrix N_0 equal to -1 while the last value is related to the case of complex eigenvalues $\widetilde{\lambda}_k$ $(k=1,\ldots,4)$ on the unit circle, with $\widetilde{\lambda}_1=\widetilde{\lambda}_2=1/\widetilde{\lambda}_3=1/\widetilde{\lambda}_4$. Stability and bifurcation conditions in the case of soft impact will be the subject of further numerical investigations.

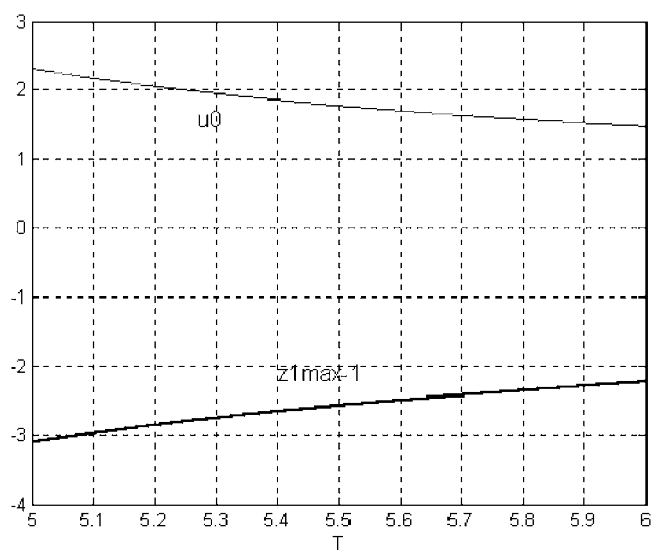

Fig. 5 Actual periodic solution (rigid impact, forced system)

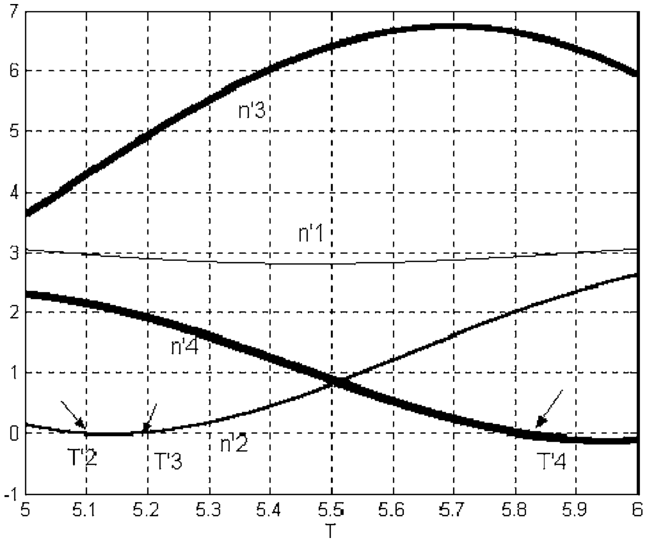

Fig. 6 Bifurcation values of the period $T=2\pi/\omega$ (rigid impact, forced system)

Appendix A: Discussion About the Existence of Periodic Motions (Soft Impact, Unforced System)

The second solution arising from conditions (25) is defined by:

$$PX_1 = 0, \quad \det(P) = 0, \quad X_1 = X_2, \quad z_{1c} = 1$$

$$P = (P_{ij}) = [H_2^{-1}(H_1 + I) + \Gamma_2^{-1}(\Gamma_1 + I)] \tag{A1}$$

From $PX_1=0$ and $X_1=\binom{X_{11}}{X_{12}}$ where $X_{11}=0$ and $X_{12}\neq 0$, we deduce:

$$P_{12} = 0, \quad P_{22} = 0 \tag{A2}$$

These two equations give discrete values of τ and T independent of the initial conditions. y, u, w are defined by three scalar equations deduced from the conditions:

$$X_1 = X_2, \quad X_{21} = 0$$

$$X_2 = \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix} \tag{A3}$$

Let us introduce

$$P_1 = \begin{pmatrix} \alpha_1 & a \\ \alpha_2 & b \end{pmatrix}, \quad P_2 = \begin{pmatrix} \beta_1 & a \\ \beta_2 & b \end{pmatrix} \tag{A4}$$

where $(\alpha_1,\alpha_2,\beta_1,\beta_2,a,b)$ are deduced from the definition (23) of P_1 and P_2 and the property (A2). From (A3), we deduce

$$(H_1 - \Gamma_1)z_0 + (H_2 + \Gamma_2)\dot{z}_0 = (H_1 - I)d_0 \tag{A5}$$

$$-yC_{12} + uC_{13} + wC_{14} = C_{11} + 1 \tag{A6}$$

From $H_1+I=H_2P_1$, $\Gamma_1-I=\Gamma_2P_2$, (A5) gives:

$$H_2\xi + \Gamma_2\eta = (H_1 - I)d_0 \tag{A7}$$

$$\xi = P_1z_0 + \dot{z}_0 = \begin{pmatrix} \alpha_1 + p \\ \alpha_2 + q \end{pmatrix},$$

$$\eta = P_2z_0 + \dot{z}_0 = \begin{pmatrix} \beta_1 + p \\ \beta_2 + q \end{pmatrix}, \quad \begin{matrix} p = u + ay \\ q = w + by \end{matrix}$$

From (A7) it is possible to deduce $Q=\binom{p}{q}$:

$$Q = (H_2 + \Gamma_2)^{-1}[(H_1 - I)d_0 - H_2\xi_0 - \Gamma_2\eta_0], \quad \xi_0 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

$$\eta_0 = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{A8}$$

The condition (A6) can also be expressed in terms of p and q:

$$C_{13}p + C_{14}q = C_{11} - 1 \tag{A9}$$

Equations (A8) and (A9) provide 3 scalar equations for the determination of the two parameters p and q. The condition of compatibility for this system is given by:

$$\det\begin{pmatrix} C_{13} & C_{14} & C_{11} - 1 \\ H_{13} & H_{14} & \tilde{L}_1 \\ H_{23} + C_{23} & H_{24} + C_{24} & \tilde{L}_2 \end{pmatrix} = 0$$

$$\tilde{L}_1 = d_2H_{12} + (1 - H_{11})(1 - d_1)$$

$$\tilde{L}_2 = (d_1 - 1)H_{21} + d_2(H_{22} - 1) + C_{21}$$

This compatibility condition is not fulfilled in the general case and the problem has no solution.

Appendix B: Properties of the N_o Matrix (Rigid Impact, Forced System)

$$\det(\tilde{B} - \lambda\tilde{A}) = \begin{vmatrix} C_{22} - \lambda & -C_{23} & C_{24} & \omega l_2 \\ C_{12}(1 - \lambda) & C_{13}(\lambda - 1) & C_{14}(\lambda + 1) & \omega l_1(\lambda + 1) \\ 1 - \lambda C_{22} & \lambda C_{23} & \lambda C_{24} & \omega l_2\lambda \\ -\tilde{\chi}C_{12} & \tilde{\chi}C_{13} & -\tilde{\chi}C_{14} & 1 - \lambda - \omega l_1\tilde{\chi} \end{vmatrix} \tag{B1}$$

$$\det(\tilde{A} - \lambda\tilde{B}) = \begin{vmatrix} 1 - \lambda C_{22} & \lambda C_{23} & -\lambda C_{24} & -\lambda\omega l_2 \\ C_{12}(1 - \lambda) & C_{13}(\lambda - 1) & -C_{14}(\lambda + 1) & -\omega l_1(\lambda + 1) \\ -\lambda + C_{22} & -C_{23} & -C_{24} & -\omega l_2 \\ \lambda\tilde{\chi}C_{12} & -\lambda\tilde{\chi}C_{13} & \lambda\tilde{\chi}C_{14} & 1 - \lambda + \omega l_1\tilde{\chi}\lambda \end{vmatrix} \tag{B2}$$

or,

$$\det(\tilde{A} - \lambda\tilde{B}) = \begin{vmatrix} 1 - \lambda C_{22} & \lambda C_{23} & -\lambda C_{24} & -\lambda\omega l_2 \\ C_{12}(1 - \lambda) & C_{13}(\lambda - 1) & -C_{14}(\lambda + 1) & -\omega l_1(\lambda + 1) \\ -\lambda + C_{22} & -C_{23} & -C_{24} & -\omega l_2 \\ \tilde{\chi}C_{12} & -\tilde{\chi}C_{13} & -\tilde{\chi}C_{14} & 1 - \lambda + \omega l_1\tilde{\chi} \end{vmatrix} \tag{B3}$$

It is not difficult to show, after some permutations of rows and columns in (B3), that $\det(\tilde{A}-\lambda\tilde{B})\equiv\det(\tilde{B}-\lambda\tilde{A})$. It results that $\tilde{P}(\lambda)\equiv\lambda^2Q(\tilde{\sigma})$, $Q(\tilde{\sigma})\equiv\tilde{\sigma}^2+a_3\tilde{\sigma}+a_2-2$, $\tilde{\sigma}=\lambda+1/\lambda$. Let us assume that the roots $\tilde{\lambda}_k$ (k=1,..,4) of $\tilde{P}(\lambda)$ are on the unit circle: $\tilde{\lambda}_1=e^{i\theta_1}$, $\tilde{\lambda}_2=e^{i\theta_2}$, $\tilde{\lambda}_3=1/\tilde{\lambda}_1$, $\tilde{\lambda}_4=1/\tilde{\lambda}_2$. The roots of $Q(\tilde{\sigma})$ in this case are $\tilde{\sigma}_j=2\cos\theta_j$, $(j=1, 2)$. We deduce that all the eigenvalues of the matrix N_0 are on the unit circle if the roots of $Q(\tilde{\sigma})$ are real and between −2 and 2. From this, we deduce the conditions (55).

Appendix C: Obtention of the $\tilde{A}_3$ Matrix (Soft Impact, Forced System)

$$\tilde{A}_2 = (\tilde{C} - \bar{p}\bar{M}_2)\tilde{A}_1 + (\omega\tilde{q}\tilde{\chi}_2 - \bar{\chi}_2\tilde{p})\bar{M}_1,$$

$$\tilde{N}_2 = (\tilde{C} - \bar{p}\bar{M}_2)\tilde{N}_1 + \tilde{\chi}_1\tilde{\chi}_2\tilde{q} - \chi_3\tilde{p}$$

$$\tilde{M}_3 = \omega(\bar{M}_1 - \bar{M}_2\tilde{A}_1), \quad \tilde{N}_3 = 1 + \omega(\chi_1 - \chi_3 - \bar{M}_2\tilde{N}_1)$$

$$p = \begin{pmatrix} -w_0 \\ \tilde{p}_2 \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} \tilde{p}_{32} \\ \tilde{p}_4 \end{pmatrix}, \quad q = \begin{pmatrix} q_{12} \\ q_2 \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} q_{32} \\ q_4 \end{pmatrix}, \quad \bar{p} = \tilde{p} + \omega\tilde{q}$$

$$\chi_1 = \frac{q_{11}}{u_0}, \quad \tilde{\chi}_1 = 1 + \omega\chi_1, \quad \chi_2 = \tilde{p}_{31} + \omega q_{31}, \quad \bar{\chi}_2 = \omega\frac{q_{31}}{\chi_2},$$

$$\tilde{\chi}_2 = 1 - \bar{\chi}_2, \quad \chi_3 = \frac{\bar{\chi}_2\tilde{\chi}_1}{\omega}$$

$$\tilde{H} = \begin{pmatrix} H_{22}H_{23}H_{24} \\ H_{32}H_{11}H_{12} \\ H_{42}H_{21}H_{22} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C_{22}C_{23}C_{24} \\ C_{32}C_{11}C_{12} \\ C_{42}C_{21}C_{22} \end{pmatrix},$$

$$\bar{M}_1 = \frac{1}{u_0}(H_{12}H_{13}H_{14}), \quad \bar{M}_2 = \frac{1}{\chi_2}(C_{12}C_{13}C_{14})$$

$$\tilde{A}_1 = \tilde{H} + p\bar{M}_1, \quad \tilde{N}_1 = q + \chi_1 p \qquad \text{(C1)}$$

References

[1] Shaw, S. W., and Holmes, P. J., 1983, "A Periodically Forced Piecewise Linear Oscillator," J. Sound Vib., **90**(1), pp. 129–155.

[2] Shaw, S. W., and Holmes, P. J., 1983, "A Periodically Forced Impact Oscillator with Large Dissipation," J. Appl. Mech., **50**, pp. 849–857.

[3] Hindmarsh, M. B., and Jeffries, D. J., 1984, "On the Motions of the Impact Oscillator," J. Phys. A, **17**, pp. 1791–1803.

[4] Peterka, F., 2001, "Dynamics of Oscillator with Soft Impacts," *Proc. of the ASME 2001 Design Engineering Technical Conferences* (CDROM), Pittsburgh, USA.

[5] Aidanpan, J. O, and Gupta, R. D., 1993, "Periodic and Chaotic Behavior of a Threshold-Limited Two Degree of Freedom System," J. Sound Vib., **165**(2), pp. 305–327.

[6] Luo, G. W., and Xie, J. H, 1998, "Hopf Bifurcation of a Two Degree of Freedom Vibro-Impact System," J. Sound Vib., **213**(3), pp. 391–408.

[7] Valente, A. X., McClamroch, N. H., and Mezie, I., 2003, "Hybrid Impact of Two Coupled Oscillators that can Impact a Fixed Stop," Int. J. Non-Linear Mech., **38**, pp. 677–689.

[8] Natsiavas, S., 1993, "Dynamics of Multiple-Degree-of-Freedom Oscillators with Colliding Components," J. Sound Vib., **165**(3), pp. 439–453.

[9] Pascal, M., Stepanov, S., and Hassan, S., 2006, "An Analytical Investigation of the Periodic Motions of a Two Degree of Freedom Oscillator with Elastic Obstacles," J. Comput. Methods Sci. Eng. (to be published).

[10] Pascal, M., "Analytical Investigation of the Dynamics of a Non Linear Structure with Two Degrees of Freedom," submitted to ASME 2005 IDETC/CIE, September 2005, Long Beach, CA.